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# Learning Market Equilibria Using Performative Prediction: Balancing Efficiency and Privacy

Raphaël Taisant<sup>1</sup>, Mandar Datar<sup>2</sup>, H el ene Le Cadre<sup>1</sup> and Eitan Altman<sup>2</sup>

**Abstract**—We consider a peer-to-peer electricity market modeled as a network game, where End Users (EUs) minimize their cost by computing their demand and generation while satisfying a set of local and coupling constraints. Their nominal demand constitutes sensitive information, that they might want to keep private. We prove that the network game admits a unique Variational Equilibrium, which depends on the private information of all the EUs. A data aggregator is introduced, which aims to learn the EUs’ private information. The EUs might have incentives to report biased and noisy readings to preserve their privacy, which creates shifts in their strategies. Relying on performative prediction, we define a decision-dependent game  $\mathcal{G}^{\text{stoch}}$  to couple the network game with a data market. Two variants of the Repeated Stochastic Gradient Method (RSGM) are proposed to compute the Performatively Stable Equilibrium solution of  $\mathcal{G}^{\text{stoch}}$ , that outperform RSGM with respect to efficiency gap minimization, privacy preservation, and convergence rates in numerical simulations.

## I. INTRODUCTION

### A. Problem Statement

In distributed control system literature, there is a usual trade-off between cost minimization and privacy preservation: to obtain a better outcome, each End User (EU) in the system relies on its own information and the information the others have reported. However, the EUs might be reluctant to release sensitive information because of privacy limitations or because, by acting *strategically* on their readings – therefore providing fake information, they might decrease their cost. The former situation might give rise to moral hazard, which could generate inefficiencies at the system level. To penalize free-rider behaviors and incentivize the EUs to minimize the variance of their readings, we aim to define contracts which remunerate the EUs as increasing functions of the data quality while discriminating between the EUs depending on their local privacy breach.

In energy systems, data – and the information it contains – is becoming an asset that the agents might want to acquire to learn market equilibria [1], [2]. For example, in many demand-response programs, the aggregator uses the EUs’ historical energy consumption to determine their nominal demand profiles, and design incentives to flatten peak times. The nominal demand profiles are then used to compute incentives depending on how much they curtail relative to their nominal profiles. However, EUs can act strategically

on the data they share. For example, energy consumers may artificially inflate their nominal demand prior to a demand-response program event, in order to receive larger payments under the program (see [3] and references therein). In this paper, we want to quantify the EUs’ trade-off between *privacy preservation*, i.e., minimum information release, and *indistinguishability*, i.e., the inability for the aggregator to determine whether the EUs’ readings are biased or not, and the *efficiency loss* for the market, under different coordination schemes.

### B. Main Contributions

We consider a peer-to-peer electricity trading problem formulated as a network game, where strategic EUs want to minimize their cost by computing their demand (flexible around a nominal value), controllable generation, and trades with their peers, while satisfying a set of local constraints including the local balance of supply and demand, and trading reciprocity constraints that couple the EUs’ optimization problems. Their nominal demand constitutes sensitive information, that is kept private to the EUs. Following [1], [2] and the electricity market literature, the EUs’ utility functions are assumed additive and quadratic. We prove that under mild assumptions, the network game admits a unique Variational Equilibrium (VE), which is efficient and stable. In addition, we prove that, at the VE, the EUs’ best-responses depend on the private information of *all* the EUs. A Data Aggregator (DA) might then be introduced, which aims to learn the EUs’ private information. Indeed the EUs might have incentives to report fake readings which might create shifts in the EUs’ strategies. Relying on performative prediction, we define a decision-dependent game to couple the peer-to-peer trading problem with a data market. On the data market, the DA designs a privacy contract incentivizing the EUs to report their private information truthfully. The EUs’ readings, demand and controllable generations are then determined as unique solutions of the decision-dependent game, relying on Nash (NE) and Performatively Stable (PSE) Equilibria. Finally, we propose three gradient method-based algorithms to compute NE and PSE, under different coordination schemes between the DA and the EUs, that we test against the Repeated Stochastic Gradient Method (RSGM) from [4]. Numerical results are presented.

### C. Reviewed Literature

Over the last few years, there has been a growing body of literature on data markets. On the market design side, Agarwal et al. propose an innovative design involving robust

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real-time matching, to efficiently buy and sell data [5], while Shilov et al. quantify the impact of privacy constraints on the market equilibrium [2]. On the pricing side, Cai et al. in [6] study a data market with many strategic data sources and one DA. The DA aims to build an estimator of a function, thanks to the readings reported by the sources. The quality of the readings depends on the level of effort exerted by the sources, which determines the variance of the sample as a decreasing function of the effort exerted, and the aggregator wants to design incentives to obtain the best possible estimator. Cai et al. prove that there exists a unique NE, which is also a dominant strategy equilibrium. Westenbroek et al. extend this result in [7] by taking into account the non-rivalry nature of data and considering multiple DAs among which the same data can be freely reproduced and shared. In the same vein, Gast et al. in [8] propose a non-cooperative game model of strategic data sources, which optimize the variance of the noise level of their readings to minimize their cost, which can be split into a disclosure cost and an estimation cost based on linear regression. The impact of strategic information is quantified through explicit bounds on the price of stability. The above works differ from ours in that we explicitly take into account the dependence of the reading distributions in the decision variables of the EUs. Furthermore, our work differs significantly from the framework of  $\epsilon$ -differential privacy, because in differential privacy, noise is added by a principal to the output of a computation which is subsequently publicly released [9], [10], and the principal performing the computation is trusted (i.e., it does not act strategically). In contrast, we do not assume that the EU is trusted, which motivates input perturbations that can be related to the more recent notion of local differential privacy, though ignoring the strategic aspect of the input perturbation [11]. Finally, some papers study the trade-off between the accuracy of equilibrium computation and privacy or fairness guarantees [12], [13].

This paper is organized as follows. In Sec. II, we formulate the peer-to-peer electricity trading problem as a non-cooperative network game. Relying on performative prediction, we define a decision-dependent game to couple the network game with a data market, in Sec. III. Algorithms for equilibrium computation are tested in Sec. IV. We conclude in Sec. V.

*Notation:*  $\mathbb{E}_X[f]$  specifies that the expectation of the function  $f(\cdot)$  is taken with respect to the random variable  $X$ . Vectors and matrices will be represented by bold letters. The stack of the  $N$  vectors  $\mathbf{x}_1, \dots, \mathbf{x}_N$  is denoted  $\mathbf{x} \triangleq \text{col}(\mathbf{x}_1, \dots, \mathbf{x}_N)$ .  $\mathbf{x}_{-n}$  is the stack of the vectors containing the decision variables of all the agents in  $\mathcal{N}$  except  $n$ . For a closed set  $\mathcal{C} \subseteq \mathbb{R}^m$ , the mapping  $\text{proj}_{\mathcal{C}} : \mathbb{R}^m \rightarrow \mathcal{C}$  denotes the projection onto  $\mathcal{C}$ , i.e.,  $\text{proj}_{\mathcal{C}}(\mathbf{x}) \triangleq \text{argmin}_{\mathbf{y} \in \mathcal{C}} \|\mathbf{y} - \mathbf{x}\|$ .  $\text{SOL}(P)$  will denote the set of solutions of Problem  $P$ .

## II. A NETWORK GAME MODEL

### A. Description of the Agents

1) *The End Users:* We let  $\mathcal{N}$  be a set of  $N$  EUs, forming a directed graph  $\Gamma \triangleq (\mathcal{N}, E)$ , where  $E \subseteq \mathcal{N} \times \mathcal{N}$  is the set

of links between the EUs. Let  $\Gamma_n$  be the set of out-neighbors of EU  $n$ . Furthermore, we make the assumption that at node 0, i.e., the root node,  $\Gamma_0 = \mathcal{N} \setminus \{0\}$ .

Each EU  $n$  updates the vector of decision variables  $\mathbf{x}_n \triangleq (D_n, G_n^c, (q_{mn})_{m \in \Gamma_n}) \in \mathcal{X}_n$ , which contains its demand, controllable generation, and bilateral trades with its neighbors, where  $\mathcal{X}_n \subseteq \mathbb{R}^{m_n}$ ,  $m_n = |\Gamma_n| + 2$ , is the strategy set of EU  $n$  that captures only local constraints. In addition, we impose the following convention on EU  $n$ 's bilateral trades:  $q_{mn} \geq 0$  means that  $n$  buys  $q_{mn}$  from  $m$ ; while  $q_{mn} < 0$  means that  $n$  sells  $-q_{mn}$  to  $m$ . Coupling constraints are added to capture bilateral trading reciprocity between EU  $n$  and its neighbors such that EU  $n$ 's strategy set takes the form:  $\mathcal{C}_n(\mathbf{x}_{-n}) \triangleq \left\{ \mathbf{x}_n = (D_n, G_n^c, (q_{mn})_{m \in \Gamma_n}) \in \mathcal{X}_n \mid q_{mn} = -q_{nm}, \forall m \in \Gamma_n \right\}$ , with  $\mathcal{X}_n \triangleq \left\{ \mathbf{x}_n = (D_n, G_n^c, (q_{mn})_{m \in \Gamma_n}) \in \mathbb{R}^{m_n} \mid \underline{D}_n \leq D_n \leq \overline{D}_n, \underline{G}_n^c \leq G_n^c \leq \overline{G}_n^c, D_n \leq G_n^c + G_n^{nc} + Q_n \right\}$  where  $\underline{D}_n$  and  $\overline{D}_n$  are lower and upper-bounds on  $n$ 's demand,  $G_n^{nc}$  is the uncontrollable generation of EU  $n$  (solar PV panels generation), and  $Q_n \triangleq \sum_{m \in \Gamma_n} q_{mn}$  is the net import at node  $n$ . The constraint  $D_n \leq G_n^c + G_n^{nc} + Q_n$  captures the fact that demand at node  $n$  should not exceed the sum of local generation and net import.

Typically, the EUs' utility functions  $J_n(\cdot)$  depend on some parameters that are sensitive, and that they might not want to disclose [2], [1]. We define  $\theta_n \in \Theta_n \subseteq \mathbb{R}$  as the private information of EU  $n$ . Let  $J_n : \mathbb{R}^{m_n} \times \Theta_n \rightarrow \mathbb{R}$  be the utility function of EU  $n$ . Following [1], we define EU  $n$ 's utility (in the cost minimization sense) as the difference between the sum of the total cost incurred from the bilateral trades with its neighbours and the controllable generation cost, and its usage benefit  $U_n(\cdot)$  ( $-U_n(\cdot)$  is the consumption cost of EU  $n$ ):

$$J_n(\mathbf{x}_n, \theta_n) = \sum_{m \in \Gamma_n} c_{nm} q_{mn} - U_n(D_n, \theta_n) + C_n^G(G_n^c), \quad (1)$$

where  $\mathbf{c}_n > \mathbf{0}$  is a known vector of parameters capturing EU  $n$ 's preferences (willingness-to-pay or taxes), and  $-U_n(\cdot)$  is its consumption cost. The controllable generation cost is assumed to be quadratic in  $G_n^c$ , i.e.,  $C_n^G(G_n^c) = \frac{a_n}{2} G_n^c{}^2 + b_n G_n^c + c_n$  where  $a_n, b_n, c_n$  are positive input coefficients.

We assume  $U_n(\cdot)$  to be a quadratic function in EU  $n$ 's demand, i.e., we can write:

$$U_n(D_n, \theta_n) = -\tilde{a}_n (D_n - \theta_n)^2 + \tilde{b}_n, \quad (2)$$

where  $\tilde{a}_n, \tilde{b}_n$  are positive input coefficients, and  $\theta_n$  is EU  $n$ 's nominal demand. EU  $n$ 's usage benefit is a strictly concave function in its demand  $D_n$ . It leads to the following interpretation: by being flexible around its nominal demand, e.g., agreeing to slightly reduce its demand around its target value, EU  $n$  will decrease its usage benefit but it might also decrease its bilateral trading or controllable generation costs.

2) *The Market Operator:* The economic dispatch is classically obtained through a centralized clearing, i.e., the market operator aims to determine the vector  $\mathbf{x}$  that minimizes the social cost defined as the sum of the EUs' costs, i.e.,  $SC(\mathbf{x}, \boldsymbol{\theta}) \triangleq \sum_n J_n(\mathbf{x}_n, \boldsymbol{\theta}_n)$  under network coupling constraints, i.e., it solves the following optimization problem:

$$\min_{\mathbf{x}} SC(\mathbf{x}, \boldsymbol{\theta}), \quad (3a)$$

$$s.t. \mathbf{x} \in \mathcal{C} \triangleq \prod_n \mathcal{C}_n(\mathbf{x}_{-n}). \quad (3b)$$

We prove in [1] that the solution of (3) can be reached through a distributed clearing, where each EU  $n \in \mathcal{N}$  solves the parametrized optimization problem:

$$\min_{\mathbf{x}_n \in \mathcal{C}_n(\mathbf{x}_{-n})} J_n(\mathbf{x}_n, \theta_n). \quad (4)$$

The latter setting gives rise to a non-cooperative game  $\mathcal{G}$  between the EUs.

3) *The Data Aggregator*: The goal of the DA is to collect readings from the EUs to build an estimate  $\mathbf{x}^a$  of the EUs' decisions  $\mathbf{x}$  at equilibrium and use it to measure the quality of the readings. To this end, it builds contracts with the EUs, and pays  $p_n^a(\cdot)$  to each EU  $n$ , which is an increasing function in the accuracy of  $\mathbf{x}_n^a$  with respect to  $\mathbf{x}_n$ .

### B. Existence, Uniqueness and Stability of the Variational Equilibrium

To generalize the notion of NE in the presence of coupling constraints, we use the Generalized Nash Equilibrium (GNE). It is defined as a strategy profile vector  $\mathbf{x}$  such that  $\mathbf{x}_n$  solves the system  $\text{KKT}_n$  for each  $n$  (see Appendix A). The coupling constraint  $q_{mn} = -q_{nm}$  written in the problem of  $n$  (respectively  $m$ ,  $\forall m \in \Gamma_n$ ) is associated to the duals  $\zeta_{nm}$  (respectively  $\zeta_{mn}$ ). In this work, we focus on the special case of a GNE known as a Variational Equilibrium (VE), i.e., a solution  $\mathbf{x} = (\mathbf{x}_n)_n$  such that  $\mathbf{x}_n$  solves the system  $\text{KKT}_n$  for each  $n$  and, in addition, such that the duals of the coupling constraints are equal, i.e.,  $\zeta_{nm} = \zeta_{mn}, \forall n \in \mathcal{N}, \forall m \in \Gamma_n$ . Set  $\mathbf{F}_x(\mathbf{x}) \triangleq \left( \nabla_{\mathbf{x}_n} J_n(\mathbf{x}_n, \theta_n) \right)_n$ .

*Definition 1 (Variational Equilibrium [14])*: The Variational Inequality problem  $\text{VI}(\mathbf{F}_x, \mathcal{C})$  consists in finding a vector  $\mathbf{x}^* \in \mathcal{C}$  such that  $(\mathbf{y} - \mathbf{x}^*)^T \mathbf{F}_x(\mathbf{x}^*) \geq 0, \forall \mathbf{y} \in \mathcal{C}$ . A solution of  $\text{VI}(\mathbf{F}_x, \mathcal{C})$  is called a VE.

*Proposition 1*: If no energy surplus is available at node  $n$ , then EU  $n$ 's best-response strategy  $\mathbf{x}_n^*$  at equilibrium depends linearly on the nominal demands of *all* the EUs, i.e.,  $\mathbf{x}_n^* = \mathbf{A}_{n,n}\theta_n + \mathbf{A}_{n,-n}\boldsymbol{\theta}_{-n} + \left[ \mathbf{B}_{n,n}G_n^{nc} + \mathbf{B}_{n,-n} \sum_{n' \neq n} G_{n'}^{nc} \right] + \mathbf{C}_n$ , with  $\mathbf{A}_{n,n}$ ,  $\mathbf{B}_{n,n}$ ,  $\mathbf{B}_{n,-n}$ ,  $\mathbf{C}_n$  vectors and  $\mathbf{A}_{n,-n}$  matrix of appropriate dimensions; otherwise  $\mathbf{x}_n^*$  depends only on the EU  $n$ 's own nominal demand,  $\theta_n$ .

*Proof*: See Appendix A. ■

*Remark 2.1*: The coefficients of the best-responses are not known by the EUs because they rely on coefficients  $(\tilde{a}_n, a_n)_n$  that the other EUs might not want to share.

*Assumption 1*: Demand and supply balance each other at each node, i.e.,  $D_n = G_n^c + G_n^{nc} + Q_n, \forall n \in \mathcal{N}$ .

*Assumption 2*: We assume that there exists at least one EU  $n \in \mathcal{N}$  such that at the VE,  $D_n^* \neq 0$  or  $G_n^{c*} \neq 0$ .

*Theorem 1*: Under Asm. 2, the game  $\mathcal{G}$  admits a unique VE, which is in addition efficient.

*Proof*: See Appendix B. ■

Equilibrium stability is a sufficient condition for distributed learning methods to converge in [15].

*Definition 2 (Stability [15])*: An equilibrium point  $\mathbf{x}^* \in \mathcal{C}$  is said to be variationally stable (or simply stable) if there exists a neighborhood  $\mathcal{U}$  of  $\mathbf{x}^*$  such that  $(\mathbf{y} - \mathbf{x}^*)^T \mathbf{F}_x(\mathbf{y}) \geq 0$  for all  $\mathbf{y} \in \mathcal{U}$ . In particular, if this property holds for all  $\mathbf{y} \in \mathcal{C}$ , we say that  $\mathbf{x}^*$  is globally stable.

*Proposition 2*: The VE solution of  $\mathcal{G}$  is globally stable.

*Proof*: In Thm. 1 proof, we check that  $\mathcal{G}$  belongs to the class of generalized potential games (GPG) [16, Sec. 2] for which the EUs' utility functions are aligned along the game's potential whose minimizers are VEs. From [15, Rk. 1], VEs solutions of convex potential games are automatically stable in the sense of Definition 2. ■

The impact on  $(c_{nm})_{n,m}$  on  $\mathcal{G}$  equilibria has been analyzed in [1]. In case  $(c_{nm})_{n,m}$  capture taxes, it seems natural to take them homogeneous among EUs belonging to same community.

*Assumption 3*: We assume that the EUs' preferences for trading are uniform and normalized, i.e.,  $c_{nm} = 1, \forall n, m$ .

We notice that under Asms. 1 and 3, controlling demand and generation allows to control the net import. Therefore  $Q$  will be treated as a dependent decision variable in the rest of the work. Furthermore, replacing the  $(q_{mn})_m$  by  $Q_n$  allows to use the linear dependence of  $\mathbf{x}_n, \forall n$  in  $\boldsymbol{\theta}$  (see Appendix A).

### III. COUPLING NETWORK GAME AND DATA MARKET AS A DECISION-DEPENDENT GAME

Decomposition methods are another way to compute equilibrium with minimum information exchange [17], [18]. Nevertheless, works in this category do not distinguish between local and global objectives. In this paper, we focus on the performative prediction setting with decision-dependent distributions which has been introduced in [4] as a game-theoretic model for decision-dependent learning in the presence of competition. It leads to a weaker solution concept than Nash Equilibrium, but allows to endogenously capture the drift (performative effect) caused by the strategic information contained in the EUs' readings on the market equilibrium, and leads to faster converging algorithms.

#### A. Balancing Cost and Privacy

We introduce randomization in the reading sent by EU  $n$  to the DA, which takes the form  $\tilde{\theta}_n = \hat{\theta}_n + \varepsilon_n$ . The first part of the reading captures the ability of EU  $n$  to add a deterministic bias to its reported information. The second part is a noise following a randomized mechanism  $M(\cdot)$ , added in order to preserve privacy. We assume that the noise in the readings is independent across the EUs [7].

*Assumption 4*: For each EU  $n \in \mathcal{N}$ ,  $\varepsilon_n$  follows a centered Gaussian distribution with variance  $V_n$ . Furthermore, the random variables  $(\varepsilon_n)_{n \in \mathcal{N}}$  are independent.

To incentivize the EUs to report truthful readings that can later be monetized through targeted advertisements, the DA comes into play. The DA defines a payoff contract to each EU; the payment is decreasing in the distance between the EUs' strategy  $\mathbf{x}_n$  and the estimate  $\mathbf{x}_n^a \triangleq (D_n^a, G_n^a)$  built by

the DA, defined as follows:

$$\mathbf{x}_n^a = \mathbf{A}_n^a \tilde{\boldsymbol{\theta}} + \mathbf{B}_n^a \mathbf{G}^{nc} + \mathbf{C}_n^a, \quad (5)$$

where  $\mathbf{A}_n^a, \mathbf{B}_n^a, \mathbf{C}_n^a$  are estimates of  $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$  introduced in Prop. 1. The form of the DA's estimator is known by all the EUs. It comes from Prop. 1, assuming that Asm. 1 holds. The contract between the EU  $n$  and the DA takes the form:

$$p_n^a(\mathbf{x}_n, \mathbf{x}_n^a, \tilde{\boldsymbol{\theta}}) = \gamma_n - \beta_n \left( (D_n - D_n^a)^2 + (G_n^c - G_n^{ca})^2 \right), \quad (6)$$

where  $\gamma_n, \beta_n$  are non-negative parameters which can be optimized by the DA to check specific market properties [7].

Each EU  $n$  chooses the neighboring input  $\hat{\theta}_n$  such that  $d(\hat{\theta}_n, \theta_n) \triangleq (\hat{\theta}_n - \theta_n)^2 \leq \alpha_n$  [19]. Let  $L_{M, \theta_n, \hat{\theta}_n}$  be the privacy loss of  $M(\cdot)$ , which follows the distribution  $\mathcal{N}\left(\frac{(\theta_n - \hat{\theta}_n)^2}{2V_n}, 2\frac{(\theta_n - \hat{\theta}_n)^2}{2V_n}\right)$  [20]. How much information can be extracted from the reading  $\hat{\theta}_n$  is measured by the expected privacy loss of the mechanism  $M(\cdot)$ :  $I_n(\theta_n, \hat{\theta}_n) \triangleq -\mathbb{E}_{\varepsilon_n} \left[ L_{M, \theta_n, \hat{\theta}_n} \right]$ , which coincides with the Kullback-Leibler divergence (relative entropy) between  $M(\cdot)$ 's output distributions on  $\theta_n$  and  $\hat{\theta}_n$ . Under Asm. 4, the Kullback-Leibler divergence takes the closed form:  $I_n(\theta_n, \hat{\theta}_n, V_n) = \frac{(\hat{\theta}_n - \theta_n)^2}{2V_n}$ .

### B. Formulation of the Stochastic Game

In the data market, EU  $n$ 's extended utility is split in three terms. The first term captures the cost it bears in  $\mathcal{G}$ ; the second term measures the cost of information release, while the third term is the contract payment it receives from the DA:

$$\begin{aligned} \tilde{J}_n(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \boldsymbol{\varepsilon}) &= Q_n + \tilde{a}_n (D_n - \theta_n)^2 - \tilde{b}_n + \frac{a_n}{2} G_n^{c2} + \\ &+ b_n G_n^c + c_n + c_n^I \frac{(\hat{\theta}_n - \theta_n)^2}{2V_n} - \gamma_n + \beta_n ((D_n - D_n^a)^2 \\ &+ (G_n^c - G_n^{ca})^2) + \xi \left( \sum_{m=1}^N Q_m \right)^2, \end{aligned} \quad (7)$$

with  $c_n^I, \xi$  non-negative parameters. From (5), since  $D_n^a, G_n^{ca}$  depend on  $\boldsymbol{\theta}, \boldsymbol{\varepsilon}$ , and  $I_n(\cdot)$  on  $\hat{\boldsymbol{\theta}}, \mathbf{V}$ ,  $\tilde{J}_n(\cdot)$  also depends on  $\boldsymbol{\theta}, \mathbf{V}, \boldsymbol{\varepsilon}$ . Under Asms. 1 and 3, the bilateral trading reciprocity constraint at each node can be equivalently reformulated as  $\sum_{n=1}^N Q_n = 0$ . To avoid coupling between the feasibility sets of the EUs, we put the constraint as a penalty in their utility function, with a weight  $\xi \geq 0$  that is the same for all the EUs. The penalty  $\xi$  calibration is discussed in Lemma 1, later.

The first term in  $Q$  in (7) refers to the benefit/cost due to the trades, while the second one encourages the EUs to ensure trading reciprocity among the community. We introduce  $\mathcal{V} \triangleq \text{diag}(\mathbf{V})$ . Each EU  $n$  solves:

$$\min_{\mathbf{x}_n, \hat{\theta}_n, V_n} \mathbb{E}_{\boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \mathcal{V})} \left[ \tilde{J}_n(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \boldsymbol{\varepsilon}) \right], \quad (8a)$$

$$s.t. \quad \underline{D}_n \leq D_n \leq \bar{D}_n, \quad (8b)$$

$$\underline{G}_n^c \leq G_n^c \leq \bar{G}_n^c, \quad (8c)$$

$$D_n = G_n^c + G_n^{mc} + Q_n, \quad (8d)$$

$$d(\hat{\theta}_n, \theta_n) \leq \alpha_n, \quad (8e)$$

$$\underline{V}_n \leq V_n \leq \bar{V}_n. \quad (8f)$$

This gives rise to the stochastic game  $\mathcal{G}^{\text{stoch}}$ . Each EU wants to ensure a minimum privacy level, so it imposes a lower bound  $\underline{V}_n$  on  $V_n$ . To limit the instability of the market,

the DA imposes an upper bound  $\bar{V}_n$  on  $V_n$ . This upper bound does not need to be shared by the EUs, instead it might be encoded by the DA in the communication interface deployed in the EUs' nodes when communicating with the DA.

The game between the EUs, which involves the DA as a coordinator of the data market, takes place as follows:

- Each EU  $n$  reports a reading  $\tilde{\theta}_n = \hat{\theta}_n + \varepsilon_n$ , with  $\varepsilon_n \sim \mathcal{N}(0, V_n)$ , to the DA by solving (8) with respect to  $\hat{\theta}_n, V_n$ . The readings  $\tilde{\boldsymbol{\theta}}$  are then broadcasted to all the EUs by the DA.
- Each EU  $n$  computes its  $\mathbf{x}_n$  by solving (8).

Note that depending on the coordination level between the DA and EUs, steps a) and b) can take place *simultaneously* or *sequentially*.

We define  $\mathcal{Y}_n$  as the feasible set of Problem (8). We denote  $\mathbf{y}_n \triangleq (\mathbf{x}_n, \hat{\theta}_n, V_n) \in \mathcal{Y}_n$  the decision variable of EU  $n$  in  $\mathcal{G}^{\text{stoch}}$ .

### C. Main Results

**Definition 3:** A Nash Equilibrium (NE) is a strategy profile  $\mathbf{y}^* = (\mathbf{y}_n^*)_n \in \mathcal{Y}$  such that for all  $n \in \mathcal{N}$ :

$$\mathbf{y}_n^* \in \underset{\mathbf{y}_n \in \mathcal{Y}_n}{\text{argmin}} \mathbb{E}_{\boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \text{diag}(\mathbf{V}_n, \mathbf{V}_{-n}^*))} \left[ \tilde{J}_n(\mathbf{y}_n, \mathbf{y}_{-n}^*, \boldsymbol{\varepsilon}) \right].$$

There is another solution concept which is more amenable to numerical methods [4], and reduces to Performatively Stable points (PSEs). The idea of the PSE is to decouple the effect of a decision  $\mathbf{y}$  on the integrand  $\tilde{J}_n(\mathbf{y}_n, \mathbf{y}_{-n}, \boldsymbol{\varepsilon})$  and on the distribution  $\mathcal{N}(\mathbf{0}, \mathcal{V})$ . Any vector  $\mathbf{V}^s$  induces a static game wherein the distribution for EU  $n$  is fixed at  $\mathcal{N}(\mathbf{0}, \mathcal{V}^s)$ , that is:

**Definition 4:** [4] A Performatively Stable Equilibrium (PSE) is a strategy profile  $\mathbf{y}^s = (\mathbf{y}_n^s)_n \in \mathcal{Y}$  such that for all  $n \in \mathcal{N}$ :

$$\mathbf{y}_n^s \in \underset{\mathbf{y}_n \in \mathcal{Y}_n}{\text{argmin}} \mathbb{E}_{\boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \mathcal{V}^s)} \left[ \tilde{J}_n(\mathbf{y}_n, \mathbf{y}_{-n}^s, \boldsymbol{\varepsilon}) \right]. \quad (9)$$

**Proposition 3:** The utility function  $\mathbb{E}_{\boldsymbol{\varepsilon}} \left[ \tilde{J}_n(\mathbf{y}_n, \mathbf{y}_{-n}, \boldsymbol{\varepsilon}) \right]$  is strongly convex in  $\mathbf{y}_n, \forall n$ .

**Proof:** We explicit the expectation of EU  $n$ 's utility function:  $\mathbb{E}_{\boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \mathcal{V})} \left[ \tilde{J}_n(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \boldsymbol{\varepsilon}) \right] = (D_n - G_n^c - G_n^{nc}) + \tilde{a}_n (D_n - \theta_n)^2 - \tilde{b}_n + \frac{a_n}{2} G_n^{c2} + b_n G_n^c + c_n + c_n^I \frac{(\hat{\theta}_n - \theta_n)^2}{2V_n} - \gamma_n + \beta \left( (D_n - D_n^a(\hat{\boldsymbol{\theta}}))^2 + (G_n^c - G_n^a(\hat{\boldsymbol{\theta}}))^2 + \sum_{m=1}^N (A_n^a)_{(1,m)}^2 V_m + \sum_{m=1}^N (A_n^a)_{(2,m)}^2 V_m \right) + \xi \left( \sum_{m=1}^N D_m - G_m^c - G_m^{mc} \right)^2$ . Using the former expression, we compute the partial derivative  $\mathbf{F}_n \triangleq \mathbf{F}_{\mathbf{y}_n}$  and the partial Hessian  $\mathbf{H}_n \triangleq \mathbf{H}_{\mathbf{y}_n}$  of  $\mathbb{E}_{\boldsymbol{\varepsilon}} \left[ \tilde{J}_n(\cdot) \right]$  with respect to  $\mathbf{y}_n$ . We get:

$$\mathbf{H}_n = \begin{bmatrix} \frac{c_n^I}{V_n} + 2\beta_n \left( (A_n^a)_{1,n}^2 + (A_n^a)_{1,n}^2 \right) \frac{-c_n^I (\hat{\theta}_n - \theta)}{V_n^2} & -2\beta_n (A_n^a)_{1,n} & -2\beta_n (A_n^a)_{2,n} & 0 \\ \frac{-c_n^I (\hat{\theta}_n - \theta)^2}{V_n^2} & \frac{c_n^I (\hat{\theta}_n - \theta)}{V_n^3} & 0 & 0 \\ -2\beta_n (A_n^a)_{1,n} & 0 & 2\tilde{a}_n + 2\beta_n + 2\xi & -2\xi \\ -2\beta_n (A_n^a)_{2,n} & 0 & -2\xi & a_n + 2\beta_n + 2\xi \end{bmatrix}.$$

We note that  $\mathbf{z}^T \mathbf{H}_n \mathbf{z} > \mathbf{0}, \forall \mathbf{z} \in \mathcal{Y}_n$ , i.e.,  $\mathbb{E}_{\boldsymbol{\varepsilon}} \left[ \tilde{J}_n(\cdot) \right]$  is strictly convex in  $\mathbf{y}_n$ , which will be used later in the proof. In a second time, we prove the strong monotonicity of  $\mathbf{F}_n$ , i.e., that  $\langle \mathbf{F}_n(\mathbf{y}_n) - \mathbf{F}_n(\mathbf{y}'_n), \mathbf{y}_n - \mathbf{y}'_n \rangle \geq \alpha \|\mathbf{y}_n - \mathbf{y}'_n\|^2, \forall \mathbf{y}_n, \mathbf{y}'_n \in \mathcal{Y}_n$ . Using the multidimensional Mean Value Theorem on the convex set  $\mathcal{Y}_n$ , we obtain the existence of a  $\mathbf{z} \in [\mathbf{y}_n, \mathbf{y}'_n]$ , i.e., component-wise  $z_i \in [(\mathbf{y}_n)_i, (\mathbf{y}'_n)_i] \forall i$ , such that

$\langle \mathbf{F}_n(\mathbf{y}_n) - \mathbf{F}_n(\mathbf{y}'_n), \mathbf{y}_n - \mathbf{y}'_n \rangle = (\mathbf{y}_n - \mathbf{y}'_n)^T \cdot \nabla_n \mathbf{F}_n(\mathbf{z}) \cdot (\mathbf{y}_n - \mathbf{y}'_n)$ . We then use the Spectral Theorem on  $\nabla_n \mathbf{F}_n = \mathbf{H}_n$  to obtain the existence of  $\mathbf{D}$ , a diagonal matrix such that  $\mathbf{H}_n(\mathbf{z}) = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ , and  $\mathbf{P}$  the orthogonal transition matrix. By replacing  $\mathbf{H}_n$  by its expression, identifying the variable in the new base, and using the property that the product of a vector by an orthogonal matrix preserves the Euclidean norm of the vector, we obtain:  $(\mathbf{y}_n - \mathbf{y}'_n)^T \cdot \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \cdot (\mathbf{y}_n - \mathbf{y}'_n) \geq \min_i \{\lambda_i(\mathbf{H}_n(\mathbf{z}))\} \|\mathbf{y}_n - \mathbf{y}'_n\|^2$  where the  $(\lambda_i(\mathbf{H}_n(\mathbf{z})))_{i \in [1,4]}$  are the eigenvalues of  $\mathbf{H}_n(\mathbf{z})$ . We obtain the  $\alpha$ -strong monotonicity by taking the minimum over all the feasible set of the eigenvalues of  $\mathbf{H}_n$  (which are positive due to the strict convexity), with  $\alpha = \min_{\mathbf{z} \in \mathcal{Y}_n} \min_{i \in [1,4]} \lambda_i(\mathbf{H}_n(\mathbf{z}))$ . Since all the conditions listed in [14] are met, we conclude that  $\mathbb{E}_\varepsilon [\tilde{J}_n(\cdot)]$  is strongly convex. ■

Let  $\nabla_n h(\mathbf{y})$  be the partial derivative of  $h(\cdot)$  with respect to  $\mathbf{y}_n$ .

*Lemma 1:* If the penalty coefficient  $\xi$  in  $\tilde{J}_n(\cdot)$  is bigger than the Lagrangian multiplier of the constraint  $(\sum_{m=1}^N Q_m)^2 = 0$ , then the penalized game has the same solutions as  $\mathcal{G}^{\text{stoch}}$ .

*Proof:* We adapt the proof of [21, Thm. 2.2]. We call  $J_n^\#(\cdot)$  the function  $\tilde{J}_n(\cdot)$  without the penalty:  $J_n^\#(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon) = J_n(\mathbf{x}_n, \theta_n) + c_n^I I_n(\theta_n, \hat{\theta}_n, V_n) - p_n^a(\mathbf{x}_n, \mathbf{x}_n^a, \hat{\boldsymbol{\theta}})$ . We set  $h(\mathbf{y}) \triangleq (\sum_{m=1}^N Q_m)^2$  and  $\phi$  is the Lagrangian multiplier related to the constraint  $h(\mathbf{y}) = 0$ . We denote  $\hat{\mathbf{y}}$  the point that minimizes  $J_n^\#(\cdot)$  over  $\mathcal{Y}$  and satisfies the constraint  $h(\hat{\mathbf{y}}) = 0$ . For any  $\mathbf{y} \in \mathcal{Y}$ , we have:

$$J_n^\#(\mathbf{y}) + \xi h(\mathbf{y}) \geq J_n^\#(\mathbf{y}) + \phi h(\mathbf{y}) \quad (10a)$$

$$\geq J_n^\#(\mathbf{y}) + \phi (h(\hat{\mathbf{y}}) + \nabla_n h(\hat{\mathbf{y}})^T (\mathbf{y} - \hat{\mathbf{y}})), \quad (10b)$$

$$= J_n^\#(\mathbf{y}) + \phi \nabla_n h(\hat{\mathbf{y}})^T (\mathbf{y} - \hat{\mathbf{y}}), \quad (10c)$$

$$= J_n^\#(\mathbf{y}) - \nabla_n J_n^\#(\hat{\mathbf{y}})^T (\mathbf{y} - \hat{\mathbf{y}}), \quad (10d)$$

$$\geq J_n^\#(\hat{\mathbf{y}}) = J_n^\#(\hat{\mathbf{y}}) + \xi h(\hat{\mathbf{y}}). \quad (10e)$$

To go from (10a) to (10b), we use the convexity of  $h(\cdot)$ . To go from (10c) to (10d), we use the stationarity condition. To go from (10d) to (10e), we use the convexity of  $J_n^\#(\cdot)$ , which can be proved by invoking similar arguments as in the proof of the strict convexity of  $\mathbb{E}_\varepsilon [\tilde{J}_n(\cdot)]$  in Proposition 3. Hence,  $\hat{\mathbf{y}}$  also minimizes the penalized problem. ■

Since for all  $n$ ,  $\mathbb{E}_\varepsilon [\tilde{J}_n(\cdot)]$  is strongly convex and the feasibility set  $\mathcal{Y}_n$  is closed and convex, we derive the following result.

*Corollary 1:* There exists a unique NE solution of  $\mathcal{G}^{\text{stoch}}$ .

*Theorem 2:* If the DA's estimate  $\mathbf{x}^a$  is perfect for the game  $\mathcal{G}$ , i.e.,  $\mathbf{x}^a(\boldsymbol{\theta}, \mathbf{G}^{nc}) = \mathbf{x}^G$  with  $\mathbf{x}^G \in \text{SOL}(\mathcal{G})$ , then the NE solution of  $\mathcal{G}^{\text{stoch}}$  is efficient.

*Proof:* We denote  $\mathbf{x}^G$  the VE solution of the game  $\mathcal{G}$ . We have,  $\forall \hat{\boldsymbol{\theta}}, \mathbf{V}, \mathbf{x}_n$ :  $J_n(\mathbf{x}^G) \leq J_n(\mathbf{x}_n) + c_n^I \frac{(\hat{\theta}_n - \theta_n)^2}{2V_n} + \beta_n ((D_n - D_n^a(\hat{\boldsymbol{\theta}}))^2 + (G_n - G_n^a(\hat{\boldsymbol{\theta}}))^2) + \xi (\sum_{m=1}^N Q_m)^2 \implies \Pi_n(\mathbf{x}^G) + \sum_{m=1}^N K_m V_m \leq J_n(\mathbf{x}_n) + c_n^I \frac{(\hat{\theta}_n - \theta_n)^2}{2V_n} + \beta_n ((D_n - D_n^a(\hat{\boldsymbol{\theta}}))^2 + (G_n - G_n^a(\hat{\boldsymbol{\theta}}))^2) + \xi (\sum_{m=1}^N Q_m)^2 + \sum_{m=1}^N K_m V_m$  where we set  $K_m \triangleq ((\mathbf{A}_n^a)_{(1,m)}^2 + (\mathbf{A}_n^a)_{(2,m)}^2)$ . Note that the term  $-\gamma_n$  is constant and does not change the equilibrium, thus we omit it. This lower bound is reached in  $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$ ,

$V_n = \underline{V}_n \forall n$  and  $\mathbf{x}_n = \mathbf{x}_n^G$ . This means that we have proved that assuming  $\mathbf{x}^a$  is perfect for  $\mathcal{G}$ ,  $\mathbf{x}^G, \boldsymbol{\theta}, \underline{V}$  is the NE solution of  $\mathcal{G}^{\text{stoch}}$ . If we sum the former inequalities over  $n$ , we observe that  $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}, V_n = \underline{V}_n \forall n$  and  $\mathbf{x} = \mathbf{x}^G$  also minimizes the EUs' expected social cost. Thus, assuming  $\mathbf{x}^a$  is perfect for  $\mathcal{G}$ , the NE solution of  $\mathcal{G}^{\text{stoch}}$  is efficient. ■

Next, we look at the properties of the PSE.

*Proposition 4:* There exists a unique PSE solution of  $\mathcal{G}^{\text{stoch}}$ .

*Proof:* To prove this result, we rewrite  $\mathbb{E}_\varepsilon [\tilde{J}_n(\cdot)]$  as follows, to use [4, Thm. 1]:

$$\mathbb{E}_{\varepsilon \sim \mathcal{N}(0, \mathbf{V})} [\tilde{J}_n(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon)] = \mathbb{E}_{\varepsilon_n \sim \mathcal{N}(0, V_n)} [\bar{J}_n(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon_n)], \quad (11)$$

where  $\bar{J}_n(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon_n) = (D_n - G_n^c - G_n^{nc}) + \tilde{a}_n (D_n - \theta_n)^2 - \tilde{b}_n + \frac{\alpha_n}{2} G_n^c + b_n G_n^c + c_n + c_n^I \frac{(\hat{\theta}_n - \theta_n)^2}{2V_n} - \gamma_n + \beta ((D_n - D_n^a(\hat{\boldsymbol{\theta}}))^2 + (G_n^c - G_n^a(\hat{\boldsymbol{\theta}}))^2 + \sum_{m=1, m \neq n}^N (\mathbf{A}_n^a)_{(1,m)}^2 V_m + \sum_{m=1, m \neq n}^N (\mathbf{A}_n^a)_{(2,m)}^2 V_m + (\mathbf{A}_n^a)_{(1,n)}^2 \varepsilon_n^2 + (\mathbf{A}_n^a)_{(2,n)}^2 \varepsilon_n^2) + \xi (\sum_{m=1}^N D_m - G_m^c - G_m^{nc})^2$ . To prove the strong convexity of  $\bar{J}_n(\cdot)$ , we proceed the same way as to prove the strong convexity of  $\tilde{J}_n(\cdot)$ , because the partial Hessians of both objective functions are the same. We also check the  $\mathcal{C}^1$  continuity of  $\bar{J}_n(\cdot)$ , the Lipschitz-continuity of the application  $\varepsilon \mapsto \bar{J}_n(\mathbf{y}, \varepsilon)$ , and that the distribution of the  $\varepsilon$  varies in a Lipschitz way according to [4, Asm. 2]. From [4, Thm. 1], we obtain the existence and uniqueness of the PSE. ■

Assuming  $\mathcal{G}^{\text{stoch}}$  takes place sequentially,  $\hat{\theta}_n, V_n$  and  $\mathbf{x}_n$  are computed by solving two distinct parametrized optimization problems, by each EU  $n \in \mathcal{N}$ . This leads us to introduce the game  $\mathcal{G}^\#(\hat{\boldsymbol{\theta}}, \mathbf{V})$ , where each EU  $n \in \mathcal{N}$  solves:

$$\min_{\mathbf{x}_n} \mathbb{E}_{\varepsilon \sim \mathcal{N}(0, \mathbf{V})} [J_n^\#(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon)], \quad (12a)$$

$$\text{s.t. } \underline{D}_n \leq D_n \leq \bar{D}_n, \quad (12b)$$

$$\underline{G}_n^c \leq G_n^c \leq \bar{G}_n^c, \quad (12c)$$

$$D_n = G_n^c + G_n^{nc} + Q_n, \quad (12d)$$

$$\sum Q_m = 0. \quad (12e)$$

*Proposition 5:*  $\mathcal{G}^\#(\hat{\boldsymbol{\theta}}, \mathbf{V})$  is a generalized potential game (GPG),  $\forall \hat{\boldsymbol{\theta}}, \mathbf{V}$ .

*Proof:* Relying on the GPG definition introduced in [16, Sec. 2], this results follows from the fact that the KKTs of  $\mathcal{G}^\#(\hat{\boldsymbol{\theta}}, \mathbf{V})$  coincide with the KKTs of the minimization problem with  $\sum_n \mathbb{E}_\varepsilon [J_n^\#(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon)]$  in the objective, under the concatenation of the constraints (12b)-(12d),  $\forall n$ , and (12e). ■

## IV. SIMULATIONS

Numerical results are presented on a three-node network, with  $\boldsymbol{\theta} = [0.23, 0.32, 0.55]$  and  $\mathbf{G}^{nc} = [0.1, 0.2, 0.4]$ .  $\mathcal{G}^{\text{stoch}}$  parameters are available on GitHub [22].

To compute the NE, we formulate (8) as a deterministic optimization problem. This is possible because the  $(\varepsilon_n)_n$  are independent and  $\varepsilon_n \sim \mathcal{N}(0, V_n), \forall n$ . To compute the NE and the social optimum (SO), we apply a classic Gradient Descent Method (GDM). To compute the PSE, we use the Repeated Stochastic Gradient Method (RSGM) [4]. In RSGM, having initialized  $\mathbf{y}^0$ , at each step  $t$ , each EU

samples its  $\varepsilon_n$ , and updates its variable  $\mathbf{y}_n^t$  using the formula:

$$\mathbf{y}_n^{t+1} = \text{proj}_{\mathcal{Y}_n}(\mathbf{y}_n^t - \eta \nabla_n \bar{J}_n(\mathbf{y}^t, \varepsilon_n^t)), \quad (13)$$

where  $\bar{J}_n(\cdot)$  is defined in Prop. 4 proof. We used a backtracking line search for the choice of the step size  $\eta$ .

However, RSGM requires that the EUs share their decision variables with the other EUs at each iteration. Also, depending on the structure of the game, it might not be possible to reformulate the objective function  $\mathbb{E}_{\varepsilon \sim \mathcal{D}(\mathbf{y})}[\tilde{J}_n(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon)]$ , in the form  $\mathbb{E}_{\varepsilon_n \sim \mathcal{D}_n(\mathbf{y})}[\bar{J}_n(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon_n)]$ ,  $\forall n$  – for example, in case the  $(\varepsilon_n)_n$  are correlated. For this reason, we propose an alternative algorithm, Coupled Privacy Repeated Stochastic Gradient Method (CP-RSGM), in which at each time step  $t$ , each EU  $n$  samples its  $\varepsilon_n^t$  and shares its  $\hat{\boldsymbol{\theta}}_n^t$  with the DA, which then broadcasts it to the other EUs. In fact, in our game  $\tilde{J}_n(\mathbf{y}^t, \varepsilon^t) \equiv \tilde{J}_n(\mathbf{x}_n^t, \hat{\boldsymbol{\theta}}_n^t, \mathbf{V}_n^t, \boldsymbol{\theta}^t)$ . Each EU updates its decision variables following the rule:

$$\mathbf{y}_n^{t+1} = \text{proj}_{\mathcal{Y}_n}(\mathbf{y}_n^t - \eta \nabla_n \tilde{J}_n(\mathbf{y}^t, \varepsilon^t)). \quad (14)$$

In CP-RSGM, the update of readings parameters, demand and generation variables is done simultaneously. In case where the readings parameters on the one side, demand and generation variables on the other side need to be updated sequentially, we introduce another algorithm, Privacy Split Repeated Stochastic Gradient Method (PS-RSGM).

**Algorithm** Privacy Split Repeated Stochastic Gradient Method (PS-RSGM)

- 1: Find  $\mathbf{x}^{\mathcal{G}} \in \text{SOL}(\mathcal{G})$ .
- 2: Find  $\hat{\boldsymbol{\theta}}_n^{\#}, \mathbf{V}_n^{\#} = \text{argmin}_{\hat{\boldsymbol{\theta}}_n, \mathbf{V}_n} \mathbb{E}_{\varepsilon}[\tilde{J}_n(\mathbf{x}_n^{\mathcal{G}}, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon)]$  under constraints (8e), (8f)  $\forall n$ . Set  $\mathbf{x} = \mathbf{x}^{\mathcal{G}}$  and compute a solution with CP-RSGM.
- 3: Find  $\mathbf{x}_n^{\#} = \text{argmin}_{\mathbf{x}_n} \mathbb{E}_{\varepsilon \sim \mathcal{N}(\mathbf{0}, \mathbf{V}^{\#})}[J_n^{\#}(\mathbf{x}_n, \hat{\boldsymbol{\theta}}^{\#}, \mathbf{V}^{\#}, \varepsilon)] \forall n$  under constraints (8b), (8c) and  $\sum_{n=1}^N (D_n - G_n^c - G_n^{nc}) = 0$ .  $J_n^{\#}(\cdot)$  is define above (10).

In Fig. 1, we plot the final value of  $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$  reached by GDM, RSGM, CP and PS-RSGM, considering 50 different initializations of  $\mathbf{y}^0$ . We proved that the NE and the PSE solutions of  $\mathcal{G}^{\text{stoch}}$  are unique. However, CP-RSGM and PS-RSGM only enable to approximate the PSE. This is due to the sampling on  $\varepsilon$  and to the fact that we impose an upper bound on the number of iterations. We depict the resulting neighborhoods as the convex hulls of the set of approximate PSEs that can be reached by CP and PS-RSGM.

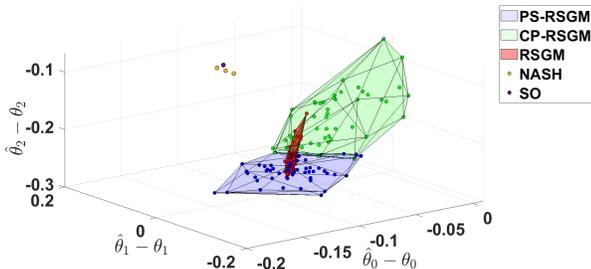


Fig. 1. Final value  $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$  reached by GDM, RSGM, CP, PS-RSGM for 50 different starting points.

We observe that the NE is very close to the SO. This gap can be explained from Thm. 2 as resulting from the estimation bias of the DA estimator. We compare the speed of convergence of RSGM, CP and PS-RSGM in  $\hat{\boldsymbol{\theta}}$ , in Fig. 2.

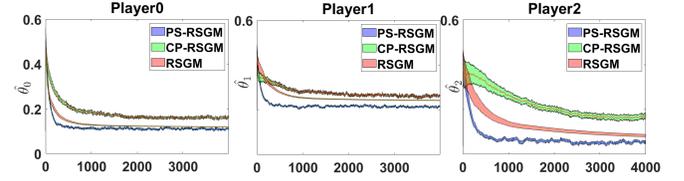


Fig. 2. 95% confidence interval for  $\hat{\boldsymbol{\theta}}$  while learning the PSE.

To build this plot, we ran the algorithm for 50 random starting points, and plot a 95% confidence interval for each step of the algorithm. Due to the sampling on  $\varepsilon$  at each step, the algorithms only converge in expectation. We observe that PS-RSGM converges faster than the two other algorithms, and that CP-RSGM and PS-RSGM might provide upper and lower bounds on the approximate PSE computed by RSGM.

Let  $\mathbf{y}^{\text{so}}$  be the SO solution of the optimization problem with objective the sum over  $n$  of the expected utilities (12a), denoted  $SC(\cdot)$ , under the concatenation of the constraints (8b)-(8f) for all the EUs. In an equilibrium  $\mathbf{y}$ , we evaluate the efficiency gap as the difference between the social cost in the equilibrium and the optimal social cost:  $SC(\mathbf{y}) - SC(\mathbf{y}^{\text{so}})$ . We plot the distribution of the efficiency gap for the GDM, RSGM, CP and PS-RSGM in Fig. 3. The efficiency gap of the NE is close to 0, i.e., the NE is a desirable equilibrium to reach. The PS-RSGM and the CP-RSGM have a 5% lower efficiency gap than the RSGM. The total information released by the EUs on the market is defined as  $\sum_n I_n(\hat{\theta}_n, \mathbf{V}_n)$ . The smaller it is, the less information is released. The total information released per algorithm is illustrated in Fig. 4. We summarize the performance of each algorithm in Tab. I.

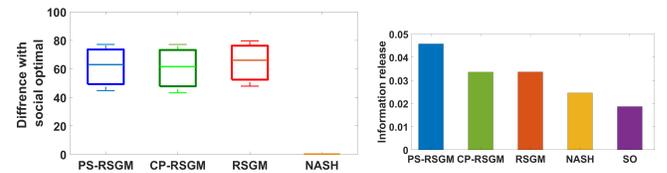


Fig. 3. Box plots of the efficiency gaps for each algorithm.

Fig. 4. Total information released by the EUs for each algorithm.

TABLE I

COMPARISON OF THE ALGORITHMS FOR EQUILIBRIUM LEARNING.

Algo.	Coordination	Eff. Gap	Info. Release	$\sum_n d(\hat{\theta}_n, \theta_n)$	Eq.
GDM	simultaneous	0.26	0.025	0.077	NE
RSGM	simultaneous	66.06	0.034	0.452	PSE
CP-RSGM	simultaneous	61.61	0.034	0.378	PSE
PS-RSGM	sequential	62.99	0.046	0.490	PSE

GDM leads to the smallest efficiency gap, information release and bias. However, it requires full coordination between the DA and EUs, which is very hard to impose in practice. CP-RSGM generates a smaller efficiency gap than RSGM, that we interpret as benchmark, for almost the same information release but with smaller bias, i.e.,

higher indistinguishability for the EUs, and requires less coordination among the EUs. Allowing sequentiality in the DA and EUs' decisions through PS-RSGM leads to a smaller variance (more information release) and larger bias (less indistinguishability) than RSGM and CP-RSGM.

## V. CONCLUSION

Relying on performative prediction, the coupling between network game and data market is formulated as a decision-dependent game. Nash and Performatively stable equilibria are characterized, and two variants (CP, PS-RSGM) of the Repeated Stochastic Gradient Method (RSGM) are proposed to compute equilibria. In our numerical experiments, CP-RSGM outperforms RSGM, leading to smaller efficiency gap and better privacy preservation for the end users. PS-RSGM achieves faster convergence rates than CP-RSGM, but its sequential implementation requires more information release and less indistinguishability for the end users.

In an extension of this paper, we will propose another variant of RSGM, relying on two-time-scale stochastic approximation [23]. The goal will be at each iteration  $t$  to apply a gradient-play update to compute  $\hat{\theta}^t, \mathbf{V}^t$ , and then, a second time gradient-play to compute  $\mathbf{x}^t$  taking as input  $\hat{\theta}^t, \mathbf{V}^t$ . Convergence proofs and convergence in terms of time of all the algorithms will be detailed. In addition, the algorithms will be tested on larger networks. Finally, we will provide a method for specifying a Pareto dominating solution that depends on the variance or hiding ability of the EUs.

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## APPENDIX

### A. Proof of Proposition 1

For any EU  $n \in \mathcal{N}$ , Problem (4) can be formulated as follows:

$$\min_{D_n, G_n^c, (q_{mn})_m} J_n(\mathbf{x}_n, \theta_n), \quad (15a)$$

$$s.t. \quad \underline{D}_n \leq D_n \leq \bar{D}_n, \quad (\underline{\mu}_n, \bar{\mu}_n) \quad (15b)$$

$$\underline{G}_n^c \leq G_n^c \leq \bar{G}_n^c, \quad (\underline{\nu}_n, \bar{\nu}_n) \quad (15c)$$

$$q_{mn} = -q_{nm}, \quad (\zeta_{nm}) \quad (15d)$$

$$D_n \leq G_n^c + G_n^{mc} + Q_n. \quad (\lambda_n) \quad (15e)$$

To compute an equilibrium solution of  $\mathcal{G}$ , we derive the KKT conditions of the Lagrangian function  $\mathcal{L}_n(\cdot)$  with respect to  $D_n, G_n^c, q_{mn}, \forall m \in \Gamma_n, m \neq n$ , the first order stationarity conditions write down as follows:

$$\frac{\partial \mathcal{L}_n}{\partial D_n} = 2\bar{a}_n(D_n - \theta_n) + \lambda_n - \underline{\mu}_n + \bar{\mu}_n = 0, \quad (16a)$$

$$\frac{\partial \mathcal{L}_n}{\partial G_n^c} = a_n G_n^c + b_n - \underline{\nu}_n + \bar{\nu}_n - \lambda_n = 0, \quad (16b)$$

$$\frac{\partial \mathcal{L}_n}{\partial q_{mn}} = c_{nm} + \zeta_{nm} - \lambda_n = 0, \forall m \in \Gamma_n, m \neq n. \quad (16c)$$

From (16c), we obtain:

$$\lambda_n = c_{nm} + \zeta_{nm}, \forall m \in \Gamma_n. \quad (17)$$

Then, from (17), we infer that at a VE:

$$\zeta_{nm} = \lambda_n - c_{nm} = \lambda_m - c_{mn}, \forall m \in \Gamma_n. \quad (18)$$

From (16a), we infer that at equilibrium:

$$D_n^* = \theta_n - \frac{1}{2\bar{a}_n} \left( \lambda_n + (\bar{\mu}_n - \underline{\mu}_n) \right). \quad (19)$$

From (16b), we infer that at equilibrium:

$$G_n^{c*} = -\frac{b_n}{a_n} + \frac{1}{a_n} \left( \lambda_n - (\bar{\nu}_n - \underline{\nu}_n) + (\bar{\mu}_n - \underline{\mu}_n) \right). \quad (20)$$

Complementarity constraints are:

$$\underline{\mu}_n \geq 0 \perp D_n \leq D_n, \quad (21a)$$

$$\bar{\mu}_n \geq 0 \perp D_n \leq \bar{D}_n, \quad (21b)$$

$$\underline{\nu}_n \geq 0 \perp G_n^c \leq G_n^c, \quad (21c)$$

$$\bar{\nu}_n \geq 0 \perp G_n^c \leq \bar{G}_n^c, \quad (21d)$$

$$\lambda_n \geq 0 \perp D_n \leq G_n^c + G_n^{nc} + Q_n. \quad (21e)$$

Complementarity constraint (21e) implies that  $\lambda_n(D_n - G_n^c - G_n^{nc} - Q_n) = 0$ , which leads us to distinguish between two cases depending on whether constraint (15e) is binding.

**Case 1** [No surplus at node  $n$ ] In case  $\lambda > 0$  then

$$Q_n^* = D_n^* - G_n^{c*} - G_n^{nc}, \quad (22)$$

i.e., there is no surplus at node  $n$ . By substitution of (19), (20) in the net import at node  $n$ ,  $Q_n$ , and using the fact that  $\sum_n Q_n = 0$ , we obtain:

$$\begin{aligned} & \sum_n \left[ \theta_n - \frac{1}{2\bar{a}_n} (\bar{\mu}_n - \underline{\mu}_n) + \frac{b_n}{a_n} + \frac{1}{a_n} (\bar{\nu}_n - \underline{\nu}_n) - G_n^{nc} \right] \\ &= \sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right) \lambda_n. \end{aligned} \quad (23)$$

But, from (18), we have the following relationship:

$$\lambda_n = c_{n0} - c_{0n} + \lambda_0, \forall n \in \Gamma_0. \quad (24)$$

By substitution in (23), we obtain a closed form expression for  $\lambda_0$ :

$$\begin{aligned} \lambda_0 &= \frac{\sum_n \left[ \theta_n - \frac{1}{2\bar{a}_n} (\bar{\mu}_n - \underline{\mu}_n) + \frac{b_n}{a_n} + \frac{1}{a_n} (\bar{\nu}_n - \underline{\nu}_n) - G_n^{nc} \right]}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)} \\ &+ \frac{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right) (c_{0n} - c_{n0})}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)}. \end{aligned} \quad (25)$$

To simplify the notations, let

$$K \triangleq \frac{\sum_n \left[ -\frac{1}{2\bar{a}_n} (\bar{\mu}_n - \underline{\mu}_n) + \frac{b_n}{a_n} + \frac{1}{a_n} (\bar{\nu}_n - \underline{\nu}_n) \right]}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)} + \frac{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right) (c_{0n} - c_{n0})}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)}.$$

Substituting  $\lambda_0$  closed form (25) in (24), and then later in (19), (20), (22), we get:

$$\begin{aligned} \lambda_n &= \frac{\sum_n \theta_n}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)} - \frac{\sum_n G_n^{nc}}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)} + K + (c_{n0} - c_{0n}), \\ D_n^* &= \left( 1 - \frac{1}{2\bar{a}_n} \frac{\sum_{m \neq n} \theta_m}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)} \right) \theta_n - \frac{1}{2\bar{a}_n} \left( \frac{\sum_{m \neq n} \theta_m}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)} \right) \\ &- \frac{\sum_n G_n^{nc}}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)} + K + (c_{n0} - c_{0n}) + (\bar{\mu}_n - \underline{\mu}_n), \\ G_n^{c*} &= \frac{1}{a_n} \left( \frac{\sum_n \theta_n}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)} - \frac{\sum_n G_n^{nc}}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)} \right) + K \\ &+ (c_{n0} - c_{0n}) - (\bar{\nu}_n - \underline{\nu}_n) + (\bar{\mu}_n - \underline{\mu}_n) - \frac{b_n}{a_n}, \text{ and} \\ Q_n^* &= \left( 1 - \frac{1}{2\bar{a}_n} \frac{\sum_{m \neq n} \theta_m}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)} - \frac{1}{a_n} \frac{\sum_{m \neq n} \theta_m}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)} \right) \theta_n \\ &- \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right) \frac{\sum_{m \neq n} \theta_m}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)} - \left( 1 - \frac{1}{2\bar{a}_n} \frac{\sum_{m \neq n} \theta_m}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)} \right) \\ &- \frac{1}{a_n} \frac{\sum_{m \neq n} \theta_m}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)} \right) G_n^{nc} - \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right) \left( -\frac{\sum_{m \neq n} G_m^{nc}}{\sum_n \left( \frac{1}{2\bar{a}_n} + \frac{1}{a_n} \right)} \right) \end{aligned}$$

$+K + (c_{n0} - c_{0n}) + (\bar{\mu}_n - \underline{\mu}_n) + \frac{(\bar{\nu}_n - \underline{\nu}_n)}{a_n} + \frac{b_n}{a_n}$ . We observe that  $D_n^*, G_n^{c*}, Q_n^*$  at equilibrium depend linearly on the target demands of *all* the EUs.

**Case 2** [Possible surplus at node  $n$ ] In case  $\lambda_n = 0$ , then by substitution in (19), (20), (22) we obtain that  $D_n^* = \theta_n - \frac{1}{2\bar{a}_n} (-b_n + (\underline{\mu}_n - \underline{\mu}_n))$ ,  $G_n^{c*} = -\frac{b_n}{a_n} - \frac{1}{a_n} (\bar{\nu}_n - \underline{\nu}_n)$ , and  $Q_n^* = \theta_n - \frac{1}{2\bar{a}_n} (-b_n + (\underline{\mu}_n - \underline{\mu}_n)) + \frac{b_n}{a_n} + \frac{1}{a_n} (\bar{\nu}_n - \underline{\nu}_n) - G_n^{nc}$ .

In both cases, Prop. 1 statement follows from the closed form expressions of  $D_n^*, G_n^{c*}, Q_n^*, \forall n \in \mathcal{N}$  at equilibrium.

## B. Proof of Theorem 1

In the non-cooperative game  $\mathcal{G}$ , the strategy set of each EU is convex and compact; thus, the existence of a NE follows from [24, Thm. 3]. Now let us consider that there exists two VEs  $\hat{x}$  and  $\bar{x}$  solutions of the game  $\mathcal{G}$ . As both  $\hat{x}$  and  $\bar{x}$  points are equilibrium they must satisfy the KKT conditions, multiplying first order condition with  $[\bar{x}_n - \hat{x}_n]$  at point  $\hat{x}_n$  and  $\bar{x}_n$  for each  $n$ , we get:  $\left[ (2\bar{a}_n - (\hat{D}_n - \theta_n)) - (2\bar{a}_n - (\bar{D}_n - \theta_n)) \right] (\hat{D}_n - \bar{D}_n) = 0$ ,  $+ \left[ (\hat{\lambda}_n - \hat{\underline{\mu}}_n + \hat{\bar{\mu}}_n) - (\bar{\lambda}_n - \bar{\underline{\mu}}_n + \bar{\bar{\mu}}_n) \right] (\hat{D}_n - \bar{D}_n) = 0$ ,  $\left[ (a_n \hat{G}_n^c + b_n) - (a_n \bar{G}_n^c + b_n) \right] (\hat{G}_n^c - \bar{G}_n^c) - \left[ (\hat{\underline{\nu}}_n + \hat{\bar{\nu}}_n - \hat{\lambda}_n) - (\bar{\underline{\nu}}_n + \bar{\bar{\nu}}_n - \bar{\lambda}_n) \right] (\hat{G}_n^c - \bar{G}_n^c) = 0$ ,  $[c_{nm} - c_{nm}] (\hat{q}_{nm} - \bar{q}_{nm}) + \left[ (\hat{\zeta}_{nm} - \hat{\lambda}_n) - (\bar{\zeta}_{nm} - \bar{\lambda}_n) \right] (\hat{q}_{nm} - \bar{q}_{nm}) = 0, \forall m \in \Gamma_n, m \neq n$ . Summing over all the  $n$  gives us:

$$(\hat{x} - \bar{x})^T [\mathbf{F}_x(\hat{x}) - \mathbf{F}_x(\bar{x})] + (\hat{x} - \bar{x})^T [\hat{\Lambda} - \bar{\Lambda}] = 0, \quad (26)$$

where  $\hat{\Lambda}$  and  $\bar{\Lambda}$  represent matrices with Lagrangian multipliers from KKT at point  $\hat{x}$  and  $\bar{x}$  respectively. We observe that:

$$(\hat{x} - \bar{x})^T [\mathbf{F}_x(\hat{x}) - \mathbf{F}_x(\bar{x})] + (\hat{x} - \bar{x})^T [\hat{\Lambda} - \bar{\Lambda}] > 0, \quad (27)$$

which contradicts the assumption that there exist two VEs. In the above equation to prove that first term is positive, we show that game  $\mathcal{G}$  is strongly monotone, i.e.:

$$(\mathbf{y} - \mathbf{x})^T [\mathbf{F}_x(\mathbf{y}) - \mathbf{F}_x(\mathbf{x})] > 0, \forall \mathbf{x} \neq \mathbf{y}, \quad (28)$$

under Asm. 2.

Consider the pseudo Hessian matrix  $\mathbf{H}$  of utilities of the EUs where each  $n^{th}$  row and  $m^{th}$  column block matrix is given as  $H_{n,m} = \left[ \frac{\partial \mathbf{F}_{Q_n}}{\partial \mathbf{x}_m} \right]$ , i.e., in details:

$$H_{n,m} = \begin{cases} \begin{bmatrix} 2\bar{a}_n & 0 & 0 \\ 0 & a_n & 0 \\ 0 & 0 & 0 \end{bmatrix} & \text{if } n = m, \\ \mathbf{0}_{3 \times 3} & \text{if } n \neq m. \end{cases} \quad (29)$$

However, we observe that the matrix  $\mathbf{H}$  is a block diagonal matrix with each diagonal matrix positive semi-definite, which implies that matrix  $\mathbf{H}$  is positive semi-definite and in our case  $\frac{\partial \mathbf{F}_{Q_n}}{\partial Q_n}$  is constant for all  $n \in \mathcal{N}$ , thus inequality (28) follows from [24, Thm. 6] and the game  $\mathcal{G}$  is strongly monotone. The efficiency of the VE follows from the generalized potential game structure of  $\mathcal{G}$ , which results from the coincidence of the KKTs of (3) and  $\mathcal{G}$ .