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Gregor Gantner, Martin Vohralík

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# INEXPENSIVE POLYNOMIAL-DEGREE- AND NUMBER-OF-HANGING-NODES-ROBUST EQUILIBRATED FLUX A POSTERIORI ESTIMATES FOR ISOGEOMETRIC ANALYSIS

GREGOR GANTNER AND MARTIN VOHRALÍK

ABSTRACT. We consider isogeometric discretizations of the Poisson model problem, focusing on high polynomial degrees and strong hierarchical refinements. We derive a posteriori error estimates by equilibrated fluxes, i.e., vector-valued mapped piecewise polynomials lying in the  $\mathbf{H}(\text{div})$  space which appropriately approximate the desired divergence constraint. Our estimates are constant-free in the leading term, locally efficient, and robust with respect to the polynomial degree. They are also robust with respect to the number of hanging nodes arising in adaptive mesh refinement employing hierarchical B-splines. Two partitions of unity are designed, one with larger supports corresponding to the mapped splines, and one with small supports corresponding to mapped piecewise affine polynomials. The equilibration is only performed on the small supports, avoiding the higher computational price of equilibration on the large supports or even a global system solve. Thus, the derived estimates are also as inexpensive as possible. An abstract framework for such a setting is developed, whose application to a specific situation only requests a verification of a few clearly identified assumptions. Numerical experiments illustrate the theoretical developments.

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## 1. INTRODUCTION

A posteriori error estimates for standard finite element methods (FEM) are nowadays well understood [AO00, Rep08, Ver13]. On the one hand, they allow to assess the quality of the computed approximation, and, on the other hand, they indicate where to refine the underlying mesh of the computational domain. Among the existing error estimators, those based on equilibrated fluxes [PS47, LL83, DM99, BS08] have the advantage that they provide a *guaranteed* upper bound for the approximation error (guaranteed and constant-free in the leading term). The estimator from [BS08] even turns out to be robust with respect to the polynomial degree [BPS09], i.e., it provides also a lower bound with an efficiency constant that does *not* depend on the polynomial degree. This result has been recently generalized in several directions; see [EV15, EV20] and the references therein. We also mention [DEV16], which covers standard FEM on triangular/rectangular meshes with hanging nodes, and its generalization to arbitrarily many hanging nodes [ESV17].

In this work, we aim to generalize this result to isogeometric analysis (IGA) [HCB05, BBdVC<sup>+</sup>06, CHB09]. The central idea of IGA is to use the same ansatz functions for the representation of the problem geometry in computer-aided design (CAD) and for the discretization of the partial differential equation (PDE). While the CAD standard for spline representation in a multivariate setting relies on tensor-product B-splines, several extensions of the B-spline model have emerged that allow for adaptive refinement, e.g., hierarchical splines [VGJS11, GJS12], (analysis-suitable) T-splines [SZBN03, SLSH12, BdVBSV13], or LR-splines [DLP13, JKD14]; see also [JRK15, HKMP17, BGG<sup>+</sup>22] for a comparison of these approaches.

**1.1. Available results.** To steer an adaptive refinement, rigorous a posteriori error estimators have been developed. Assuming a certain admissibility condition of the employed meshes, the works [BG16b, GP20] generalize the weighted residual error estimator from standard FEM to IGA with hierarchical splines and analysis-suitable T-splines, respectively. It has even been proved that the corresponding adaptive algorithms converge at optimal algebraic rate with respect to the number of mesh elements [BG17, GHP17, GP20], see also the recent overview article [BGG<sup>+</sup>22]. However, the reliability and efficiency constants depend on the employed polynomial degree, which is indeed witnessed in numerical experiments, see, e.g., [BGG<sup>+</sup>22] for hierarchical splines. Similarly, for the hierarchical spline space from [BG16a], the work [BG18] derives a weighted residual estimator being the sum of indicators associated to basis functions instead of elements. It is shown to be reliable for *arbitrary* hierarchical meshes, with an unknown reliability constant that again particularly depends on the used polynomial degree  $p$ .

In the spirit of [Rep99, Rep00a, Rep00b], the works [KT15a, KT15b, Mat18] present guaranteed fully computable upper bounds of the approximation error for tensor-product splines and hierarchical splines, respectively. A second estimate is also derived, giving a lower bound of the error. However, to compute these so-called functional-type estimators,

a *global* minimization problem may need to be solved or an  $\mathbf{H}(\text{div})$  flux not in the equilibrium with the load may be employed, and the efficiency of the upper bound, as well as the reliability of the lower bound, are theoretically unclear. In the recent work [TCHM19], a non- $\mathbf{H}(\text{div})$  approximation of an equilibrated flux is constructed for tensor-product splines in extension of the the concepts of [LL83]. It only requires to solve locally, on the knot spans, a pair of a low-order problem for the normal fluxes together with a high-order problem for the equilibrated flux approximation. A generalization to hierarchical splines is briefly sketched and a corresponding numerical example is provided. While this yields a fully computable and approximately guaranteed upper bound on the error, efficiency of the resulting estimator is unclear.

**1.2. The present contribution.** In the present work, we address IGA discretizations by hierarchical splines of arbitrary degree and smoothness, on arbitrary underlying meshes. We construct an equilibrated flux in three stages, taking inspiration from [DEV16, ESV17]. First, we employ the partition of unity, on the whole computational domain, by mapped scaled hierarchical B-splines  $\psi_{\mathbf{a}}$ , as in [BG16a]. On each support of  $\psi_{\mathbf{a}}$ , we consider the coarsest tensor mesh containing the local mesh and solve a discrete minimization problem following Definition 6.1. This is a primal lowest- (first-)order problem yielding the scalar-valued auxiliary residual lifting  $r_h^{\mathbf{a}}$ . Then, we consider the partition of unity, on the support of  $\psi_{\mathbf{a}}$ , by the mapped piecewise affine hat functions  $\psi_{\mathbf{b}}$ . We solve a dual discrete minimization problem of high order on each (small) vertex patch in the support of  $\psi_{\mathbf{b}}$ . This is described in Definition 6.2 and yields the vector-valued equilibrated flux contribution  $\sigma_h^{\mathbf{a},\mathbf{b}}$ . We note that all the above local problems can be solved independently from each other, allowing for efficient parallelization. We finally sum the contributions  $\sigma_h^{\mathbf{a},\mathbf{b}}$  in Definition 6.5. The resulting equilibrated flux  $\sigma_h$  yields a fully computable guaranteed (constant-free in the leading term) upper bound on the unknown error; see Proposition 7.2. This bound is also locally and globally efficient, i.e., it is a lower bound for the (local or global) error up to oscillation terms; see respectively Proposition 7.4 and Proposition 7.6. The involved constant is robust with respect to the polynomial degree of the used hierarchical splines and the number of hanging nodes of the underlying hierarchical meshes, but not necessarily with respect to the smoothness of the splines. That said, in all given numerical experiments, the effectivity index (the ratio of estimator and error) is, *independently* of the smoothness, close to one.

**1.3. Outline.** The manuscript is organized as follows. We introduce the model problem and its Galerkin discretization in Section 2. Sections 3 and 4 then contain a detailed description of the discretization space formed by hierarchical B-splines. This technical description might be skipped in a first reading, since all our results are only based on a couple of abstract assumptions collected in Section 5, where we also verify them in the hierarchical B-splines setting. Section 6 introduces our inexpensive flux equilibration in the present context. The a posteriori error estimates are then derived in Section 7, where we prove their reliability and efficiency. Numerical experiments illustrating the theoretical findings are collected in Section 8. Finally, Appendix A proves the broken polynomial extension property that is central for the polynomial-degree robustness in the present setting.

## 2. MODEL PROBLEM AND ITS GALERKIN DISCRETIZATION

Let  $\Omega$  be an open bounded connected Lipschitz domain in  $\mathbb{R}^d$  with  $d = 2, 3$ . We consider the Poisson model problem with homogeneous Dirichlet data of finding  $u : \Omega \rightarrow \mathbb{R}$  such

that

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where  $f \in L^2(\Omega)$  is a given source term. On an arbitrary subset  $\omega \subseteq \overline{\Omega}$ , let  $(\cdot, \cdot)_\omega := \int_\omega (\cdot)(\cdot) dx$  and  $\|\cdot\|_\omega$  denote the  $L^2$ -scalar product and the corresponding norm, respectively; we also denote by  $\|\cdot\|_{\infty, \omega}$  the  $L^\infty$ -norm. Then, the weak formulation of problem (1) consists in finding  $u \in H_0^1(\Omega)$  such that

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \text{for all } v \in H_0^1(\Omega). \tag{2}$$

Let  $V_h$  be a finite-dimensional subspace of  $H_0^1(\Omega)$ . The corresponding Galerkin approximation is to find  $u_h \in V_h$  with

$$(\nabla u_h, \nabla v_h)_\Omega = (f, v_h)_\Omega \quad \text{for all } v_h \in V_h. \tag{3}$$

In this work, we will choose  $V_h$  as the space of (mapped) *hierarchical splines*. To this end, we assume that  $\Omega$  can be parametrized over  $\widehat{\Omega} := (0, 1)^d$  via a bi-Lipschitz mapping  $\mathbf{F} : \widehat{\Omega} \rightarrow \Omega$  with positive Jacobian determinant, i.e.,  $\Omega = \mathbf{F}(\widehat{\Omega})$ ,  $\mathbf{F} \in W^{1, \infty}(\widehat{\Omega})$ ,  $\mathbf{F}^{-1} \in W^{1, \infty}(\Omega)$ , and  $\det(D\mathbf{F}) > 0$ .

### 3. HIERARCHICAL SPLINES

In this section, we first describe the piecewise polynomial space of multivariate splines in the parameter domain  $\widehat{\Omega}$ . We then introduce its hierarchical extension, covering highly graded local mesh refinement. Finally, the space  $V_h$  from (3) will be given by the transformation of the latter one by the mapping  $\mathbf{F}$ .

**3.1. Multivariate splines in the parameter domain  $(0, 1)^d$ .** We recall here the standard definition of multivariate splines in the parameter domain  $(0, 1)^d$ ; for a detailed introduction, we refer, e.g., to [dB86, dB01, Sch07].

Let the integer  $p \geq 1$  be a fixed positive polynomial degree and let

$$\mathcal{K}_h = (\mathcal{K}_{1(h)}, \dots, \mathcal{K}_{d(h)})$$

be a fixed  $d$ -dimensional vector of  $p$ -open knot vectors, i.e., for each spatial dimension  $1 \leq i \leq d$ ,

$$\mathcal{K}_{i(h)} = (t_{i(h),0}, \dots, t_{i(h),N_{i(h)}+p})$$

is a  $p$ -open knot vector in  $[0, 1]$ , which means that

$$0 = t_{i(h),0} = \dots = t_{i(h),p} < t_{i(h),p+1} \leq t_{i(h),p+2} \leq \dots \leq t_{i(h),N_{i(h)}} = \dots = t_{i(h),N_{i(h)}+p} = 1.$$

Moreover, we assume that each of the interior knots  $t_{i(h),j} \in (0, 1)$  appears at most with multiplicity  $p$ . By definition, the boundary knots 0 and 1 have multiplicity  $p + 1$ . An example is given in Figure 1, with polynomial degree  $p = 2$ , number of knots minus  $p$  minus one  $N_{i(h)} = 9$  (this will later correspond to the dimension of the B-splines space), and a varying multiplicity.

We define the resulting one-dimensional meshes

$$\widehat{\mathcal{T}}_{i(h)} := \{[t_{i(h),j-1}, t_{i(h),j}] : j \in \{1, \dots, N_{i(h)} + p\} \wedge t_{i(h),j-1} < t_{i(h),j}\}, \quad i = 1, \dots, d,$$

as well as the resulting tensor mesh

$$\widehat{\mathcal{T}}_h := \{\widehat{T}_1 \times \dots \times \widehat{T}_d : \widehat{T}_i \in \widehat{\mathcal{T}}_{i(h)} \text{ for all } i \in \{1, \dots, d\}\}.$$

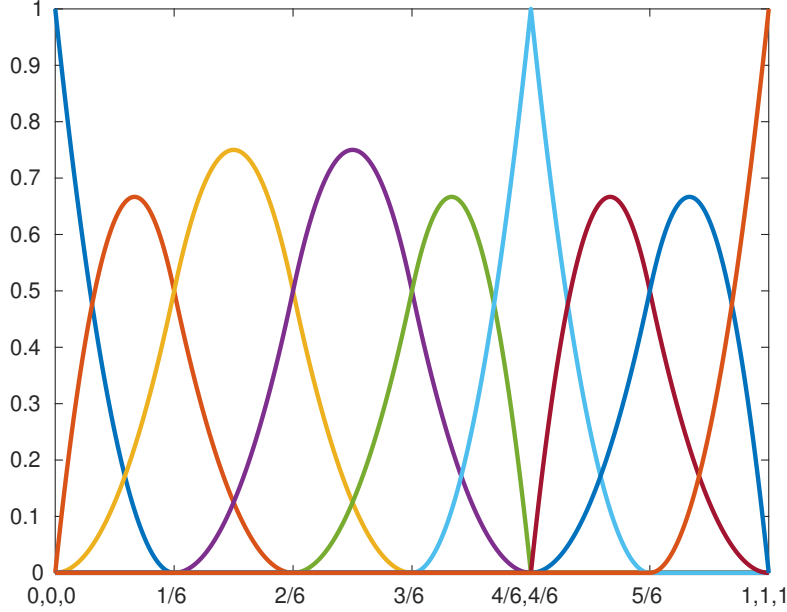


FIGURE 1. The B-splines of degree  $p = 2$  corresponding to the knot vector  $\mathcal{K} = (0, 0, 0, 1/6, 2/6, 3/6, 4/6, 4/6, 5/6, 1, 1, 1)$  are depicted. They are at least  $C^1$  at the knots  $1/6, 2/6, 3/6,$  and  $5/6$ , and at least  $C^0$  at  $4/6$ .

By  $S^p(\mathcal{K}_{i(h)})$ , we denote the set of all corresponding *univariate splines*, i.e., the set of all  $\widehat{\mathcal{T}}_{i(h)}$ -piecewise univariate polynomials of degree  $p$  that are  $p - \#t_{i(h),j}$ -times continuously differentiable at any interior knot  $t_{i(h),j}$ , where  $\#t_{i(h),j}$  denotes the corresponding multiplicity of  $t_{i(h),j}$  within  $\mathcal{K}_{i(h)}$ . Assuming for example that all interior multiplicities are equal to  $p$ ,  $S^p(\mathcal{K}_{i(h)})$  is just the space of all continuous  $\widehat{\mathcal{T}}_{i(h)}$ -piecewise univariate polynomials of degree  $p$ . At the other extreme, when all interior multiplicities are equal to 1, then  $S^p(\mathcal{K}_{i(h)})$  is the space of univariate polynomials of degree  $p$  with continuous derivatives up to order  $p - 1$ . We refer again to Figure 1 for an example.

A basis of  $S^p(\mathcal{K}_{i(h)})$ , of dimension  $N_{i(h)}$ , is given by the set of *B-splines*

$$\{B_{i(h),j,p} : j \in \{1, \dots, N_{i(h)}\}\},$$

where the B-splines  $B_{i(h),j,p}$  are recursively defined for all points  $t \in (0, 1)$  via

$$B_{i(h),j,0}(t) := \begin{cases} 1 & \text{if } t_{i(h),j} \leq t < t_{i(h),j+1}, \\ 0 & \text{else,} \end{cases} \quad (4a)$$

and

$$B_{i(h),j,p}(t) := \frac{t - t_{i(h),j}}{t_{i(h),j+p} - t_{i(h),j}} B_{i(h),j,p-1}(t) + \frac{t_{i(h),j+p+1} - t}{t_{i(h),j+p+1} - t_{i(h),j+1}} B_{i(h),j+1,p-1}(t) \quad (4b)$$

with the formal convention that  $\cdot/0 := 0$ . Figure 1 gives an illustrative example, where the  $N_{i(h)} = 9$  basis functions are depicted. It is easy to see that the support is

$$\text{supp}(B_{i(h),j,p}) = [t_{i(h),j}, t_{i(h),j+p+1}] \quad (5)$$

and we also remark that

$$0 \leq B_{i(h),j,p} \leq 1. \quad (6)$$

We abbreviate  $\mathbf{p} := (p, \dots, p) \in \mathbb{N}^d$ . The space  $S^{\mathbf{p}}(\mathcal{K}_h)$  of *multivariate splines* is defined as tensor-product of the univariate spline spaces. Note that each function in  $S^{\mathbf{p}}(\mathcal{K}_h)$  is a

$\widehat{\mathcal{T}}_h$ -piecewise multivariate polynomial of degree  $\mathbf{p}$ . Again, assuming for example that all interior knots have multiplicity  $p$ ,  $S^{\mathbf{p}}(\mathcal{K}_h)$  is just the space of all continuous  $\widehat{\mathcal{T}}_h$ -piecewise multivariate polynomials of degree  $\mathbf{p}$ , while multiplicity 1 of all interior knots yields the space of all  $\widehat{\mathcal{T}}_h$ -piecewise multivariate polynomials of degree  $\mathbf{p}$  with continuous derivatives up to order  $p - 1$ . Clearly, the set of tensor-products of the univariate B-splines

$$\{B_{h,\mathbf{j},\mathbf{p}} : \mathbf{j} \in \Pi_{i=1}^d \{1, \dots, N_{i(h)}\}\} \quad \text{with} \quad B_{h,\mathbf{j},\mathbf{p}} := B_{1(h),j_1,p} \otimes \dots \otimes B_{d(h),j_d,p}, \quad (7)$$

provides a basis of  $S^{\mathbf{p}}(\mathcal{K}_h)$ .

**3.2. Hierarchical splines in the parameter domain  $(0, 1)^d$ .** We now introduce hierarchical splines, which are defined on a hierarchical mesh and are essentially coarse splines on coarse mesh elements and fine splines on fine mesh elements. For a detailed introduction, we refer, e.g., to the seminal work [VGJS11].

Let  $p$  be a fixed positive polynomial degree as above and let  $\mathcal{K}_0 = (\mathcal{K}_{1(0)}, \dots, \mathcal{K}_{d(0)})$  be a fixed initial  $d$ -dimensional vector of  $p$ -open knot vectors in  $[0, 1]$  with interior multiplicities less than or equal to  $p$ , as in Section 3.1. Recall that we consider  $d = 2, 3$ . We set  $\mathcal{K}_{\text{uni}(0)} := \mathcal{K}_0$  and recursively define  $\mathcal{K}_{\text{uni}(\ell+1)}$  for  $\ell \in \mathbb{N}_0$  as the uniform  $h$ -refinement of  $\mathcal{K}_{\text{uni}(\ell)}$  with fixed multiplicity  $m \in \mathbb{N}$ , i.e., obtained by inserting the knot  $(t_{i(\text{uni}(\ell)),j-1} + t_{i(\text{uni}(\ell)),j})/2$  to the knots  $\mathcal{K}_{i(\text{uni}(\ell))}$  with multiplicity  $m$  whenever  $t_{i(\text{uni}(\ell)),j-1} < t_{i(\text{uni}(\ell)),j}$ . We stress that these spline spaces are nested in the sense that

$$S^{\mathbf{p}}(\mathcal{K}_{\text{uni}(\ell)}) \subset S^{\mathbf{p}}(\mathcal{K}_{\text{uni}(\ell+1)}) \subset C^0((0, 1)^d), \quad (8)$$

where the last relation follows from the assumption that multiplicities of interior knots is less than or equal to  $p$ .

We say that a set

$$\widehat{\mathcal{T}}_h \subseteq \bigcup_{\ell \in \mathbb{N}_0} \widehat{\mathcal{T}}_{\text{uni}(\ell)}$$

is a *hierarchical mesh* if it is a partition of  $[0, 1]^d$  in the sense that  $\bigcup \widehat{\mathcal{T}}_h = [0, 1]^d$ , where the intersection of two different elements  $\widehat{T} \neq \widehat{T}'$  with  $\widehat{T}, \widehat{T}' \in \widehat{\mathcal{T}}_h$  has ( $d$ -dimensional) measure zero. Since  $\widehat{\mathcal{T}}_{\text{uni}(\ell)} \cap \widehat{\mathcal{T}}_{\text{uni}(\ell')} = \emptyset$  for  $\ell, \ell' \in \mathbb{N}_0$  with  $\ell \neq \ell'$ , we can define for any element  $\widehat{T} \in \widehat{\mathcal{T}}_h$ ,

$$\text{level}(\widehat{T}) := \ell \in \mathbb{N}_0 \quad \text{with} \quad \widehat{T} \in \widehat{\mathcal{T}}_{\text{uni}(\ell)}.$$

For an illustrative example of a hierarchical mesh, see Figure 2. In particular, any uniformly refined tensor mesh  $\widehat{\mathcal{T}}_{\text{uni}(\ell)}$  with  $\ell \in \mathbb{N}_0$  is a hierarchical mesh.

For a hierarchical mesh  $\widehat{\mathcal{T}}_h$ , we define a corresponding nested sequence  $(\widehat{\Omega}_h^\ell)_{\ell \in \mathbb{N}_0}$  of closed subsets of  $[0, 1]^d$  as

$$\widehat{\Omega}_h^\ell := \bigcup_{\ell' \geq \ell} (\widehat{\mathcal{T}}_h \cap \widehat{\mathcal{T}}_{\text{uni}(\ell')}),$$

i.e.,  $\widehat{\Omega}_h^\ell$  covers all elements with level greater than or equal to  $\ell$ . Note that there exists a minimal integer  $L_h$  such that  $\widehat{\Omega}_h^\ell = \emptyset$  for all  $\ell \geq L_h$ . It holds that

$$\widehat{\mathcal{T}}_h = \bigcup_{\ell \in \mathbb{N}_0} \{\widehat{T} \in \widehat{\mathcal{T}}_{\text{uni}(\ell)} : \widehat{T} \subseteq \widehat{\Omega}_h^\ell \wedge \widehat{T} \not\subseteq \widehat{\Omega}_h^{\ell+1}\}. \quad (9)$$

In the literature, one usually assumes that one is given the sequence  $(\widehat{\Omega}_h^\ell)_{\ell \in \mathbb{N}_0}$  and defines the corresponding hierarchical mesh via (9).



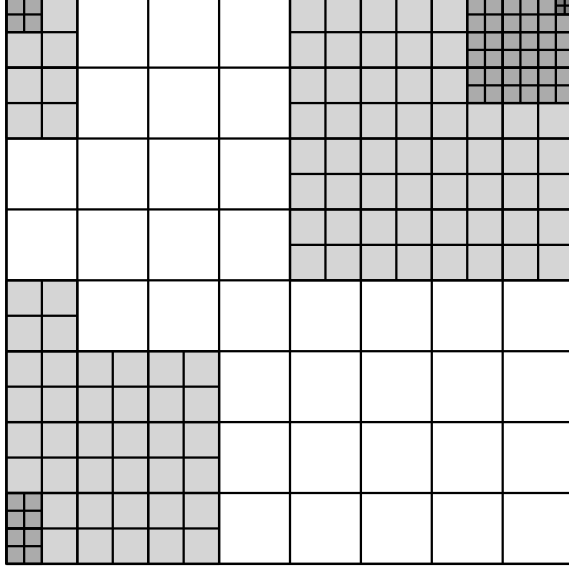


FIGURE 2. A two-dimensional hierarchical mesh  $\widehat{\mathcal{T}}_h$  with level of all elements less than 4. Levels 0 to 3 are respectively highlighted in white, light gray, gray, and dark gray, also denoting the corresponding domains  $\widehat{\Omega}_h^0 \setminus \widehat{\Omega}_h^1$ ,  $\widehat{\Omega}_h^1 \setminus \widehat{\Omega}_h^2$ ,  $\widehat{\Omega}_h^2 \setminus \widehat{\Omega}_h^3$ ,  $\widehat{\Omega}_h^3 \setminus \widehat{\Omega}_h^4$ ;  $\widehat{\Omega}_h^4 = \emptyset$ .

We introduce the *hierarchical basis*

$$\{B_{\mathbf{a}} : \mathbf{a} \in \mathcal{I}_h\} \quad \text{with} \quad \mathcal{I}_h := \bigcup_{\ell \in \mathbb{N}_0} \left\{ (\text{uni}(\ell), \mathbf{j}, \mathbf{p}) : \mathbf{j} \in \prod_{i=1}^d \{1, \dots, N_i(\text{uni}(\ell))\} \right. \quad (10)$$

$$\left. \wedge \text{supp}(B_{\text{uni}(\ell), \mathbf{j}, \mathbf{p}}) \subseteq \widehat{\Omega}_h^\ell \wedge \text{supp}(B_{\text{uni}(\ell), \mathbf{j}, \mathbf{p}}) \not\subseteq \widehat{\Omega}_h^{\ell+1} \right\},$$

where we recall the definition (7) of a multivariate B-spline. Figure 3 gives an illustrative example. Its elements are referred to as (multivariate) *hierarchical B-splines*. For  $\mathbf{a} \in \mathcal{I}_h$ , the level of the corresponding hierarchical B-spline is well defined

$$\text{level}(B_{\mathbf{a}}) := \ell \in \mathbb{N}_0 \quad \text{with} \quad \mathbf{a} = (\text{uni}(\ell), \mathbf{j}, \mathbf{p}). \quad (11)$$

It is easy to check that if  $\widehat{\mathcal{T}}_h$  is a tensor mesh, and hence coincides with some  $\widehat{\mathcal{T}}_{\text{uni}(\ell)}$ , then the hierarchical basis and the standard tensor-product B-spline basis are the same. One can prove that the hierarchical B-splines are linearly independent; see, e.g., [VGJS11, Lemma 2]. They span the space of *hierarchical splines*

$$S^{\mathcal{P}}(\mathcal{K}_0, m, \widehat{\mathcal{T}}_h) := \text{span}(\{B_{\mathbf{a}} : \mathbf{a} \in \mathcal{I}_h\}). \quad (12)$$

The hierarchical basis and the mesh  $\widehat{\mathcal{T}}_h$  are compatible in the following sense: For all  $B_{\mathbf{a}}$ ,  $\mathbf{a} \in \mathcal{I}_h$ , the corresponding support can be written as union of elements in  $\widehat{\mathcal{T}}_{\text{uni}(\text{level}(B_{\mathbf{a}}))}$ ; i.e.,

$$\text{supp}(B_{\mathbf{a}}) = \bigcup \{ \widehat{T} \in \widehat{\mathcal{T}}_{\text{uni}(\text{level}(B_{\mathbf{a}}))} : \widehat{T} \subseteq \text{supp}(B_{\mathbf{a}}) \}.$$

Each such element  $\widehat{T} \in \widehat{\mathcal{T}}_{\text{uni}(\text{level}(B_{\mathbf{a}}))}$  with  $\widehat{T} \subseteq \text{supp}(B_{\mathbf{a}}) \subseteq \widehat{\Omega}_h^{\text{level}(B_{\mathbf{a}})}$  satisfies that  $\widehat{T} \in \widehat{\mathcal{T}}_h$  or  $\widehat{T} \subseteq \widehat{\Omega}_h^{\text{level}(B_{\mathbf{a}})+1}$ . In either case, we see that  $\widehat{T}$  can be written as union of elements in



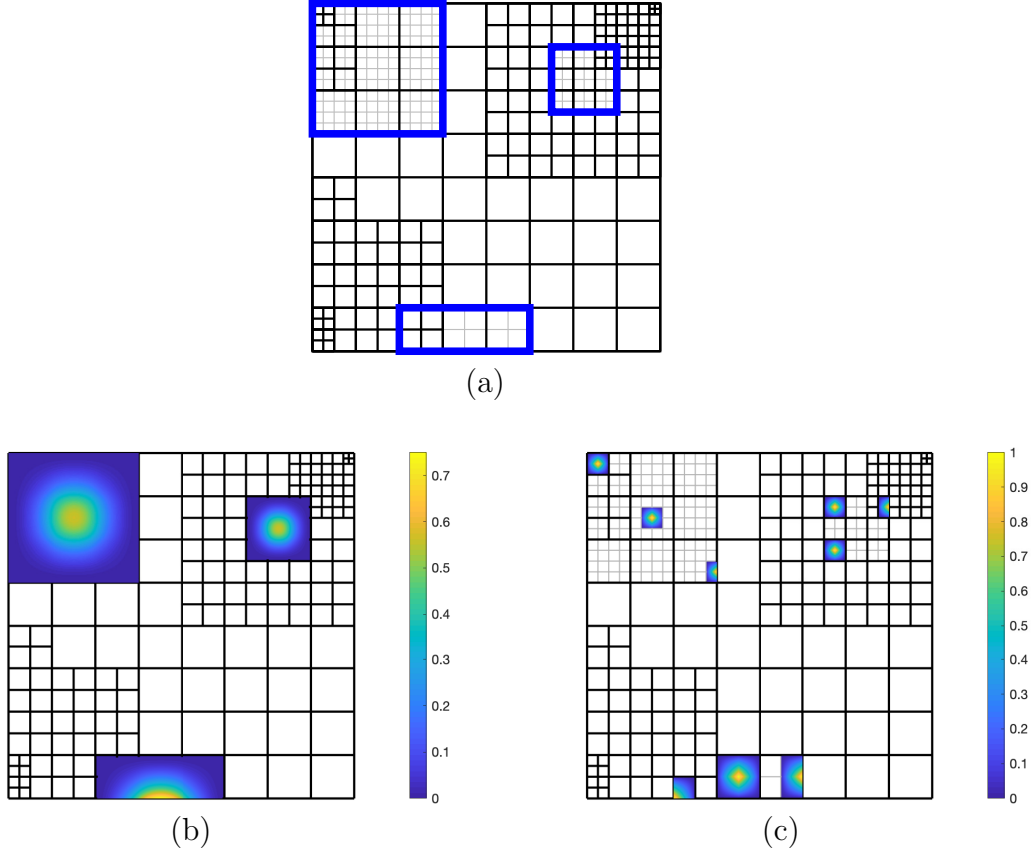


FIGURE 3. A two-dimensional hierarchical mesh  $\widehat{\mathcal{T}}_h$  (of Figure 2) is depicted in black. Assuming that all interior knots have multiplicity 1, in (a), the support of three hierarchical B-splines of degree  $p = 2$  is highlighted in blue; two hierarchical B-splines of level 0 (left and bottom), and one hierarchical B-spline of level 1 (right). The corresponding local tensor meshes  $\widehat{\mathcal{T}}_{\mathbf{a}}$  are indicated in gray. In (b), these three hierarchical B-splines are depicted. For  $\bar{p} = 2$ , these functions also belong (up to scaling) to the partition of unity  $\{\psi_{\mathbf{a}} \circ \mathbf{F} : \mathbf{a} \in \mathcal{V}_h\}$  defined in Section 4.1. Finally, in (c), for each of them, three functions of the corresponding local partition of unity  $\{\psi_{\mathbf{b}} \circ \mathbf{F} : \mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}\}$  defined over the local tensor meshes are depicted.

$\widehat{\mathcal{T}}_h$  with level greater or equal to  $\text{level}(B_{\mathbf{a}})$ . Altogether, we have that

$$\text{supp}(B_{\mathbf{a}}) = \bigcup_{\ell \geq \text{level}(B_{\mathbf{a}})} \{\widehat{T} \in \widehat{\mathcal{T}}_h \cap \widehat{\mathcal{T}}_{\text{uni}(\ell)} : \widehat{T} \subseteq \text{supp}(B_{\mathbf{a}})\}. \quad (13)$$

Moreover,  $\text{supp}(B_{\mathbf{a}})$  must contain at least one element of  $\text{level}(B_{\mathbf{a}})$ . Otherwise one would get the contradiction  $\text{supp}(B_{\mathbf{a}}) \subseteq \widehat{\Omega}_h^{\text{level}(B_{\mathbf{a}})+1}$ .

We now present the following characterization of hierarchical splines, which has been verified in [SM16, Section 3],

$$S^{\mathcal{P}}(\mathcal{K}_0, m, \widehat{\mathcal{T}}_h) = \{\widehat{v}_h : (0, 1)^d \rightarrow \mathbb{R} : \widehat{v}_h|_{(0,1)^d \setminus \widehat{\Omega}_h^{\ell+1}} \in S^{\mathcal{P}}(\mathcal{K}_{\text{uni}(\ell)})|_{(0,1)^d \setminus \widehat{\Omega}_h^{\ell+1}} \text{ for all } \ell \in \mathbb{N}_0\}. \quad (14)$$

In particular, each hierarchical spline is a  $\widehat{\mathcal{T}}_h$ -piecewise tensor-product polynomial of degree  $\mathbf{p}$ . Put into words, hierarchical splines are coarse splines on coarse mesh elements, and they are fine splines on fine mesh elements.

Finally, we say that a hierarchical mesh  $\widehat{\mathcal{T}}_h$  is *finer* than another hierarchical  $\widehat{\mathcal{T}}_H$  if  $\widehat{\mathcal{T}}_h$  is obtained from  $\widehat{\mathcal{T}}_H$  via iterative dyadic bisection. Formally, this can be stated as  $\widehat{\Omega}_H^\ell \subseteq \widehat{\Omega}_h^\ell$  for all  $\ell \in \mathbb{N}_0$ . In this case, (14) shows that the corresponding hierarchical spline spaces are nested, i.e.,

$$S^{\mathbf{p}}(\mathcal{K}_0, m, \widehat{\mathcal{T}}_H) \subseteq S^{\mathbf{p}}(\mathcal{K}_0, m, \widehat{\mathcal{T}}_h). \quad (15)$$

In particular, we see that

$$S^{\mathbf{p}}(\mathcal{K}_{\text{uni}(0)}) \subseteq S^{\mathbf{p}}(\mathcal{K}_0, m, \widehat{\mathcal{T}}_h) \subseteq S^{\mathbf{p}}(\mathcal{K}_{\text{uni}(L_h-1)}).$$

**3.3. Hierarchical splines in the physical domain  $\Omega$ .** If  $\widehat{\mathcal{T}}_h$  is a hierarchical mesh in the parameter domain  $(0, 1)^d$ , we set

$$\mathcal{T}_h := \{\mathbf{F}(\widehat{T}) : \widehat{T} \in \widehat{\mathcal{T}}_h\}.$$

In this context,  $h \in L^\infty(\Omega)$  denotes the mesh size function defined by  $h|_T := \text{diam}(T)$  for all  $T \in \mathcal{T}_h$ . Moreover, we set

$$S^{\mathbf{p}}(\mathcal{K}_0, m, \mathcal{T}_h) := \{\widehat{v} \circ \mathbf{F}^{-1} : \widehat{v} \in S^{\mathbf{p}}(\mathcal{K}_0, m, \widehat{\mathcal{T}}_h)\}.$$

The conforming ansatz space for the Galerkin discretization (3) is then defined as

$$V_h := S^{\mathbf{p}}(\mathcal{K}_0, m, \mathcal{T}_h) \cap H_0^1(\Omega). \quad (16)$$

#### 4. PARTITIONS OF UNITY AND PATCHWISE SPACES

In this section, we prepare the necessary material for defining and analyzing our equilibrated fluxes in the IGA context later. We particularly design a partition of unity based on hierarchical B-splines and define continuous- and discrete-level local spaces.

**4.1. Partitions of unity based on hierarchical splines.** We now first construct a partition of unity on  $\Omega$  consisting of hierarchical B-splines with the smallest-possible polynomial degree  $\bar{p}$  but sufficient smoothness to be contained in  $S^{\mathbf{p}}(\mathcal{K}_0, m, \mathcal{T}_h)$ . Subsequently, on the local tensor meshes of the support of each of these hierarchical B-splines, we form a partition of unity by piecewise affine hat functions.

Let  $\bar{p} \leq p$  be a supplementary polynomial degree and let

$$\bar{\mathcal{K}}_0 = (\bar{\mathcal{K}}_{1(0)}, \dots, \bar{\mathcal{K}}_{d(0)})$$

be a fixed  $d$ -dimensional vector of  $\bar{p}$ -open knot vectors

$$\bar{\mathcal{K}}_{i(0)} = (\bar{t}_{i(0),0}, \dots, \bar{t}_{i(0),\bar{N}_{i(0)}+\bar{p}})$$

such that  $(\bar{t}_{i(0),\bar{p}}, \dots, \bar{t}_{i(0),\bar{N}_{i(0)}})$  is a subsequence of  $(t_{i(0),p}, \dots, t_{i(0),N_{i(0)}})$  which is obtained by reducing multiplicities of the latter knots to at least one and

$$0 = \bar{t}_{i(0),0} = \dots = \bar{t}_{i(0),\bar{p}} \quad \text{and} \quad \bar{t}_{i(0),\bar{N}_{i(0)}} = \dots = \bar{t}_{i(0),\bar{N}_{i(0)}+\bar{p}} = 1.$$

In particular, the tensor mesh corresponding to  $\bar{\mathcal{K}}_0$  coincides with the initial tensor-mesh  $\widehat{\mathcal{T}}_0$  corresponding to  $\mathcal{K}_0$ . To guarantee that

$$S^{\bar{p}}(\bar{\mathcal{K}}_{i(0)}) \subseteq S^{\mathbf{p}}(\mathcal{K}_{i(0)}), \quad (17)$$

we further suppose that

$$p - \#t_{i(0),j} \leq \bar{p} - \#\bar{t}_{i(0),j} \quad (18)$$

for all interior knots  $t_{i(0),j}$  in  $(0, 1)$  (which determines the smoothness of the considered splines), where  $\#$  denotes the multiplicity within  $\overline{\mathcal{K}}_{i(0)}$ . Next, we set  $\overline{\mathcal{K}}_{\text{uni}(0)} := \overline{\mathcal{K}}_0$  and recursively define  $\overline{\mathcal{K}}_{\text{uni}(\ell+1)}$  for  $\ell \in \mathbb{N}_0$  as the uniform  $h$ -refinement of  $\overline{\mathcal{K}}_{\text{uni}(\ell)}$  with fixed multiplicity  $\overline{m} \in \mathbb{N}$  such that

$$p - m \leq \overline{p} - \overline{m}. \quad (19)$$

In words, the knots of  $\overline{\mathcal{K}}_{\text{uni}(\ell+1)}$  in  $[0, 1]$  are the knots of  $\overline{\mathcal{K}}_{\text{uni}(\ell)}$  in  $[0, 1]$  plus the points  $(t_{i(\text{uni}(\ell)),j-1} + t_{i(\text{uni}(\ell)),j})/2$  with multiplicity  $\overline{m}$  if  $t_{i(\text{uni}(\ell)),j-1} < t_{i(\text{uni}(\ell)),j}$  for  $j \in \{\overline{p} + 1, \dots, \overline{N}_{i(\text{uni}(\ell))}\}$ . While the analysis below does not rely on this, we will always choose the smallest-possible polynomial degree  $\overline{p}$  together with  $\overline{m} := 1$ . If, for example, all knots in  $\mathcal{K}_0$  have the same multiplicity 1 and  $m = 1$ , i.e., the corresponding splines are  $C^{p-1}$  along initial as well as new lines, one can only choose  $\overline{p} = p$  and  $\overline{m} = 1$ . If all knots in  $\mathcal{K}_0$  have the same multiplicity  $p$  and  $m = p$ , i.e., the corresponding splines are only  $C^0$  along initial as well as new lines, one can choose  $\overline{p} \leq p$  and  $\overline{m} \leq \overline{p}$  arbitrarily, which leads us to  $\overline{p} = 1$  and  $\overline{m} = 1$ .

Let again  $\widehat{\mathcal{T}}_h$  be a hierarchical mesh with corresponding sets  $(\widehat{\Omega}_h^\ell)_{\ell \in \mathbb{N}_0}$  as in Section 3.2. With  $\overline{\mathbf{p}} := (\overline{p}, \dots, \overline{p})$  and the B-splines  $\overline{B}_{(\text{uni}(\ell), \mathbf{j}, \overline{\mathbf{p}})}$  defined as in (7), we can define a second hierarchical basis

$$\{\overline{B}_{\mathbf{a}} : \mathbf{a} \in \overline{\mathcal{I}}_h\} \quad \text{with} \quad \overline{\mathcal{I}}_h := \bigcup_{\ell \in \mathbb{N}_0} \left\{ (\text{uni}(\ell), \mathbf{j}, \overline{\mathbf{p}}) : \mathbf{j} \in \Pi_{i=1}^d \{1, \dots, \overline{N}_{i(\text{uni}(\ell))}\} \right. \\ \left. \wedge \text{supp}(\overline{B}_{\text{uni}(\ell), \mathbf{j}, \overline{\mathbf{p}}}) \subseteq \widehat{\Omega}_h^\ell \wedge \text{supp}(\overline{B}_{\text{uni}(\ell), \mathbf{j}, \overline{\mathbf{p}}}) \not\subseteq \widehat{\Omega}_h^{\ell+1} \right\}. \quad (20)$$

For  $\mathbf{a} \in \overline{\mathcal{I}}_h$ , we define  $\text{level}(\overline{B}_{\mathbf{a}})$  as in (11). Again, [SM16, Section 3] gives an explicit characterization for the spanned space of hierarchical splines

$$S^{\mathbf{p}}(\overline{\mathcal{K}}_0, \overline{m}, \widehat{\mathcal{T}}_h) := \text{span}(\{\overline{B}_{\mathbf{a}} : \mathbf{a} \in \overline{\mathcal{I}}_h\}) \\ = \{\widehat{v}_h : (0, 1)^d \rightarrow \mathbb{R} : \widehat{v}_h|_{(0,1)^d \setminus \widehat{\Omega}_h^{\ell+1}} \in S^{\mathbf{p}}(\overline{\mathcal{K}}_{\text{uni}(\ell)})|_{(0,1)^d \setminus \widehat{\Omega}_h^{\ell+1}} \text{ for all } \ell \in \mathbb{N}_0\}. \quad (21)$$

Our assumptions on the knot multiplicities, which imply the nestedness  $S^{\overline{\mathbf{p}}}(\overline{\mathcal{K}}_{\text{uni}(\ell)}) \subseteq S^{\mathbf{p}}(\mathcal{K}_{\text{uni}(\ell)})$  for all  $\ell \in \mathbb{N}_0$ , thus give that

$$S^{\overline{\mathbf{p}}}(\overline{\mathcal{K}}_0, \overline{m}, \widehat{\mathcal{T}}_h) \subseteq S^{\mathbf{p}}(\mathcal{K}_0, m, \widehat{\mathcal{T}}_h). \quad (22)$$

Now  $1 \in S^{\overline{\mathbf{p}}}(\overline{\mathcal{K}}_0, \overline{m}, \widehat{\mathcal{T}}_h)$  yields the existence of a partition of unity on the parameter domain

$$1 = \sum_{\mathbf{a} \in \overline{\mathcal{I}}_h} c_{\mathbf{a}} \overline{B}_{\mathbf{a}}.$$

One can prove that the coefficients  $c_{\mathbf{a}} \in \mathbb{R}$  satisfy that

$$0 \leq c_{\mathbf{a}} \leq 1 \quad (23)$$

for all  $\mathbf{a} \in \overline{\mathcal{I}}_h$ ; see, e.g., [BG16a, Lemma 3.2]. Consequently, we can define

$$\psi_{\mathbf{a}} := (c_{\mathbf{a}} \overline{B}_{\mathbf{a}}) \circ \mathbf{F}^{-1} \quad \text{for all } \mathbf{a} \in \mathcal{V}_h := \{\mathbf{a} \in \overline{\mathcal{I}}_h : c_{\mathbf{a}} > 0\} \quad (24)$$

and observe that the  $\psi_{\mathbf{a}}$  form a *partition of unity* on the physical domain

$$\sum_{\mathbf{a} \in \mathcal{V}_h} \psi_{\mathbf{a}} = 1 \quad \text{in } \Omega.$$

Henceforth, we call  $\mathcal{V}_h$  the set of nodes and  $\mathbf{a} \in \mathcal{V}_h$  a *node*. For further use, we abbreviate  $\omega_{\mathbf{a}} := \text{int}(\text{supp}(\psi_{\mathbf{a}}))$  as well as  $\widehat{\omega}_{\mathbf{a}} := \mathbf{F}^{-1}(\omega_{\mathbf{a}})$  for all  $\mathbf{a} \in \mathcal{V}_h$ ; we will use the terminology *large patches* for  $\omega_{\mathbf{a}}$  or  $\widehat{\omega}_{\mathbf{a}}$ . Figure 3 gives an illustrative example.

Below, we will also crucially use a second partition of unity on each large patch  $\omega_{\mathbf{a}}$ . Let  $\widehat{\mathcal{T}}_{\mathbf{a}}$  be the smallest *uniform* tensor mesh refinement of  $\{\widehat{T} \in \widehat{\mathcal{T}}_h : \widehat{T} \subseteq \text{supp}(B_{\mathbf{a}})\}$ , i.e.,

$$\widehat{\mathcal{T}}_{\mathbf{a}} := \{\widehat{T} \in \widehat{\mathcal{T}}_{\text{uni}(\ell_{\mathbf{a}})} : \widehat{T} \subseteq \text{supp}(B_{\mathbf{a}})\}$$

with  $\ell_{\mathbf{a}} := \max\{\text{level}(\widehat{T}) : \widehat{T} \in \widehat{\mathcal{T}}_h \wedge \widehat{T} \subseteq \text{supp}(B_{\mathbf{a}})\}$ , and  $\mathcal{T}_{\mathbf{a}}$  the corresponding mesh of the large patch  $\omega_{\mathbf{a}}$  in the physical domain; see again Figure 3. Moreover, let  $P^1(\widehat{\mathcal{T}}_{\mathbf{a}})$  be the set of all  $\widehat{\mathcal{T}}_{\mathbf{a}}$ -piecewise tensor-product polynomials of degree  $\mathbf{1} := (1, \dots, 1)$  and

$$P^1(\mathcal{T}_{\mathbf{a}}) := \{\widehat{v} \circ \mathbf{F}^{-1} : \widehat{v} \in P^1(\widehat{\mathcal{T}}_{\mathbf{a}})\}. \quad (25)$$

Finally, let  $\mathcal{V}_h^{\mathbf{a}}$  be the set of all vertices in the local mesh  $\mathcal{T}_{\mathbf{a}}$ . We denote by  $\psi_{\mathbf{b}}$  the hat function associated with the vertex  $\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}$ ; this is the unique function in  $P^1(\mathcal{T}_{\mathbf{a}})$  taking value 1 in the vertex  $\mathbf{b}$  and 0 in all other vertices from  $\mathcal{V}_h^{\mathbf{a}}$ . Observe that the  $\psi_{\mathbf{b}}$  form a *partition of unity* on the large patches  $\omega_{\mathbf{a}}$

$$\sum_{\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}} \psi_{\mathbf{b}} = 1 \quad \text{in } \omega_{\mathbf{a}}.$$

We abbreviate  $\omega_{\mathbf{b}} := \text{int}(\text{supp}(\psi_{\mathbf{b}}))$  as well as  $\widehat{\omega}_{\mathbf{b}} := \mathbf{F}^{-1}(\omega_{\mathbf{b}})$  for all  $\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}$ , for which we use the name *small patches*. Figure 3 gives again an illustrative example.

**4.2. Patchwise Sobolev spaces.** For a node  $\mathbf{a} \in \mathcal{V}_h$ , define a local Sobolev space on the large patch  $\omega_{\mathbf{a}}$  as

$$H_*^1(\omega_{\mathbf{a}}) := \begin{cases} \{v \in H^1(\omega_{\mathbf{a}}) : (v, 1)_{\omega_{\mathbf{a}}} = 0\} & \text{if } \psi_{\mathbf{a}} \in H_0^1(\Omega), \\ \{v \in H^1(\omega_{\mathbf{a}}) : v = 0 \text{ on } \partial\omega_{\mathbf{a}} \setminus \psi_{\mathbf{a}}^{-1}(\{0\})\} & \text{else.} \end{cases} \quad (26)$$

This is the mean-value-free subspace of  $H^1(\omega_{\mathbf{a}})$  in the interior of  $\Omega$ , and the trace-free (on that part of  $\partial\omega_{\mathbf{a}}$  where  $\psi_{\mathbf{a}}$  is nonzero) subspace of  $H^1(\omega_{\mathbf{a}})$  close to the boundary of  $\Omega$ . For vector-valued functions, we will use

$$\mathbf{H}_0(\text{div}, \omega_{\mathbf{a}}) := \begin{cases} \{\mathbf{v} \in \mathbf{H}(\text{div}, \omega_{\mathbf{a}}) : \mathbf{v} \cdot \mathbf{n}_{\omega_{\mathbf{a}}} = 0 \text{ on } \partial\omega_{\mathbf{a}}\} & \text{if } \psi_{\mathbf{a}} \in H_0^1(\Omega), \\ \{\mathbf{v} \in \mathbf{H}(\text{div}, \omega_{\mathbf{a}}) : \mathbf{v} \cdot \mathbf{n}_{\omega_{\mathbf{a}}} = 0 \text{ on } \partial\omega_{\mathbf{a}} \cap (\psi_{\mathbf{a}})^{-1}(\{0\})\} & \text{else,} \end{cases} \quad (27)$$

where  $\mathbf{n}_{\omega_{\mathbf{a}}}$  denotes the outer normal vector on  $\partial\omega_{\mathbf{a}}$  and  $\mathbf{v} \cdot \mathbf{n}_{\omega_{\mathbf{a}}}$  is understood in the appropriate weak sense.

For a node  $\mathbf{a} \in \mathcal{V}_h$  and a vertex  $\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}$ , we also define some spaces on the small patches  $\omega_{\mathbf{b}}$ . In particular, we let

$$L_*^2(\omega_{\mathbf{b}}) := \begin{cases} \{v \in L^2(\omega_{\mathbf{b}}) : (v, 1)_{\omega_{\mathbf{b}}} = 0\} & \text{if } \psi_{\mathbf{a}}\psi_{\mathbf{b}} \in H_0^1(\Omega), \\ L^2(\omega_{\mathbf{b}}) & \text{else,} \end{cases} \quad (28)$$

where  $\psi_{\mathbf{b}}$  is identified with its extension by zero onto  $\Omega$ . We also define

$$H_*^1(\omega_{\mathbf{b}}) := \begin{cases} \{v \in H^1(\omega_{\mathbf{b}}) : (v, 1)_{\omega_{\mathbf{b}}} = 0\} & \text{if } \psi_{\mathbf{a}}\psi_{\mathbf{b}} \in H_0^1(\Omega), \\ \{v \in H^1(\omega_{\mathbf{b}}) : v = 0 \text{ on } \partial\omega_{\mathbf{b}} \setminus (\psi_{\mathbf{a}}\psi_{\mathbf{b}})^{-1}(\{0\})\} & \text{else} \end{cases} \quad (29)$$

and

$$\mathbf{H}_0(\operatorname{div}, \omega_{\mathbf{b}}) := \begin{cases} \{\mathbf{v} \in \mathbf{H}(\operatorname{div}, \omega_{\mathbf{b}}) : \mathbf{v} \cdot \mathbf{n}_{\omega_{\mathbf{b}}} = 0 \text{ on } \partial\omega_{\mathbf{b}}\} & \text{if } \psi_{\mathbf{a}}\psi_{\mathbf{b}} \in H_0^1(\Omega), \\ \{\mathbf{v} \in \mathbf{H}(\operatorname{div}, \omega_{\mathbf{b}}) : \mathbf{v} \cdot \mathbf{n}_{\omega_{\mathbf{b}}} = 0 \text{ on } \partial\omega_{\mathbf{b}} \\ \quad \cap (\psi_{\mathbf{a}}\psi_{\mathbf{b}})^{-1}(\{0\})\} & \text{else.} \end{cases} \quad (30)$$

Finally, for  $\mathbf{a} \in \mathcal{V}_h$ ,  $\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}$ , and  $\omega \in \{\omega_{\mathbf{a}}, \omega_{\mathbf{b}}\}$ , define the Poincaré–Friedrichs constant as the minimal constant  $C_{\text{PF}}(\omega) > 0$  such that

$$\|v\|_{\omega} \leq \operatorname{diam}(\omega) C_{\text{PF}}(\omega) \|\nabla v\|_{\omega} \quad \text{for all } v \in H_*^1(\omega). \quad (31)$$

Note that  $C_{\text{PF}}(\omega)$  only depends on the shape of  $\omega$  and  $\partial\omega$  where respectively  $\psi_{\mathbf{a}}$  or  $\psi_{\mathbf{a}}\psi_{\mathbf{b}}$  is nonzero; if  $\psi_{\mathbf{a}} \in H_0^1(\Omega)$  or  $\psi_{\mathbf{a}}\psi_{\mathbf{b}} \in H_0^1(\Omega)$  and for convex  $\omega$ , there in particular holds  $C_{\text{PF}}(\omega) \leq 1/\pi$  (“interior” cases). In the other, “boundary”, cases,  $C_{\text{PF}}(\omega) \leq 1$  when there exists a unit vector  $\mathbf{m}$  such that the straight semi-line of direction  $\mathbf{m}$  originating at (almost) any point in  $\omega$  hits the boundary  $\partial\omega$  there where respectively  $\psi_{\mathbf{a}}$  or  $\psi_{\mathbf{a}}\psi_{\mathbf{b}}$  is nonzero, cf., e.g., [Voh05, VV12] and the references therein.

**4.3. Patchwise discrete subspaces.** For a node  $\mathbf{a} \in \mathcal{V}_h$ , let us define the  $H_*^1(\omega_{\mathbf{a}})$ -conforming subspace of the mapped piecewise affine functions  $P^1(\mathcal{T}_{\mathbf{a}})$  from (25) as

$$V_h^{\mathbf{a}} := P^1(\mathcal{T}_{\mathbf{a}}) \cap H_*^1(\omega_{\mathbf{a}}). \quad (32)$$

For a node  $\mathbf{a} \in \mathcal{V}_h$  and a vertex  $\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}$ , define the meshes of the small patches as  $\mathcal{T}_{\mathbf{b}} := \{T \in \mathcal{T}_{\mathbf{a}} : T \subseteq \bar{\omega}_{\mathbf{b}}\}$ ,  $\widehat{\mathcal{T}}_{\mathbf{b}} := \{\mathbf{F}^{-1}(T) : T \in \mathcal{T}_{\mathbf{b}}\}$ ; note that the elements  $\widehat{T} \in \widehat{\mathcal{T}}_{\mathbf{b}}$  are rectangles for  $d = 2$  and cuboids for  $d = 3$ . Let  $P^{\tilde{\mathbf{p}}}(\widehat{\mathcal{T}}_{\mathbf{b}})$  be the space of all  $\widehat{\mathcal{T}}_{\mathbf{b}}$ -piecewise polynomials of some fixed degree  $\tilde{\mathbf{p}} = (\tilde{p}, \dots, \tilde{p})$  in each coordinate,  $\tilde{p} \geq 0$ , and let  $L_*^2(\widehat{\omega}_{\mathbf{b}})$  be defined as in (28) with  $\omega_{\mathbf{b}}$  replaced by  $\widehat{\omega}_{\mathbf{b}}$ . We then define the local space

$$Q_h^{\mathbf{a}, \mathbf{b}} := \{\widehat{q}_h \circ \mathbf{F}^{-1} : \widehat{q}_h \in P^{\tilde{\mathbf{p}}}(\widehat{\mathcal{T}}_{\mathbf{b}}) \cap L_*^2(\widehat{\omega}_{\mathbf{b}})\}. \quad (33)$$

Note that  $Q_h^{\mathbf{a}, \mathbf{b}}$  is in general *not* contained in  $L_*^2(\omega_{\mathbf{b}})$ , the exception being when the mapping  $\mathbf{F}$  is affine. The mean-value constraint in (28), though, makes it a constrained subspace of mapped piecewise polynomials from  $P^{\tilde{\mathbf{p}}}(\widehat{\mathcal{T}}_{\mathbf{b}})$ . With the set  $P^{\tilde{\mathbf{p}}}(\widehat{\mathcal{T}}_h)$  of all  $\widehat{\mathcal{T}}_h$ -piecewise polynomials of degree  $\tilde{\mathbf{p}}$ , we also define the global (unconstrained) space via the mapping  $\mathbf{F}$  as

$$Q_h := \{\widehat{q}_h \circ \mathbf{F}^{-1} : \widehat{q}_h \in P^{\tilde{\mathbf{p}}}(\widehat{\mathcal{T}}_h)\}.$$

Let

$$\mathbf{RT}^{\tilde{\mathbf{p}}}(\widehat{\mathcal{T}}_{\mathbf{b}}) := \begin{cases} P^{\tilde{\mathbf{p}}+(1,0)}(\widehat{\mathcal{T}}_{\mathbf{b}}) \times P^{\tilde{\mathbf{p}}+(0,1)}(\widehat{\mathcal{T}}_{\mathbf{b}}) & \text{if } d = 2, \\ P^{\tilde{\mathbf{p}}+(1,0,0)}(\widehat{\mathcal{T}}_{\mathbf{b}}) \times P^{\tilde{\mathbf{p}}+(0,1,0)}(\widehat{\mathcal{T}}_{\mathbf{b}}) \times P^{\tilde{\mathbf{p}}+(0,0,1)}(\widehat{\mathcal{T}}_{\mathbf{b}}) & \text{if } d = 3 \end{cases}$$

be the usual broken (elementwise) Raviart–Thomas space on the rectangular/cuboid mesh  $\widehat{\mathcal{T}}_{\mathbf{b}}$ , see, e.g., [BBF13]. Define the vector-valued contravariant Piola transform

$$\Phi(\cdot) := (\det(D\mathbf{F})^{-1}(D\mathbf{F})(\cdot)) \circ \mathbf{F}^{-1}, \quad (34)$$

see, e.g., [EG21, Chapter 9]. We then set

$$\mathbf{V}_h^{\mathbf{a}, \mathbf{b}} := \mathbf{RT}_h^{\mathbf{a}, \mathbf{b}} \cap \mathbf{H}_0(\operatorname{div}, \omega_{\mathbf{b}}), \quad (35)$$

where  $\mathbf{RT}_h^{\mathbf{a}, \mathbf{b}}$  is the space  $\mathbf{RT}^{\tilde{\mathbf{p}}}(\widehat{\mathcal{T}}_{\mathbf{b}})$  mapped by the Piola transform

$$\mathbf{RT}_h^{\mathbf{a}, \mathbf{b}} := \mathbf{RT}^{\tilde{\mathbf{p}}}(\mathcal{T}_{\mathbf{b}}) := \{\Phi(\widehat{\mathbf{v}}_h) : \widehat{\mathbf{v}}_h \in \mathbf{RT}^{\tilde{\mathbf{p}}}(\widehat{\mathcal{T}}_{\mathbf{b}})\}. \quad (36)$$

## 5. ABSTRACT ASSUMPTIONS

In this section, we attempt to describe as clearly as possible the underlying principles of a posteriori error analysis by equilibrated fluxes in the present context. For this purpose, we identify six abstract assumptions under which our subsequent analysis can be carried out. We then immediately verify these assumptions in the particular context of Sections 2–4.

As for the general setting, we merely need to assume:

**Assumption 5.1.**  $\widehat{\Omega}$  is an open bounded connected Lipschitz domain in  $\mathbb{R}^d$ ,  $\mathbf{F}$  is a bi-Lipschitz mapping,  $\Omega := \mathbf{F}(\widehat{\Omega})$ , and  $V_h$  is an arbitrary subspace of  $H_0^1(\Omega)$ .

As for the partitions of unity, the minimalist assumptions are (note that we do not require that the partitions are non-negative, i.e.,  $\psi_{\mathbf{a}}, \psi_{\mathbf{b}} \geq 0$ ):

**Assumption 5.2.** There is a finite index set  $\mathcal{V}_h$  and functions  $\psi_{\mathbf{a}}$  such that

$$\{\psi_{\mathbf{a}} : \mathbf{a} \in \mathcal{V}_h\} \subset W^{1,\infty}(\Omega) \quad (37a)$$

form a partition of unity over  $\Omega$  in the sense that

$$\sum_{\mathbf{a} \in \mathcal{V}_h} \psi_{\mathbf{a}} = 1 \quad \text{in } \Omega. \quad (37b)$$

The interior  $\omega_{\mathbf{a}}$  of the support of any  $\psi_{\mathbf{a}}$  is a connected Lipschitz domain with  $|\omega_{\mathbf{a}}| > 0$ . Moreover,

$$\{\psi_{\mathbf{a}} : \mathbf{a} \in \mathcal{V}_h\} \cap H_0^1(\Omega) \subset V_h. \quad (37c)$$

**Assumption 5.3.** For any node  $\mathbf{a} \in \mathcal{V}_h$ , there is a finite set of vertices  $\mathcal{V}_h^{\mathbf{a}}$  and functions  $\psi_{\mathbf{b}}$  such that

$$\{\psi_{\mathbf{b}} : \mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}\} \subset W^{1,\infty}(\omega_{\mathbf{a}}) \quad (38a)$$

form a partition of unity over  $\omega_{\mathbf{a}}$  in that

$$\sum_{\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}} \psi_{\mathbf{b}} = 1 \quad \text{in } \omega_{\mathbf{a}}. \quad (38b)$$

The interior  $\omega_{\mathbf{b}}$  of the support of any  $\psi_{\mathbf{b}}$  is a connected Lipschitz domain with  $|\omega_{\mathbf{b}}| > 0$ . Moreover, for  $H_*^1(\omega_{\mathbf{a}})$  given by (26), all  $\psi_{\mathbf{b}}$  such that  $\psi_{\mathbf{a}}\psi_{\mathbf{b}} \in H_0^1(\Omega)$  are contained up to additive constants in a finite-dimensional subspace  $V_h^{\mathbf{a}} \subset H_*^1(\omega_{\mathbf{a}})$ , i.e.,

$$\{\psi_{\mathbf{b}} : \mathbf{b} \in \mathcal{V}_h^{\mathbf{a}} \wedge \psi_{\mathbf{a}}\psi_{\mathbf{b}} \in H_0^1(\Omega)\} \subset \begin{cases} \{v_h + c : v_h \in V_h^{\mathbf{a}}, c \in \mathbb{R}\} & \text{if } \psi_{\mathbf{a}} \in H_0^1(\Omega) \\ V_h^{\mathbf{a}} & \text{else.} \end{cases} \quad (38c)$$

Next, we recall the space  $H_*^1(\omega_{\mathbf{b}})$  from (29), the Poincaré–Friedrichs inequality (31), and assume the following set of estimates:

**Assumption 5.4.** *There exist generic positive constants  $C_1, \dots, C_6 > 0$  such that for all  $\mathbf{a} \in \mathcal{V}_h$  and  $\mathbf{b} \in \mathcal{V}_h^\mathbf{a}$ , it holds that*

$$\|\psi_\mathbf{a}\|_{\infty, \omega_\mathbf{a}} \leq C_1, \quad (39a)$$

$$\|\nabla \psi_\mathbf{a}\|_{\infty, \omega_\mathbf{a}} \text{diam}(\omega_\mathbf{a}) C_{\text{PF}}(\omega_\mathbf{a}) \leq C_2, \quad (39b)$$

$$\|\nabla \psi_\mathbf{a}\|_{\infty, \omega_\mathbf{b}} \text{diam}(\omega_\mathbf{b}) C_{\text{PF}}(\omega_\mathbf{b}) \leq C_3, \quad (39c)$$

$$\sup_{\mathbf{x} \in \omega_\mathbf{a}} \#\{\mathbf{b}' \in \mathcal{V}_h^\mathbf{a} : \mathbf{x} \in \omega_{\mathbf{b}'}\} \leq C_4, \quad (39d)$$

$$\|\psi_\mathbf{b}\|_{\infty, \omega_\mathbf{b}} \leq C_5, \quad (39e)$$

$$\|\nabla \psi_\mathbf{b}\|_{\infty, \omega_\mathbf{b}} \text{diam}(\omega_\mathbf{b}) C_{\text{PF}}(\omega_\mathbf{b}) \leq C_6. \quad (39f)$$

Recall the space  $\mathbf{H}_0(\text{div}, \omega_\mathbf{b})$  from (30), the subspace  $L_*^2(\omega_\mathbf{b})$  of  $L^2(\omega_\mathbf{b})$  containing functions with mean value zero if  $\psi_\mathbf{a}\psi_\mathbf{b} \in H_0^1(\Omega)$  from (28), and the similar space  $L_*^2(\widehat{\omega}_\mathbf{b})$  in the parameter domain. Recall also the contravariant Piola transform  $\Phi$  from (34). For local flux equilibration, we will rely on discrete spaces  $Q_h^{\mathbf{a}, \mathbf{b}}$  and  $\mathbf{V}_h^{\mathbf{a}, \mathbf{b}}$  satisfying the following:

**Assumption 5.5.** *For any node  $\mathbf{a} \in \mathcal{V}_h$  and any vertex  $\mathbf{b} \in \mathcal{V}_h^\mathbf{a}$ , there are finite-dimensional subspaces*

$$Q_h^{\mathbf{a}, \mathbf{b}} \subset L^2(\omega_\mathbf{b}) \quad \text{and} \quad \mathbf{V}_h^{\mathbf{a}, \mathbf{b}} \subset \mathbf{H}_0(\text{div}, \omega_\mathbf{b}) \quad (40a)$$

*satisfying the compatibility condition in the parameter domain  $\widehat{\Omega}$*

$$\begin{aligned} \nabla \cdot \widehat{\mathbf{V}}_h^{\mathbf{a}, \mathbf{b}} &= \widehat{Q}_h^{\mathbf{a}, \mathbf{b}} \\ &\text{with} \end{aligned} \quad (40b)$$

$$\widehat{Q}_h^{\mathbf{a}, \mathbf{b}} := \{q_h \circ \mathbf{F} : q_h \in Q_h^{\mathbf{a}, \mathbf{b}}\} \quad \text{and} \quad \widehat{\mathbf{V}}_h^{\mathbf{a}, \mathbf{b}} := \{\Phi^{-1}(\mathbf{v}_h) : \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}, \mathbf{b}}\}.$$

*Moreover, we suppose the existence of a global space  $Q_h \subset L^2(\Omega)$  such that*

$$Q_h|_{\omega_\mathbf{b}} \subseteq Q_{h,c}^{\mathbf{a}, \mathbf{b}} := \begin{cases} \{q_h + c : q_h \in Q_h^{\mathbf{a}, \mathbf{b}}, c \in \mathbb{R}\} & \text{if } \psi_\mathbf{a}\psi_\mathbf{b} \in H_0^1(\Omega), \\ Q_h^{\mathbf{a}, \mathbf{b}} & \text{else.} \end{cases} \quad (40c)$$

**Remark 5.6.** *By the properties of the contravariant Piola transform  $\Phi$  from (34), see, e.g., [EG21, Lemma 9.13], there holds  $\widehat{\mathbf{V}}_h^{\mathbf{a}, \mathbf{b}} \subset \mathbf{H}_0(\text{div}, \widehat{\omega}_\mathbf{b}) = \Phi^{-1}(\mathbf{H}_0(\text{div}, \omega_\mathbf{b}))$ . Thus, for any  $\widehat{\mathbf{v}}_h \in \widehat{\mathbf{V}}_h^{\mathbf{a}, \mathbf{b}}$ , there holds  $(\nabla \cdot \widehat{\mathbf{v}}_h, 1)_{\widehat{\omega}_\mathbf{b}} = 0$  if  $\psi_\mathbf{a}\psi_\mathbf{b} \in H_0^1(\Omega)$ , so that (40b) implies that functions in  $\widehat{Q}_h^{\mathbf{a}, \mathbf{b}}$  have mean value zero when  $\psi_\mathbf{a}\psi_\mathbf{b} \in H_0^1(\Omega)$ , i.e.,*

$$\widehat{Q}_h^{\mathbf{a}, \mathbf{b}} \subset L_*^2(\widehat{\omega}_\mathbf{b}). \quad (41)$$

*In contrast,  $Q_h^{\mathbf{a}, \mathbf{b}} = \{\widehat{q}_h \circ \mathbf{F}^{-1} : \widehat{q}_h \in \widehat{Q}_h^{\mathbf{a}, \mathbf{b}}\}$  does not satisfy the mean value condition if  $\psi_\mathbf{a}\psi_\mathbf{b} \in H_0^1(\Omega)$  and thus is only a subspace of  $L^2(\omega_\mathbf{b})$  and not of  $L_*^2(\omega_\mathbf{b})$  (unless the mapping  $\mathbf{F}$  is affine). The space  $Q_h^{\mathbf{a}, \mathbf{b}}$ , though, is a constrained space, as it maps the constraint (41). The space  $Q_{h,c}^{\mathbf{a}, \mathbf{b}}$  from (40c) then contains no constraint, independently of whether  $\psi_\mathbf{a}\psi_\mathbf{b} \in H_0^1(\Omega)$  or not. Finally, the role of the global space  $Q_h \subset L^2(\Omega)$  will be prominent below: please note that it is related to neither the node  $\mathbf{a}$ , nor to the vertex  $\mathbf{b}$ ; this forces the local spaces  $Q_{h,c}^{\mathbf{a}, \mathbf{b}}$  to contain patch-independent “base-blocks”. In practice,  $Q_h|_{\omega_\mathbf{b}} = Q_{h,c}^{\mathbf{a}, \mathbf{b}}$  only for uniform mesh refinement but not, for example, in the setting of Figure 3, where a strict inclusion holds.*

Finally, we will essentially employ the following broken polynomial extension property.



**Assumption 5.7.** *There exist a generic constant  $C_{\text{st}} \geq 1$ , as well as a superspace  $\mathbf{RT}_h^{a,b} \subset [L^2(\omega_b)]^d$  verifying  $\mathbf{V}_h^{a,b} \subseteq \mathbf{RT}_h^{a,b}$  for all nodes  $\mathbf{a} \in \mathcal{V}_h$  and vertices  $\mathbf{b} \in \mathcal{V}_h^a$ , such that*

$$\min_{\substack{\mathbf{v}_h \in \mathbf{V}_h^{a,b} \\ \nabla \cdot \mathbf{v}_h = g_h}} \|\mathbf{v}_h + \boldsymbol{\tau}_h\|_{\omega_b} \leq C_{\text{st}} \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\text{div}, \omega_b) \\ \nabla \cdot \mathbf{v} = g_h}} \|\mathbf{v} + \boldsymbol{\tau}_h\|_{\omega_b} \quad (42)$$

for all  $g_h \in \nabla \cdot \mathbf{V}_h^{a,b}$  and all  $\boldsymbol{\tau}_h \in \mathbf{RT}_h^{a,b}$ .

We now verify that the above assumptions are satisfied in our IGA context:

**Proposition 5.8.** *Assumptions 5.1–5.7 are satisfied in the context of Sections 2–4. In particular, the constants in Assumption 5.4 can be taken as  $C_1 = 1$ ,  $C_4 = 2^d$ ,  $C_5 = 1$  and such that  $C_2, C_3, C_6$ , and  $C_{\text{st}}$  only depend on the shapes in  $\widehat{\mathcal{T}}_0$  as well as the mapping  $\mathbf{F}$  via  $\max\{\|D\mathbf{F}\|_{\infty, \widehat{\Omega}}, \|(D\mathbf{F})^{-1}\|_{\infty, \widehat{\Omega}}\}$ ;  $C_2$  and  $C_3$  additionally depend on the supplementary polynomial degree  $\bar{p}$  from (18)–(19).*

*Proof.* Assumptions 5.1–5.3 are immediately satisfied with the choices made in Sections 2–4, in particular fixing the space  $V_h^a$  following (32).

We now verify Assumption 5.4. Thanks to (6) and (23), inequality (39a) is satisfied with the constant  $C_1 = 1$ , whereas (39d) and (39e) hold easily with respectively  $C_4 = 2^d$  and  $C_5 = 1$ . Next, (39b), (39c), and (39f) follow from the facts that  $\|\nabla \psi_a\|_{\infty, \omega_a} \lesssim \bar{p} \text{diam}(\omega_a)^{-1}$ ,  $\text{diam}(\omega_b) \leq \text{diam}(\omega_a)$ , and  $\|\nabla \psi_b\|_{\infty, \omega_b} \lesssim \text{diam}(\omega_b)^{-1}$ , see, e.g., [BdVBSV14, Equation (2.7)], where we also use that  $C_{\text{PF}}(\omega_a), C_{\text{PF}}(\omega_b) \lesssim 1$  (see, e.g., [VV12, Corollary 2.2]). Here,  $A \lesssim B$  means that  $A \leq CB$  for a hidden constant  $C > 0$  depending only on the shapes of the elements in  $\widehat{\mathcal{T}}_0$  as well as on  $\max\{\|D\mathbf{F}\|_{\infty, \widehat{\Omega}}, \|(D\mathbf{F})^{-1}\|_{\infty, \widehat{\Omega}}\}$ .

We now turn to Assumption 5.5. Properties (40a) and (40c) are trivially satisfied for the spaces defined in Section 4.3, whereas the compatibility condition (40b) is the construction requirement of the Raviart–Thomas space, see [BBF13].

We finally address Assumption 5.7. For the spaces defined in Section 4.3, this result is proved in the parameter domain in [BPS09, Theorems 5 and 7], see also [EV20, Theorem 2.5 and Corollary 3.3]. We give a proof in the physical domain in Appendix A, where the resulting constant  $C_{\text{st}}$  depends only on the shapes of the elements in  $\widehat{\mathcal{T}}_0$  and  $\max\{\|D\mathbf{F}\|_{\infty, \widehat{\Omega}}, \|(D\mathbf{F})^{-1}\|_{\infty, \widehat{\Omega}}\}$ . We stress that  $C_{\text{st}}$  is independent of the polynomial degree  $\bar{p}$ .  $\square$

**Remark 5.9.** *Recall that the supplementary polynomial degree  $\bar{p}$  from Section 4.1 depends on the considered smoothness but not necessarily on the polynomial degree  $p$ . In this sense,  $C_2, C_3$  in Proposition 5.8 are independent of the polynomial degree  $p$ . We recall in particular that  $\bar{p} = 1$  can be taken for  $C^0$  splines, and in general  $\bar{p} = k + 1$  for  $C^k$  splines. We admit, though, that  $\bar{p} = p$  for  $C^{p-1}$  splines, see the discussion in Section 4.1.*

## 6. INEXPENSIVE EQUILIBRATION

Let the abstract assumptions of Section 5 be satisfied, let  $u$  solve (2), and let  $u_h$  solve (3). The partition of unity functions  $\psi_a$  from Assumption 5.2, in our setting the hierarchical B-splines  $\psi_a$  of polynomial degree  $\bar{\mathbf{p}} = (\bar{p}, \dots, \bar{p})$  and multiplicity  $\bar{m}$  from (24), lead to

$$(\nabla u_h, \nabla \psi_a)_{\omega_a} = (f, \psi_a)_{\omega_a} \quad \text{for all } \mathbf{a} \in \mathcal{V}_h \text{ with } \psi_a \in H_0^1(\Omega), \quad (43)$$

which is an immediate consequence of (3) and (37c). Thus, in view of (2),

$$(\nabla(u - u_h), \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \quad \text{for all } \mathbf{a} \in \mathcal{V}_h \text{ with } \psi_{\mathbf{a}} \in H_0^1(\Omega). \quad (44)$$

This orthogonality is sufficient to localize the error  $\|\nabla(u - u_h)\|_{\Omega}$  (or, equivalently, the dual norm of the residual) over the large patches  $\omega_{\mathbf{a}}$ , see [BMV20] and the references therein, and then a flux equilibration can be easily devised. The issue, however, is that this would lead to an expensive equilibration with *higher-order* (related to the polynomial degree  $p$ ) mixed finite spaces on the *large patches*  $\omega_{\mathbf{a}}$ . Our goal below is to design a much *less expensive equilibration*, with some inexpensive (typically piecewise affine) solve on the large patches  $\omega_{\mathbf{a}}$  followed by higher-order (related to  $p$ ) mixed finite element solves on the *small patches*  $\omega_{\mathbf{b}}$  only. In order to achieve it, we crucially rely on the discrete patchwise spaces from Section 5/Section 4.3.

**6.1. Inexpensive (lowest-order) residual lifting on the large patches  $\omega_{\mathbf{a}}$ .** Our first step is to construct a discrete residual function  $r_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ , where, recalling from Assumption 5.3,  $V_h^{\mathbf{a}}$  in (32) merely consists of mapped piecewise affine functions in the context of hierarchical B-splines:

**Definition 6.1.** For all nodes  $\mathbf{a} \in \mathcal{V}_h$ , let  $r_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$  be such that

$$(\nabla r_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = (f, v_h \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla(v_h \psi_{\mathbf{a}}))_{\omega_{\mathbf{a}}} \quad \text{for all } v_h \in V_h^{\mathbf{a}}. \quad (45)$$

Thus, (45) is an *inexpensive* scalar-valued low-order local problem on the larger patches  $\omega_{\mathbf{a}}$ , lifting the  $\psi_{\mathbf{a}}$ -weighted residual, see the discussions in [CF00, BPS09, EV15]. We note that (45) is a finite-dimensional version of the problem: find  $r^{\mathbf{a}} \in H_*^1(\omega_{\mathbf{a}})$  such that

$$(\nabla r^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = (f, v \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla(v \psi_{\mathbf{a}}))_{\omega_{\mathbf{a}}} \quad \text{for all } v \in H_*^1(\omega_{\mathbf{a}}). \quad (46)$$

This can be equivalently written as: find  $r^{\mathbf{a}} \in H_*^1(\omega_{\mathbf{a}})$  such that

$$(\nabla r^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h, \nabla v)_{\omega_{\mathbf{a}}} = (f \psi_{\mathbf{a}} - \nabla u_h \cdot \nabla \psi_{\mathbf{a}}, v)_{\omega_{\mathbf{a}}} \quad \text{for all } v \in H_*^1(\omega_{\mathbf{a}}). \quad (47)$$

From (47), we see that  $-(\nabla r^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h)$  lies in  $\mathbf{H}_0(\text{div}, \omega_{\mathbf{a}})$  with the divergence equal to  $f \psi_{\mathbf{a}} - \nabla u_h \cdot \nabla \psi_{\mathbf{a}}$ .

**6.2. Projection-like operators for general bi-Lipschitz mappings  $\mathbf{F}$ .** In order to proceed, we need to define some projection-like operators. Define the scalar Piola transform by

$$\tilde{\Phi}(\cdot) := (\det(D\mathbf{F})^{-1}(\cdot)) \circ \mathbf{F}^{-1}. \quad (48)$$

Paired to the vector-valued Piola transformation  $\Phi$  from (34), they satisfy the identity

$$\tilde{\Phi}(\nabla \cdot (\cdot)) = \nabla \cdot \Phi(\cdot),$$

see, e.g., [EG21, Lemma 9.6]. Hence, from assumptions (40a) and (40b), there holds

$$\nabla \cdot \mathbf{V}_h^{\mathbf{a}, \mathbf{b}} = \tilde{\Phi}(\widehat{Q}_h^{\mathbf{a}, \mathbf{b}}) \subset L_*^2(\omega_{\mathbf{b}}).$$

With the  $L^2$ -orthogonal projection  $\Pi_{\widehat{Q}_h^{\mathbf{a}, \mathbf{b}}} : L^2(\widehat{\omega}_{\mathbf{b}}) \rightarrow \widehat{Q}_h^{\mathbf{a}, \mathbf{b}}$ , we define the mapping

$$\Upsilon_{Q_h^{\mathbf{a}, \mathbf{b}}} : L^2(\omega_{\mathbf{b}}) \rightarrow \nabla \cdot \mathbf{V}_h^{\mathbf{a}, \mathbf{b}}, \quad g \mapsto \tilde{\Phi}(\Pi_{\widehat{Q}_h^{\mathbf{a}, \mathbf{b}}}(\tilde{\Phi}^{-1}g)). \quad (49)$$

It is easy to check that  $\Upsilon_{Q_h^{\mathbf{a}, \mathbf{b}}}g$  is the unique element in  $\nabla \cdot \mathbf{V}_h^{\mathbf{a}, \mathbf{b}} = \tilde{\Phi}(\widehat{Q}_h^{\mathbf{a}, \mathbf{b}})$  with

$$(\Upsilon_{Q_h^{\mathbf{a}, \mathbf{b}}}g, q_h)_{\omega_{\mathbf{b}}} = (g, q_h)_{\omega_{\mathbf{b}}} \quad \text{for all } q_h \in Q_h^{\mathbf{a}, \mathbf{b}}. \quad (50)$$

In general, though, since  $\nabla \cdot \mathbf{V}_h^{\mathbf{a},\mathbf{b}} = \tilde{\Phi}(\widehat{Q}_h^{\mathbf{a},\mathbf{b}}) \neq Q_h^{\mathbf{a},\mathbf{b}}$  (unless the mapping  $\mathbf{F}$  is affine),  $\Upsilon_{Q_h^{\mathbf{a},\mathbf{b}}}$  is *not* the  $L^2$ -orthogonal projection onto  $Q_h^{\mathbf{a},\mathbf{b}}$ . We define  $\Upsilon_{Q_{h,c}^{\mathbf{a},\mathbf{b}}}$  and  $\Upsilon_{Q_h}$  analogously as

$$g \mapsto \tilde{\Phi}(\Pi_{\widehat{Q}_{h,c}^{\mathbf{a},\mathbf{b}}}(\tilde{\Phi}^{-1}g)) \quad \text{and} \quad g \mapsto \tilde{\Phi}(\Pi_{\widehat{Q}_h}(\tilde{\Phi}^{-1}g)), \quad (51)$$

where  $\Pi$  still stands for the  $L^2$ -orthogonal projection and  $\widehat{Q}_{h,c}^{\mathbf{a},\mathbf{b}} := \{q_h \circ \mathbf{F} : q_h \in Q_{h,c}^{\mathbf{a},\mathbf{b}}\}$ ,  $\widehat{Q}_h := \{q_h \circ \mathbf{F} : q_h \in Q_h\}$ . As above,  $\Upsilon_{Q_{h,c}^{\mathbf{a},\mathbf{b}}}$  is the unique element in  $\tilde{\Phi}(\widehat{Q}_{h,c}^{\mathbf{a},\mathbf{b}})$  such that

$$(\Upsilon_{Q_{h,c}^{\mathbf{a},\mathbf{b}}}g, q_h)_{\omega_b} = (g, q_h)_{\omega_b} \quad \text{for all } q_h \in Q_{h,c}^{\mathbf{a},\mathbf{b}} \quad (52)$$

and similarly for  $\Upsilon_{Q_h}$ .

**6.3. Equilibrated flux on the small patches  $\omega_b$ .** Let  $r_h^{\mathbf{a}}$  be given by Definition 6.1 and recall the operator  $\Upsilon_{Q_h^{\mathbf{a},\mathbf{b}}}$  from Section 6.2. Then our equilibrated flux on the small patches  $\omega_b$  is given by:

**Definition 6.2.** For all nodes  $\mathbf{a} \in \mathcal{V}_h$  and all vertices  $\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}$ , let

$$\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}} := \underset{\substack{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a},\mathbf{b}} \\ \nabla \cdot \mathbf{v}_h = \Upsilon_{Q_h^{\mathbf{a},\mathbf{b}}}(f\psi_{\mathbf{a}}\psi_{\mathbf{b}} - \nabla u_h \cdot \nabla(\psi_{\mathbf{a}}\psi_{\mathbf{b}}) - \nabla r_h^{\mathbf{a}} \cdot \nabla\psi_{\mathbf{b}})}}{\operatorname{argmin}} \|\mathbf{v}_h + \psi_{\mathbf{b}}(\psi_{\mathbf{a}}\nabla u_h + \nabla r_h^{\mathbf{a}})\|_{\omega_b}. \quad (53)$$

From (47), we know that  $-(\nabla r^{\mathbf{a}} + \psi_{\mathbf{a}}\nabla u_h)$  lies in  $\mathbf{H}_0(\operatorname{div}, \omega_{\mathbf{a}})$  with the divergence equal to  $f\psi_{\mathbf{a}} - \nabla u_h \cdot \nabla\psi_{\mathbf{a}}$ . Thus, its cut-off by  $\psi_{\mathbf{b}}$ ,  $-\psi_{\mathbf{b}}(\psi_{\mathbf{a}}\nabla u_h + \nabla r^{\mathbf{a}})$ , lies in  $\mathbf{H}_0(\operatorname{div}, \omega_{\mathbf{b}})$  with the divergence equal to  $f\psi_{\mathbf{a}}\psi_{\mathbf{b}} - \nabla u_h \cdot \nabla(\psi_{\mathbf{a}}\psi_{\mathbf{b}}) - \nabla r^{\mathbf{a}} \cdot \nabla\psi_{\mathbf{b}}$ . It can be characterized implicitly by

$$\underset{\substack{\mathbf{v} \in \mathbf{H}_0(\operatorname{div}, \omega_b) \\ \nabla \cdot \mathbf{v} = f\psi_{\mathbf{a}}\psi_{\mathbf{b}} - \nabla u_h \cdot \nabla(\psi_{\mathbf{a}}\psi_{\mathbf{b}}) - \nabla r^{\mathbf{a}} \cdot \nabla\psi_{\mathbf{b}}}}{\operatorname{argmin}} \|\mathbf{v} + \psi_{\mathbf{b}}(\psi_{\mathbf{a}}\nabla u_h + \nabla r^{\mathbf{a}})\|_{\omega_b}.$$

The flux  $\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}}$  from (53) is then its discrete approximation.

The following lemma guarantees the existence and uniqueness of the minimizer of Definition 6.2:

**Lemma 6.3.** *There exists a unique minimizer  $\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}}$  of (53).*

*Proof.* Let us abbreviate

$$g := f\psi_{\mathbf{a}}\psi_{\mathbf{b}} - \nabla u_h \cdot \nabla(\psi_{\mathbf{a}}\psi_{\mathbf{b}}) - \nabla r_h^{\mathbf{a}} \cdot \nabla\psi_{\mathbf{b}} \quad (54)$$

and

$$\boldsymbol{\tau} := \psi_{\mathbf{b}}(\psi_{\mathbf{a}}\nabla u_h + \nabla r_h^{\mathbf{a}}).$$

By definition of  $\Upsilon_{Q_h^{\mathbf{a},\mathbf{b}}}$ ,  $\Upsilon_{Q_h^{\mathbf{a},\mathbf{b}}}g \in \nabla \cdot \mathbf{V}_h^{\mathbf{a},\mathbf{b}}$ , so that the minimization set is nonempty, and existence and uniqueness follow by standard convexity arguments, see, e.g., [BBF13]. In more details, problem (53) is equivalent to finding  $\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}}$  with  $\nabla \cdot \boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}} = \Upsilon_{Q_h^{\mathbf{a},\mathbf{b}}}g$  such that

$$(\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}}, \mathbf{v}_h)_{\omega_b} = -(\boldsymbol{\tau}, \mathbf{v}_h)_{\omega_b} \quad \text{for all } \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a},\mathbf{b}} \text{ with } \nabla \cdot \mathbf{v}_h = 0, \quad (55)$$

which is a square linear system. Existence and uniqueness thus follow when  $\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}} = 0$  for zero data. Let thus  $g = 0$  and  $\boldsymbol{\tau} = \mathbf{0}$ . Since  $\Upsilon_{Q_h^{\mathbf{a},\mathbf{b}}}0 = 0$  follows from (50), we can take  $\mathbf{v}_h = \boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}}$  in (55), which implies  $\|\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}}\|_{\omega_b} = 0$  and thus  $\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}} = 0$ .  $\square$

The following is an important extension of Lemma 6.3:

**Lemma 6.4.** *The projection  $\Upsilon_{Q_h^{a,b}}$  in (53) can be replaced by  $\Upsilon_{Q_{h,c}^{a,b}}$ , i.e.,*

$$(\Upsilon_{Q_h^{a,b}} - \Upsilon_{Q_{h,c}^{a,b}})(f\psi_a\psi_b - \nabla u_h \cdot \nabla(\psi_a\psi_b) - \nabla r_h^a \cdot \nabla\psi_b) = 0. \quad (56)$$

*Proof.* From (40c), the spaces  $Q_h^{a,b}$  and  $Q_{h,c}^{a,b}$  only differ if  $\psi_a\psi_b \in H_0^1(\Omega)$ , see Remark 5.6, and this only by constants removing the constraint. Let thus  $\psi_a\psi_b \in H_0^1(\Omega)$  and recall the notation (54). From  $V_h^{a,b} \subset \mathbf{H}_0(\text{div}, \omega_b)$ , (30), and (53),  $0 = (\nabla \cdot \boldsymbol{\sigma}_h^{a,b}, 1)_{\omega_b} = (\Upsilon_{Q_h^{a,b}} g, 1)_{\omega_b}$ . Taking into account (50) and (52), we thus only need to verify the Neumann compatibility condition  $(g, 1)_{\omega_b} = 0$ . Hence, test  $g$  with 1 and obtain that

$$(g, 1)_{\omega_b} = (f\psi_a, \psi_b)_{\omega_b} - (\nabla u_h \cdot \nabla\psi_a, \psi_b)_{\omega_b} - (\psi_a \nabla u_h, \nabla\psi_b)_{\omega_b} - (\nabla r_h^a, \nabla\psi_b)_{\omega_b}. \quad (57)$$

We consider two cases.

First, let  $\psi_a \in H_0^1(\Omega)$ . Then, we use that  $\text{supp}(\psi_b) \subseteq \text{supp}(\psi_a)$  from (38a), (45) with the test function  $v_h = \psi_b - |\omega_a|^{-1}(\psi_b, 1)_{\omega_a}$  having zero mean value on  $\omega_a$  (which is possible due to (38c)), and (43) to see for the last term in (57) that

$$\begin{aligned} (\nabla r_h^a, \nabla\psi_b)_{\omega_b} &= (\nabla r_h^a, \nabla(\psi_b - |\omega_a|^{-1}(\psi_b, 1)_{\omega_a}))_{\omega_a} \\ &\stackrel{(45)}{=} (f\psi_a - \nabla u_h \cdot \nabla\psi_a, \psi_b - |\omega_a|^{-1}(\psi_b, 1)_{\omega_a})_{\omega_a} - (\psi_a \nabla u_h, \nabla\psi_b)_{\omega_a} \\ &\stackrel{(43)}{=} (f\psi_a - \nabla u_h \cdot \nabla\psi_a, \psi_b)_{\omega_b} - (\psi_a \nabla u_h, \nabla\psi_b)_{\omega_b}. \end{aligned}$$

With (57), we then see that  $(g, 1)_{\omega_b} = 0$ .

It remains to consider  $\psi_a \notin H_0^1(\Omega)$ . In this case, we obtain again with  $\text{supp}(\psi_b) \subseteq \text{supp}(\psi_a)$  and (45) with the test function  $v_h = \psi_b$  (which is possible due to (38c)) that

$$\begin{aligned} (\nabla r_h^a, \nabla\psi_b)_{\omega_b} &= (\nabla r_h^a, \nabla\psi_b)_{\omega_a} \stackrel{(45)}{=} (f\psi_a - \nabla u_h \cdot \nabla\psi_a, \psi_b)_{\omega_a} - (\psi_a \nabla u_h, \nabla\psi_b)_{\omega_a} \\ &= (f\psi_a - \nabla u_h \cdot \nabla\psi_a, \psi_b)_{\omega_b} - (\psi_a \nabla u_h, \nabla\psi_b)_{\omega_b}. \end{aligned}$$

With (57), we again see that  $(g, 1)_{\omega_b} = 0$ , which concludes the proof.  $\square$

**6.4. Equilibrated flux on the large patches and the final equilibrated flux.** Our equilibrated flux on the large patches  $\omega_a$  and the final equilibrated flux are given by:

**Definition 6.5.** *Define the patchwise fluxes, for all  $\mathbf{a} \in \mathcal{V}_h$ ,*

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \sum_{\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}} \boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}} \quad (58a)$$

*and the equilibrated flux*

$$\boldsymbol{\sigma}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}}. \quad (58b)$$

We first discuss the contributions  $\boldsymbol{\sigma}_h^{\mathbf{a}}$ , which can be seen as discrete approximations of  $-(\nabla r^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h)$  from (46). Recalling (27), we have.

**Lemma 6.6.** *For all  $\mathbf{a} \in \mathcal{V}_h$ , it holds that  $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}_0(\text{div}, \omega_a)$  and*

$$(\nabla \cdot \boldsymbol{\sigma}_h^{\mathbf{a}}, q_h)_{\omega_a} = (f\psi_{\mathbf{a}} - \nabla u_h \cdot \nabla\psi_{\mathbf{a}}, q_h)_{\omega_a} \quad \text{for all } q_h \in Q_h. \quad (59)$$

*Proof.* By Definition 6.2, we have  $\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}} \in V_h^{\mathbf{a},\mathbf{b}} \subset \mathbf{H}_0(\text{div}, \omega_b)$  for all  $\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}$ . If  $\psi_{\mathbf{a}} \in H_0^1(\Omega)$ , then  $\psi_{\mathbf{a}}\psi_{\mathbf{b}} \in H_0^1(\Omega)$ , and  $\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}} \cdot \mathbf{n}_{\omega_b} = 0$  on  $\partial\omega_b$  from (30), so that  $\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}} \cdot \mathbf{n}_{\omega_a} = 0$  on  $\partial\omega_a$  by virtue of  $\text{supp}(\psi_{\mathbf{b}}) \subseteq \text{supp}(\psi_{\mathbf{a}})$ , which we assume in (38a). If  $\psi_{\mathbf{a}} \notin H_0^1(\Omega)$ , then  $\psi_{\mathbf{a}}\psi_{\mathbf{b}} \in H_0^1(\Omega)$  may still hold, in which case the previous reasoning applies. If  $\psi_{\mathbf{a}} \notin H_0^1(\Omega)$  and  $\psi_{\mathbf{a}}\psi_{\mathbf{b}} \notin H_0^1(\Omega)$ , then the homogeneous Neumann (no flow) boundary condition only

applies on  $\partial\omega_b \cap (\psi_a\psi_b)^{-1}(\{0\})$  in (30), and, in the sum of contributions over all  $\mathbf{b} \in \mathcal{V}_h^a$ , on  $\partial\omega_a \cap (\psi_a)^{-1}(\{0\})$ . Thus  $\boldsymbol{\sigma}_h^a \in \mathbf{H}_0(\text{div}, \omega_a)$ .

To see the second point, we use again definitions (53) and (58a), along with the partition of unity property (38b). Let  $q_h \in Q_h$  be fixed. Then

$$\begin{aligned} (\nabla \cdot \boldsymbol{\sigma}_h^a, q_h)_{\omega_a} &= \sum_{\mathbf{b} \in \mathcal{V}_h^a} (\nabla \cdot \boldsymbol{\sigma}_h^{a,\mathbf{b}}, q_h)_{\omega_b} \\ &\stackrel{(53)}{=} \sum_{\mathbf{b} \in \mathcal{V}_h^a} (\Upsilon_{Q_h^{a,\mathbf{b}}} (f\psi_a\psi_b - \nabla u_h \cdot \nabla(\psi_a\psi_b) - \nabla r_h^a \cdot \nabla\psi_b), q_h)_{\omega_b} \\ &\stackrel{(52)}{=} \sum_{\mathbf{b} \in \mathcal{V}_h^a} (f\psi_a\psi_b - \nabla u_h \cdot \nabla(\psi_a\psi_b) - \nabla r_h^a \cdot \nabla\psi_b, q_h)_{\omega_b} \\ &\stackrel{(38b)}{=} (f\psi_a - \nabla u_h \cdot \nabla\psi_a, q_h)_{\omega_a}, \end{aligned}$$

where we have also crucially used Lemma 6.4 on the second line and the inclusion property (40c) on the third line.  $\square$

We now show that  $\boldsymbol{\sigma}_h$  is indeed an equilibrated flux:

**Lemma 6.7.** *It holds that  $\boldsymbol{\sigma}_h \in \mathbf{H}(\text{div}, \Omega)$  with*

$$(\nabla \cdot \boldsymbol{\sigma}_h, q_h)_\Omega = (f, q_h)_\Omega \quad \text{for all } q_h \in Q_h, \quad (60)$$

or equivalently  $\Upsilon_{Q_h}(f - \nabla \cdot \boldsymbol{\sigma}_h) = 0$ .

*Proof.* Definition 6.5, (27), and Lemma 6.6 immediately imply that  $\boldsymbol{\sigma}_h \in \mathbf{H}(\text{div}, \Omega)$ . To see the second point, we use again Lemma 6.6 and the partition of unity property (37b), giving

$$(\nabla \cdot \boldsymbol{\sigma}_h, q_h)_\Omega = \sum_{\mathbf{a} \in \mathcal{V}_h} (\nabla \cdot \boldsymbol{\sigma}_h^{\mathbf{a}}, q_h)_{\omega_a} \stackrel{(59)}{=} \sum_{\mathbf{a} \in \mathcal{V}_h} (f\psi_a - \nabla u_h \cdot \nabla\psi_a, q_h)_{\omega_a} \stackrel{(37b)}{=} (f, q_h)_\Omega,$$

which concludes the proof.  $\square$

## 7. A POSTERIORI ERROR ESTIMATES

We are now ready to present our a posteriori error estimates.

**7.1. Reliability.** With the dual norm

$$\|\cdot\|_{H^{-1}(\Omega)} := \sup_{\substack{v \in H_0^1(\Omega) \\ \|\nabla v\|_\Omega = 1}} (\cdot, v)_\Omega,$$

one immediately gets the following reliability result:

**Proposition 7.1.** *Let the abstract assumptions of Section 5 be satisfied, let  $u$  solve (2), and let  $u_h$  solve (3). Let the equilibrated flux be given by Definitions 6.1, 6.2, and 6.5. Then*

$$\|\nabla(u - u_h)\|_\Omega \leq \|\boldsymbol{\sigma}_h + \nabla u_h\|_\Omega + \|(1 - \Upsilon_{Q_h})(f - \nabla \cdot \boldsymbol{\sigma}_h)\|_{H^{-1}(\Omega)}. \quad (61)$$

*Proof.* The weak solution definition (2), the fact that  $\boldsymbol{\sigma}_h \in \mathbf{H}(\text{div}, \Omega)$ , the Green theorem, and Lemma 6.7 show

$$\begin{aligned}
\|\nabla(u - u_h)\|_\Omega &= \sup_{\substack{v \in H_0^1(\Omega) \\ \|\nabla v\|_\Omega = 1}} ((f, v)_\Omega - (\nabla u_h, \nabla v)_\Omega) \\
&= \sup_{\substack{v \in H_0^1(\Omega) \\ \|\nabla v\|_\Omega = 1}} ((\nabla \cdot \boldsymbol{\sigma}_h, v)_\Omega - (\nabla u_h, \nabla v)_\Omega + (f - \nabla \cdot \boldsymbol{\sigma}_h, v)_\Omega) \\
&= \sup_{\substack{v \in H_0^1(\Omega) \\ \|\nabla v\|_\Omega = 1}} (-(\boldsymbol{\sigma}_h + \nabla u_h, \nabla v)_\Omega + ((1 - \Upsilon_{Q_h})(f - \nabla \cdot \boldsymbol{\sigma}_h), v)_\Omega) \\
&\leq \|\boldsymbol{\sigma}_h + \nabla u_h\|_\Omega + \|(1 - \Upsilon_{Q_h})(f - \nabla \cdot \boldsymbol{\sigma}_h)\|_{H^{-1}(\Omega)}.
\end{aligned}$$

□

Let  $\|\cdot\|_2$  denote the spectral norm of a square matrix. In the particular situation of Section 4.3, the  $\|\cdot\|_{H^{-1}(\Omega)}$  in (61) can be further estimated by a computable weighted  $L^2$ -norm, forming a data oscillation term.

**Proposition 7.2.** *Suppose that  $Q_h$  contains the space of piecewise constants with respect to some mesh  $\mathcal{T}_h$  of  $\Omega$ , as is the case in Section 4.3. Assume that for all  $T \in \mathcal{T}_h$ ,  $\widehat{T} := \mathbf{F}^{-1}(T)$  is convex. Then, there holds that*

$$\|\nabla(u - u_h)\|_\Omega \leq \|\boldsymbol{\sigma}_h + \nabla u_h\|_\Omega + \text{osc}_h^{\text{rel}}, \quad (62a)$$

where

$$\text{osc}_h^{\text{rel}} := \left( \sum_{T \in \mathcal{T}_h} \text{osc}_h^{\text{rel}}(T)^2 \right)^{1/2} \quad \text{with} \quad \text{osc}_h^{\text{rel}}(T) := \frac{C_{\text{rel}}}{\pi} \text{diam}(\widehat{T}) \|(1 - \Upsilon_{Q_h})(f - \nabla \cdot \boldsymbol{\sigma}_h)\|_T \quad (62b)$$

and

$$C_{\text{rel}} := \|\det(D\mathbf{F})\|_{\infty, \widehat{\Omega}}^{1/2} \|\det(D\mathbf{F})^{-1}\|_{\infty, \widehat{\Omega}}^{1/2} \sup_{\widehat{x} \in \widehat{\Omega}} \|D\mathbf{F}(\widehat{x})\|_2. \quad (62c)$$

*Proof.* Let  $v \in H_0^1(\Omega)$  with  $\|\nabla v\|_\Omega = 1$  and let  $v_h$  be its  $L^2(\Omega)$ -orthogonal projection onto the space of  $\mathcal{T}_h$ -piecewise constants. Recall from Lemma 6.7 that  $\Upsilon_{Q_h}(f - \nabla \cdot \boldsymbol{\sigma}_h) = 0$ . Hence, the Cauchy–Schwarz inequality and the Poincaré inequality as in (31) (with constant  $C_P(T)$ ) show that

$$\begin{aligned}
((1 - \Upsilon_{Q_h})(f - \nabla \cdot \boldsymbol{\sigma}_h), v)_\Omega &= (f - \nabla \cdot \boldsymbol{\sigma}_h, v - v_h)_\Omega = \sum_{T \in \mathcal{T}_h} (f - \nabla \cdot \boldsymbol{\sigma}_h, v - v_h)_T \\
&\leq \sum_{T \in \mathcal{T}_h} \|f - \nabla \cdot \boldsymbol{\sigma}_h\|_T \|v - v_h\|_T \leq \sum_{T \in \mathcal{T}_h} \|f - \nabla \cdot \boldsymbol{\sigma}_h\|_T \text{diam}(T) C_P(T) \|\nabla v\|_T \\
&\leq \left( \sum_{T \in \mathcal{T}_h} \text{diam}(T)^2 C_P(T)^2 \|f - \nabla \cdot \boldsymbol{\sigma}_h\|_T^2 \right)^{1/2}.
\end{aligned}$$

According to [VV12, Corollary 2.2], it further holds for  $\widehat{T} := \mathbf{F}^{-1}(T)$  that

$$\text{diam}(T) C_P(T) \leq \text{diam}(\widehat{T}) C_P(\widehat{T}) \|\det(D\mathbf{F})\|_{\infty, \widehat{T}}^{1/2} \|\det(D\mathbf{F})^{-1}\|_{\infty, \widehat{T}}^{1/2} \sup_{\widehat{x} \in \widehat{T}} \|D\mathbf{F}(\widehat{x})\|_2,$$

where  $C_P(\widehat{T})$  denotes the Poincaré constant of  $\widehat{T}$ , which is smaller or equal to  $1/\pi$  as we assumed that  $\widehat{T}$  is convex. □

**Remark 7.3.** *If there also holds the converse inclusion in (40c), i.e.,  $Q_{h,c}^{a,b} \subseteq Q_h$  (where elements in  $Q_{h,c}^{a,b}$  are extended by zero outside of  $\omega_b$ ), then  $\nabla \cdot \boldsymbol{\sigma}_h \in \tilde{\Phi}(Q_h \circ \mathbf{F})$  and thus  $(1 - \Upsilon_{Q_h})\nabla \cdot \boldsymbol{\sigma}_h = 0$ . For the spaces of Section 4.3, the converse inclusion is satisfied for uniform refinements  $\mathcal{T}_h$  of  $\mathcal{T}_0$  but not in general. In this case, the second term in (62a) is, for smooth  $f$ , of order  $\mathcal{O}(h^{\bar{p}+2})$ , where  $h$  denotes the maximal diameter in  $\mathcal{T}_h$ , cf. [DEV16, Equation (3.12b)] and [EV15, Remark 3.6].*

**7.2. Efficiency.** To prove efficiency, we will crucially rely on the patchwise Sobolev spaces from Section 4.2 and Assumption 5.7. We will employ the residual function  $r^{\mathbf{a}} \in H_*^1(\omega_{\mathbf{a}})$  from (46). The next proposition states local efficiency of the equilibrated flux estimator from Proposition 7.2:

**Proposition 7.4.** *Let  $\mathcal{T}_h$  be a mesh of  $\Omega$ , as is the case in Section 4.3. For all  $T \in \mathcal{T}_h$ , it holds that*

$$\begin{aligned} \|\boldsymbol{\sigma}_h + \nabla u_h\|_T \leq & \sum_{\substack{\mathbf{a} \in \mathcal{V}_h \\ |\omega_{\mathbf{a}} \cap T| > 0}} \left( 2\sqrt{1 + (C_1 + C_2)^2} C_4 C_{\text{veff}} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}} \right. \\ & \left. + \sqrt{2} C_4 C_{\text{veff}} \text{osc}_h^{\text{eff}}(\omega_{\mathbf{a}}, T) \right), \end{aligned} \quad (63a)$$

where

$$\begin{aligned} \text{osc}_h^{\text{eff}}(\omega_{\mathbf{a}}, T) := & \left( \sum_{\substack{\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}} \\ |\omega_{\mathbf{b}} \cap T| > 0}} (\text{diam}(\omega_{\mathbf{b}})^2 C_{\text{PF}}(\omega_{\mathbf{b}})^2 \|(1 - \Upsilon_{Q_{h,c}^{\mathbf{a},\mathbf{b}}})(f\psi_{\mathbf{a}}\psi_{\mathbf{b}} - \nabla u_h \cdot \nabla(\psi_{\mathbf{a}}\psi_{\mathbf{b}}))\|_{\omega_{\mathbf{b}}}^2 \right. \\ & \left. + \|(1 - \Pi_{\mathbf{RT}_h^{\mathbf{a},\mathbf{b}}})(\psi_{\mathbf{a}}\psi_{\mathbf{b}}\nabla u_h)\|_{\omega_{\mathbf{b}}}^2 \right)^{1/2} \end{aligned} \quad (63b)$$

and  $\Pi_{\mathbf{RT}_h^{\mathbf{a},\mathbf{b}}} : L^2(\omega_{\mathbf{b}}) \rightarrow \mathbf{RT}_h^{\mathbf{a},\mathbf{b}}$  denotes the  $L^2$ -orthogonal projection. With  $C_{\mathbf{F}} := \|\det(D\mathbf{F})\|_{\infty, \hat{\Omega}} \|\det(D\mathbf{F})^{-1}\|_{\infty, \hat{\Omega}}$ , the constant  $C_{\text{veff}}$  is explicitly given as

$$C_{\text{veff}} := 2C_{\text{st}} \max \{ C_3 C_5 + C_1(C_5 + C_6), C_5 + (C_5 + C_6) + C_{\mathbf{F}} C_6 \}. \quad (63c)$$

In the setting of Section 4, all involved constants depend themselves only on the space dimension  $d$ , the polynomial degree  $\bar{p}$  from Section 4.1 (which itself depends only on the considered smoothness, see Remark 5.9), and  $\max\{\|D\mathbf{F}\|_{\infty, \hat{\Omega}}, \|(D\mathbf{F})^{-1}\|_{\infty, \hat{\Omega}}\}$ . They do not depend on the polynomial degrees  $p$  and  $\tilde{p}$ .

*Proof.* We prove the assertion in seven steps.

**Step 1:** Definition (58b), the partition of unity property (37b), and the triangle inequality show that

$$\|\boldsymbol{\sigma}_h + \nabla u_h\|_T \stackrel{(58b)}{=} \stackrel{(37b)}{=} \left\| \sum_{\substack{\mathbf{a} \in \mathcal{V}_h \\ |\omega_{\mathbf{a}} \cap T| > 0}} (\boldsymbol{\sigma}_h^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h) \right\|_T \leq \sum_{\substack{\mathbf{a} \in \mathcal{V}_h \\ |\omega_{\mathbf{a}} \cap T| > 0}} \|\boldsymbol{\sigma}_h^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_T.$$



**Step 2:** Next, we bound each summand separately. Let  $\mathbf{a} \in \mathcal{V}_h$  with  $|\omega_{\mathbf{a}} \cap T| > 0$ . Then, definition (58a) and the partition of unity property (38b) together with the Cauchy–Schwarz inequality give that

$$\begin{aligned}
\|\boldsymbol{\sigma}_h^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_T &\stackrel{(58a)}{=} \left\| \sum_{\substack{\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}} \\ |\omega_{\mathbf{b}} \cap T| > 0}} (\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}} + \psi_{\mathbf{a}} \psi_{\mathbf{b}} \nabla u_h) \right\|_T \\
&\stackrel{(39d)}{\leq} \sqrt{C_4} \left( \sum_{\substack{\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}} \\ |\omega_{\mathbf{b}} \cap T| > 0}} \|\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}} + \psi_{\mathbf{a}} \psi_{\mathbf{b}} \nabla u_h\|_T^2 \right)^{1/2} \\
&\stackrel{(39e)}{\leq} \sqrt{2C_4} \left( \sum_{\substack{\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}} \\ |\omega_{\mathbf{b}} \cap T| > 0}} (\|\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}} + \psi_{\mathbf{b}}(\psi_{\mathbf{a}} \nabla u_h + \nabla r_h^{\mathbf{a}})\|_T^2 + C_5^2 \|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{b}}}^2) \right)^{1/2}.
\end{aligned} \tag{64}$$

**Step 3:** We estimate the first summand of (64). To this end, let  $\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}$  with  $|\omega_{\mathbf{b}} \cap T| > 0$ . As in Lemma 6.3, we abbreviate  $g := f\psi_{\mathbf{a}}\psi_{\mathbf{b}} - \nabla u_h \cdot \nabla(\psi_{\mathbf{a}}\psi_{\mathbf{b}}) - \nabla r_h^{\mathbf{a}} \cdot \nabla\psi_{\mathbf{b}}$  as well as  $\boldsymbol{\tau} := \psi_{\mathbf{b}}(\psi_{\mathbf{a}} \nabla u_h + \nabla r_h^{\mathbf{a}})$ . Then, definition (53) (or its equivalent formulation (55), which allows to stick in the projector  $\Pi_{\mathbf{RT}_h^{\mathbf{a},\mathbf{b}}}$  with  $\mathbf{V}_h^{\mathbf{a},\mathbf{b}} \subseteq \mathbf{RT}_h^{\mathbf{a},\mathbf{b}}$  together with (42) give that

$$\begin{aligned}
\|\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}} + \psi_{\mathbf{b}}(\psi_{\mathbf{a}} \nabla u_h + \nabla r_h^{\mathbf{a}})\|_T &\leq \|\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}} + \boldsymbol{\tau}\|_{\omega_{\mathbf{b}}} \\
&\stackrel{(53)}{=} \min_{\substack{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a},\mathbf{b}} \\ \nabla \cdot \mathbf{v}_h = \Upsilon_{Q_h^{\mathbf{a},\mathbf{b}}}(g)}} \|\mathbf{v}_h + \Pi_{\mathbf{RT}_h^{\mathbf{a},\mathbf{b}}}(\boldsymbol{\tau})\|_{\omega_{\mathbf{b}}} \stackrel{(42)}{\leq} C_{\text{st}} \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\text{div}, \omega_{\mathbf{b}}) \\ \nabla \cdot \mathbf{v} = \Upsilon_{Q_h^{\mathbf{a},\mathbf{b}}}(g)}} \|\mathbf{v} + \Pi_{\mathbf{RT}_h^{\mathbf{a},\mathbf{b}}}(\boldsymbol{\tau})\|_{\omega_{\mathbf{b}}} \\
&\stackrel{(39e)}{\leq} C_{\text{st}} \left( \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\text{div}, \omega_{\mathbf{b}}) \\ \nabla \cdot \mathbf{v} = \Upsilon_{Q_h^{\mathbf{a},\mathbf{b}}}(g)}} \|\mathbf{v} + \boldsymbol{\tau}\|_{\omega_{\mathbf{b}}} + \|(1 - \Pi_{\mathbf{RT}_h^{\mathbf{a},\mathbf{b}}})(\psi_{\mathbf{a}} \psi_{\mathbf{b}} \nabla u_h)\|_{\omega_{\mathbf{b}}} + C_5 \|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{b}}} \right).
\end{aligned}$$

With a primal–dual equivalence as in [EV15, Corollary 3.16] or in [EV20, Corollary 3.6], we further see that

$$\begin{aligned}
\min_{\substack{\mathbf{v} \in \mathbf{H}_0(\text{div}, \omega_{\mathbf{b}}) \\ \nabla \cdot \mathbf{v} = \Upsilon_{Q_h^{\mathbf{a},\mathbf{b}}}(g)}} \|\mathbf{v} + \boldsymbol{\tau}\|_{\omega_{\mathbf{b}}} &\stackrel{[EV15]}{=} \sup_{\substack{\mathbf{v} \in H_*^1(\omega_{\mathbf{b}}) \\ \|\nabla \mathbf{v}\|_{\omega_{\mathbf{b}}} = 1}} \left( -(\boldsymbol{\tau}, \nabla \mathbf{v})_{\omega_{\mathbf{b}}} + (g, \mathbf{v})_{\omega_{\mathbf{b}}} - ((1 - \Upsilon_{Q_h^{\mathbf{a},\mathbf{b}}})g, \mathbf{v})_{\omega_{\mathbf{b}}} \right) \\
&= \sup_{\substack{\mathbf{v} \in H_*^1(\omega_{\mathbf{b}}) \\ \|\nabla \mathbf{v}\|_{\omega_{\mathbf{b}}} = 1}} \left( (f\psi_{\mathbf{a}} - \nabla u_h \cdot \nabla \psi_{\mathbf{a}}, \mathbf{v}\psi_{\mathbf{b}})_{\omega_{\mathbf{b}}} - (\psi_{\mathbf{a}} \nabla u_h + \nabla r_h^{\mathbf{a}}, \nabla(\mathbf{v}\psi_{\mathbf{b}}))_{\omega_{\mathbf{b}}} \right. \\
&\quad \left. - ((1 - \Upsilon_{Q_h^{\mathbf{a},\mathbf{b}}})(f\psi_{\mathbf{a}}\psi_{\mathbf{b}} - \nabla u_h \cdot \nabla(\psi_{\mathbf{a}}\psi_{\mathbf{b}}) - \nabla r_h^{\mathbf{a}} \cdot \nabla\psi_{\mathbf{b}}), \mathbf{v})_{\omega_{\mathbf{b}}} \right).
\end{aligned} \tag{65}$$

$$- ((1 - \Upsilon_{Q_h^{\mathbf{a},\mathbf{b}}})(f\psi_{\mathbf{a}}\psi_{\mathbf{b}} - \nabla u_h \cdot \nabla(\psi_{\mathbf{a}}\psi_{\mathbf{b}}) - \nabla r_h^{\mathbf{a}} \cdot \nabla\psi_{\mathbf{b}}), \mathbf{v})_{\omega_{\mathbf{b}}}. \tag{66}$$

**Step 4:** Recalling (29), we note that for all  $v \in H_*^1(\omega_{\mathbf{b}})$ ,

$$\begin{aligned}
(f\psi_{\mathbf{a}}, \mathbf{v}\psi_{\mathbf{b}})_{\omega_{\mathbf{b}}} &= (f, \underbrace{\mathbf{v}\psi_{\mathbf{a}}\psi_{\mathbf{b}}}_{\in H_0^1(\Omega)})_{\omega_{\mathbf{b}}} \stackrel{(2)}{=} (\nabla u, \nabla(\mathbf{v}\psi_{\mathbf{a}}\psi_{\mathbf{b}}))_{\omega_{\mathbf{b}}} \\
&= (\nabla u \cdot \nabla \psi_{\mathbf{a}}, \mathbf{v}\psi_{\mathbf{b}})_{\omega_{\mathbf{b}}} + (\psi_{\mathbf{a}} \nabla u, \nabla(\mathbf{v}\psi_{\mathbf{b}}))_{\omega_{\mathbf{b}}}.
\end{aligned}$$

With the Poincaré–Friedrichs inequality (31) and Assumption 5.4, the latter equality allows us to further estimate the term in (65)

$$\begin{aligned}
& \sup_{\substack{v \in H_*^1(\omega_b) \\ \|\nabla v\|_{\omega_b} = 1}} ((f\psi_a - \nabla u_h \cdot \nabla \psi_a, v\psi_b)_{\omega_b} - (\psi_a \nabla u_h + \nabla r_h^a, \nabla(v\psi_b))_{\omega_b}) \\
&= \sup_{\substack{v \in H_*^1(\omega_b) \\ \|\nabla v\|_{\omega_b} = 1}} ((\nabla(u - u_h) \cdot \nabla \psi_a, v\psi_b)_{\omega_b} + (\psi_a \nabla(u - u_h) - \nabla r_h^a, \nabla(v\psi_b))_{\omega_b}) \\
&\leq \|\nabla(u - u_h)\|_{\omega_b} \|\nabla \psi_a\|_{\infty, \omega_b} \sup_{\substack{v \in H_*^1(\omega_b) \\ \|\nabla v\|_{\omega_b} = 1}} \|v\psi_b\|_{\omega_b} \\
&\quad + (\|\psi_a\|_{\infty, \omega_b} \|\nabla(u - u_h)\|_{\omega_b} + \|\nabla r_h^a\|_{\omega_b}) \sup_{\substack{v \in H_*^1(\omega_b) \\ \|\nabla v\|_{\omega_b} = 1}} \|\nabla(v\psi_b)\|_{\omega_b} \\
&\leq \|\nabla(u - u_h)\|_{\omega_b} (\|\nabla \psi_a\|_{\infty, \omega_b} \|\psi_b\|_{\infty, \omega_b} \text{diam}(\omega_b) C_{\text{PF}}(\omega_b)) \\
&\quad + (\|\psi_a\|_{\infty, \omega_b} \|\nabla(u - u_h)\|_{\omega_b} + \|\nabla r_h^a\|_{\omega_b}) (\|\nabla \psi_b\|_{\infty, \omega_b} \text{diam}(\omega_b) C_{\text{PF}}(\omega_b) + \|\psi_b\|_{\infty, \omega_b}) \\
&\stackrel{5.4}{\leq} (C_3 C_5 + C_1(C_5 + C_6)) \|\nabla(u - u_h)\|_{\omega_b} + (C_5 + C_6) \|\nabla r_h^a\|_{\omega_b}.
\end{aligned}$$

**Step 5:** To estimate the term in (66), we use that  $\Upsilon_{Q_h^{a,b}g} = \Upsilon_{Q_{h,c}^{a,b}g}$  from Lemma 6.4, (48), (51), the Poincaré–Friedrichs inequality (31), and (39f) to infer that

$$\begin{aligned}
& \sup_{\substack{v \in H_*^1(\omega_b) \\ \|\nabla v\|_{\omega_b} = 1}} ((1 - \Upsilon_{Q_h^{a,b}})(f\psi_a\psi_b - \nabla u_h \cdot \nabla(\psi_a\psi_b) - \nabla r_h^a \cdot \nabla \psi_b), v)_{\omega_b} \\
&\stackrel{(48)}{\leq} \stackrel{(51)}{\leq} (\|(1 - \Upsilon_{Q_{h,c}^{a,b}})(f\psi_a\psi_b - \nabla u_h \cdot \nabla(\psi_a\psi_b))\|_{\omega_b} \\
&\quad + \|\det(D\mathbf{F})\|_{\infty, \hat{\omega}_b} \|\det(D\mathbf{F})^{-1}\|_{\infty, \hat{\omega}_b} \|\nabla r_h^a\|_{\omega_b} \|\nabla \psi_b\|_{\infty, \omega_b}) \text{diam}(\omega_b) C_{\text{PF}}(\omega_b) \\
&\stackrel{(39f)}{\leq} \text{diam}(\omega_b) C_{\text{PF}}(\omega_b) \|(1 - \Upsilon_{Q_{h,c}^{a,b}})(f\psi_a\psi_b - \nabla u_h \cdot \nabla(\psi_a\psi_b))\|_{\omega_b} + C_{\mathbf{F}} C_6 \|\nabla r_h^a\|_{\omega_b}.
\end{aligned}$$

Together with Steps 3–4 and the Cauchy–Schwarz inequality, we conclude that

$$\begin{aligned}
& \|\sigma_h^{a,b} + \psi_b(\psi_a \nabla u_h + \nabla r_h^a)\|_T^2 \\
&\leq C_{\text{veff}}^2 \left( \|\nabla(u - u_h)\|_{\omega_b}^2 + \|\nabla r_h^a\|_{\omega_b}^2 + \|(1 - \Pi_{\mathbf{RT}_h^{a,b}})(\psi_a\psi_b \nabla u_h)\|_{\omega_b}^2 \right. \\
&\quad \left. + \text{diam}(\omega_b)^2 C_{\text{PF}}(\omega_b)^2 \|(1 - \Upsilon_{Q_{h,c}^{a,b}})(f\psi_a\psi_b - \nabla u_h \cdot \nabla(\psi_a\psi_b))\|_{\omega_b}^2 \right). \tag{67}
\end{aligned}$$

**Step 6:** As a final auxiliary step, we bound  $\|\nabla r_h^a\|_{\omega_a}$ . Since  $r_h^a$  from (45) is the Galerkin approximation of  $r^a$  from (46), and crucially employing (26), we see with the variational formulation (2) that

$$\begin{aligned}
\|\nabla r_h^a\|_{\omega_a} &\leq \|\nabla r^a\|_{\omega_a} \stackrel{(46)}{=} \sup_{\substack{v \in H_*^1(\omega_a) \\ \|\nabla v\|_{\omega_a} = 1}} ((f, \underbrace{v\psi_a}_{\in H_0^1(\Omega)})_{\omega_a} - (\nabla u_h, \nabla(v\psi_a))_{\omega_a}) \\
&\stackrel{(2)}{=} \sup_{\substack{v \in H_*^1(\omega_a) \\ \|\nabla v\|_{\omega_a} = 1}} (\nabla(u - u_h), \nabla(v\psi_a))_{\omega_a} \leq \|\nabla(u - u_h)\|_{\omega_a} \sup_{\substack{v \in H_*^1(\omega_a) \\ \|\nabla v\|_{\omega_a} = 1}} \|\nabla(v\psi_a)\|_{\omega_a}.
\end{aligned}$$

Relying on the Poincaré inequality (31) and with (39b) and (39a), we can bound the supremum

$$\sup_{\substack{v \in H_*^1(\omega_a) \\ \|\nabla v\|_{\omega_a} = 1}} \|\nabla(v\psi_a)\|_{\omega_a} \leq \sup_{\substack{v \in H_*^1(\omega_a) \\ \|\nabla v\|_{\omega_a} = 1}} (\|\nabla\psi_a\|_{\infty, \omega_a} \|v\|_{\omega_a} + \|\psi_a\|_{\infty, \omega_a} \|\nabla v\|_{\omega_a}) \stackrel{(39a)}{\stackrel{(39b)}}{\leq} C_1 + C_2. \quad (68)$$

**Step 7:** Putting all steps together, also using that  $C_5 \leq C_{\text{veff}}$ , we obtain that

$$\begin{aligned} & \|\sigma_h + \nabla u_h\|_T \\ & \stackrel{1,2}{\leq} \sum_{\substack{\mathbf{a} \in \mathcal{V}_h \\ |\omega_a \cap T| > 0}} \sqrt{2C_4} \left( \sum_{\substack{\mathbf{b} \in \mathcal{V}_h^a \\ |\omega_b \cap T| > 0}} (\|\sigma_h^{\mathbf{a}, \mathbf{b}} + \psi_b(\psi_a \nabla u_h + \nabla r_h^{\mathbf{a}})\|_T^2 + C_5^2 \|\nabla r_h^{\mathbf{a}}\|_{\omega_b}^2) \right)^{1/2} \\ & \stackrel{(67)}{\leq} \sqrt{2C_4} \sum_{\substack{\mathbf{a} \in \mathcal{V}_h \\ |\omega_a \cap T| > 0}} \left( \sum_{\substack{\mathbf{b} \in \mathcal{V}_h^a \\ |\omega_b \cap T| > 0}} (C_{\text{veff}}^2 (\|\nabla(u - u_h)\|_{\omega_b}^2 + \|\nabla r_h^{\mathbf{a}}\|_{\omega_b}^2) \right. \\ & \quad \left. + \|(1 - \Pi_{\mathbf{RT}_h^{\mathbf{a}, \mathbf{b}}})(\psi_a \psi_b \nabla u_h)\|_{\omega_b}^2 \right. \\ & \quad \left. + \text{diam}(\omega_b)^2 C_{\text{PF}}(\omega_b)^2 \|(1 - \Upsilon_{Q_{h,c}^{\mathbf{a}, \mathbf{b}}})(f\psi_a \psi_b - \nabla u_h \cdot \nabla(\psi_a \psi_b))\|_{\omega_b}^2 + C_5^2 \|\nabla r_h^{\mathbf{a}}\|_{\omega_b}^2 \right)^{1/2} \\ & \stackrel{(39d)}{\stackrel{(63b)}}{\leq} 2C_4 C_{\text{veff}} \sum_{\substack{\mathbf{a} \in \mathcal{V}_h \\ |\omega_a \cap T| > 0}} (\|\nabla(u - u_h)\|_{\omega_a}^2 + \|\nabla r_h^{\mathbf{a}}\|_{\omega_a}^2)^{1/2} + \sqrt{2C_4} C_{\text{veff}} \sum_{\substack{\mathbf{a} \in \mathcal{V}_h \\ |\omega_a \cap T| > 0}} \text{osc}_h^{\text{eff}}(\omega_a, T) \\ & \stackrel{6}{\leq} 2\sqrt{1 + (C_1 + C_2)^2} C_4 C_{\text{veff}} \sum_{\substack{\mathbf{a} \in \mathcal{V}_h \\ |\omega_a \cap T| > 0}} \|\nabla(u - u_h)\|_{\omega_a} + \sqrt{2C_4} C_{\text{veff}} \sum_{\substack{\mathbf{a} \in \mathcal{V}_h \\ |\omega_a \cap T| > 0}} \text{osc}_h^{\text{eff}}(\omega_a, T), \end{aligned}$$

which concludes the proof.  $\square$

**Remark 7.5.** In the situation of Section 4,  $\text{osc}_h^{\text{eff}}(\omega_a, T)$  of (63b) vanishes if  $\mathbf{F}$  is affine,  $\mathcal{T}_h$  is a uniform refinement of  $\mathcal{T}_0$ ,  $f$  is a  $\mathcal{T}_h$ -piecewise polynomial of some degree  $\mathbf{q} = (q, \dots, q)$  with  $q \geq 0$ , and  $\tilde{p} \geq \max\{q + \bar{p} + 1, p + \bar{p} + 1\}$ . Note that  $\tilde{p} \geq p + \bar{p} + 1$  in particular implies that  $\psi_a \psi_b \nabla u_h + \psi_b \nabla r_h^{\mathbf{a}} \in \mathbf{RT}_h^{\mathbf{a}, \mathbf{b}}$ .

We now turn towards the global efficiency. In order to achieve robustness with respect to the strength of the hierarchical refinement (the number of hanging nodes), we do not straightforwardly use the element-related result of Proposition 7.4 but rather resort to its patch-related variant. With an assumption on the maximal overlap by the patches  $\omega_a$  (not limiting the strength of the hierarchical refinement, see Remark 7.7), our global efficiency result is:

**Proposition 7.6.** Let  $\mathcal{T}_h$  be a mesh of  $\Omega$ , as is the case in Section 4.3. Let  $C_{\text{over}} > 0$  be a constant such that

$$\sup_{\mathbf{x} \in \Omega} \#\{\mathbf{a} \in \mathcal{V}_h : \mathbf{x} \in \omega_a\} \leq C_{\text{over}}. \quad (69a)$$

Then, there holds that

$$\begin{aligned} \|\boldsymbol{\sigma}_h + \nabla u_h\|_{\Omega}^2 &\leq 4C_{\text{over}}^2 C_4^2 C_{\text{veff}}^2 ((C_1 + C_2)^2 + 1) \|\nabla(u - u_h)\|_{\Omega}^2 \\ &\quad + 4C_{\text{over}} C_4 C_{\text{veff}}^2 \sum_{\mathbf{a} \in \mathcal{V}_h} \text{osc}_h^{\text{eff}}(\omega_{\mathbf{a}})^2, \end{aligned} \quad (69b)$$

where

$$\begin{aligned} \text{osc}_h^{\text{eff}}(\omega_{\mathbf{a}})^2 &:= \sum_{\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}} (\text{diam}(\omega_{\mathbf{b}})^2 C_{\text{PF}}(\omega_{\mathbf{b}})^2 \|(1 - \Upsilon_{Q_{h,c}^{\mathbf{a},\mathbf{b}}})(f\psi_{\mathbf{a}}\psi_{\mathbf{b}} - \nabla u_h \cdot \nabla(\psi_{\mathbf{a}}\psi_{\mathbf{b}}))\|_{\omega_{\mathbf{b}}}^2 \\ &\quad + \|(1 - \Pi_{\mathbf{RT}_h^{\mathbf{a},\mathbf{b}}})(\psi_{\mathbf{a}}\psi_{\mathbf{b}}\nabla u_h)\|_{\omega_{\mathbf{b}}}^2). \end{aligned} \quad (69c)$$

*Proof.* Proceeding as in the proof of Proposition 7.4 while relying on Definition 6.5 and the partitions of unity (37b) and (38b) together with the finite overlap assumptions (39d) and (69a), we see

$$\begin{aligned} \|\boldsymbol{\sigma}_h + \nabla u_h\|_{\Omega}^2 &\stackrel{(37b)}{=} \left\| \sum_{\mathbf{a} \in \mathcal{V}_h} (\boldsymbol{\sigma}_h^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h) \right\|_{\Omega}^2 \stackrel{(69a)}{\leq} C_{\text{over}} \sum_{\mathbf{a} \in \mathcal{V}_h} \|\boldsymbol{\sigma}_h^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}}^2 \\ &\stackrel{(38b)}{=} C_{\text{over}} \sum_{\mathbf{a} \in \mathcal{V}_h} \left\| \sum_{\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}} (\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}} + \psi_{\mathbf{a}}\psi_{\mathbf{b}}\nabla u_h) \right\|_{\omega_{\mathbf{a}}}^2 \\ &\stackrel{(58a)}{\leq} C_{\text{over}} C_4 \sum_{\mathbf{a} \in \mathcal{V}_h} \sum_{\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}} \|\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{b}} + \psi_{\mathbf{a}}\psi_{\mathbf{b}}\nabla u_h\|_{\omega_{\mathbf{b}}}^2. \end{aligned} \quad (39d)$$

We now employ (67) plus the triangle and the Cauchy–Schwarz inequalities. Also using (39e) and the fact that  $C_5 \leq C_{\text{veff}}$ , this leads to

$$\|\boldsymbol{\sigma}_h + \nabla u_h\|_{\Omega}^2 \leq C_{\text{over}} C_4 4C_{\text{veff}}^2 \sum_{\mathbf{a} \in \mathcal{V}_h} \left( \sum_{\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}} (\|\nabla(u - u_h)\|_{\omega_{\mathbf{b}}}^2 + \|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{b}}}^2) + \text{osc}_h^{\text{eff}}(\omega_{\mathbf{a}})^2 \right),$$

and we are left to treat the first two terms. The finite overlap assumptions (39d) and (69a) again imply

$$\sum_{\mathbf{a} \in \mathcal{V}_h} \sum_{\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}} \|\nabla(u - u_h)\|_{\omega_{\mathbf{b}}}^2 \stackrel{(39d)}{\leq} C_4 \sum_{\mathbf{a} \in \mathcal{V}_h} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}^2 \stackrel{(69a)}{\leq} C_{\text{over}} C_4 \|\nabla(u - u_h)\|_{\Omega}^2.$$

Reasoning similarly and also employing the two estimates from Step 6 of the proof of Proposition 7.4, we see

$$\begin{aligned} \sum_{\mathbf{a} \in \mathcal{V}_h} \sum_{\mathbf{b} \in \mathcal{V}_h^{\mathbf{a}}} \|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{b}}}^2 &\stackrel{(39d)}{\leq} C_4 \sum_{\mathbf{a} \in \mathcal{V}_h} \|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}^2 \leq C_4 (C_1 + C_2)^2 \sum_{\mathbf{a} \in \mathcal{V}_h} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}^2 \\ &\stackrel{(69a)}{\leq} C_{\text{over}} C_4 (C_1 + C_2)^2 \|\nabla(u - u_h)\|_{\Omega}^2, \end{aligned}$$

which altogether gives (69b).  $\square$

**Remark 7.7.** For (scaled) hierarchical B-splines  $\psi_{\mathbf{a}}$  of degree  $\bar{p}$  on  $\mathcal{H}$ -admissible meshes  $\widehat{\mathcal{T}}_h$  of class  $\mu$  introduced along with suitable refinement algorithms in [GHP17] for  $\mu = 2$  and in [BGV18] for  $\mu \geq 2$ , the upper bound  $C_{\text{over}}$  from (69a) only depends on the polynomial degree  $\bar{p}$  (which itself only depends on the considered smoothness) and the grading parameter  $\mu$ . Meshes with arbitrarily many hanging nodes satisfying assumption (69a) with the constant  $C_{\text{over}}$  only depending on the polynomial degree  $\bar{p}$  are considered in the numerics Section 8 below.

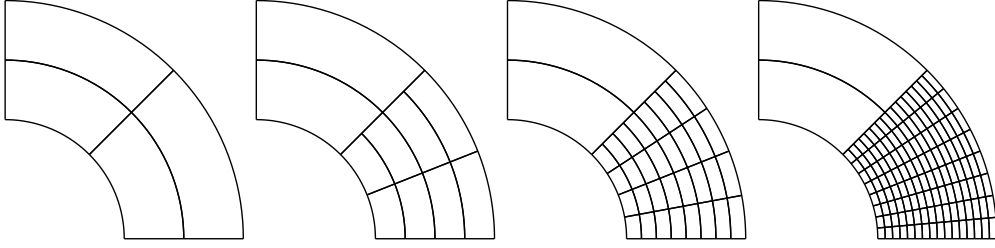


FIGURE 4. The domain  $\Omega$  considered in Section 8 along with the initial mesh  $\mathcal{T}_0$  and the first three artificial refinements.

## 8. NUMERICAL EXPERIMENTS

We consider problem (1) on the quarter ring depicted in Figure 4,

$$\Omega := \{r(\cos(\varphi), \sin(\varphi)) : r \in (1/2, 1) \wedge \varphi \in (0, \pi/2)\}$$

with NURBS parametrization  $\mathbf{F}$  as in [GHP17, Section 6.3], and prescribe the exact solution

$$u(x, y) = xy \sin(4\pi(x^2 + y^2)). \quad (70)$$

For polynomial degrees  $p \in \{1, \dots, 5\}$  and multiplicities  $m \in \{1, p\}$ , we define the initial knot vectors by

$$\mathcal{K}_{1(0)} := \mathcal{K}_{2(0)} := \left( \underbrace{0, \dots, 0}_{(p+1)\text{-times}}, \underbrace{1/2, \dots, 1/2}_m, \underbrace{1, \dots, 1}_{(p+1)\text{-times}} \right),$$

leading to piecewise  $p$ -degree polynomials with  $C^{p-m}$  smoothness. As the corresponding polynomial degree  $\bar{p}$  for the partition of unity by the  $\psi_{\mathbf{a}}$  in Section 4.1, we choose  $\bar{p} := p + 1 - m$  and the corresponding multiplicity  $\bar{m} := \bar{p} - p + m = 1$  following (19), so that the  $\psi_{\mathbf{a}}$  are mapped piecewise  $\bar{p}$ -degree polynomials of class  $C^{\bar{p}-1}$ . The polynomial degree  $\tilde{p}$  for the flux equilibration in Section 4.3 is chosen in  $\{p+1, p+2\}$ . Following Remark 7.5, ignoring temporarily the source term  $f$ , this would only imply  $\text{osc}_h^{\text{eff}}(\omega_{\mathbf{a}}, T) = \text{osc}_h^{\text{eff}}(\omega_{\mathbf{a}}) = 0$  in Propositions 7.4 and 7.6 if  $\mathbf{F}$  was affine (which is not the case here) and  $\tilde{p} \geq p + \bar{p} + 1$  (which is only the case here for  $m = p$ , i.e.,  $C^0$ , but not higher-smoothness splines). Nevertheless, both choices  $\tilde{p} \in \{p+1, p+2\}$  seem to perform numerically well in the considered test case also for high-smoothness cases, up to  $m = 1$ , corresponding to  $C^{p-1}$  splines.

We consider three different refinements of the initial mesh: 1) uniform refinement, where in each step, all elements in the parameter domain are bisected in both directions; 2) adaptive refinement, where in each step, a minimal set of elements  $\mathcal{M}_h \subseteq \mathcal{T}_h$  is marked via the Dörfler marking

$$\theta \sum_{T \in \mathcal{T}_h} (\|\boldsymbol{\sigma}_h + \nabla u_h\|_T^2 + \text{osc}_h^{\text{rel}}(T)^2) \leq \sum_{T \in \mathcal{M}_h} (\|\boldsymbol{\sigma}_h + \nabla u_h\|_T^2 + \text{osc}_h^{\text{rel}}(T)^2)$$

with  $\theta = 0.5$  and subsequently refined via the refinement strategy from [GHP17] (see also Remark 7.7); and 3) artificial refinement enforcing an unlimited number of hanging nodes, where in each step, all elements in the parameter domain  $[0, 1]^2$  that are contained in  $[0, 1/2] \times [0, 1]$  are bisected in both directions; see Figure 4. In each case, new knots have multiplicity  $m$ .

The resulting effectivity indices as function of the number of mesh elements  $N$  in  $\mathcal{T}_h$  are displayed in Figures 5–7. Recall that the efficiency constant in (69b) may theoretically depend on the space dimension  $d$ , the polynomial degree  $\bar{p}$  (which itself depends on

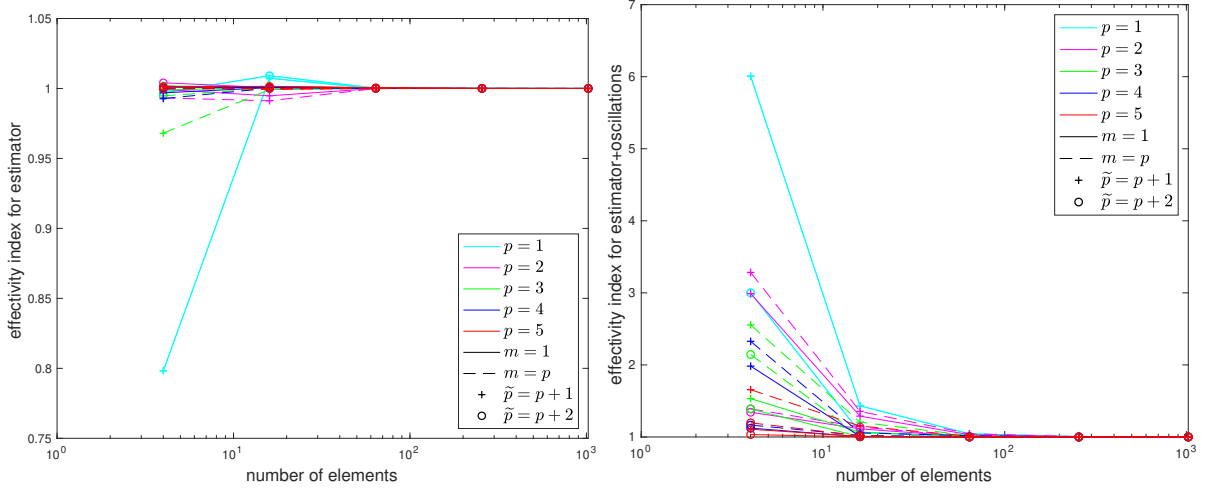


FIGURE 5. Effectivity indices  $\frac{\|\boldsymbol{\sigma}_h + \nabla u_h\|_{\Omega}}{\|\nabla(u - u_h)\|_{\Omega}}$  (left) and  $\frac{\|\boldsymbol{\sigma}_h + \nabla u_h\|_{\Omega} + \text{osc}_h^{\text{rel}}}{\|\nabla(u - u_h)\|_{\Omega}}$  (right) corresponding to the problem of Section 8 with uniform mesh refinement, polynomial degrees  $p \in \{1, \dots, 5\}$ , multiplicities  $m \in \{1, p\}$ .

the considered smoothness, see Remark 5.9), and  $\max\{\|D\mathbf{F}\|_{\infty, \hat{\Omega}}, \|(D\mathbf{F})^{-1}\|_{\infty, \hat{\Omega}}\}$  (see Propositions 7.4 and 7.6). At least in this example, though, the dependence on  $\bar{p}$  is *not* observed, and the equilibrated flux estimator  $\|\boldsymbol{\sigma}_h + \nabla u_h\|_{\Omega}$  (plus oscillation terms) seems to be not only, as proven, robust with respect to the polynomial degrees  $p$  and  $\tilde{p}$ , but also with respect to the smoothness incarnated in  $\bar{p}$ . The increase of the effectivity indices on adaptively refined meshes with  $\tilde{p} = p + 1$  in the right part of Figure 6 is only because of data oscillation, as we discuss below. Moreover, as theoretically shown in Proposition 7.6, we also numerically observe in Figure 7 the robustness with respect to the strength of the hierarchical refinement (number of hanging nodes); recall in this respect from Remark 7.7 that the overlap constant  $C_{\text{over}}$  from (69a) again only depends on  $\bar{p}$ . Additional numerical experiments were carried out for an exact solution  $u(x, y) = \sin(2\pi x) \sin(2\pi y)$  on the square (not displayed), with similar results.

The “data oscillation” terms  $\text{osc}_h^{\text{rel}}$  from (62b) are displayed separately in Figure 8, again as function of the number of mesh elements  $N$  in  $\mathcal{T}_h$ . For the present smooth solution (70), one expects the error  $\|\nabla(u - u_h)\|_{\Omega}$  to decay as  $\mathcal{O}(h^p) \approx \mathcal{O}(N^{-p/2})$  for uniform mesh refinement. Recall from Remark 7.3 that  $\text{osc}_h^{\text{rel}}$  are expected to decay in this case as  $\mathcal{O}(h^{\tilde{p}+2})$ , i.e. as  $\mathcal{O}(N^{-p/2-3/2})$  or  $\mathcal{O}(N^{-p/2-2})$  for respectively  $\tilde{p} = p + 1$  and  $\tilde{p} = p + 2$ . Figure 8 only concerns adaptive mesh refinement, where  $\|\nabla(u - u_h)\|_{\Omega}$  is still expected to decay as  $\mathcal{O}(N^{-p/2})$ . We do not have here theoretical indications for  $\text{osc}_h^{\text{rel}}$ , but we observe at least  $\mathcal{O}(N^{-p/2})$  in most cases, even though the chosen  $\tilde{p}$  is theoretically inappropriate for higher smoothness as discussed above. For uniform mesh refinement (not displayed), we indeed observe  $\mathcal{O}(h^{\tilde{p}+2})$ , following Remark 7.3.

## APPENDIX A. PROOF OF THE BROKEN POLYNOMIAL EXTENSION PROPERTY

In this section, we verify Assumption 5.7 for the spaces defined in Section 4.3. Recall the Piola transformations from (34) and (48)

$$\Phi(\cdot) := (\det(D\mathbf{F})^{-1}(D\mathbf{F})(\cdot)) \circ \mathbf{F}^{-1} \quad \text{and} \quad \tilde{\Phi}(\cdot) := (\det(D\mathbf{F})^{-1}(\cdot)) \circ \mathbf{F}^{-1}$$

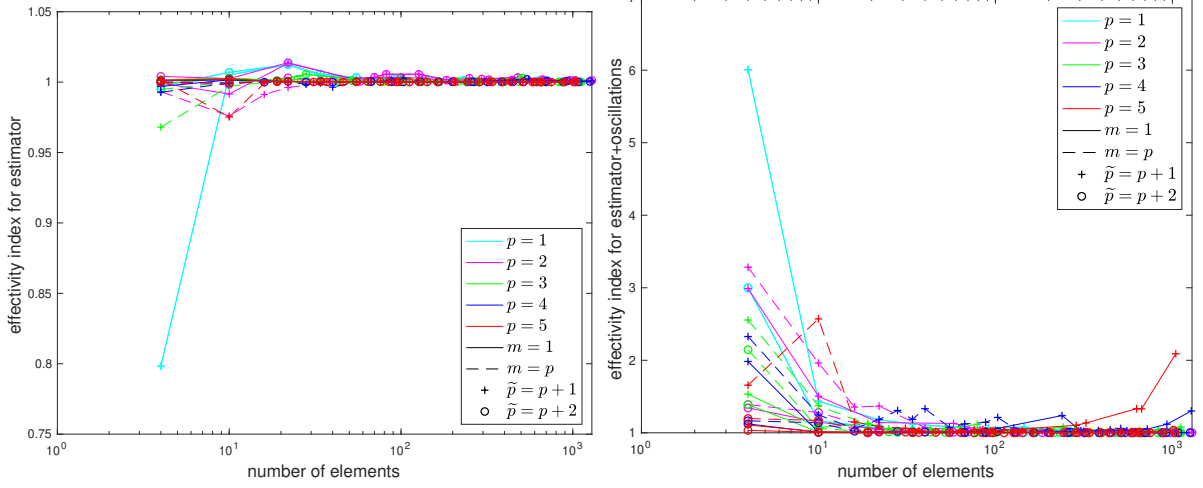


FIGURE 6. Effectivity indices  $\frac{\|\sigma_h + \nabla u_h\|_{\Omega}}{\|\nabla(u - u_h)\|_{\Omega}}$  (left) and  $\frac{\|\sigma_h + \nabla u_h\|_{\Omega} + \text{osc}_h^{\text{rel}}}{\|\nabla(u - u_h)\|_{\Omega}}$  (right) corresponding to the problem of Section 8 with adaptive mesh refinement, polynomial degrees  $p \in \{1, \dots, 5\}$ , multiplicities  $m \in \{1, p\}$ .

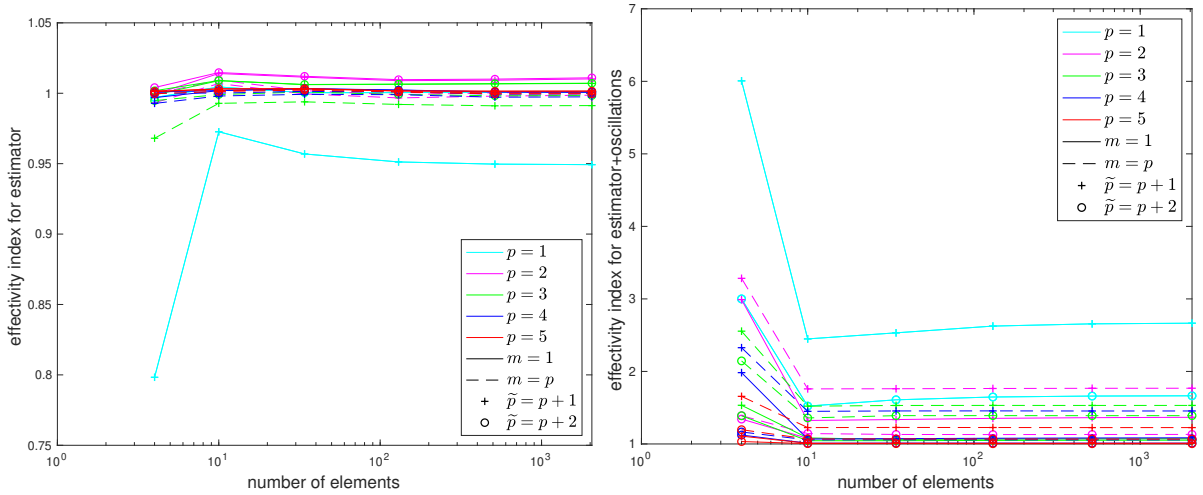


FIGURE 7. Effectivity indices  $\frac{\|\sigma_h + \nabla u_h\|_{\Omega}}{\|\nabla(u - u_h)\|_{\Omega}}$  (left) and  $\frac{\|\sigma_h + \nabla u_h\|_{\Omega} + \text{osc}_h^{\text{rel}}}{\|\nabla(u - u_h)\|_{\Omega}}$  (right) corresponding to the problem of Section 8 with artificial refinement enforcing a high number of hanging nodes, polynomial degrees  $p \in \{1, \dots, 5\}$ , multiplicities  $m \in \{1, p\}$ .

and the identity

$$\tilde{\Phi}(\nabla \cdot (\cdot)) = \nabla \cdot \Phi(\cdot); \quad (71)$$

see, e.g., [EG21, Lemma 9.6]. Following the imposition of the Neumann boundary condition in the space  $\mathbf{H}_0(\text{div}, \omega_b)$  in (30), we denote by  $\mathcal{F}_b^N$  the boundary faces of the local mesh  $\mathcal{T}_b$  if  $\psi_a \psi_b \in H_0^1(\Omega)$  and such boundary faces of  $\mathcal{T}_b$  where  $F \subset (\psi_a \psi_b)^{-1}(\{0\})$  for  $\psi_a \psi_b \notin H_0^1(\Omega)$ . By  $\mathcal{F}_b^{\text{int}}$ , we denote the interior faces of  $\mathcal{T}_b$ . Analogously, we write  $\widehat{\mathcal{F}}_b^N$  and  $\widehat{\mathcal{F}}_b^{\text{int}}$  for the corresponding faces on the parameter mesh  $\widehat{\mathcal{T}}_b$ . Finally, we write  $\nabla_b \cdot (\cdot)$  for the  $\mathcal{T}_b$ - or  $\widehat{\mathcal{T}}_b$ -piecewise divergence operator.



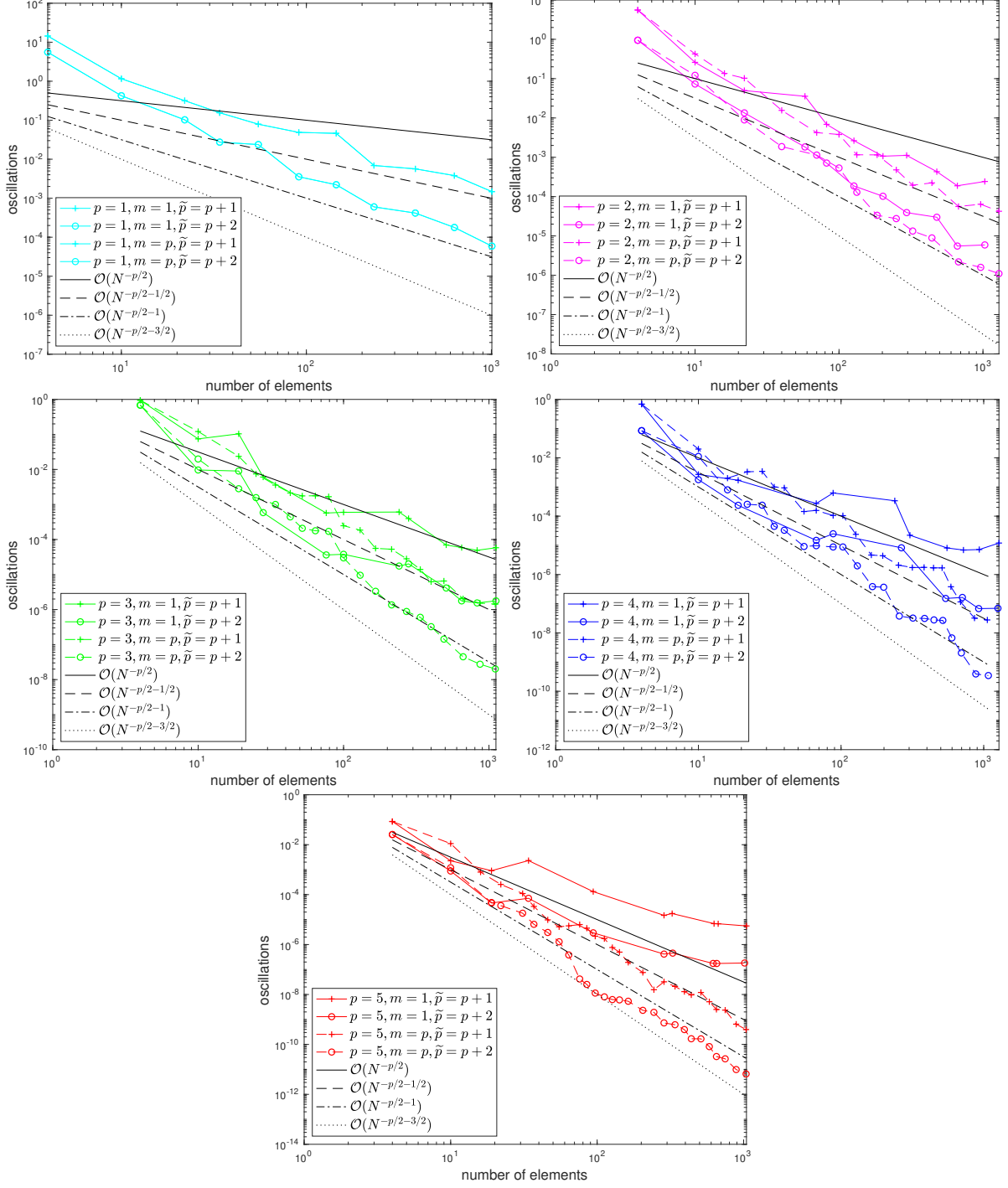


FIGURE 8. Oscillation terms  $\text{osc}_h^{\text{rel}}$  from (62b) corresponding to the problem of Section 8 with adaptive mesh refinement, polynomial degrees  $p \in \{1, \dots, 5\}$ , multiplicities  $m \in \{1, p\}$ .

Substituting in (42)  $\mathbf{v}_h + \boldsymbol{\tau}_h = \mathbf{w}_h$  and  $\mathbf{w}_h = \Phi(\hat{\mathbf{w}}_h)$  and recalling (35) shows that

$$\begin{aligned}
 \min_{\substack{\mathbf{v}_h \in \mathbf{V}_h^{a,b} \\ \nabla \cdot \mathbf{v}_h = g_h}} \|\mathbf{v}_h + \boldsymbol{\tau}_h\|_{\omega_b} &= \min_{\mathbf{w}_h \in \mathbf{RT}_h^{a,b}} \|\mathbf{w}_h\|_{\omega_b} = \min_{\hat{\mathbf{w}}_h \in \mathbf{RT}^{\tilde{p}}(\hat{\mathcal{T}}_b)} \|\Phi(\hat{\mathbf{w}}_h)\|_{\omega_b}. \\
 \mathbf{w}_h \cdot \mathbf{n}_{\omega_b}|_F &= \boldsymbol{\tau}_h \cdot \mathbf{n}_{\omega_b}|_F =: r_F \quad \forall F \in \mathcal{F}_b^N \\
 [\mathbf{w}_h \cdot \mathbf{n}_{\omega_b}]_F &= [\boldsymbol{\tau}_h \cdot \mathbf{n}_{\omega_b}]_F =: r_F \quad \forall F \in \mathcal{F}_b^{\text{int}} \\
 \nabla_b \cdot \mathbf{w}_h|_T &= (g_h + \nabla_b \cdot \boldsymbol{\tau}_h)|_T =: r_T \quad \forall T \in \mathcal{T}_b
 \end{aligned} \tag{72}$$

With  $\mathbf{n}_{\widehat{\omega}_b}$  denoting the outer normal vector on  $\partial\widehat{\omega}_b$ , elementary analysis, cf. [EG21, Lemma 9.11], provides the relation

$$(D\mathbf{F}^\top \circ \mathbf{F}^{-1}) \mathbf{n}_{\omega_b} = \mathbf{n}_{\widehat{\omega}_b} \circ \mathbf{F}^{-1} |(D\mathbf{F}^\top \circ \mathbf{F}^{-1}) \mathbf{n}_{\omega_b}|,$$

and thus

$$\begin{aligned} \Phi(\widehat{\mathbf{w}}_h) \cdot \mathbf{n}_{\omega_b} &= \left( (\det(D\mathbf{F})^{-1} \widehat{\mathbf{w}}_h^\top (D\mathbf{F})^\top) \circ \mathbf{F}^{-1} \right) \mathbf{n}_{\omega_b} \\ &= \left( \widehat{\mathbf{w}}_h \cdot \mathbf{n}_{\widehat{\omega}_b} \frac{|(D\mathbf{F}^\top \circ \mathbf{F}^{-1}) \mathbf{n}_{\omega_b}|}{\det(D\mathbf{F})} \right) \circ \mathbf{F}^{-1}. \end{aligned}$$

Hence, with  $\widehat{F} := \mathbf{F}^{-1}(F)$  and  $\widehat{\tau}_h := \Phi^{-1}(\tau_h)$ , the first equation in the last minimum of (72) is equivalent to  $\widehat{\mathbf{w}}_h \cdot \mathbf{n}_{\widehat{\omega}_b}|_{\widehat{F}} = \widehat{\tau}_h \cdot \mathbf{n}_{\widehat{\omega}_b}|_{\widehat{F}} =: r_{\widehat{F}}$ , and the second equation is equivalent to  $[\widehat{\mathbf{w}}_h \cdot \mathbf{n}_{\widehat{\omega}_b}]|_{\widehat{F}} = [\widehat{\tau}_h \cdot \mathbf{n}_{\widehat{\omega}_b}]|_{\widehat{F}} =: r_{\widehat{F}}$ . The identity (71) shows that the third equation is equivalent to  $\nabla_b \cdot \widehat{\mathbf{w}}_h|_{\widehat{T}} = \widehat{\Phi}^{-1}(g_h + \nabla_b \cdot \tau_h) =: r_{\widehat{T}}$ . As  $\|\Phi(\widehat{\mathbf{w}}_h)\|_{\omega_b} \lesssim \|\widehat{\mathbf{w}}_h\|_{\widehat{\omega}_b}$  (with a hidden constant depending only on  $\|D\mathbf{F}\|_{\infty, \widehat{\Omega}}$ ), we can formulate (72) in the parameter domain

$$\min_{\substack{\mathbf{v}_h \in \mathbf{V}_h^{a,b} \\ \nabla \cdot \mathbf{v}_h = g_h}} \|\mathbf{v}_h + \tau_h\|_{\omega_b} \lesssim \min_{\substack{\widehat{\mathbf{w}}_h \in \mathbf{RT}^{\widehat{p}}(\widehat{\mathcal{T}}_b) \\ \widehat{\mathbf{w}}_h \cdot \mathbf{n}_{\widehat{\omega}_b}|_{\widehat{F}} = r_{\widehat{F}} \quad \forall \widehat{F} \in \widehat{\mathcal{F}}_b^N \\ [\widehat{\mathbf{w}}_h \cdot \mathbf{n}_{\widehat{\omega}_b}]|_{\widehat{F}} = r_{\widehat{F}} \quad \forall \widehat{F} \in \widehat{\mathcal{F}}_b^{\text{int}} \\ \nabla_b \cdot \widehat{\mathbf{w}}_h|_{\widehat{T}} = r_{\widehat{T}} \quad \forall \widehat{T} \in \widehat{\mathcal{T}}_b}} \|\widehat{\mathbf{w}}_h\|_{\widehat{\omega}_b}. \quad (73)$$

Finally, an application of [BPS09, Theorems 5 and 7], see also [EV20, Theorem 2.5 and Corollary 3.3], yields that

$$\min_{\substack{\widehat{\mathbf{w}}_h \in \mathbf{RT}^{\widehat{p}}(\widehat{\mathcal{T}}_b) \\ \widehat{\mathbf{w}}_h \cdot \mathbf{n}_{\widehat{\omega}_b}|_{\widehat{F}} = r_{\widehat{F}} \quad \forall \widehat{F} \in \widehat{\mathcal{F}}_b^N \\ [\widehat{\mathbf{w}}_h \cdot \mathbf{n}_{\widehat{\omega}_b}]|_{\widehat{F}} = r_{\widehat{F}} \quad \forall \widehat{F} \in \widehat{\mathcal{F}}_b^{\text{int}} \\ \nabla_b \cdot \widehat{\mathbf{w}}_h|_{\widehat{T}} = r_{\widehat{T}} \quad \forall \widehat{T} \in \widehat{\mathcal{T}}_b}} \|\widehat{\mathbf{w}}_h\|_{\widehat{\omega}_b} \leq C_{\text{st}} \min_{\substack{\widehat{\mathbf{w}} \in \mathbf{H}(\text{div}, \widehat{\mathcal{T}}_b) \\ \widehat{\mathbf{w}} \cdot \mathbf{n}_{\widehat{\omega}_b}|_{\widehat{F}} = r_{\widehat{F}} \quad \forall \widehat{F} \in \widehat{\mathcal{F}}_b^N \\ [\widehat{\mathbf{w}} \cdot \mathbf{n}_{\widehat{\omega}_b}]|_{\widehat{F}} = r_{\widehat{F}} \quad \forall \widehat{F} \in \widehat{\mathcal{F}}_b^{\text{int}} \\ \nabla_b \cdot \widehat{\mathbf{w}}|_{\widehat{T}} = r_{\widehat{T}} \quad \forall \widehat{T} \in \widehat{\mathcal{T}}_b}} \|\widehat{\mathbf{w}}\|_{\widehat{\omega}_b},$$

for a generic constant  $C_{\text{st}}$  only depending on the shapes of the elements in  $\widehat{\mathcal{T}}_0$ . Here,  $\mathbf{H}(\text{div}, \widehat{\mathcal{T}}_b)$  is the space of  $\widehat{\mathcal{T}}_b$ -piecewise  $\mathbf{H}(\text{div})$ -functions on  $\widehat{\omega}_b$ . The required compatibility condition

$$\sum_{\widehat{T} \in \widehat{\mathcal{T}}_b} (r_{\widehat{T}}, 1)_{\widehat{T}} - \sum_{\widehat{F} \in \widehat{\mathcal{F}}_b^N \cup \widehat{\mathcal{F}}_b^{\text{int}}} (r_{\widehat{F}}, 1)_{\widehat{F}} = 0$$

in case of  $\psi_a \psi_b \in H_0^1(\Omega)$  follows from the assumption that  $g_h$ , and thus  $\widehat{\Phi}^{-1}(g_h)$ , has integral mean zero then. Finally,

$$\min_{\substack{\widehat{\mathbf{w}} \in \mathbf{H}(\text{div}, \widehat{\mathcal{T}}_b) \\ \widehat{\mathbf{w}} \cdot \mathbf{n}_{\widehat{\omega}_b}|_{\widehat{F}} = r_{\widehat{F}} \quad \forall \widehat{F} \in \widehat{\mathcal{F}}_b^N \\ [\widehat{\mathbf{w}} \cdot \mathbf{n}_{\widehat{\omega}_b}]|_{\widehat{F}} = r_{\widehat{F}} \quad \forall \widehat{F} \in \widehat{\mathcal{F}}_b^{\text{int}} \\ \nabla_b \cdot \widehat{\mathbf{w}}|_{\widehat{T}} = r_{\widehat{T}} \quad \forall \widehat{T} \in \widehat{\mathcal{T}}_b}} \|\widehat{\mathbf{w}}\|_{\widehat{\omega}_b} \lesssim \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\text{div}, \omega_b) \\ \nabla \cdot \mathbf{v} = g_h}} \|\mathbf{v} + \tau_h\|_{\omega_b}$$

with a hidden constant depending only on  $\|(D\mathbf{F})^{-1}\|_{\infty, \widehat{\Omega}}$ , as in (72) and (73). This concludes the proof.

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INSTITUTE OF ANALYSIS AND SCIENTIFIC COMPUTING, TU WIEN, WIEDNER HAUPTSTRASSE 8-10, 1040 VIENNA, AUSTRIA ([GREGOR.GANTNER@ASC.TUWIEN.AC.AT](mailto:GREGOR.GANTNER@ASC.TUWIEN.AC.AT)).

INRIA, 2 RUE SIMONE IFF, 75589 PARIS, FRANCE & CERMICS, ECOLE DES PONTS, 77455 MARNE-LA-VALLÉE, FRANCE ([MARTIN.VOHRALIK@INRIA.FR](mailto:MARTIN.VOHRALIK@INRIA.FR)).