



Strategic Resource Pricing and Allocation in a 5G Network Slicing Stackelberg Game

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they contribute to the infrastructure utilization. The prices are also locational, being differentiated by cell and resource. Relying on a service market where the prices are automatically adjusted according to supply and demand, Lieto et al. proved that the dynamic trading of resources enables service providers to reduce their costs [4], and therefore maximize their profits.

Relying on network slicing, we consider a market design where a set of SPs lease their respective networks from InP and employ the network slicing mechanism to request the resources required for their service provision. We assume that the SPs offer a particular service to users, and the resources inventory available with SPs characterizes their service performance. The users are free to choose their SP. Their decisions are made based on the service satisfaction attained from SPs. Furthermore, the SPs collect revenue by providing the service to their customers. Assuming a dynamic resource sharing mechanism and that SPs are utility maximizing players, it is highly expected that selfish SPs may exhibit strategic behaviour. For example, they might strategically distribute their shares on the resources conditioned on the trade-off between the quality of service (QoS) they want to offer and the congestion perceived by the users. In this work, we focus on (1) building a game-theoretic model of the communication service market where SPs negotiate with InP to request resources and compete with one another to serve a pool of end-users, (2) developing a dynamic resource allocation and pricing mechanism under a competitive environment.

Related work: There is a large part of the literature dedicated to the design of communication service markets. Broadly, communication service markets have been studied as a two-stage noncooperative game involving three types of participants: Infrastructure provider (InP), Service provider (SP)¹ and End-users (EU). In the first level of the game, SPs (buyers) lease the resources from the InPs (sellers), negotiating for resource prices and resource quantity. In the second level, SPs (buyers) use the acquired resources from InPs to offer a certain service to their end-users (buyers). At this level, SPs decide on their service price and the scheduling of resources, while EUs make their subscription decisions. In [5], SPs' strategic decision over their service pricing scheme has been analyzed as a Cournot game. In [6], Korcak et al. considered that the quality of service (QoS) achieved by the SP's users depends on the number of subscribers of that SP, and users' choice behavior can be analyzed relying on evolutionary game theory (EGT). Li et al. in [7] integrated both the users' choice evolution and the SPs pricing scheme. They formulated the resulting problem as a Stackelberg game. The SPs, interpreted as leaders, strategically decide the price to attract the users, and the users, seen as followers choose the SPs to maximize their service satisfaction level. Also, the number of subscribers of the SPs depends on the perceived QoS and, consequently, on their resources availability. Focusing on competitive aspects, the SPs can act strategically when computing their resource demand, giving rise to a simultaneous noncooperative game [8]. In [9], D'Oro et al. used a similar Stackelberg game

formulation for resource allocation and orchestration in the network functions virtualization scenario. In [10], Azouzi et al. considered that the competition between SPs takes place both in prices and in the QoS. In practice, SPs may not have complete information about the other SPs resources. Dealing with such an incomplete information setting, Li et al. in [11] studied SPs' pricing strategies relying on a Bayesian game formulation, where SPs compute their prices based on their beliefs about the resource availability. Li et al. also considered the possibility that the SPs can coordinate and analyzed the impact of cooperation on the pricing scheme. In all the above works, the SPs lease the resources from the InP and compete to serve EUs, which is also the case in our work. However, our work innovates in that the resources are shared using a slice-based dynamic sharing mechanism. Moreover, in our case, resources are spatially distributed, and service offered in a particular cell can only be supported by the resources available within that cell. In communication networks, one of the well-known scheme for resource allocation is the auction-based allocation [12], e.g., Kelly's mechanism. Datar et al. in [13], as well as Tun et al. in [14] proposed multi-bidding Kelly's mechanism-based resource allocation for 5G slicing. They showed that Kelly's mechanism leads to a fair and efficient resource allocation both at slices and EUs levels. Our work departs from the auction-based mechanism like [13]–[14], where agents' bids are unbounded.

In follow up work to [3], Zheng et al. in [15] considered the network slicing under stochastic loads and applied SCPA based resource sharing scheme. They modeled the resource sharing scheme as a noncooperative game and proved that slices achieve efficient statistical multiplexing at the Nash equilibrium. Guijarro et al. in [16] designed a communication service market where SPs employ the SCPA mechanism to request the resources from InP. They analyzed the economic impact of network slicing on the market. In [17], an automated negotiation mechanism is defined relying on an aggregative game that enables the slice tenants to dynamically trade the radio resources and customize their slices on instantaneous demands, which help tenants achieve higher profits. Our paper is closely related to [16]. The main novelty of our work lies in the fact that we consider multi-resource service provisioning, contrary to most articles dealing with communication service market design, which, to the best of our knowledge, only deals with radio resources.

In our paper, we leveraged the Tullock contest (TC) framework [18] to model the competition between slices. This framework has been extensively used in the communication network literature, to model the interactions between competitive agents. To mention a few, in [19], the competition between social media users for visibility over the timeline was modeled as a TC. Luo et al. in [20] proposed a TC based incentive mechanism for crowdsourcing. The TC framework has been applied to the multipath TCP network utility maximization problem [21]. In [22], Altman et al. studied the multi-cryptocurrency blockchain from a game-theoretic perspective, where the competition between the miners is framed as a TC. To the best of our knowledge, the theoretical results on the TC framework and its applications in literature only deal with a

¹In many articles, the terms Mobile virtual network operator (MVNO), slice tenant, or Mobile service provider (MSP) are used without distinction to designate SP.

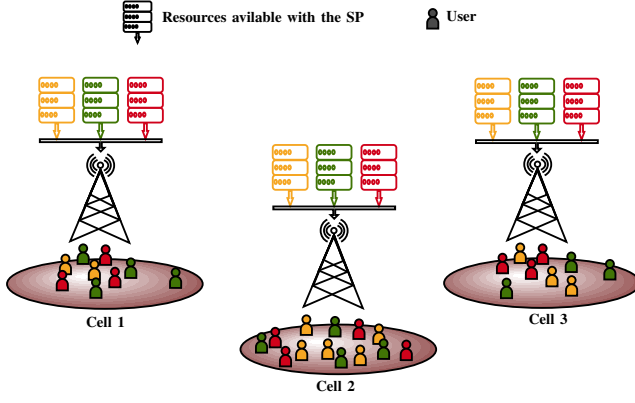


Fig. 1. service providers *i.e.*, (slices) compete to offer a certain service to geographically distributed pool of users

single resource case. We extend the TC framework to a multi-resource scenario, and thus our results also contribute to the theoretical literature on the TC framework.

We list below the key contributions of our article.

Main Contributions:

- 1) We propose a business model for the SPs, where the SPs deploy network slices for their business and lease their respective resources through network slicing mechanism (*i.e.*, dynamic sharing). The SPs compete with one another, with respect to their QoS, to serve end-users.
- 2) We model the interactions between the SPs and users as a Stackelberg game, where SPs act as leaders and users as followers.
- 3) We model the decision-making process among the users with replicator dynamics.
- 4) We show that the noncooperative game induced by the competition between SPs (that can be interpreted as a multi-resource Tullock rent-seeking game) admits a unique Nash equilibrium (NE). Thus, our theoretical results also contribute to the study of the Tullock rent-seeking game.
- 5) In a market design where the InP prices the resources at their optimal values, we implement resource pricing through two different approaches.
- 6) First, when service providers possess finite monetary budgets, we propose the implementation of pricing through a trading post mechanism.
- 7) Assuming a single resource is available in the above case, we show analytically that the game induced by the trading post mechanism admits a unique NE.
- 8) When budgets do not bind service providers for their demand, we prove that resource pricing can be implemented by solving a generalized Nash equilibrium problem, involving coupling constraints capturing the network finite capacity.
- 9) Finally, we provide two online learning algorithms and a fully distributed proximal based algorithm to reach NE and variational equilibrium solutions of the post trading mechanism and pricing game respectively.

The rest of the paper is organised as follows: Section II introduces the system model. Section III details the game-

theoretic model of competition between the SPs. In Section IV, we describe the Stackelberg game between SPs and EUs. In Section V, we analyse the existence and uniqueness of the NE. Section VI introduces the resource pricing and market equilibrium. In Section VII, we provide two (semi-decentralized) online learning algorithms and a fully distributed algorithm to compute market equilibria. In Section VII, we report on numerical results. A concluding section ends the paper.

TABLE I
MAIN NOTATIONS USED THROUGHOUT THE PAPER

$\mathcal{C} := \{1, \dots, C\}$	\triangleq	Set of base stations or cells
$\mathcal{S} := \{1, \dots, S\}$	\triangleq	Set of slices (tenants)
\mathcal{M}^c	\triangleq	Set of resources at base station c
N^c	\triangleq	Number of users in cell c
ν_s^c	\triangleq	Utility of user associated with SP s in cell c
q_c^s	\triangleq	Quality of service offered by SP s in cell c
n_s^c	\triangleq	Number (subscribers) users associated with slice s in cell c
$d_s^c := (d_{sm}^c)_{m \in \mathcal{M}^c}$	\triangleq	Bundle of resources available with slice s in cell c
d_{sm}^c	\triangleq	Amount of resource type m available with slice s in cell c
D_m^c	\triangleq	Capacity of resource type m at base station c
ω_m^c	\triangleq	Price per unit resource of type m at base station c
p_s	\triangleq	Service fees charge by slice s to users
R_s	\triangleq	SP s expected revenue
U_s	\triangleq	SP s profit
B_s	\triangleq	Budget available with slice s
K_s	\triangleq	Local constraint function
$\tau_{s,s'}^c(n^c, U^c)$	\triangleq	Revision protocol which defines the switching rate at which users in cell c switch their choice from SP s to SP s'
A_s	\triangleq	Slice association probability function
b_{sm}^c	\triangleq	Bid by SP s to resource m at cell c
λ_{sm}^c	\triangleq	Lagrange multipliers of the capacity based coupling constraints
r_s	\triangleq	Ratio of the coupling constraint dual variable at r-normalized Nash equilibrium (λ_{sm}^c) over dual variable evaluated by SP s (λ_{sm}^c)
y_s	\triangleq	Auxiliary variable which accumulates discounted gradient for SP s
α_n	\triangleq	Discounting factor or step size
$h_s(b)$	\triangleq	Regularization function or a penalty function
ζ	\triangleq	Vanilla ADMM penalty term
β	\triangleq	Proximal approximation penalty term

Notation: Let \mathbb{R}^n indicate the set of n dimension real vectors, and \mathbb{R}_+^n its nonnegative orthant. $\|\cdot\|$ represents the Euclidean norm. Given a vector x , x^T denotes its transpose. Let $\text{col}(x_1, \dots, x_N) := [x_1^T, \dots, x_N^T]$. For a closed set $\mathcal{F} \subseteq \mathbb{R}^n$, the mapping $\text{proj}_{\mathcal{F}} : \mathbb{R}^n \rightarrow \mathcal{F}$ denotes the projection onto \mathcal{F} , *i.e.*, $\text{proj}_{\mathcal{F}}(x) := \arg \min_{y \in \mathcal{F}} \|y - x\|$. Depending on the context, $|\cdot|$ will denote the absolute value of a scalar or the cardinal of a set. Table I summarizes the main notation used in the paper.

II. SYSTEM MODEL

We consider a market design, where in the first stage, a set of SPs, \mathcal{S} , leases their respective networks from the InP and employs a network slicing mechanism to request the resources required for their service provision. In stage two, the SPs (sellers) use the leased resources and compete to maximize their number of end-users (buyers). Specifically, we assume the InP owns a network that consists of a set of base stations or cells, \mathcal{C} . Each base station at different locations accommodates multiple types of resources such as bandwidth, CPU, memory, etc. Users are spread across the network, let N^c be the number of users present in cell c . We assume that the service offered by the SP in a particular cell can only be supported by the resources available within that cell.

A. User Model

We assume that all the users need the same type of service, and they achieve their demand by subscribing to one of the SPs. We consider that each user is opportunistic and free to switch from one SP to another, which is equivalent to choose one slice among the set of available slices at his base station. The user chooses the slice (or, equivalently, the SP) that offers the better trade-off, i.e., the higher QoS at the lowest price. We model the utility of each user served by SP $s \in \mathcal{S}$ in cell c as [11]

$$\nu_s^c(n_s^c, q_s^c, p_s) = \log\left(\frac{q_s^c}{n_s^c}\right) - p_s, \quad (1)$$

where q_s^c is the QoS of SP s in cell c , n_s^c is number of users connected to SP s while p_s is the subscription fees charged by SP s for its service. We assume that the service fees charged by each SP is the same across all cells. Here the use of a logarithmic²(concave) function as the user's utility in QoS means that the users' satisfaction level saturates as the QoS increases, which is coherent with the economic principle of diminishing marginal returns. In turn, the SP QoS depends on the resources inventory available to him. We assume each SP applies a scheduling policy to distribute his resources among the users, in order to achieve equal QoS among them, in the long run.

B. Service Provider Model

We assume that the SPs aim at maximizing their number of subscribers, n_s^c , by attracting users with a better QoS and lower price. We assume that the QoS provided by each SP depends on the resource inventory available at the slice and is defined according to the relation $q_s^c \triangleq q_s^c(d_s^c)$. Let $d_s^c \triangleq (d_{sm}^c)_{m \in \mathcal{M}^c}$ denote a bundle of resources available with SP s . d_{sm}^c captures the amount of resource type m acquired by SP s at cell c . We assume that for all $c \in \mathcal{C}$ and $s \in \mathcal{S}$, the function $q_s^c(d_s^c)$ is concave non decreasing in d_s^c . This assumption is classical

in economics, reflecting the principle of diminishing marginal returns.

Each SP collects revenue from the fees paid by his subscribers. The expected revenue of SP s over the network is defined as

$$R_s = \left(\sum_{c \in \mathcal{C}} p_s n_s^c \right). \quad (2)$$

Each SP needs to pay for the resources he leases from the InP. Let ω_m^c be the price per unit of resource of type m charged by the InP, at base station c . The total cost each SP s needs to pay to the InP for resource activation is therefore $\sum_{c \in \mathcal{C}} \sum_{m \in \mathcal{M}^c} \omega_m^c d_{sm}^c$. We define the profit gained by SPs as a quasi linear utility function

$$U_s = R_s - \sum_{c \in \mathcal{C}} \sum_{m \in \mathcal{M}^c} \omega_m^c d_{sm}^c. \quad (3)$$

Depending on the SPs' budgets, we consider two possible cases in the following.

In **Case I**, we assume that each SP s has a finite budget B_s , which captures the market (purchasing) power of the SP. Another relevant interpretation in the context of network slicing is that it represents the SP's priority or a fixed share of the available resource pool, such that $\sum_{s \in \mathcal{S}} B_s = 1$. In this case, each SP s must satisfy $\sum_{c \in \mathcal{C}} \sum_{m \in \mathcal{M}^c} \omega_m^c d_{sm}^c = B_s$.

In **Case II**, we assume that SP s has no bound over his budget. His strategy set is defined as the set of vectors d_s such that $d_{sm}^c \geq 0, \forall m \in \mathcal{M}^c, \forall c \in \mathcal{C}$. Additional linear local constraints can be included in the form $K_s(d_s) \leq 0$.

III. GAME THEORETIC MODEL

We assume that each user is opportunistic, and takes decisions to maximize his utility. From (1), we observe that the utility of each user depends on the total number of users of the SP. On the one hand, as the number of users connected to the SP increases, the utility of the user decreases. Therefore, the decision made by each user is also influenced by decisions taken by the other users. On the other hand, SPs maximize their revenues by attracting the maximum number of users. Naturally, each SP anticipates the users' behaviour while computing their strategy. Therefore, it is highly expected that users and SPs exhibit strategic behaviors. In our work, the SPs take selfish decisions while anticipating the rational reactions of the users. We model the interactions between the users and the SPs as a (two-stage) Stackelberg game, where the SPs act as leaders while users react rationally as followers, computing their best responses to the signal sent by the SPs. In the first stage, SPs compete in terms of QoS to attract the maximum number of users. In the second stage, users optimally select their SPs to maximize their utility given prices and QoS offered by SPs.

In classical game theory, Nash equilibrium is the most popular solution concept to analyze noncooperative game solutions. This concept is based on the assumptions that each player has an exact knowledge about all other players' strategies at the equilibrium, and no player has an incentive to deviate from his

²The logarithm function also signifies that the SPs achieve the proportional fair allocation between the user in the long run

own strategy at equilibrium. In many cases, knowing the exact information about all other players equilibrium strategies is a strong assumption, particularly when there are many users, and information about the strategy profile of all opponents is rarely perfectly known. In light of these limitations, we model the interactions between the users as a population game that extends the formulation of a noncooperative game by incorporating the notion of population.

A. Population game \mathcal{E} among Users

The population game provides an alternative to the classical equilibrium approach by involving a dynamic model. Unlike a single-play game or repeated games where all agents take their decisions simultaneously and repetitions occur at regular time periods, in a population game, each agent revises his decision sporadically, and the decision made by the revising agent only depends on the current system state and available payoff opportunities. Now for each cell c , we define the population game \mathcal{E}_c

- Population: set of users $\mathcal{N}^c := \{1 \dots N^c\}$ in cell c .
- Strategy: it is the choice of SP $s \in \mathcal{S}$ that each user in cell c opts to join.
- Utility: the utility achieved by each user of slice $s \in \mathcal{S}$ is equal to ν_s^c .

In an population game, each agent revises his decision occasionally after some random duration of time. Whenever an agent reconsiders his decision, it depends on the system state and payoff opportunity available at that time. A general model of decision of the game is based on the concept of revision protocol. It is a mapping that translates the current population state (i.e., distribution of user) and available payoff (i.e., utilities (1)) into a switching rate which determines when users might update their choice of provider. Let $\tau_{s,s'}^c(n^c, U^c)$ be the revision protocol which defines the switching rate at which users switch their choice from SP s to SP s' given population state $n^c = [n_1^c \dots n_S^c]$ and utility vector $\nu^c = [\nu_1^c \dots \nu_S^c]$. Let $\mathbb{N}^c = \{n^c | \sum_{s \in \mathcal{S}} n_s^c = N^c\}$ defines the set of all possible population states. Population game \mathcal{E} with revision protocol τ generates a continuous time evolutionary process on set \mathbb{N}^c defined as

$$\dot{n}_s^c = \sum_{s'} n_{s'}^c \tau_{s',s}^c - n_s^c \sum_{s'} \tau_{s,s'}^c. \quad (4)$$

The first-term in the right-hand side of equation (4) measures the rate at which users connect to SP s . The second term measures the rate at which the portion of the population connected to SP s disconnects. A different choice of revision protocol results in different dynamics. In this work, we assume that the users follow the pairwise proportional imitation behavior, e.g., after every random interval of time, each user interacts with his opponents (i.e., other users), and only if users meet an opponent with a higher utility than his own, he imitates the opponent with a probability proportional to the utility difference. The switching rate at which users in cell c

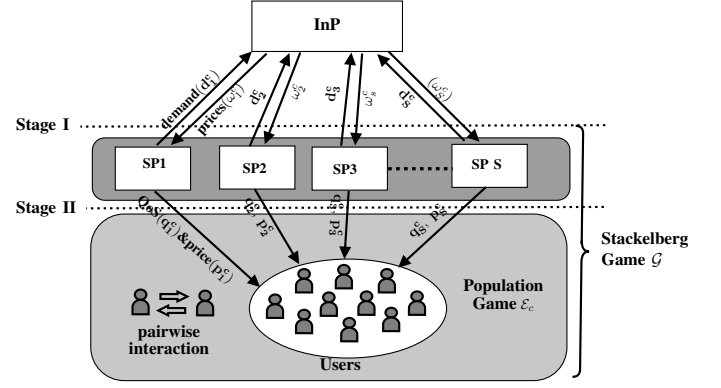


Fig. 2. service providers and end users interactions as a stackelberg game

switch from SP s to SP s' takes the form

$$\tau_{s,s'}^c = \frac{n_{s'}^c}{N^c} [\nu_{s'}^c - \nu_s^c]_+. \quad (5)$$

After replacing $\tau_{s,s'}^c$ in (4) with (5) and after some analytical calculations detailed in Appendix D we get the replicator dynamics

$$\dot{n}_s^c = n_s^c \left[\nu_s^c - \frac{1}{N^c} \sum_{s'} n_{s'}^c \nu_{s'}^c \right]. \quad (6)$$

An Evolutionarily Stable Strategy (ESS) characterizes the equilibrium solution concept for population games. Once the evolutionary process reaches an ESS, the population state will not change. It is defined as the fixed point of the dynamical system defined through equation (4).

Proposition 1. For all $c \in \mathcal{C}$ and for any bundle of resources available with SP s , the replicator equation (6) admits a unique evolutionary equilibrium \hat{n}_s . Moreover, the number of users \hat{n}_s^c in cell c associated with SP s at the equilibrium point can be defined as

$$\hat{n}_s^c = \frac{N^c q_s^c e^{-p_s}}{\sum_{s' \in \mathcal{S}} q_{s'}^c e^{-p_{s'}}}. \quad (7)$$

Proof. The replicator equation (6) is nothing but a set of ordinary differential equations (ODE). The equilibrium is the stationary point of ODE. Hence, to show that the replicator dynamics admits a unique equilibrium point, it is sufficient to show that the right-hand side of (6) is continuously differentiable and that it admits a unique stationary point [23]. Replacing $\nu_s^c, \nu_{s'}^c$ from (1) in (6), we derive analytically the equilibrium point expression (7). A detailed proof is provided in Appendix A. \square

IV. STACKELBERG GAME BETWEEN SERVICE PROVIDERS AND USERS

In our game-theoretic formulation of the communication service market, the SPs are the leaders, and the users are the followers. We have proved in Section III that the equilibrium of the population game \mathcal{E}_c between the users admits a unique

solution, and the distribution of users at the equilibrium point is derived relying on the closed form expression (7). In this section, we model the interactions between the SPs as a noncooperative game.

We note that the analytical expression of the number of users \hat{n}_s^c in cell c of SP s at equilibrium is very similar to a contest success function from the well known TC framework [24]. The TC framework is commonly used in the economics literature to model strategic interactions between two or more competing agents. The basic contest framework consists of competing agents who expend costly resources to win a prize (a contest). Given the efforts exerted by all the agents, the probability of an agent i winning a prize is defined by the contest success function (CSF). Typically, the CSF function is defined as $\rho_i(x) = \frac{(x_i)^r}{\sum_{i'} (x_{i'})^r}$ where x_i is the effort made by a agent i and r is a parameter. For example, $r = 1$ is the well know lottery and $r \rightarrow \infty$ defines the all-pay auction.

In the communication market context, the SPs compete to attract users to their services by exerting effort on costly resources. The resources acquired by SPs further reflect their service quality (a higher QoS is seen as a desirable attribute in the process of SP selection). Thus, in our case, the CSF can be considered as the probability that any SP successfully attracts an end-user. We call it the *slice association probability function* A_s . It is the probability that given resources expended by all SPs, a user will associate with SP s . For our model, we rely on a more general and multi-resource CSF function or slice association probability function

$$A_s^c(d^c, p) = \frac{f_s^c(d_s^c, p_s)}{\sum_{s' \in \mathcal{S}} f_{s'}^c(d_{s'}^c, p_{s'})}. \quad (8)$$

Remark 1. Bernstein and Federgruen proposed a very well known general equilibrium model, named as attraction model, for industries with price and service competition in [25]. It is very similar to our slice association probability function.

In (8), the number of potential users in each cell as well as the slice association probability for each slice, might vary from cell to cell. The expected number of users choosing SP s is defined as

$$\sum_{c \in \mathcal{C}} N^c A_s^c(d^c, p) = \sum_{c \in \mathcal{C}} \frac{N^c f_s^c(d_s^c, p_s)}{\sum_{s' \in \mathcal{S}} f_{s'}^c(d_{s'}^c, p_{s'})}. \quad (9)$$

Incorporating (7) and (2) in (3), we get

$$U_s(d_s, d_{-s}) = \sum_{c \in \mathcal{C}} p_s \frac{N^c f_s^c(d_s^c, p_s) e^{-p_s}}{\sum_{s' \in \mathcal{S}} f_{s'}^c(d_{s'}^c, p_{s'}) e^{-p_{s'}}} - \sum_{m \in \mathcal{M}^c} \omega_m^c d_{s,m}^c. \quad (10)$$

In this work, we set

$$f_s^c(d_s^c, p_s) = q_s^c(d_s^c) e^{-p_s}. \quad (11)$$

We assume that SPs are selfish, and that each SP aims at maximizing his profit. They take into account the decisions of the other SPs when computing their own decision. To theoretically analyze the outcome of these strategic interactions, we define the noncooperative game $\mathcal{G} \triangleq \langle \mathcal{S}, (\mathcal{F}_s)_{s \in \mathcal{S}}, (U_s)_{s \in \mathcal{S}} \rangle$

as follows:

- Player set: the set of service providers \mathcal{S} .
- Strategy: the vector of resource demand $d_s = (d_s^1, \dots, d_s^C)$ where d_s^c is the amount of resource to be requested by each base station c . The strategy set for each SP s is \mathcal{F}_s .
- Utility: the utility of each SP s is defined as U_s .

We study the competition between SPs in terms of QoS, i.e., how SPs strategically spend their budget on the resources to attract the maximum number of users and, in turn, maximize their profits. The SPs' profit depends on both their individual decision and the decision taken by the other SPs. Let d_s be the vector of strategy of SP s , $d_{-s} \triangleq \text{col}((d_{s'})_{s' \neq s})$ is the stack vector which contains the vector of strategies of all the SPs in \mathcal{S} except s . The decision problem of each SP s is defined as

$$Q_s \quad \underset{d_s \in \mathcal{F}_s}{\text{maximize}} \quad U_s(d_s, d_{-s}).$$

To study the outcome of the noncooperative game \mathcal{G} , we recall the solution concept of Nash equilibrium (NE)

Definition 1. [26] A strategy profile $d^* = (d_1^*, \dots, d_S^*)$ is a Nash equilibrium of the game \mathcal{G} if

$$\forall s \in \mathcal{S}, U_s(d_s^*, d_{-s}^*) \geq U_s(d_s, d_{-s}^*), \forall d_s \in \mathcal{F}_s. \quad (12)$$

Here, (d_s, d_{-s}^*) denotes the strategy profile with s^{th} element equals d_s and all other elements equal $d_{s'}^*$ (for any $s' \neq s$).

In the next section, we analyze the existence and uniqueness of the Nash equilibrium (NE) solution of the noncooperative game \mathcal{G} .

V. EXISTENCE AND UNIQUENESS OF THE NASH EQUILIBRIUM

In this section, we establish the existence and uniqueness of the NE of game \mathcal{G} . To prove the uniqueness of the NE, we rely on the concept of diagonally strict concavity (DSC) introduced by Rosen [27]. Intuitively, DSC is a generalization of the idea of convexity to a noncooperative game setting.

Definition 2 (Diagonal strict concavity [27]). A game with profiles of strategies d and profiles of utility functions U is called diagonally strict concave (DSC) for a given vector r if for every distinct \bar{d} and \hat{d} ,

$$\left[g(\bar{d}, r) - g(\hat{d}, r) \right] (\bar{d} - \hat{d})' < 0, \quad (13)$$

with g the concatenation of the weighted gradients of the players' utility functions

$$g(d, r) = [r_1 \nabla_1 U_1(d), r_2 \nabla_2 U_2(d), \dots, r_S \nabla_S U_S(d)], \quad (14)$$

where $\nabla_s U_s(d)$ denotes the gradient of utility of player s with respect to his own strategy d_s

Theorem 1. The game \mathcal{G} admits a unique NE.

Proof. The utility of each SP in-game \mathcal{G} is continuous, increasing, and concave, while the strategy space for each SP is convex and compact. Therefore, the existence of an

equilibrium for the game follows from [27], Thm.1. To prove the NE uniqueness, we note that if the players' utilities in the game \mathcal{G} satisfy the DSC property, then \mathcal{G} admits a unique NE (see [27], Thm.2).

Let $G(d, r)$ be the Jacobian of $g(d, r)$ with respect to d , where d is any profile of strategies. In order to prove the strict DSC of $g(d, r)$, from [27], Thm.6, we note that it is sufficient to prove that the symmetrized version of the pseudo-Jacobian, i.e., $\hat{G}(d, r) \triangleq G(d, r) + G(d, r)'$, is negative definite over the domain of interest. To show that $\hat{G}(d, r)$ is negative definite, we must prove that the following three conditions hold simultaneously:

C 1. each $U_s(d)$ is a regular strictly concave function of d_s (i.e., its Hessian is negative definite).

C 2. each $U_s(d)$ is convex in d_{-s} .

C 3. there is some $r > 0$ such that function $\sigma(d, r) = \sum_s r_s U_s(d)$ is concave in d .

The negative definiteness of $[G(d, r) + G'(d, r)]$ follows from [28], Lem.1. We first consider the case of a single base station c and show that $\hat{G}^c(d, r)$ is negative definite for this case. We compute the Hessian ($H_s U_s^c$) of utility of any SP s with respect to SP s owns strategy

$$H_s U_s^c = -2 \frac{p_s \sum_{s' \in \mathcal{S}, s' \neq s} f_{s'}^c}{\left(\sum_{s' \in \mathcal{S}} f_{s'}^c \right)^3} \left[(\nabla_s f_s^c)^T \nabla f_s^c - H_s(f_s^c) \sum_{s' \in \mathcal{S}} f_{s'}^c \right] \quad (15)$$

On the right hand side of (15), matrix $(\nabla_s f_s^c)^T \nabla f_s^c$ is positive semi-definite, where $\nabla_s f_s^c$ is the gradient row vector of f_s^c with respect to SP s 's own strategy d_s^c , $H_s(f_s^c)$ is the Hessian of f_s^c with respect to d_s^c and it is negative definite as f_s^c is concave. Thus, the Hessian of utility $H_s U_s^c$ is negative definite and satisfies the first condition **C1**.

We still need to show that the utility of each SP s is convex in the strategy of all other SPs. For that purpose, consider the Hessian of utility of SP s with respect to strategy of all other SPs

$$H_{-s} U_s = 2 \frac{f_s^c}{\left(\sum_{s' \in \mathcal{S}} f_{s'}^c \right)^3} [M_s^c - \text{diag}_{-s} \{H(f_u^c)\}], \quad (16)$$

where is M_s^c block matrix and uv^{th} block is defined as

$$M_{su}^c = (\nabla_u f_u^c)^T \nabla_v f_v^c \text{ where } u, v \neq s, u, v, s \in \mathcal{S}. \quad (17)$$

$\nabla_u f_u^c$ is the gradient row vector of f_u^c with respect to SP s 's own strategy and $\text{diag}_{-s} \{H(f_u^c)\}$ is the block diagonal matrix with block u where $H(f_u^c)$ is the Hessian of f_u^c with respect to strategy vector of u itself $\forall u, u \neq s, u \in \mathcal{S}$. In right hand side of equation (16) matrix M_s^c is positive definite and the block diagonal matrix $\text{diag}_{-s} \{H(f_u^c)\}$ is negative definite as the each diagonal matrix element. $H(f_u^c)$ is negative definite, thus $H_{-s} U_s$ is positive definite, which satisfies the condition **C2**.

Finally, by choosing $r_s = \frac{1}{p_s} \forall s \in \mathcal{S}$ we check that $\sigma(d, r) \triangleq \sum_s r_s U_s(d)$ is concave in d , therefore satisfying the condition **C3**.

We now want to extend the previous proof to the multi-base station case. We have already shown that \hat{G}^c is negative definite for any single base station c . For C base stations consider a \hat{G} symmetrized version of the pseudo Jacobian, after arranging columns and rows we get (see [29], Cor.2)

$$(\hat{G}) = \text{diag} \{ \hat{G}^1, \dots, \hat{G}^c, \dots, \hat{G}^C \}.$$

The above \hat{G} matrix is negative definite as each diagonal matrix is negative definite, which proves the DSC property holds for the multi-cell setting. By applying [27], Thm.2, we prove that the NE d^* solution of the game \mathcal{G} is unique. \square

VI. RESOURCE PRICING AND EQUILIBRIUM

We have shown in the previous section that there exists a unique NE solution of the noncooperative game \mathcal{G} . We assume that the capacity of the resource released by the InP in each cell is finite. Given the per-unit prices for resources decided by the InP, the total resource requested by the SPs at the NE of \mathcal{G} may violate the infrastructure capacity. Thus, the InP's primary concern is how to efficiently allocate the finite capacity constrained resources to competing SPs. The desired allocation must satisfy all the SPs' constraints and simultaneously maintain high resource utilization. In this regard, we assume that the InP optimizes the unit price of each resource such that at the NE of the game \mathcal{G} each SP utilizes his entire budget and no resource remains leftover, i.e., the total demand of resources matches the available infrastructure capacity. In market economics, this pricing problem is formulated as a market clearing problem, e.g., a Fisher market, where the market prices are settled in such a way that the amount of resources requested by the buyers is equal to the amount of resources supplied by the sellers. We propose two approaches, introduced in Section II B, to deal with this challenge depending on whether the SPs' budget is binding.

One way to compute the market equilibrium is through Walras' "tâtonnement" process, i.e., if the demand exceeds the resource capacity, the market operator increases the resource's price. Conversely, the market operator decreases the resource's price when the demand is smaller than the resource capacity. The process is repeated until demand equals supply (resource capacity). The disadvantage of this approach is that its outcome (known as a general equilibrium) relies on the strong assumption of perfect competition, which in practice does not hold. To overcome this limitation, we use the approach introduced by Shapley and Shubik in their pioneering work [30], also known as trading post share constrained proportional allocation (SCPA) scheme[3]. Now we formally define the trading post mechanism.

A. Trading post mechanism

In the trading-post mechanism, each player (i.e., SP) places a bid on each type of resource. Once all SPs have placed their

bids, each resource type's price is determined by the total bids placed for that resource. Precisely, let SP s submits a bid b_{sm}^c to resource m at cell c . The price per unit of resource m at cell c is then set to $\frac{\sum_{s \in \mathcal{S}} b_{sm}^c}{D_m^c}$. Accordingly, SP s receives a fraction of d_{sm}^c in return to his spending of b_{sm}^c

$$d_{sm}^c = \begin{cases} \frac{b_{sm}^c D_m^c}{\sum_{u \in \mathcal{S}} b_{um}^c} & \text{if } b_{sm}^c > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$

After replacing d_{sm}^c in (3) and (Q_s) in terms of bids, the decision problem of each SP s can be written as follows

$$\begin{aligned} \hat{Q}_s \quad & \text{maximize} \quad U_s(b_s, b_{-s}), \\ & \text{subject to} \quad \sum_{c \in \mathcal{C}} \sum_{m \in \mathcal{M}^c} b_{s,m}^c \leq B_s, b_{s,m}^c \geq 0. \end{aligned}$$

We may consider two possible behaviors for the SPs. First, they are *price takers*, i.e., they accept the prices decided by the price setter (market operator), and they only act strategically in terms of QoS by optimizing their demand in the bundle resources. Second, SPs are *price makers*, i.e., they anticipate the effect of their demand on the price of the bundle of resources.

The trading post mechanism induces a new noncooperative game $\hat{\mathcal{G}}$ defined as follows:

- Player set: the set of SPs \mathcal{S} .
- Strategy: the vector of bids $b_s = [b_s^1, \dots, b_s^C]$ where b_s^c is the bid to be submitted to cell c . SP s strategy set is $\mathcal{F}_s \triangleq \left\{ b_s | b_{sm}^c \geq 0, \forall m \in \mathcal{M}^c, c \in \mathcal{C}, \sum_{c \in \mathcal{C}} \sum_{m \in \mathcal{M}^c} b_{s,m}^c = B_s \right\}$.
- Utility: The utility of each SP s is U_s .

To study the outcome of the mechanism, we consider the standard notion of NE, applied to the trading post mechanism

Definition 3. A multi-bid strategy $b^* = (b_1^*, \dots, b_S^*)$ is called a NE of the game $\hat{\mathcal{G}}$ if

$$\forall s \in \mathcal{S}, U_s(b_s^*, b_{-s}^*) \geq U_s(b_s, b_{-s}^*), b_s \in \mathcal{F}_s. \quad (19)$$

Here, (b_s, b_{-s}^*) denotes the strategy vector with s^{th} element equals b_s and all other elements equal b_v^* (for any $v \neq s$).

For the proposed mechanism, a NE solution of game $\hat{\mathcal{G}}$ constitutes a stable bidding policy where each SP maximizes his utility and the InP implements the resource allocation mechanism (18).

We investigate conditions for the existence and uniqueness of the NE solution of the game $\hat{\mathcal{G}}$. This requires complex calculations. Thus, to keep the analysis tractable, we restrict the problem to a single resource (radio resource). We assume that the QoS provided by SP s in cell c is given by $q_s^c \triangleq (d_s^c)^{\rho_s^c}$ where ρ_s^c is the sensitivity parameter and $0 < \rho_s^c \leq 1$. Such a type of function has been used in [16] to model the effect of users sensity towards their service provider selection. We replace $q_s^c = (d_s^c)^{\rho_s^c}$ in (11) and from (8) we get

$$A_s^c(d^c, p) = \frac{(d_s^c)^{\rho_s^c} e^{-p_s}}{\sum_{s' \in \mathcal{S}} (d_{s'}^c)^{\rho_{s'}^c} e^{-p_{s'}}}. \quad (20)$$

Proposition 2. If for a single resource, the QoS provided by SP s in cell c is defined by $q_s^c = (d_s^c)^{\rho_s^c}$ and $0 < \rho_s^c \leq 1$, then the game $\hat{\mathcal{G}}$ admits unique NE.

Proof. If the QoS provided by SP s in cell c is defined by $q_s^c = (d_s^c)^{\rho_s^c}$ and $0 < \rho_s^c \leq 1$, then the SPs' utility functions satisfy the three conditions **C1**, **C2** and **C3**. The detailed proof is provided in Appendix B The rest of the proof is the same as the proof of Theorem 1. \square

Moving ahead, we compare the profit gained by SPs at the NE of the game with the baseline static proportional allocation scheme (SS), i.e., the allocation where each resource is allocated to a SP s in proportion to his budget, e.g., $\frac{B_s}{\sum_{s' \in \mathcal{S}} B_{s'}}$

Proposition 3. For two service providers, the revenue gained under a dynamic resource sharing scheme is at least equal to the revenue gained under a proportional allocation scheme

Proof. The proof is provided in Appendix C. \square

We have seen in the first part of this section, that when SPs are constrained by budgets, the resource pricing can be implemented by a trading post mechanism. However, this mechanism requires a third-party player (market operator) to centralize the bids made by all the SPs, and thus can only lead to semi-decentralized implementations. Furthermore, the network capacity constraints are only implicitly taken into account through the budget constraint.

In the next section, we design a pricing and resource allocation scheme for **Case II** introduced in Section II B, that explicitly takes into account network capacity constraints and can be implemented in a fully distributed way. **Case II** gives rise to a generalized Nash equilibrium problem (GNEP) involving global coupling constraints, which take into account the network finite capacities. To solve the GNEP, we rely on a variational reformulation of the noncooperative game, which leads to a unique variational equilibrium (VE). Using that property, we implement two algorithms to compute the VE: the first one requires an extended game reformulation of the GNEP and is based on asymmetric projected gradient descent methods; the second one relies on an extension of the alternating direction method of multipliers (ADMM).

B. Pricing Game

We consider a noncooperative game where, similar to game \mathcal{G} , each SP aims at maximizing his profit by requesting resources under a set of local constraints that are not binded by a finite budget. However, we now assume that the SPs take into account the infrastructure capacity while requesting resources, therefore giving rise to a global coupling constraint for each cell and each resource available within that cell

$$\sum_{s \in \mathcal{S}} d_{sm}^c \leq D_m^c, \forall c \in \mathcal{C}, m \in \mathcal{M}, \quad (21)$$

Let $\tilde{\mathcal{F}}_s \triangleq \{d_s | d_{sm}^c \geq 0, \forall m \in \mathcal{M}^c, c \in \mathcal{C}, K_s(d_s) \leq 0\}$. The decision problem faced by each SP in this new noncoopera-

tive game can be formulated as a parametrized optimization problem with local and global coupling constraints

$$\begin{aligned} Q_s \quad & \text{maximize} \quad R_s(d_s, d_{-s}), \\ & d_s \in \tilde{\mathcal{F}}_s \\ \text{subject to} \quad & \sum_{s \in \mathcal{S}} d_{sm}^c \leq D_m^c, \forall c \in \mathcal{C}, m \in \mathcal{M}, (\lambda_{sm}^c) \end{aligned} \quad (22)$$

where λ_{sm}^c at the right of (22) and between brackets, is the Lagrange multiplier (shadow price) of the coupling constraint (22).

We define a new noncooperative game $\mathcal{G}_p \triangleq \langle \mathcal{S}, (\tilde{\mathcal{F}}_s)_{s \in \mathcal{S}}, (R_s)_{s \in \mathcal{S}} \rangle$, where the set of players and utility is the same as in game \mathcal{G} . However, the strategy set of the players are coupled through the capacity constraint (22), giving rise to a GNEP. Consider the generalized Nash equilibrium (GNE) as the solution to this game.

Definition 4. [31] A strategy profile $d^* = (d_1^*, \dots, d_S^*)$ is called a GNE of the game \mathcal{G}_p if

$$\forall s \in \mathcal{S}, R_s(d_s^*, d_{-s}^*) \geq R_s(d_s, d_{-s}^*), \quad (23)$$

$$d_s \in \tilde{\mathcal{F}}_s, d_{sm}^c \geq 0, \forall m, c \text{ and } \sum_{s \in \mathcal{S}} d_{sm}^c \leq D_m^c, \forall c \in \mathcal{C}, m \in \mathcal{M}$$

Due to coupling, solving directly \mathcal{G}_p requires coordination among possibly all SPs, which might be hard to enforce in practice. To solve \mathcal{G}_p , we will make use of the duality approach as a natural way to obtain a hierarchical decomposition of the GNEP. To that purpose, we start by characterizing the GNE solutions of game \mathcal{G}_p in terms of KKTs [31]: any strategy profile \bar{d} is a GNE of the game \mathcal{G}_p if and only if it satisfies the KKT conditions, which are: $\forall s \in \mathcal{S}, \forall c \in \mathcal{C}, \forall m \in \mathcal{M}^c$,

$$\begin{cases} \frac{\partial R_s}{\partial d_{sm}^c}(\bar{d}) = \lambda_{sm}^c, \\ \lambda_{sm}^c \left(\sum_{s \in \mathcal{S}} \bar{d}_{sm}^c - D_m^c \right) = 0, \\ \text{with } \lambda_{sm}^c \geq 0, \bar{d}_s \in \tilde{\mathcal{F}}_s. \end{cases} \quad (24)$$

In the above KKT conditions, we are primarily interested in λ_{sm}^c , the Lagrange multipliers of (22), as these Lagrange multipliers can be interpreted as shadow prices for the resource allocation and can be used in the game \mathcal{G}_p as the evaluations by the SPs of the prices charged by the InP per resource unit. However, notice that if implemented without coordination, the Lagrange multipliers for each SP are different, resulting in possibly discriminatory pricing. Moreover, there can be multiple possible GNEs. In fact, there are infinite GNEs solutions to \mathcal{G}_p in this case. Nevertheless, in the following discussion, we show that there exists an equilibrium solution to \mathcal{G}_p with a special characteristic: it is unique and gives rise to the same valuation among the players. Rosen [27] has introduced concept of such equilibrium in his seminal work and called it as normalized Nash equilibrium

Definition 5. A r -normalized equilibrium point is such that there exists a $\lambda_m^c > 0$ associated to each resource at each cell

so that for all customers $\lambda_{sm}^c = \lambda_m^c / r_s$, for a suitable vector of nonnegative coefficients vector r .

It is very common in the literature, to relate normalized Nash equilibrium to the concept of variational equilibrium (VE) [31]. We will use both concepts without distinction in the following. The parameters $\{r_1 \dots r_S\}$ intuitively show the proportion of a burden on SP s for satisfying the coupling constraints among all other service providers in the set. Notice, λ_m^c is the same for all the SP and thus can be treated as the base price. Next, we prove that such r -normalized Nash equilibrium (variational equilibrium), is unique for game \mathcal{G}_p .

Corollary 1. The Pricing Game \mathcal{G}_p admits unique normalized equilibrium for $r_s = 1/p_s$.

Proof. In the proof of Theorem 1, we have shown that the Game \mathcal{G} has the DSC property $\forall c \in \mathcal{C}$ and $\forall m \in \mathcal{M}$ and any $\omega_m^c \geq 0$. The utilities of players in the Game \mathcal{G}_p (i.e. revenues) are the same as in \mathcal{G} with $\omega_m^c = 0$. Hence, the Pricing Game \mathcal{G}_p also satisfies the DSC property. The proof is a consequence of [27], Thm.4. \square

Theorem 2. Every r -normalized Nash equilibrium of the Pricing Game \mathcal{G}_p with shadow prices $\lambda_{sm}^c = \frac{\lambda_m^c}{r_s}$ for all $s \in \mathcal{S}, m \in \mathcal{M}^c, c \in \mathcal{C}$ is a NE for the corresponding Game \mathcal{G} with $\omega_m^c = \lambda_m^c, \forall m \in \mathcal{M}^c, \forall c \in \mathcal{C}$.

Proof. We reformulate \mathcal{G}_p using an augmented system-like utility function, that we call the Nash game (NG)-game utility function [32]. The NG utility function is defined as a two argument function

$$\tilde{R}(d; x) \triangleq \sum_{s=1}^S r_s R_s(d_{-s}, x_s), \forall x \in \tilde{\mathcal{F}} \triangleq \prod_s \tilde{\mathcal{F}}_s, \quad (25)$$

where $x \triangleq (x_s)_s$ and d_{-s} defined as before. We note that Definition 5 can be equivalently formulated with respect to the NG utility function. A vector $d^* \in \tilde{\mathcal{F}}$ is called a NE solution of this game if its NG utility function \tilde{R} satisfies

$$\tilde{R}(d^*; d^*) \geq \tilde{R}(d^*; x) \quad \forall x \in \tilde{\mathcal{F}} \quad (26)$$

Above condition can equivalently be written as follows for given d_{-s}^*

$$\sum_{s=1}^S r_s R_s(d_{-s}^*; d_s^*) \geq \sum_{s=1}^S r_s R_s(d_{-s}^*; x_s), \forall x \in \tilde{\mathcal{F}}. \quad (27)$$

Note that the NG utility function \tilde{R} is separable in the second argument x for any given first argument d^* . The existence of NE is guaranteed by Theorem 2 [32]. Now to extend the NG utility function formulation to coupled constrained game, i.e., pricing game, we use the fact that the pricing game is related to a constrained maximization of NG utility function with respect to the second argument keeping the first argument as a fixed point solution. Consider that the SPs maximize their revenue subject to coupled constraints $g_i(d) \leq 0, i \in \mathcal{I} \triangleq \mathcal{C} \times \mathcal{M}$ where $g_i(d) \triangleq D_m^c - \sum_{s'} d_{s'm}^c$. Then

$$\tilde{R}(d^*; d^*) \geq \tilde{R}(d^*; x) \quad x \in \mathcal{F}, \tilde{g}(d_{-s}^*, x_s) \leq 0, \quad (28)$$

where $\tilde{g}(d^*; x) = \sum_{s=1}^S g(d_{-s}^*, x_s)$. We introduce the augmented Lagrangian function of the constrained NG utility maximization problem, with λ a Lagrange multiplier vector

$$\tilde{L}(d; x; \lambda) \triangleq \tilde{R}(d; x) + \lambda^T \tilde{g}(d; x). \quad (29)$$

In our case $\forall s \in \mathcal{S}$, R_s is increasing concave and continuously differential, and g is affine. Thus, all the constraints are active at the equilibrium. If d^* is an equilibrium solution of the pricing game \mathcal{G}_p , then by [32], Lem.2, there exists a unique $\lambda^* > 0$ such that $\nabla_d \tilde{L}(d, x, \lambda^*) = 0$ and d^* maximizes the Lagrangian \tilde{L} , over $x \in \tilde{\mathcal{F}}$ as a fixed point.

Relying on the duality framework, we prove that we can decompose the coupled constrained game \mathcal{G}_p into the equivalent game with no coupled constraints, and indeed the equivalent game coincides with the noncooperative game \mathcal{G} with $\omega_m^c = \lambda_m^c$. To that purpose, we consider the dual cost function $D(\lambda)$ defined as

$$D(\lambda) \triangleq \tilde{L}(d^*; d^*; \lambda). \quad (30)$$

Equivalently, relying on the notion of fixed point, the dual cost can be written as

$$D(\lambda) \triangleq \left[\max_{x \in \Omega} \tilde{L}(d; x; \lambda) \right] \Big|_{x=d}. \quad (31)$$

The dual NG can then be defined as the minimization of the dual cost function

$$D^* = \min_{\lambda \geq 0} D(\lambda). \quad (32)$$

The Lagrangian function \tilde{L} is separable over each SP, thus, the dual function can be separately written for each player as

$$D(\lambda) \triangleq \sum_{s \in \mathcal{S}} \left[\max_{x_s \in \Omega_s} L_s(d_{-s}; x_s; \lambda) \right] \Big|_{x_s=d_s} \quad (33)$$

$$= \sum_{s \in \mathcal{S}} L_s(u_{-s}^*(\lambda), u_s^*(\lambda), \lambda), \quad (34)$$

where

$$L_s(d_{-s}; x_s; \lambda) = r_s R_s(d_{-s}; x_s) + \lambda^T g(d_{-s}; x_s). \quad (35)$$

From [32], Thm.3, we prove that $D(\lambda)$ can be obtained by solving the relaxed game with utility function L_s and no coupled constraints. Indeed, that relaxed game is the game \mathcal{G} with $\omega = \lambda$, which concludes the proof. \square

This approach enables us to reformulate the GNEP \mathcal{G}_p , as a lower-level noncooperative Nash game with utility function L_s (U_s) and a higher-level optimization problem for coordination.

C. Extended Pricing Game

Paccagnan et al. addressed decentralized computation of variational equilibrium (VE) for aggregative games with quadratic utility functions [33], [34]. They relaxed the coupling constraints of the generalized Nash equilibrium problem by including a penalty term in the original utility functions. A VE is

then computed applying asymmetric gradient algorithms with constant step size. The purpose of the penalty term is to assign large penalties to deviations from the constraints. The penalty reformulation helps avoid the high computational complexity of conventional optimization reformulations or the requirement of projection steps. Traditionally, drawbacks is that penalty method convergence might be quite sensitive on selecting penalty parameters. To overcome this issue, we follow the formulation proposed in the [35]. We consider a game with $S + 1$ players, where the first S players are the SPs and the $(S + 1)^{\text{th}}$ player is the InP, who controls the λ price vector. We define the decision problem of the InP, Q_{S+1} as below

$$Q_{S+1} \quad \underset{\lambda \geq 0}{\text{maximize}} \quad \sum_c \sum_m \lambda_m^c \left(\sum_s d_{sm}^c - D_m^c \right). \quad (36)$$

The idea behind using $\sum_c \sum_m \lambda_m^c (\sum_s d_{sm}^c - D_m^c)$ as the utility for InP in the above decision problem, is that it solves complementary condition from KKT (24). For the remaining S players the decision problem is

$$Q_s \quad \underset{d_s \geq 0}{\text{maximize}} \quad R_s(d_s, d_{-s}) - \frac{1}{r_s} \sum_c \sum_m \lambda_m^c d_{sm}^c \quad \forall s \in \mathcal{S}.$$

We call $\mathcal{Q}_+ \triangleq \{Q_1, \dots, Q_{S+1}\}$ the extended pricing game. The difference between the extended pricing game and the pricing game \mathcal{G}_p is that in the former, there are no coupled constraints – complementary conditions are treated as the utility of an additional player (InP).

Proposition 4. *If \bar{d} is a r -normalized equilibrium of the pricing game, then there exists $\bar{\lambda} \geq 0$ such that $(\bar{d}, \bar{\lambda})$ is an equilibrium of the extended pricing game.*

Proof. We have already proved that the pricing game \mathcal{G}_p is monotone on $\tilde{\mathcal{F}}$, which implies that the extended pricing game is also monotone on $\tilde{\mathcal{F}} \times \mathbb{R}$, the proof follows from [35], Prop.4. \square

VII. ALGORITHMS TO COMPUTE MARKET EQUILIBRIA

In this section, we introduce two semi-decentralized algorithms to compute the equilibria solutions of the trading post mechanism and extended pricing game, respectively. Computational and privacy issues might limit the implementation of such algorithms on medium to large-scale problems. To mitigate these issues, we propose a fully distributed proximal algorithm, inspired from the inexact-ADMM, to compute the VE of the pricing game \mathcal{G}_p .

A. Semi-Decentralized Learning Algorithms

We have proved in Section V that \mathcal{G} admits a unique equilibrium for any price vector decided by the InP. A similar result also holds for VI-A when dealing with a single resource. However, we still need to check whether tenants can reach this equilibrium in a decentralized fashion. In this regard, we propose the use of the dual averaging or mirror-descent method suggested for continuous action convex games [36]. We proceed by describing the dual averaging method. In

the dual averaging method, each player, *i.e.*, SP s estimates his marginal utility or utility gradient with respect to his own strategy. To increase their utilities, the players need to take action along the direction of their utility gradient while maintaining their action in the feasible action space. In order to achieve this, each player s at each time step n accumulates his discounted utility gradient in some auxiliary variable y_s

$$y_s(n+1) = [y_s(n) + \alpha_n \nabla_{b_s} U_s(b_s(n), b_{-s}(n))]. \quad (\text{A1})$$

In the above equation α_n denotes the discount factor or step size. Once the discounted gradient has been accumulated, every SP s uses his own updated value of the auxiliary variable, y_s , to take the next feasible action

$$b_s(n+1) = Q_s(y_s). \quad (37)$$

In turn, each SP s maps the recent value of auxiliary variable y_s to his decision space \mathcal{F}_s using the mapping $Q_s(y_s)$, *e.g.*, Q_s can be interpreted as a projection map. The map $Q_s(y_s)$ is defined more generically as

$$Q_s(y_s) = \operatorname{argmax}_{b_s \in \mathcal{F}_s} \{ \langle y_s(n), b_s \rangle - h_s(b_s) \}, \quad (\text{A2})$$

where $h_s(b)$ is a regularization function, also called penalty function, over the feasible action set \mathcal{F}_s . The penalty $h_s(b)$ aims to force the algorithm to converge within the interior of the feasible domain set. Different definitions of the regularization functions induce different maps. For instance, the use of l_2 norm $h_s(\cdot) = \|\cdot\|$ as a regularizer, results in the well-known Euclidean projection map.

For the game \mathcal{G} where service providers actions are bounded by the their budgets, we use the Gibbs entropy function as a regularization function

$$h_s(b_s) \triangleq \sum_{c \in \mathcal{C}} \sum_{m \in \mathcal{M}} b_{sm}^c \log(b_{sm}^c). \quad (38)$$

We replace $h_s(b_s)$ in equation (A2) by the entropic regularization function and after some calculation we get the exponential mapping

$$b_{sm}^c = \frac{B_i \exp(y_{sm}^c)}{\sum_{c \in \mathcal{C}} \sum_{k \in \mathcal{M}} \exp(y_{sk}^c)}. \quad (39)$$

The induced map $Q_s(y_s)$ is similar to the well-know Logit map, where each player distributes his budget (weights) to different resources depending on exponential of accumulated discounted gradients.

Theorem 3. *If Algorithm 1 satisfies the required conditions for step size sequence, *e.g.*, $\sum_{n=0}^{\infty} \alpha_n = +\infty, \alpha_n \rightarrow 0$ as $n \rightarrow +\infty$, then it converges to the unique NE of the Game \mathcal{G} .*

Proof. The proposed exponential algorithm is the special case of the dual averaging algorithm. If the NE of the any continuous action convex game is strictly r-variationally stable, then the converges of the dual averaging algorithm to a unique NE of the game is guaranteed by [36], Thm. 4.6. Hence to prove the convergence of the proposed algorithm, it is sufficient to show that the unique NE of game \mathcal{G} is strictly r-variationally

Algorithm 1 Online Learning Algorithm for $\hat{\mathcal{G}}$

Require: $\sum_{n=0}^{\infty} \alpha_n = +\infty, \alpha_n \rightarrow 0$ as $n \rightarrow +\infty$

- 1: **repeat** $n = 1, 2, \dots$,
- 2: **for each** SP $s \in \mathcal{S}$
- 3: Observe gradient of utility and update
- 4: $y_s = [y_s + \alpha_n \nabla_{b_s} U_s(b_s, b_{-s})]$
- 5: **end for**
- 6: **for each** SP $s \in \mathcal{S}$
- 7: **for each** cell $x \in \mathcal{C}$ and resource $m \in \mathcal{M}^c$
- 8: Play $b_{sm}^c \leftarrow \frac{B_s \exp(y_{sm}^c)}{\sum_{c \in \mathcal{C}} \sum_{k \in \mathcal{M}} \exp(y_{sk}^c)}$.
- 9: **end for**
- 10: **end for**
- 11: **until** $\|b(n) - b(n-1)\| \leq \epsilon$

stable. The unique NE \hat{b} to the any convex game is strictly r-variationally stable if $\forall b_s \in \mathcal{F}_s$

$$\sum_{s \in \mathcal{S}} r_s \nabla_s U_s(b)(b_s - \hat{b}_s) < 0. \quad (40)$$

As we have already shown in section V, the SPs' utility functions in game \mathcal{G} satisfy the DSC for $r_s = \frac{1}{p_s}, \forall s \in \mathcal{S}$

$$\sum_{s \in \mathcal{S}} r_s [\nabla_s U_s(b) - \nabla_s U_s(\hat{b})] (b_s - \hat{b}_s) < 0. \quad (41)$$

We know that for any continuous action convex game, a feasible point \hat{b} is a NE of the game if and only if

$$\sum_{s \in \mathcal{S}} r_s \nabla_s U_s(\hat{b})(b_s - \hat{b}_s) \leq 0. \quad (42)$$

Inequalities (42) and (41) imply (40), which proves that the unique NE of game \mathcal{G} is strictly r-variationally stable and then by [36], Thm. 4.6, Algorithm 1 converges to the unique NE of game \mathcal{G} . \square

For **Case II** when the SPs have no bound on their budgets, we have proved in the Section VI B that the resource pricing scheme can be set up by solving the GNEP \mathcal{G}_p . Furthermore, we have also shown that the VE solution to \mathcal{G}_p can be computed as the solution of an extended pricing game \mathcal{Q}_+ . Now, we provide an online semi-decentralized learning algorithm that enables the SPs and the InP to reach the VE of \mathcal{G}_p . In the proposed semi-decentralized algorithm, we leverage on the framework from [37]: the first S players, *i.e.*, the SPs, follow similar steps as in Algorithm 1. However, an $(S+1)^{th}$ player, *i.e.*, the InP, accumulates the augmented discounted gradients of his utility in the auxiliary variable y_{S+1}

$$y_{S+1} = \lambda_m^c + \alpha_n \left[\left(\sum_{s \in \mathcal{S}} x_{sm}^c - C_m^c \right) - \theta_n \lambda_n \right]. \quad (43)$$

Here rationale behind adding an extra term is that the original game is strictly monotone, and thus convergence is guaranteed in that case. However, the extended pricing game is just monotone and therefore, to make the algorithm converge to an equilibrium point, an additional term must be included [37]. InP updates the market price by projecting the stored auxiliary

Algorithm 2 Online Learning Algorithm for \mathcal{Q}_+

Require: $\sum_{n=0}^{+\infty} \alpha_n = +\infty, \alpha_n \rightarrow 0$ as $n \rightarrow +\infty$

- 1: **repeat** $n = 1, 2, \dots$,
- 2: **for each** SP $s \in \mathcal{S}$
- 3: Observe gradient of utility and update
- 4: $y_s \leftarrow [y_s + \alpha_n \nabla_{d_s} U_s(d_s, d_{-s}, \omega_s)]$
- 5: **end for**
- 6: **for each** SP $s \in \mathcal{S}$
- 7: $d_s \leftarrow \text{proj}_{\mathcal{D}_s} [y_s]$
- 8: **end for**
- 9: InP update the resource prices
- 10: **for each** Cell $c \in \mathcal{C}$
- 11: **for each** Resource $m \in \mathcal{M}$ update the base price
- 12: $\lambda_m^c \leftarrow \max [0, \lambda_m^c + \alpha_n (\sum_{s \in \mathcal{S}} x_{sm}^c - C_m^c)]$
- 13: **end for**
- 14: **end for**
- 15: **until** $\|(x(n), \omega(n)) - (x(n-1), \omega(n-1))\| \leq \epsilon$

variable on the positive orthant

$$\lambda_m^c \leftarrow \text{proj}_{\mathbb{R}_{\geq 0}} \left(\lambda_m^c + \alpha_n \left[\left(\sum_{s \in \mathcal{S}} x_{sm}^c - C_m^c \right) - \theta_n \lambda_n \right] \right). \quad (44)$$

Theorem 4. [37] *If Algorithm 2 satisfies the required conditions for step size sequence, e.g., $\sum_{n=0}^{+\infty} \alpha_n = +\infty, \alpha_n \rightarrow 0$ as $n \rightarrow +\infty$ and for an augmented sequence $\theta_n, \frac{\sum_{n=1}^N \alpha_n \theta_n}{\sum_{n=1}^N \alpha_n} \rightarrow 0, N \rightarrow 0$, then it converges to the unique equilibrium of the extended pricing game.*

B. A Distributed Proximal Algorithm

We assume a fully connected communication graph between the SPs, e.g., $\Gamma_s \triangleq \mathcal{S} \setminus \{s\}, \forall s \in \mathcal{S}$. We want to compute the r -normalized Nash equilibrium solution of \mathcal{G}_p relying on a fully distributed algorithm. To that purpose, we set $x_s^s \triangleq d_s$ as SP s 's own action, x_{-s}^s as SP s 's estimate of the other SPs' actions, and $x^s \triangleq \text{col}(x_s^s, x_{-s}^s)$ as the concatenation of SP s 's own action and estimate of the others' actions. Let $\bar{\mathcal{F}}_s \triangleq \{x_s^s | x_s^s \geq 0, \omega^T x_s^s = B_s\}$ be the strategy set of SP s . Following [38], [39], we decompose the pricing game \mathcal{G}_p per agent. Some slack variables $(v^{ss'})_{s,s'}$ and $(w_{ss'})_{s,s'}$ are introduced to guarantee the coincidence of the local copies. Let M_{S-1} be the matrix made of $S-1$ blocks, each one of them containing the Identity matrix of size $\sum_c |\mathcal{M}^c| \times \sum_c |\mathcal{M}^c|$. Each SP s solves the local optimization problem

$$\min_{\lambda_s \geq 0, (v_{ss'})_{s'}, x_s^s \in \bar{\mathcal{F}}_s, (v^{ss'})_{s'}} \max_{x_{-s}^s \in \bar{\mathcal{F}}_s, (v^{ss'})_{s'}} \left[\mathcal{R}_s(x_s^s, x_{-s}^s) - \lambda_s^T (x_s^s + M_{S-1} x_{-s}^s - D) \right], \quad (45a)$$

$$\text{s.t. } x^{s'} = v^{s's''}, \forall s' \in \mathcal{S}, \forall s'' \in \Gamma_{s'}, (\alpha^{s's''}) \quad (45b)$$

$$x^{s''} = v^{s's''}, \forall s' \in \mathcal{S}, \forall s'' \in \Gamma_{s'}, (\beta^{s's''}) \quad (45c)$$

$$\lambda_{s'} = r_{s'} w_{s's''}, \forall s' \in \mathcal{S}, \forall s'' \in \Gamma_{s'}, (\gamma_{s's''}) \quad (45d)$$

$$\lambda_{s''} = r_{s''} w_{s's''}, \forall s' \in \mathcal{S}, \forall s'' \in \Gamma_{s'}, (\delta_{s's''}) \quad (45e)$$

where $r_{s'} = \frac{1}{p_{s'}}$ and $r_{s''} = \frac{1}{p_{s''}}, \forall s' \in \mathcal{S}, s'' \in \Gamma_{s'}$. Note that we use the convention to have superscript indices for primal variables, and lower script indices for duals of \mathcal{G}_p . A solution of \mathcal{G}_p is obtained by assuming that each SP s solves the partial dual optimization problem (45) and by identifying $x_s^s = d_s$ and $\lambda_s = \frac{1}{r_s} \lambda$.

Let $\zeta > 0$ be a scalar coefficient. We follow the alternating direction method of multipliers (ADMM). To that purpose, we explicit the Lagrangian function associated with (45)

$$\begin{aligned} & L_s(x^s, \{v, \alpha, \beta\}, \lambda_s, \{w, \gamma, \delta\}) \\ & := \mathcal{R}_s(x^s) - \lambda_s^T (x_s^s + M_{S-1} x_{-s}^s - D) \\ & - \sum_{s'} \sum_{s'' \in \Gamma_{s'}} \left[(\alpha^{s's''})^T (x^{s'} - v^{s's''}) + (\beta^{s's''})^T (x^{s''} - v^{s's''}) \right] \\ & + \sum_{s'} \sum_{s'' \in \Gamma_{s'}} \left[\gamma_{s's''} (\lambda_{s'} - r_{s'} w_{s's''}) + \delta_{s's''} (\lambda_{s''} - r_{s''} w_{s's''}) \right], \end{aligned}$$

and associated KKTs, which give rise to the following relationships: $\alpha^{ss'} + \beta^{ss'} = 0$ and $\gamma_{ss'} + \delta_{ss'} = 0, \forall s' \in \Gamma_s$.

To update the SPs' strategies, we rely on the augmented Lagrangian associated with (45): $\bar{L}_s(x^s, \{v, \alpha, \beta\}, \lambda_s, \{w, \gamma, \delta\}) \triangleq L_s(x^s, \{v, \alpha, \beta\}, \lambda_s, \{w, \gamma, \delta\}) - \frac{\zeta}{2} \left(\sum_{s'} \sum_{s'' \in \Gamma_{s'}} (\|x^{s'} - v^{s's''}\|^2 + \|x^{s''} - v^{s's''}\|^2) \right) + \frac{\zeta}{2} \left(\sum_{s'} \sum_{s'' \in \Gamma_{s'}} ((\lambda_{s'} - r_{s'} w_{s's''})^2 + (\lambda_{s''} - r_{s''} w_{s's''})^2) \right)$. Following vanilla ADMM, the duals in (45) are updated according to the rules

$$\alpha^{s's''}(t) = \alpha^{s's''}(t-1) + \frac{\zeta}{2} (x^{s'}(t-1) - x^{s''}(t-1)), \quad (46a)$$

$$\beta^{s's''}(t) = \beta^{s's''}(t-1) + \frac{\zeta}{2} (x^{s''}(t-1) - x^{s'}(t-1)), \quad (46b)$$

$$\gamma_{s's''}(t) = \gamma_{s's''}(t-1) + \frac{\zeta}{2} \left(\frac{\lambda_{s'}(t-1)}{r_{s'}} - \frac{\lambda_{s''}(t-1)}{r_{s''}} \right), \quad (46c)$$

$$\delta_{s's''}(t) = \delta_{s's''}(t-1) + \frac{\zeta}{2} \left(\frac{\lambda_{s''}(t-1)}{r_{s''}} - \frac{\lambda_{s'}(t-1)}{r_{s'}} \right). \quad (46d)$$

We update the slacks v, w by solving the following optimization problems

$$v^{ss'}(t) = \arg \max_{v^{ss'}} \bar{L}_s \left(x^s(t-1), \{v, \alpha(t), \beta(t)\}, \lambda_s(t-1), \right. \quad (47a)$$

$$\left. \{w(t-1), \gamma(t), \delta(t)\} \right),$$

$$w_{ss'}(t) = \arg \min_{w_{ss'}} \tilde{L}_s \left(x^s(t-1), \{v(t), \alpha(t), \beta(t)\}, \lambda_s(t-1), \right. \quad (47b)$$

$$\left. \{w, \gamma(t), \delta(t)\} \right).$$

Assuming that $\alpha^{ss'}(0) = \beta^{ss'}(0) = 0$ and $\gamma_{ss'}(0) = \delta_{ss'}(0) = 0$ and relying on (46a)-(46d), the slack update rules (47a)-(47b) give rise to the following closed form expressions

$$v^{s's''}(t) = \frac{1}{2} (x^{s'}(t-1) + x^{s''}(t-1)), \quad (48a)$$

$$w_{s's''}(t) = \frac{1}{2} \left(\frac{\lambda_{s'}(t-1)}{r_{s'}} + \frac{\lambda_{s''}(t-1)}{r_{s''}} \right). \quad (48b)$$

Set $\Phi^s \triangleq \sum_{s' \in \Gamma_s} (\alpha^{ss'} + \beta^{ss'})$ and $\Psi_s \triangleq \sum_{s' \in \Gamma_s} (\gamma_{ss'} + \delta_{ss'})$. From (46a)-(46b) and (46c)-(46d), we get that Φ and Ψ

are updated according to the rules

$$\Phi^s(t) = \Phi^s(t-1) + \zeta \sum_{s' \in \Gamma_s} (x^s(t-1) - x^{s'}(t-1)), \quad (49a)$$

$$\Psi_s(t) = \Psi_s(t-1) + \zeta \sum_{s' \in \Gamma_s} \left(\frac{\lambda_s(t-1)}{r_s} - \frac{\lambda_{s'}(t-1)}{r_{s'}} \right). \quad (49b)$$

Let $\tilde{\beta}_s > 0$ be a penalty factor for the proximal first-order approximation for $s \in \mathcal{S}$.

Following [39], from (48a)-(48b), the primal update rule for SP s is obtained by solving a local optimization problem

$$\begin{aligned} x_s^s(t) = \arg \max_{x_s^s \in \tilde{\mathcal{F}}_s} & \left\{ \nabla_{x_s^s} \mathcal{R}_s(x_s^s(t-1))^T (x_s^s - x_s^s(t-1)) \right. \\ & - \frac{1}{2\zeta|\Gamma_s|} \left[2\zeta r_s \sum_{s' \in \Gamma_s} \frac{\lambda_s(t-1) + \lambda_{s'}(t-1)}{r_s + r_{s'}} - \Psi_s(t) \right. \\ & + x_s^s(t-1) + M_{S-1} x_{-s}^s(t-1) - D \left. \right]^T (x_s^s - x_s^s(t-1)) \\ & - \frac{\tilde{\beta}_s}{2} \|x_s^s - x_s^s(t-1)\|^2 - \Phi_s^s(t)^T x_s^s \\ & \left. - \zeta \sum_{s' \in \Gamma_s} \|x_s^s - \frac{x_s^s(t-1) + x_{s'}^s(t-1)}{2}\|^2 \right\}. \end{aligned} \quad (50)$$

Dual update rule takes the form

$$\begin{aligned} \lambda_s(t) = \text{proj}_{\mathbb{R}_+^{\sum_{c \in \mathcal{M}^c} |\mathcal{M}^c|}} & \left(\frac{1}{2\zeta|\Gamma_s|} (x_s^s(t) + M_{S-1} x_{-s}^s(t-1) - D \right. \\ & \left. - \Psi_s(t) + 2\zeta r_s \sum_{s' \in \Gamma_s} \frac{\lambda_s(t-1) + \lambda_{s'}(t-1)}{r_s + r_{s'}} \right). \end{aligned} \quad (51)$$

Let $\tilde{\mathcal{F}}_{-s} \triangleq \prod_{s' \neq s} \tilde{\mathcal{F}}_{s'} \subseteq \mathbb{R}_+^{(S-1) \sum_{c \in \mathcal{M}^c} |\mathcal{M}^c|}$. It is a closed set as the product of closed sets. The mapping $\text{proj}_{\tilde{\mathcal{F}}_{-s}} : \mathbb{R}_+^{(S-1) \sum_{c \in \mathcal{M}^c} |\mathcal{M}^c|} \rightarrow \tilde{\mathcal{F}}_{-s}$ denotes the projection onto $\tilde{\mathcal{F}}_{-s}$. Update of SP s 's estimates can be obtained as

$$\begin{aligned} x_{-s}^s(t) = \text{proj}_{\tilde{\mathcal{F}}_{-s}} & \left(\frac{1}{2} (x_{-s}^s(t-1) + \frac{1}{|\Gamma_s|} \sum_{s' \in \Gamma_s} x_{-s}^{s'}(t-1)) \right. \\ & \left. - \frac{1}{2\zeta|\Gamma_s|} \Phi_{-s}^s(t) \right). \end{aligned} \quad (52)$$

Theorem 5. If $f_s^c(\cdot)$ is K_s^c Lipschitz continuous for all $s \in \mathcal{S}, c \in \mathcal{C}$, Algorithm 3 converges to the r -normalized Nash equilibrium solution to \mathcal{G}_p .

Proof. See Appendix E. \square

VIII. NUMERICAL EXPERIMENTS

In this section, we analyse numerically the dynamic resource allocation schemes proposed in our paper. In our simulations, we primarily focus on a network with two cells, C 1 and C 2, and two service providers SP 1 and SP 2, who request resources for their service provision. This setting allows us to efficiently study the dynamics of interaction between EUs and SPs, and the effect of different system parameters on the outcome of the game \mathcal{G} .

We assume there are 200 and 300 EUs present in the cell C1 and C2, respectively. First, we consider the simple case of a single resource where the QoS offered by the slices only depends on the radio resource (bandwidth). The plot in

Algorithm 3 Distributed Proximal Algorithm for \mathcal{G}_p

Require: $\zeta > 0, \tilde{\beta}_s > 0, \forall s \in \mathcal{S}, \epsilon_{stop}^{primal}, \epsilon_{stop}^{dual}, t_{max}$

- 1: $\#$ Initialization Step
 - 2: Each SP s builds initial estimate $x^s(0) \in \tilde{\mathcal{F}}$ and $\lambda_s(0) \geq 0$
 - 3: Set $\alpha^{ss'} = \beta^{ss'} = 0$ and $\gamma_{ss'} = \delta_{ss'} = 0, \forall s \in \mathcal{S}, \forall s' \in \Gamma_s$
 - 4: **while** $\epsilon_{stop}^{primal}(t) \geq \epsilon_{stop}^{primal} \vee \epsilon_{stop}^{dual}(t) \geq \epsilon_{stop}^{dual} \wedge t \leq t_{max}$
 - 5: $\#$ Communication Step
 - 6: Each SP s exchanges his previous estimate $x^s(t-1)$ and his dual Lagrange multiplier $\lambda_s(t-1)$ with his neighbors $s' \in \Gamma_s$
 - 7: $\#$ Action Step Update
 - 8: **for each** SP $s \in \mathcal{S}$
 - 9: $\Phi^s(t)$ is updated according to (49a)
 - 10: $\Psi^s(t)$ is updated according to (49b)
 - 11: $x_s^s(t)$ is updated by solving (50)
 - 12: $\lambda_s(t)$ is updated according to (51)
 - 13: $x_{-s}^s(t)$ is updated according to (52)
 - 14: **end for**
 - 15: $t = t + 1$
 - 15: **end while**
-

Fig.3(c) illustrate the impact of the price parameter on the number of EUs in the slices at the NE of \mathcal{G} . For this simulation, we assume that the price applied by the slice 1 is constant and equal to 5, and we vary the fee applied by SP 2 in the range of 0 to 10. Fig2.(c) shows the change in the distribution of EUs in the slices as function of the price applied by SP 1. In the same figure we analyze the effect of slices shares on the distribution of users at the outcome of the game. The regular line in red and blue shows the distribution of users with slice 1 and slice 2 as a function of the price provided by the slices and when the slices are assigned with 10% share and 90% share of the infrastructure, respectively. The plots with the dash line, dot line and dot-dash line are outcome when 30%, 70% and 90% of share are assigned to SP 1. Under the same settings, Fig.2(d) illustrate the impact of the price applied by the slices and their infrastructure share on their profits. As second case, we consider that the QoS provided by SP s in cell c is given by $q_s^c = (d_s^c)^{\rho_s^c}$ where ρ_s^c is sensitivity parameter and $0 < \rho_s^c \leq 1$, we vary ρ_2^c , i.e., the SP 2 sensitivity parameter in cell C 2 from 0.1 to 1. Fig.2(b) shows the comparison of profit gain by SPs at NE with the profit gained under static resource allocation scheme (SS). For the multi-resource case, we consider that the SP QoS depends on their bandwidth as well as power allocation. To be precise, we assume that the QoS is the maximum possible data rate that SP can achieve. It is given by

$$q_s = B_s \log_2 \left(1 + \frac{h^2 P_s}{N_0} \right),$$

where B_s and P_s is bandwidth and power allocated to SP s respectively, while h is the channel gain and N_0 the noise. For simulations purpose, we assume that the availability of maximum bandwidth and transmitting power at each base station is 30 Mhz and 47 dBm respectively. The prices applied

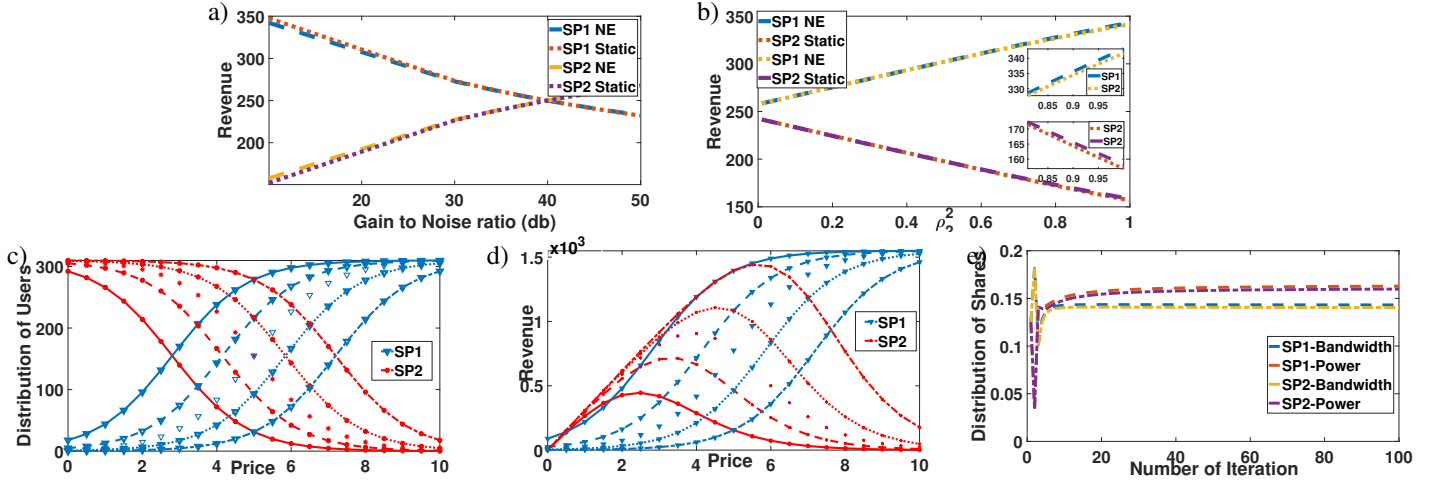


Fig. 3. Comparison between the revenue gained by the SPs at the NE of game G vs the revenue gain under static proportional allocation scheme (SS) a) for different value of SP 2 power to noise ratio at cell C2, b) for different values of ρ_2^2 . c) Distribution of users at the NE wrt fees charged by SPs. d) Revenue gained by SPs wrt fees charged by them. e) Convergence of Algorithm 1 to the NE

by each SP is constant 1, and each SP is assigned with half of the infrastructure share. For the numerical experiments, we vary the channel gain to noise ratio for SP2 at cell C2 from 10db to 50db; for each value of the channel gain to noise ratio, we compute the Nash equilibrium. The Fig.3(a) shows a comparison between the profit gain by SPs at nash equilibrium with profit gain by SPs under a static resource allocation scheme. The numerical results show a tiny difference between the SPs' profits under the dynamic resource sharing scheme and static resource allocation. In Fig.3(e), we observe the fast convergence of distributions of budgets on the different resources at different cells by SPs to a Nash equilibrium of game through an exponential learning scheme.

Next, we present the numerical results to validate the pricing and multi-resource allocation scheme for the second case where SPs have no budgets. In this scenario, particularly to define the QoS as a function of multi resources, we consider a general class of utility function known as CES (constant elasticity of substitution), mathematically defined as

$$q_s(d_s) = \left(\sum_r v_{sr}(d_{sr})^\rho \right)^{1/\rho},$$

where $\rho \in (-\infty, 0) \cup (0, 1]$ parameterizes the whole family of utility functions. For example $\rho = 1$ corresponds to linear (additive) valuations $q_s(d_s) = \sum_r v_{sr}d_{sr}$, $\rho \rightarrow 0$ corresponds to Cobb Douglas function $q_s(d_s) = \prod_r (d_{sr})^{v_{sr}}$, $\rho \rightarrow -\infty$ correspond to Leontief utilities $q_s(d_s) = \min_r \{ \frac{d_{sr}}{v_{sr}} \}$, and $D_s = (v_{s1} \dots v_{sr})$ where v_{sr} is the amount of resource type r needed by SP s to support one unit of QoS. Linear valuation signifies the perfect substitutes, representing a scenario where the resources can replace each other in utilization. Contrary, Leontief utilities represent the perfect complement scenario where one resource may have no value without the other. For instance, the CPU and computer memory are both essential for completing a computing task. CSE utility function interpolate between perfect substitutes and the

perfect complement through the parameter ρ . For numerical experiments, we consider that each SP needs three types of resources, namely, Bandwidth (Gbps), vCPU, Memory (GB). For SP1, we consider $v_{1,BW} = 10$ and $v_{1,vCPU} = 32$ and $v_{1,MEM} = 244$. Similarly for SP2 $v_{2,BW} = 10$ and $v_{2,vCPU} = 40$ and $v_{2,MEM} = 160$.

We consider that the total available capacity of bandwidth and vCPU is fixed at 40 GB and 60 units, respectively. For the memory, we vary the available capacity from 100 GB to 400 GB, and we examine the its effects on the resources' prices.

First, we consider the case where the QoS provided by SPs follows a substitutive relationship between resources, *e.g.*, we rely on the CSE function with $\rho = 0.1$. Fig.4(a) illustrates the effect of available capacity of resources on the resources price. As the total availability of memory increases. In this case, the cost of memory decreases. However, as the relationship between the resources is substitutive, we observe from the figure that a change in memory availability does not affect the price of the other resources.

Next, we consider the scenario where the SPs' QoS is defined by the CSE function with $\rho = -1.5$. In this case, the relationship between the resources is more complementary than the previous one. Fig.4(b) illustrates the effect of available capacity of resources on the resources price. As the availability of memory increases, the cost of memory decreases. However, in this case, we observe that a change in memory availability also affects the price of the other resources. The cost of bandwidth and VCPU also increases with a rise in memory's availability. An increase in the capacity of memory gives SPs room to improve their QoS, but at the cost of increasing the other related resources, Bandwidth and VCPU. Thus, it causes congestion at Bandwidth and VCPU hence resulting in a rise in their prices. Similarly, Fig.4(d) presents a change in the prices of the resources with respect to the capacity of the memory where QoS is considered a CSE function with $\rho = -2.5$. In Fig.4(d) we demonstrate the fast convergence of Algorithm 3. For simulation purposes, we consider the availability of

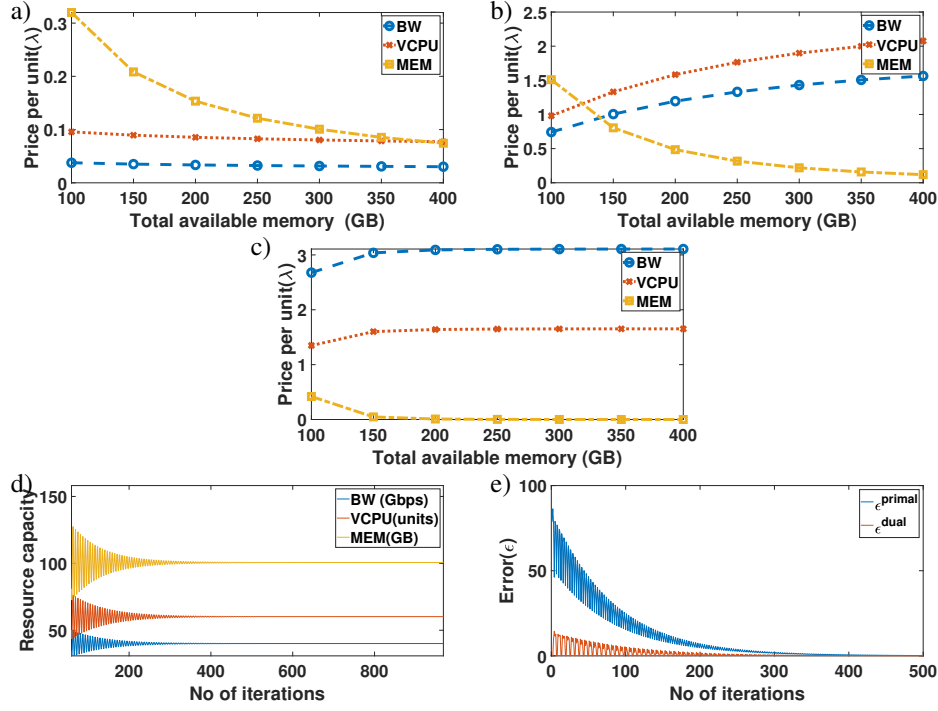


Fig. 4. Changes in the price (λ) with respect to the available memory capacity, with a) $\rho = 0.1$, b) $\rho = -1.5$, and c) $\rho = -2.5$. d) Convergence of Algorithm 3, convergence of total resources' demand to the available capacity. e) Convergence of the primal and dual errors in Algorithm 3.

Bandwidth, VCPU and Memory as 40 Gbps 60 units and 100 GB, respectively. The plot in the figure shows the exact convergence of total demand for all three resources to their available capacity. Fig.4(e) shows the convergence of error in decision variable (resources' demand) by the SPs and the convergence of error in price λ .

IX. CONCLUSION

In this work, we have considered a setting where service providers lease resources from an infrastructure provider through a network slicing mechanism and compete with one another to serve a large pool of end-users. We have shown that the interactions between the end-users and service providers can be modelled as a Stackelberg game, where the service providers act as leaders and the end-users as followers. In addition, we have proved that the competition between the service providers results in a multi-resource Tullock rent-seeking game, which admits a unique Nash equilibrium. The market price is computed by the infrastructure provider for each resource, taking into account the finite capacity of the network. To compute the market price and resource allocation, we have proposed two innovative market mechanisms. First, we have implemented a trading post mechanism taking into account the fact that the SPs have bounds on their budgets. We have proved that the noncooperative game induced by the trading post mechanism admits a unique Nash equilibrium in case a single resource is considered. We have implemented a semi-decentralized exponential learning algorithm to compute the unique Nash equilibrium of this game. However, this mechanism does not enable an explicit incorporation of the

network finite capacity constraints. To overcome that limitation, in a second design, when SPs have no bound on their budgets but take into account the network finite capacity as a global coupling constraint, we have shown that the market equilibrium can be obtained by solving a generalized Nash equilibrium problem. We have provided a dual averaging-based semi-decentralized algorithm to compute solution of the extended game reformulation of the pricing game, and a proximal inexact- ADMM based distributed algorithm that provably converges to the variational equilibrium of the pricing game. Finally, we have provided numerical results to analyse the economic properties of the two market designs, and confirm the fast convergence rate of the inexact-ADMM highlighting its practical applicability.

APPENDIX

A. Proof of Proposition 1

To find the equilibrium of the replicator dynamics defined in (6) consider $\log\left(\frac{q_s^c}{n_s^c}\right) - p_s = \log\left(\frac{q_{s'}^c}{n_{s'}^c}\right) - p_{s'}$. Taking the exponential of both sides, we obtain $\frac{q_s^c}{n_s^c} \frac{n_{s'}^c}{q_{s'}^c} = e^{p_s - p_{s'}} \Leftrightarrow \frac{q_s^c}{n_s^c} n_{s'}^c = q_{s'}^c e^{p_s - p_{s'}}$. Summing over $\forall s' \in \mathcal{S}$ gives us $\sum_{s'} \frac{q_s^c}{n_s^c} n_{s'}^c = \sum_{s'} q_{s'}^c e^{p_s - p_{s'}}$, which can be rewritten as $n_s^c = \frac{N^c q_s^c e^{-p_s}}{\sum_{s'} q_{s'}^c e^{-p_{s'}}}$, i.e., $n_s^c = \frac{N^c f_s^c(d_s^c) e^{-p_s}}{\sum_{s'} f_{s'}^c(d_{s'}^c) e^{-p_{s'}}}$.

B. Proof of Proposition 2

First, we show that the utility of each SP is concave in its own decision variable. $\frac{\partial^2 U_s}{\partial b_1^c{}^2} = \frac{A+B}{C} < 0$

$$A = - \left(\frac{b_2^c}{b_1^c + b_2^c} \right)^{\rho_2} \left(\frac{b_1^c}{b_1^c + b_2^c} \right)^{2\rho_1} \left((\rho_2^2 + \rho_2) b_1^{c2} + 2\rho_1 b_2^c(\rho_2 + 1) b_1^c + \rho_1 b_2^{c2}(\rho_1 + 1) \right) \quad (53)$$

$$B = \left(\frac{b_2^c}{b_1^c + b_2^c} \right)^{2\rho_2} \left(\frac{b_1^c}{b_1^c + b_2^c} \right)^{\rho_1} \left((\rho_2^2 - \rho_2) b_1^{c2} + 2\rho_1 d_2^c(\rho_2 - 1) b_1^c + \rho_1 b_2^{c2}(\rho_1 - 1) \right) \quad (54)$$

$$C = (b_1^c + b_2^c)^2 b_1^{c2} \left(\left(\frac{b_1^c}{b_1^c + b_2^c} \right)^{\rho_1} + \left(\frac{b_2^c}{b_1^c + b_2^c} \right)^{\rho_2} \right)^3 \quad (55)$$

Next, we show that the utility of each SP is convex with respect to the opponents' decision variable. $\frac{\partial^2 U_s}{\partial b_2^c{}^2} = \frac{G+H}{I} > 0$

$$G = - \left(\frac{b_2^c}{b_1^c + b_2^c} \right)^{\rho_2} \left(\frac{b_1^c}{b_1^c + b_2^c} \right)^{2\rho_1} \left((\rho_1^2 - \rho_1) b_2^{c2} + 2\rho_2 b_1^c(\rho_1 - 1) b_2^c + \rho_2 b_1^{c2}(\rho_2 - 1) \right) \quad (56)$$

$$H = \left(\frac{b_2^c}{b_1^c + b_2^c} \right)^{2\rho_2} \left(\frac{b_1^c}{b_1^c + b_2^c} \right)^{\rho_1} \left((\rho_1^2 + \rho_1) b_2^{c2} + 2\rho_2 b_1^c(\rho_1 + 1) b_2^c + \rho_2 b_1^{c2}(\rho_2 + 1) \right) \quad (57)$$

$$I = (b_1^c + b_2^c)^2 b_2^{c2} \left(\left(\frac{b_1^c}{b_1^c + b_2^c} \right)^{\rho_1} + \left(\frac{b_2^c}{b_1^c + b_2^c} \right)^{\rho_2} \right)^3 \quad (58)$$

C. Proof of Proposition 3

Consider that for any bid $b_2^c > 0$ submitted by SP 2 at cell c , SP 1 places a bid of $b_1^c = B_1 \frac{b_2^c}{B_2}$ at cell c . Then, the quantity of resource received by SP 1 at cell c is $d_1^c = \frac{B_1 \frac{b_2^c}{B_2}}{B_1 \frac{b_2^c}{B_2} + b_2^c} = \frac{B_1}{B_1 + B_2}$.

This proves that for any strategy played by SP there exists a strategy for the other SP such that he receives the resource in proportion to his budget.

D. Revision Protocol

Let recall the revision protocol, which defines the switching rate at which users switch their choice from SP s to SP s' given population state n $\tau_{s,s'} = n_{s'} [\nu_{s'} - \nu_s]_+$. Note that for the sake of simplicity, we omit the cell dependence (c).

Relying on the evolutionary process (4) and by substitution of the revision protocol, we get

$$\begin{aligned} \dot{n}_s &= \sum_{s'} n_{s'} \tau_{s',s} - n_s \sum_{s'} \tau_{s,s'}, \\ \dot{n}_s &= \sum_{s'} n_{s'} n_s [\nu_s - \nu_{s'}]_+ - n_s \sum_{s'} n_{s'} [\nu_{s'} - \nu_s]_+, \\ \dot{n}_s &= n_s \sum_{s'} n_{s'} [\nu_s - \nu_{s'}], \\ \dot{n}_s &= n_s \left[\nu_s - \sum_{s'} n_{s'} \nu_{s'} \right]. \end{aligned}$$

Similarly, considering the revision protocol $\tau_{s,s'} = \frac{n_{s'}}{N} [U_{s'}^c - U_s^c]_+$, we get

$$\begin{aligned} \dot{n}_s &= \sum_{s'} n_{s'} \tau_{s',s} - n_s \sum_{s'} \tau_{s,s'}, \\ \dot{n}_s &= \sum_{s'} n_{s'} \frac{n_s}{N} [\nu_s^c - \nu_{s'}^c]_+ - n_s \sum_{s'} \frac{n_{s'}}{N} [\nu_{s'}^c - \nu_s^c]_+, \\ \dot{n}_s &= n_s \sum_{s'} \frac{n_{s'}}{N} [\nu_s - \nu_{s'}], \\ \dot{n}_s^c &= n_s^c \left[\nu_s^c - \frac{1}{N} \sum_{s'} n_{s'}^c \nu_{s'}^c \right]. \end{aligned}$$

E. Proof of Theorem 5

Taking the gradient of $\mathcal{R}_s(\cdot)$ with respect to $x_s^s := d_s$, we obtain:

$$\nabla_{x_s^s} \mathcal{R}_s(d) = \sum_c p_s N^c \frac{\nabla_{x_s^s} f_s^c(d_s^c) e^{-p_s}}{(\sum_{s'} f_{s'}^c(d_{s'}^c) e^{-p_{s'}})^2} \sum_{s' \neq s} f_{s'}^c(d_{s'}^c) e^{-p_{s'}}.$$

For any $d, \tilde{d} \in \mathcal{F}$, $\|\nabla_{x_s^s} \mathcal{R}_s(d) - \nabla_{x_s^s} \mathcal{R}_s(\tilde{d})\| \leq \sum_c p_s N^c \max \left\{ \frac{\nabla_{x_s^s} f_s^c(d_s^c) e^{-p_s}}{(\sum_{s'} f_{s'}^c(d_{s'}^c) e^{-p_{s'}})^2}; \frac{\nabla_{x_s^s} f_s^c(\tilde{d}_s^c) e^{-p_s}}{(\sum_{s'} f_{s'}^c(\tilde{d}_{s'}^c) e^{-p_{s'}})^2} \right\}.$

$\|\sum_{s' \neq s} f_{s'}^c(\tilde{d}_{s'}^c) e^{-p_{s'}} - \sum_{s'} f_{s'}^c(d_{s'}^c) e^{-p_{s'}}\|$ by Hölder inequality. Then, applying Jensen's inequality, we obtain that $\|\nabla_{x_s^s} \mathcal{R}_s(d) - \nabla_{x_s^s} \mathcal{R}_s(\tilde{d})\| \leq \sum_c p_s N^c \max \left\{ \frac{\nabla_{x_s^s} f_s^c(d_s^c) e^{-p_s}}{(\sum_{s'} f_{s'}^c(d_{s'}^c) e^{-p_{s'}})^2}; \frac{\nabla_{x_s^s} f_s^c(\tilde{d}_s^c) e^{-p_s}}{(\sum_{s'} f_{s'}^c(\tilde{d}_{s'}^c) e^{-p_{s'}})^2} \right\}.$

$\sum_{s' \neq s} \|f_{s'}^c(d_{s'}^c) - f_{s'}^c(\tilde{d}_{s'}^c)\|$. If $f_s^c(\cdot)$ is K_s^c Lipschitz continuous then by setting $L_s := \sum_c p_s N^c \max \left\{ \frac{\nabla_{x_s^s} f_s^c(d_s^c) e^{-p_s}}{(\sum_{s'} f_{s'}^c(d_{s'}^c) e^{-p_{s'}})^2}; \frac{\nabla_{x_s^s} f_s^c(\tilde{d}_s^c) e^{-p_s}}{(\sum_{s'} f_{s'}^c(\tilde{d}_{s'}^c) e^{-p_{s'}})^2} \right\}$ then for any $d, \tilde{d} \in \mathcal{F}$, $\|\nabla_{x_s^s} \mathcal{R}_s(d) - \nabla_{x_s^s} \mathcal{R}_s(\tilde{d})\| \leq L_s \|d - \tilde{d}\|_1 \leq L_s \cdot \sqrt{S \cdot \sum_c |\mathcal{M}^c|} \|d - \tilde{d}\|_2$. This proves that $\nabla_{x_s^s} \mathcal{R}_s(\cdot)$ is $L_s \cdot \sqrt{S \cdot \sum_c |\mathcal{M}^c|}$ Lipschitz continuous. In addition, the coupling constraints in the pricing game \mathcal{G}_p are linear in the SPs' decision variables. Though we introduce projection operators in (51), (52), Cauchy-Schwarz inequality implies that the norm of the projection matrix can be upper bounded by 1. This enables us to derive the same upper bound and statement as in [39], Thm.1.

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