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# A Characterization of Properly Displayable Atomic Logics

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## Abstract

The general results of Ciabattoni & Ramanayake about display calculi are applied to the framework of atomic logics. We obtain an instantiation of their “ $\mathcal{I}_2$  acyclic” formulas in this framework. Then, we show that our versions of  $\mathcal{I}_2$  acyclic formulas are inductive formulas in the sense of Goranko & Vakarelov, again adapted to this framework. This allows us to provide a characterization of displayable calculi extending basic atomic logics in terms of these formulas: a calculus is properly displayable iff it is sound and complete w.r.t. a class of frames defined by  $\mathcal{I}_2$  acyclic formulas. As corollary of the admissibility of the cut rule in display calculi, we obtain sufficient conditions for obtaining conservative extensions of logics. Then, we give examples of correspondences between structural inference rules and first-order frame properties and we detail some of the computations that enable to transform one to the other. Finally, we apply our characterization theorem to propositional logic, FDE, the family of relevance logics, intuitionistic and intermediate logics and obtain novel calculi and conservativity results for these logics. In particular, we show how classical and standard inference rules reappear as simplifications of more general and abstract rules that stem from the display calculi of our atomic logics.

**Keywords** : Universal logic; correspondence theory; display calculus; atomic logics;  $\mathcal{I}_2$  acyclic formula; inductive formula; FDE; relevance logics; intermediate logics

## 1 Introduction

Despite their complicated automated proof search, display calculi introduced by Belnap [6] have the advantage that a proof of cut-elimination can be easily obtained by inspecting the shape of the inference rules of such calculi and also that adding so-called analytic inference rules to any display calculus preserves the cut-elimination property. Determining to which formulas these analytic inference rules correspond has been studied in the case of tense logics by Kracht [30] and in the general case by Ciabattoni & Ramanayake [12]. The latter introduced a general framework for dealing with display calculi and they showed that these analytic inference rules somehow correspond to so-called “ $\mathcal{I}_2$  acyclic” formulas which coincide in the framework of Palmigiano et al. with so-called “analytic inductive inequalities” [25, Appendix D]. Ciabattoni & Ramanayake also obtain a general characterization of properly displayable calculi extending any base display

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calculus in terms of these formulas. Their general theorem implies in particular the result of Kracht [30] for tense logics.

We are going to port and apply their results to our atomic and molecular logics. We will not only instantiate their general result but we will also show in this specific setting that their  $\mathcal{I}_2$  acyclic formulas are in fact inductive in the sense of Goranko & Vakarelov [24] (a result similar to [25, Proposition 82]). This implies that they are also canonical and we can therefore obtain a characterization of displayable atomic logics which is somehow ‘semantic’. To be more precise, we will prove that a calculus is properly displayable iff it is sound and complete w.r.t. a class of models defined by acyclic formulas. Moreover, we will provide algorithms to translate analytic inference rules into first-order frame conditions.

Our characterization theorem of the properly displayable atomic logics is quite general. It applies to infinitely many logics. It is therefore natural to study the outcomes that it would produce from a proof-theoretical perspective if we take non-classical logics as well as classical logic as input for that theorem. For example, propositional logic is an extension of our basic atomic logic and we can wonder what happens if we apply our theorem with the Gentzen structural rules.

We will illustrate the various correspondence algorithms that play a role in our characterization theorem with well-known structural inference rules. We will see that adding the classical Gentzen structural rules to our base display calculus  $\mathcal{P}_{\mathbf{C},\mathbf{C}+}$  (where  $\mathbf{C}$  is the set of Boolean connectives) simplifies greatly the display rules as well as the 48 structural connectives of the orbit of the binary connectives  $\rightarrow, \wedge, \vee$  and the 8 structural connectives of the orbit of the unary connective  $\neg$ . They almost all collapse and can all be expressed alone by means of the usual structural comma “,” and the structural negation “\*”. Many rules thus become redundant or useless as well as many structural connectives. We will also illustrate our general results by applying them to some well-known non-classical logics such as the family of relevance logics, first-degree entailment (FDE), intuitionistic logic and intermediate (superintuitionistic) logics. We will provide novel display calculi for some of these logics and we will rediscover some well-known calculi and results for others, in particular those regarding the conservativity of the extensions of relevance logics with Boolean negation.

**Structure of the article.** In Sections 2 and 3, we recall atomic and molecular logics, in Section 4 their Boolean versions and in Section 5 their extension/generalization with information orders. In Section 6, we reproduce almost *verbatim* some general definitions and notions about display calculi introduced by Ciabattoni & Ramanayake [12]. In Section 7, we recall the group actions on atomic connectives that play an important role in the definitions of the various calculi for atomic logics. In Section 8, we recall and slightly adapt the general framework for dealing with display calculi introduced by Ciabattoni & Ramanayake [12]. In Section 10, we introduce inductive atomic formulas which are natural adaptation of the inductive formula of Goranko & Vakarelov [24]. In Section 11, we show that acyclic formulas are inductive. In Section 12, we state our main results and our characterization theorem. Then, in Section 13 we provide examples of correspondence translations from analytic inference rules to first-order frame conditions and translations back from inductive atomic formulas to analytic inference rule to illustrate it. In Section 14, we apply our theorem to the structural inference rules of classical propositional logic and reconstruct a display calculus for it. In Sections 15 and 16, we show that FDE, relevance logics, intuitionistic logic and intermediate logics are atomic logics and provide novel display calculi or rediscover existing display calculi for these logics. Finally, we discuss related works in Section 17 and conclude.

**Note.** The article is self-contained. It is the third part of a series of articles on the proof and correspondence theory of atomic and molecular logics, starting with [4, 5]. Omitted proofs are in the appendix or in the companion articles.

## 2 Atomic logics

Molecular logics are logics whose primitive connectives are compositions of connectives of atomic logics. Atomic logics are logics in which the truth conditions of connectives are defined by first-order (FO for short) formulas of the form  $\forall x_1 \dots x_n (\pm_1 Q_1 x_1 \vee \dots \vee \pm_n Q_n x_n \vee \pm R x_1 \dots x_n x)$  or  $\exists x_1 \dots x_n (\pm_1 Q_1 x_1 \wedge \dots \wedge \pm_n Q_n x_n \wedge \pm R x_1 \dots x_n x)$  where the  $\pm_i$ s and  $\pm$  are either empty or  $\neg$ . Likewise, propositional letters are defined by first-order formulas of the form  $\pm R x$ . They are viewed as 0-ary connectives (which is why we note them R and not Q) and the  $\pm$  in front of them stands for the fact that they can stand for literals. We will represent the structure of these formulas by means of so-called *skeletons* whose various arguments capture the different features and patterns from which they can be redefined completely. But first, we introduce some notations.

**Notations.** We use the symbol  $\triangleq$  for equality by definition instead of the assignment symbol  $:=$  often encountered in the literature (the  $\Delta$  above the equality symbol  $=$  is the Greek ‘D’ of Definition).  $\mathbb{N}^*$  denotes the set of natural numbers without 0 and, for all  $m, n \in \mathbb{N}$  such that  $m \leq n$ ,  $\llbracket m; n \rrbracket$  denotes the set of natural numbers  $\{m, m+1, \dots, n\}$ .  $\mathfrak{S}_n$  denotes the group of permutations over the set  $\{1, \dots, n\}$ . Permutations are generally denoted  $\sigma, \tau$ , the identity permutation is denoted Id and  $\sigma^-$  stands for the inverse permutation of the permutation  $\sigma$ . For example, the permutation  $\sigma = (3, 1, 2)$  is the permutation that maps 1 to 3, 2 to 1 and 3 to 2. We recall that  $\mathbb{Z}/2\mathbb{Z}$  denotes the field of the integers modulo 2 (also known as the dihedral group of order 2).<sup>1</sup> When viewed as a multiplicative group, its elements will be denoted in the sequel  $+$  and  $-$  and its operation  $\cdot$  is such that  $+\cdot- = -\cdot+ = -$  and  $-\cdot- = +\cdot+ = +$ . For brevity, when we use the notation  $+, -$ , we often omit the  $\cdot$  and write for example  $+- = -$  for  $+\cdot- = -$ . See [41] for relevant details on group theory.

For example, the truth condition of the connectives  $\square$  of modal logic is the first-order formula  $\forall y (P y \vee \neg R x y)$ . Its ‘skeleton’ is  $((1, 1), +, +, -, (2, 1))$ , which we also write for better readability  $((1, 1), \forall, +, -, (2, 1))$ . The so-called ‘dimension signature’  $(1, 1)$  corresponds to the fact that this connective takes as input a formula of dimension 1, represented by the predicate P of arity 1, and yields another formula of dimension 1, because the first-order formula has a single free variable representing the state where the resulting formula is evaluated. The ‘quantification signature’  $\forall$  corresponds to the universal quantification  $\forall$  in front of the formula and is represented by  $+$  ( $-$  represents the existential quantification  $\exists$ ). The ‘tonicity signature’ is  $+$  because there is no negation in front of P. The  $-$  in the skeleton corresponds to the negation  $\neg$  in front of  $\neg R x y$ . Finally, by convention, the natural order for elements appearing in a relation is  $R y z x$  or  $R y x$ , and more generally  $R x_1 \dots x_n x$ , where the free variable  $x$  denotes the state where the formula is evaluated. So in this example,  $R y x$  is transformed into  $R x y$ , which explains the introduction of the permutation  $(2, 1) \in \mathfrak{S}_2$  which swaps  $x$  and  $y$ .

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<sup>1</sup>The group  $\mathbb{Z}/2\mathbb{Z}$  was used to define atomic connectives by Espejo-Boix [21, Definition 3]. He used both the additive and multiplicative operations of the field  $\mathbb{Z}/2\mathbb{Z}$  to reformulate the central group action for atomic logics [2, Definition 18] in terms of a matrix product over  $\mathbb{Z}/2\mathbb{Z}^{n+1}$ .

**Definition 1** (Atomic connectives and skeletons). The set  $\text{SKL}_0$  of *propositional letter skeletons* and the sets  $\text{SKL}_n$  of *skeletons of arity*  $n \in \mathbb{N}^*$  are defined as follows:

$$\begin{aligned}\text{SKL}_0 &\triangleq \mathbb{N}^* \times \mathbb{Z} / 2\mathbb{Z}^2 \\ \text{SKL}_n &\triangleq \mathbb{N}^{*n+1} \times \mathbb{Z} / 2\mathbb{Z}^{n+2} \times \mathfrak{S}_{n+1}.\end{aligned}$$

The set of *atomic skeletons* is the set  $\text{SKL} \triangleq \bigcup_{n \in \mathbb{N}} \text{SKL}_n$ . They can be represented by tuples  $(k_1, \dots, k_{n+1}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$ , or  $(k, \mathbb{A}, \pm)$  if it is a propositional letter skeleton, where  $(k_1, \dots, k_{n+1}) \in \mathbb{N}^{*n+1}$ , often denoted  $\bar{k}$ , is called the *dimension signature*,  $\mathbb{A} \in \{+, -\}$  is called the *quantification signature*,  $(\pm_1, \dots, \pm_n) \in \{+, -\}^n$  is called the *tonicity signature*,  $\pm \in \{+, -\}$  is called the *relation signature* and  $\sigma$  is called the *permutation signature*. The tuple  $(\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_n)$  is called the *signature* or *trace* of the skeleton; the tonicity and quantification signatures are in correspondence with Dunn's notion of trace [18] (see [2, Definition 15] for details). The quantification signature  $\mathbb{A}$  will often be denoted  $\forall$  if it is  $+$  and  $\exists$  if it is  $-$ . The *arity* of a propositional letter skeleton is 0 and its *dimension* is  $k$ . The *input dimensions* and (*output*) *dimension* of a connective skeleton  $\bullet \in \text{SKL}_n$  of arity  $n \in \mathbb{N}^*$  are  $k_1, \dots, k_n$  and  $k_{n+1}$  respectively.

Let  $\mathbb{I}$  be an arbitrary but fixed set; in this article we assume that  $\mathbb{N} \subseteq \mathbb{I}$ . The set  $\text{ATM}_n$  of *atomic connectives of arity*  $n \in \mathbb{N}$  is defined as follows:

$$\text{ATM}_n \triangleq \{(\bullet, i) \mid \bullet \in \text{SKL}_n, i \in \mathbb{I}\}.$$

The set of *atomic connectives* is the set  $\text{ATM} \triangleq \bigcup_{n \in \mathbb{N}} \text{ATM}_n$ . Those of arity 0,  $\text{ATM}_0$ , are also called *propositional letters*. The arity, signature, quantification signature, dimension signature, tonicity signature, relation signature, permutation signature and input and output dimensions of an atomic connective  $(\bullet, i)$  are the same as its skeleton  $\bullet$ .

If  $\mathbb{C}$  is a set of atomic connectives, its set of propositional letters is denoted  $\mathbb{C}_0$ . Propositional letters are denoted  $p, p_1, p_2, \dots, p_i$ , etc, skeletons are denoted  $\bullet, \bullet_1, \bullet_2, \dots, \bullet_i$ , etc. and connectives  $\odot, \odot_1, \odot_2, \dots, \odot_i$ , etc. The quantification signature of a connective  $\odot$  or a skeleton  $\bullet$  is denoted  $\mathbb{A}(\odot) = \mathbb{A}(\bullet)$  and the  $j^{\text{th}}$  element of the tonicity signature of  $\odot$  and  $\bullet$  is denoted  $\pm_j(\odot) = \pm_j(\bullet)$ .  $\square$

**Definition 2** (Residuated skeletons and connectives). Two atomic skeletons  $(\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$  and  $(\bar{k}', \mathbb{A}', \pm'_1, \dots, \pm'_n, \pm', \sigma')$  are *residuated* when they are of equal arity  $n \in \mathbb{N}^*$  and there is  $\tau \in \mathfrak{S}_{n+1}$  such that  $\sigma' = \tau \circ \sigma$  and  $(k'_{\tau(1)}, \dots, k'_{\tau(n+1)}) = (k_1, \dots, k_{n+1})$ . Two atomic connectives  $(\bullet, i), (\bullet', i') \in \text{ATM}_n$  are residuated when  $i = i'$  and their skeletons  $\bullet$  and  $\bullet'$  are residuated.

**Example 1.** The skeleton of the implication  $\supset$  of the Lambek calculus, represented by the FO formula  $\forall yz(\neg Py \vee Qz \vee \neg Rxyz)$ , is  $((1, 1, 1), \forall, -, +, -, (2, 3, 1))$  and the skeleton of the fusion  $\otimes$  of the Lambek calculus, represented by the FO formula  $\exists yz(Py \wedge Qz \wedge Ryzx)$ , is  $((1, 1, 1), \exists, +, +, +, \text{Id})$ . Note the permutations associated with the relations:  $(2, 3, 1)$  for  $Rxyz$  and  $\text{Id} = (1, 2, 3)$  for  $Ryzx$  because the latter order  $(y, z, x)$  is the natural order. As one can also easily notice,  $\supset$  and  $\otimes$  are residuated.

The full list of permutations of  $\mathfrak{S}_2$  and  $\mathfrak{S}_3$  as well as all unary and binary signatures of dimension  $(1, 1)$  and  $(1, 1, 1)$  are given in Fig. 1.

**Definition 3** (Atomic language). Let  $\mathbb{C} \subseteq \text{ATM}$  be a set of atomic connectives. The *atomic language*  $\mathcal{L}_{\mathbb{C}}$  associated to  $\mathbb{C}$  is the smallest set that contains the propositional letters of  $\mathbb{C}$  and that is closed under the atomic connectives of  $\mathbb{C}$  while respecting the dimensions constraint. That is,

- $\mathcal{C}_0 \subseteq \mathcal{L}_{\mathcal{C}}$ ;
- for all  $\odot \in \mathcal{C}$  of arity  $n > 0$  and of dimension signature  $(k_1, \dots, k_{n+1})$  and for all  $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathcal{C}}$  of dimensions  $k_1, \dots, k_n$  respectively, we have that  $\odot(\varphi_1, \dots, \varphi_n) \in \mathcal{L}_{\mathcal{C}}$  and  $\odot(\varphi_1, \dots, \varphi_n)$  is of dimension  $k_{n+1}$ .

Elements of  $\mathcal{L}_{\mathcal{C}}$  are called *atomic formulas* and are denoted  $\varphi, \psi, \dots$ . The *dimension* of a formula  $\varphi \in \mathcal{L}_{\mathcal{C}}$  is denoted  $k(\varphi)$ . A set of atomic connectives  $\mathcal{C}$  is *plain* if for all  $\odot \in \mathcal{C}$  of dimension signature  $(k_1, \dots, k_{n+1})$  with  $n > 0$  there are propositional letters  $p_1, \dots, p_n \in \mathcal{C}_0$  of dimensions  $k_1, \dots, k_n$  respectively. *In the sequel, we assume that all sets of connectives  $\mathcal{C}$  are plain.*

**Definition 4** (Atomic  $\mathcal{C}$ -models and  $\mathcal{C}$ -frames). Let  $\mathcal{C} \subseteq \text{ATM}$  be a set of atomic connectives. An (*atomic*)  $\mathcal{C}$ -*model* is a tuple  $\mathfrak{M} = (W, \mathfrak{R})$  where  $W$  is a non-empty set and  $\mathfrak{R}$  is a set of relations over  $W$  such that each  $n$ -ary connective  $\odot \in \mathcal{C}$  of dimension signature  $(k_1, \dots, k_{n+1})$  is associated to a  $k_1 + \dots + k_{n+1}$ -ary relation  $R_{\odot} \in \mathfrak{R}$  and such that for all connectives  $\odot, \odot' \in \mathcal{C}$  we have that  $R_{\odot} = R_{\odot'}$  iff  $\odot$  and  $\odot'$  are residuated. An *assignment* is a tuple  $(w_1, \dots, w_k) \in W^k$  for some  $k \in \mathbb{N}^*$ , generally denoted  $\bar{w}$ . The set of assignments of a  $\mathcal{C}$ -model  $\mathfrak{M}$  is denoted  $\omega(\mathfrak{M}, \mathcal{C})$ . A *pointed  $\mathcal{C}$ -model*  $(\mathfrak{M}, \bar{w})$  is a  $\mathcal{C}$ -model  $\mathfrak{M}$  together with an assignment  $\bar{w}$  and, in that case, we say that  $(\mathfrak{M}, \bar{w})$  is of *dimension*  $k$ . The class of all pointed  $\mathcal{C}$ -models is denoted  $\mathcal{C}_{\mathcal{C}}$ .

A (*pointed*) *atomic  $\mathcal{C}$ -frame* is a (pointed) atomic  $(\mathcal{C} - \text{ATM}_0)$ -model. The class of all pointed  $\mathcal{C}$ -frames is denoted  $\mathcal{F}_{\mathcal{C}}$ .

**Definition 5** (Atomic logics). Let  $\mathcal{C} \subseteq \text{ATM}$  be a set of atomic connectives and let  $\mathfrak{M} = (W, \mathfrak{R})$  be a  $\mathcal{C}$ -model. We define the *interpretation function of  $\mathcal{L}_{\mathcal{C}}$  in  $\mathfrak{M}$* , denoted  $\llbracket \cdot \rrbracket^{\mathfrak{M}} : \mathcal{L}_{\mathcal{C}} \rightarrow \bigcup_{k \in \mathbb{N}^*} W^k$ , inductively as follows: for all propositional letters  $p \in \mathcal{C}$  of dimension  $k$ , all connectives  $\odot \in \mathcal{C}$  of skeleton  $(k_1, \dots, k_{n+1}, \mathbb{E}, \pm_1, \dots, \pm_n, \pm, \sigma)$  of arity  $n > 0$ , for all  $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathcal{C}}$ ,

$$\begin{aligned} \llbracket p \rrbracket^{\mathfrak{M}} &\triangleq \begin{cases} R_p & \text{if } \pm = + \\ W^k - R_p & \text{if } \pm = - \end{cases} \\ \llbracket \odot(\varphi_1, \dots, \varphi_n) \rrbracket^{\mathfrak{M}} &\triangleq f_{\odot}(\llbracket \varphi_1 \rrbracket^{\mathfrak{M}}, \dots, \llbracket \varphi_n \rrbracket^{\mathfrak{M}}) \end{aligned}$$

where the function  $f_{\odot}$  is defined as follows. For all  $W_1 \in \mathcal{P}(W^{k_1}), \dots, W_n \in \mathcal{P}(W^{k_n}), f_{\odot}(W_1, \dots, W_n) \triangleq \{\bar{w}_{n+1} \in W^{k_{n+1}} \mid \mathcal{C}^{\odot}(W_1, \dots, W_n, \bar{w}_{n+1})\}$  where  $\mathcal{C}^{\odot}(W_1, \dots, W_n, \bar{w}_{n+1})$  is called the *truth condition* of  $\odot$  and is defined as follows:

- if  $\mathbb{E} = \forall$ : “ $\forall \bar{w}_1 \in W^{k_1} \dots \bar{w}_n \in W^{k_n}$   
( $\bar{w}_1 \uparrow_1 W_1 \vee \dots \vee \bar{w}_n \uparrow_n W_n \vee R_{\odot}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}_{n+1}$ )”;
- if  $\mathbb{E} = \exists$ : “ $\exists \bar{w}_1 \in W^{k_1} \dots \bar{w}_n \in W^{k_n}$   
( $\bar{w}_1 \uparrow_1 W_1 \wedge \dots \wedge \bar{w}_n \uparrow_n W_n \wedge R_{\odot}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}_{n+1}$ )”;

where, for all  $j \in \llbracket 1; n \rrbracket$ ,  $\bar{w}_j \uparrow_j W_j \triangleq \begin{cases} \bar{w}_j \in W_j & \text{if } \pm_j = + \\ \bar{w}_j \notin W_j & \text{if } \pm_j = - \end{cases}$  and  $R_{\odot}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_{n+1}$  holds iff  $\pm R_{\odot} \bar{w}_{\sigma(1)} \dots \bar{w}_{\sigma(n+1)}$  holds, with the notations  $+R_{\odot} \triangleq R_{\odot}$  and  $-R_{\odot} \triangleq W^{k_1 + \dots + k_{n+1}} - R_{\odot}$ . We extend the definition of the interpretation function  $\llbracket \cdot \rrbracket^{\mathfrak{M}}$  to  $\mathcal{C}$ -frames as follows: for all  $\varphi \in \mathcal{L}_{\mathcal{C}}$  and all  $\mathcal{C}$ -frames  $\mathfrak{F}$ ,

$$\llbracket \varphi \rrbracket^{\mathfrak{F}} \triangleq \bigcap \left\{ \llbracket \varphi \rrbracket^{(\mathfrak{F}, V)} \mid V \text{ a set of } n\text{-ary relations over } W \text{ such that } (\mathfrak{F}, V) \text{ is a } \mathcal{C}\text{-model} \right\}$$

Permutations of $\mathfrak{S}_2$	Unary signatures
$\tau_1 = (1, 2)$	$t_1 = ((1, 1), \exists, +)$
$\tau_2 = (2, 1)$	$t_2 = ((1, 1), \forall, +)$
	$t_3 = ((1, 1), \forall, -)$
	$t_4 = ((1, 1), \exists, -)$
Permutations of $\mathfrak{S}_3$	Binary signatures
$\sigma_1 = (1, 2, 3)$	$s_1 = ((1, 1, 1), \exists, (+, +))$
$\sigma_2 = (3, 2, 1)$	$s_2 = ((1, 1, 1), \forall, (+, -))$
$\sigma_3 = (2, 3, 1)$	$s_3 = ((1, 1, 1), \forall, (-, +))$
$\sigma_4 = (2, 1, 3)$	$s_4 = ((1, 1, 1), \forall, (+, +))$
$\sigma_5 = (3, 1, 2)$	$s_5 = ((1, 1, 1), \exists, (+, -))$
$\sigma_6 = (1, 3, 2)$	$s_6 = ((1, 1, 1), \exists, (-, +))$
	$s_7 = ((1, 1, 1), \exists, (-, -))$
	$s_8 = ((1, 1, 1), \forall, (-, -))$

Figure 1: Permutations of  $\mathfrak{S}_2$  and  $\mathfrak{S}_3$  and ‘orbits’ of unary and binary signatures

Finally, if  $\mathcal{E}_C \subseteq \mathcal{C}_C$  is a class of pointed  $C$ -models, or pointed  $C$ -frames, the *satisfaction relation*  $\Vdash \subseteq \mathcal{E}_C \times \mathcal{L}_C$  is defined as follows: for all  $\varphi \in \mathcal{L}_C$  and all  $(\mathfrak{M}, \bar{w}) \in \mathcal{E}_C$ ,  $((\mathfrak{M}, \bar{w}), \varphi) \in \Vdash$  iff  $\bar{w} \in \llbracket \varphi \rrbracket^{\mathfrak{M}}$ . We usually write  $(\mathfrak{M}, \bar{w}) \Vdash \varphi$  instead of  $((\mathfrak{M}, \bar{w}), \varphi) \in \Vdash$  and we say that  $\varphi$  is *true* in  $(\mathfrak{M}, \bar{w})$ . The logic  $(\mathcal{L}_C, \mathcal{E}_C, \Vdash)$  is the *atomic logic associated to  $\mathcal{E}_C$  and  $C$* . Logics of the form  $(\mathcal{L}_C, \mathcal{C}_C, \Vdash)$  are called *basic atomic logics*.

**Example 2.** A simple example of an atomic logic is modal logic, where  $C = \{p, \top, \perp, \neg, \wedge, \vee, \rightarrow, \diamond_i, \square_i \mid i \in \mathbb{I}\}$ . We spell this example out in some detail:

- $p$  is a proposition letter of dimension 1 and  $\top, \perp$  are the proposition letters  $((1, \exists, +), 0)$  and  $((1, \forall, -), 0)$  respectively;
- $\neg$  is the connective  $((1, 1), \exists, -, +, \text{Id}), 0$ ;
- $\wedge, \vee, \rightarrow$  are the connectives  $((1, 1, 1), \exists, +, +, +, \text{Id}), 0$ ,  $((1, 1, 1), \forall, +, +, -, \text{Id}), 0$  and  $((1, 1, 1), \forall, -, +, -, (3, 1, 2)), 0$  respectively;
- $\diamond_i, \square_i$  are the connectives  $((1, 1), \exists, +, +, (2, 1), i)$  and  $((1, 1), \forall, +, -, (2, 1), i)$  respectively;
- the  $C$ -models  $\mathfrak{M} = (W, \mathfrak{R}) \in \mathcal{E}_C$  are such that  $R_\neg \triangleq \{(w, w) \mid w \in W\}$ ,  $R_\wedge = R_\vee = R_\rightarrow \triangleq \{(w, w, w) \mid w \in W\}$ ,  $R_{\diamond_i} = R_{\square_i}$  for all  $i \in \mathbb{I}$  and  $R_\top = R_\perp = W$ .

Atomic skeleton	Truth condition	Connective in the literature
$\varphi (s_1, +, \sigma_1) \psi$	$\exists yz (y \in \llbracket \varphi \rrbracket \wedge z \in \llbracket \psi \rrbracket \wedge Ryzx)$	$\varphi \otimes \psi$ [31]
$\varphi (s_2, -, \sigma_2) \psi$	$\forall yz (y \in \llbracket \varphi \rrbracket \vee z \notin \llbracket \psi \rrbracket \vee -Rxyz)$	$\varphi / \psi$ [31]
$\varphi (s_2, -, \sigma_3) \psi$	$\forall yz (y \in \llbracket \varphi \rrbracket \vee z \notin \llbracket \psi \rrbracket \vee -Rzxy)$	$\varphi \subset \psi$ [40]
$\varphi (s_1, +, \sigma_4) \psi$	$\exists yz (y \in \llbracket \varphi \rrbracket \wedge z \in \llbracket \psi \rrbracket \wedge Rzyx)$	
$= \psi (s_1, +, \sigma_1) \varphi$		
$\varphi (s_3, -, \sigma_5) \psi$	$\forall yz (y \notin \llbracket \varphi \rrbracket \vee z \in \llbracket \psi \rrbracket \vee -Rxyz)$	$\varphi \supset \psi$ [42]
$= \psi (s_2, -, \sigma_2) \varphi$		
$\varphi (s_3, -, \sigma_6) \psi$	$\forall yz (y \notin \llbracket \varphi \rrbracket \vee z \in \llbracket \psi \rrbracket \vee -Ryxz)$	$\varphi \setminus \psi$ [31]
$= \psi (s_2, -, \sigma_3) \varphi$		

Figure 2: Binary atomic skeletons of dimension signature  $(1, 1, 1)$  of the conjunction orbit  $\mathcal{O}_{\alpha_3}(\otimes)$

With these conditions on the C-models of  $\mathcal{E}_C$ , for all  $(\mathfrak{M}, w) \in \mathcal{E}_C$ , for all  $i \in \mathbb{I}$ ,

$w \in \llbracket \Diamond_i \varphi \rrbracket^{\mathfrak{M}}$	iff	$\exists v (v \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \wedge R_{\Diamond_i} wv)$
$w \in \llbracket \Box_i \varphi \rrbracket^{\mathfrak{M}}$	iff	$\forall v (v \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \vee -R_{\Box_i} wv)$
$w \in \llbracket \wedge(\varphi, \psi) \rrbracket^{\mathfrak{M}}$	iff	$\exists vu (v \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \wedge u \in \llbracket \psi \rrbracket^{\mathfrak{M}} \wedge R_{\wedge} wvu)$
	iff	$w \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \wedge w \in \llbracket \psi \rrbracket^{\mathfrak{M}}$
$w \in \llbracket \vee(\varphi, \psi) \rrbracket^{\mathfrak{M}}$	iff	$\forall vu (v \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \vee u \in \llbracket \psi \rrbracket^{\mathfrak{M}} \vee -R_{\vee} wvu)$
	iff	$w \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \vee w \in \llbracket \psi \rrbracket^{\mathfrak{M}}$
$w \in \llbracket \rightarrow(\varphi, \psi) \rrbracket^{\mathfrak{M}}$	iff	$\forall vu (v \notin \llbracket \varphi \rrbracket^{\mathfrak{M}} \vee u \in \llbracket \psi \rrbracket^{\mathfrak{M}} \vee -R_{\rightarrow} wvu)$
	iff	$w \notin \llbracket \varphi \rrbracket^{\mathfrak{M}} \vee w \in \llbracket \psi \rrbracket^{\mathfrak{M}}$
$w \in \llbracket \neg \varphi \rrbracket^{\mathfrak{M}}$	iff	$\exists v (v \notin \llbracket \varphi \rrbracket^{\mathfrak{M}} \wedge R_{\neg} wv)$
	iff	$w \notin \llbracket \varphi \rrbracket^{\mathfrak{M}}$
$w \in \llbracket \top \rrbracket^{\mathfrak{M}}$	iff	$w \in R_{\top}$
	iff	always
$w \in \llbracket \perp \rrbracket^{\mathfrak{M}}$	iff	$w \in W - R_{\perp}$
	iff	never

**Example 3** (Dummy connectives). We can define ‘dummy’ connectives, in the sense that these connectives do not affect the truth value of the formulas that they take as argument. They will be used in the definition of molecular connectives and will serve as ‘markers’ so that we can ‘plug’ the same formula at different places of a (molecular) connective. They are defined as follows:

$$\text{ATM}_x \triangleq \{(((k, k), \exists, +, +, \text{Id}), i), (((k, k), \forall, +, -, \text{Id}), i) \mid k, i \in \mathbb{N}^*\}$$

These ‘dummy’ connectives will be denoted  $x_i^k$  in general and the set of their skeletons, denoted  $x^k$ , is denoted  $\text{SKL}_x$ . Any C-model  $\mathfrak{M} = (W, \mathfrak{R})$  containing some of these connectives is such that  $R_{x_i^k} = \{(\bar{w}, \bar{w}) \mid \bar{w} \in W^k\}$ . Hence, for all  $\varphi \in \mathcal{L}_C$  and all  $\bar{w} \in W^k$ , we have that  $\bar{w} \in \llbracket x_i^k(\varphi) \rrbracket^{\mathfrak{M}}$  iff  $\bar{w} \in \llbracket \varphi \rrbracket^{\mathfrak{M}}$ .

Other examples of atomic logics are given in Table 2 as well as in [2, 3]. The following theorem can be easily proven by polynomially reducing the satisfiability problem of basic atomic logic to the satisfiability problem of modal logic.

**Theorem 1** ([4]). *Every basic atomic logic is decidable and in PSPACE.*



### 3 Molecular Logics

Molecular logics are basically logics whose primitive connectives are compositions of atomic connectives and in which it is possible to repeat the same atomic connective at different places in the connective. That is why we call them ‘molecular’, just as molecules are compositions of atoms in chemistry.

**Definition 6** (Molecular connectives and skeletons). The set  $\text{MOL}$  of *molecular connectives* is the smallest set such that:

- $\text{ATM}_x \cup \text{ATM}_0 \subseteq \text{MOL}$ ;
- for all atomic connectives  $\odot \in \text{ATM} - \text{ATM}_x$  of arity  $n > 0$  and dimension signature  $(k_1^0, \dots, k_n^0, k)$  and all  $\otimes_1, \dots, \otimes_n \in \text{MOL}$  of output dimensions or dimensions (if they are propositional letters)  $k_1^0, \dots, k_n^0$  respectively, the string of symbols  $\odot(\otimes_1, \dots, \otimes_n)$  is a molecular connective of  $\text{MOL}$  of *output dimension*  $k$ .

If  $\otimes \in \text{MOL}$ , we define its *decomposition tree* as follows. If  $\otimes = p \in \text{ATM}_0$  or  $\otimes = x_i^k \in \text{ATM}_x$ , then its decomposition tree  $T_{\otimes}$  is the tree consisting of a single node labeled with  $p$  or  $x_i^k$  respectively. If  $\otimes = \odot(\otimes_1, \dots, \otimes_n) \in \text{MOL}$  then its decomposition tree  $T_{\otimes}$  is the tree defined inductively as follows: the root of  $T_{\otimes}$  is  $\otimes$  and it is labeled with  $\odot$  and one sets edges between that root and the roots  $\otimes_1, \dots, \otimes_n$  of the decomposition trees  $T_{\otimes_1}, \dots, T_{\otimes_n}$  respectively. A *subconnective* of  $\otimes$  is any vertex of the decomposition tree  $T_{\otimes}$ . The set of *atomic connectives associated to a set  $C$  of molecular connectives* is the set of labels different from  $x_i^k$  of the decomposition trees of the molecular connectives of  $C$ .

If  $\otimes$  is a molecular connective then the tuple  $(x_{i_1}^{k_1}, \dots, x_{i_l}^{k_l})$  of dummy connectives which appear (possibly with some repetitions) in the leaves of the decomposition tree  $T_{\otimes}$  in the in-order traversal of  $T_{\otimes}$  is called the *dummy signature* of  $\otimes$ . The same tuple in the same order but without repetition is called the *input signature* of  $\otimes$ .<sup>2</sup> The number  $l$  is called the *width* of  $\otimes$ . If  $(x_{i_1}^{k_1}, \dots, x_{i_m}^{k_m})$  is the input signature of a molecular connective  $\otimes$  of output dimension  $k$  then the *dimension signature* of  $\otimes$  is  $(k_1, \dots, k_m, k)$  and its *arity* is  $m$ . We also define the *quantification signature*  $\mathcal{A}(\otimes)$  of  $\otimes = \odot(\otimes_1, \dots, \otimes_n)$  by  $\mathcal{A}(\otimes) \triangleq \mathcal{A}(\odot)$ .

A *molecular skeleton* is defined exactly like a molecular connective except that in the above definition the term ‘connective’ has to be replaced uniformly by the term ‘skeleton’ and  $\text{ATM}$  by  $\text{SKL}$ . Likewise, the arity, dimension signature, output dimension, dummy signature, input signature, quantification signature and decomposition tree of a molecular skeleton are defined like those for molecular connectives. The molecular skeleton  $*$  associated to a molecular connective  $\otimes \in \text{MOL}$  is defined inductively as follows:

$$* \triangleq \begin{cases} \cdot & \text{if } \otimes = \odot \in \text{ATM} \\ \bullet(*_1, \dots, *_n) & \text{if } \otimes = \odot(\otimes_1, \dots, \otimes_n). \end{cases}$$

Molecular connectives will be generally denoted  $\otimes, \otimes_1, \otimes_2, \dots, \otimes_i$ , *etc.* and molecular skeletons  $*, *_1, *_2, \dots, *_i$ , *etc.* □

Note that two different molecular connectives may have the same molecular skeleton, such as  $\diamond_1(\square_1(x_1^1) \vee \square_2(x_2^1))$  and  $\diamond_2(\square_1(x_1^1) \vee \square_3(x_2^1))$ . Every atomic connective  $\odot$  of dimension signature  $(k_1, \dots, k_n, k)$  can be canonically associated to a (specific) molecular connective  $\otimes \triangleq \odot(x_1^{k_1}, \dots, x_n^{k_n})$  (the  $x_i^{k_i}$ s can be chosen differently but they must all be different and with

<sup>2</sup>For example, if the dummy signature of some molecular connective  $\otimes$  is  $(x_1^1, x_2^1, x_1^2, x_2^2, x_1^3, x_1^3)$  then its associated input signature is  $(x_1^1, x_1^2, x_2^2, x_1^3)$ .

appropriate dimensions). One needs to introduce the connective  $x_i^k$  either in order to deal with molecular connectives whose skeletons are for example of the form  $\odot(p, x_i^k)$  where  $p \in \text{ATM}_0$  is a propositional letter, or to deal with molecular connectives in which the same argument(s) appear at different places, like for example in  $\odot(x_1^k, \dots, x_1^k)$  which is in fact of arity 1.

**Definition 7** (Molecular language). Let  $\mathbf{C} \subseteq \text{MOL}$  be a set of molecular connectives. The *molecular language*  $\mathcal{L}_{\mathbf{C}}$  associated to  $\mathbf{C}$  is the smallest set such that

- the propositional letters of  $\mathbf{C}$ , denoted  $\mathbf{C}_0$ , belong to  $\mathcal{L}_{\mathbf{C}}$ ;
- for all  $\otimes \in \mathbf{C}$  of output dimension  $k$  and input signature  $(x_{i_1}^{k_1}, \dots, x_{i_m}^{k_m})$  and for all  $\varphi_1, \dots, \varphi_m \in \mathcal{L}_{\mathbf{C}}$  of dimensions  $k_1, \dots, k_m$  respectively, the string of symbols denoted  $\otimes(\varphi_1, \dots, \varphi_m)$ , which is the string of symbols  $\otimes$  where the dummy atomic connectives  $x_{i_1}^{k_1}, \dots, x_{i_m}^{k_m}$  appearing in  $\otimes$  are uniformly replaced by  $\varphi_1, \dots, \varphi_m$  respectively, belongs to  $\mathcal{L}_{\mathbf{C}}$  and  $\otimes(\varphi_1, \dots, \varphi_m)$  is of *dimension*  $k$ .

Elements of  $\mathcal{L}_{\mathbf{C}}$  are called *molecular formulas* and are denoted  $\varphi, \psi, \alpha, \dots$ . The *dimension* of a formula  $\varphi \in \mathcal{L}_{\mathbf{C}}$  is denoted  $k(\varphi)$ . We use the same abbreviations as for the atomic language. We also assume in the sequel that all our molecular connectives are plain (as in Definition 3).  $\square$

**Definition 8** (Molecular logic). Let  $\mathbf{C} \subseteq \text{MOL}$  be a set of molecular connectives. A (*molecular*)  $\mathbf{C}$ -*model*  $\mathfrak{M}$  is a  $\mathbf{C}'$ -model  $\mathfrak{M}$  where  $\mathbf{C}'$  is the set of atomic connectives associated to  $\mathbf{C}$ . The class of all pointed molecular  $\mathbf{C}$ -models is also denoted  $\mathcal{C}_{\mathbf{C}}$ , like for atomic logics.

The truth conditions for molecular connectives are defined naturally from the truth conditions of atomic connectives. We define the *interpretation function* of  $\mathcal{L}_{\mathbf{C}}$  in  $\mathfrak{M}$ , denoted  $\llbracket \cdot \rrbracket^{\mathfrak{M}} : \mathcal{L}_{\mathbf{C}} \rightarrow \bigcup_{k \in \mathbb{N}^*} W^k$ , inductively as follows: for all propositional letters  $p \in \mathbf{C}$  of skeleton  $(k, \mathbb{A}, \pm, \sigma)$ , all molecular connectives  $\odot(\otimes_1, \dots, \otimes_n) \in \mathbf{C}$  of arity  $m > 0$  and input signature  $(x_{i_1}^{k_1}, \dots, x_{i_m}^{k_m})$  and all  $k, i \in \mathbb{N}^*$ , for all  $\varphi, \varphi_1, \dots, \varphi_m \in \mathcal{L}_{\mathbf{C}}$ ,

$$\begin{aligned} \llbracket p \rrbracket^{\mathfrak{M}} &\triangleq \pm R_p \\ \llbracket x_i^k(\varphi) \rrbracket^{\mathfrak{M}} &\triangleq \llbracket \varphi \rrbracket^{\mathfrak{M}} \\ \llbracket \odot(\otimes_1, \dots, \otimes_n)(\varphi_1, \dots, \varphi_m) \rrbracket^{\mathfrak{M}} &\triangleq f_{\odot}(\llbracket \otimes_1(\varphi_1^1, \dots, \varphi_{m_1}^1) \rrbracket^{\mathfrak{M}}, \dots, \llbracket \otimes_n(\varphi_1^n, \dots, \varphi_{m_n}^n) \rrbracket^{\mathfrak{M}}) \end{aligned}$$

where for all  $j \in \llbracket 1; n \rrbracket$ , if  $(x_{i_{j_1}}^{k_{j_1}}, \dots, x_{i_{j_{m_j}}^{k_{j_{m_j}}}})$  is the input signature of  $\otimes_j$  then  $(\varphi_1^j, \dots, \varphi_{m_j}^j) = (\varphi_{j_1}, \dots, \varphi_{j_{m_j}})$ .

If  $\mathcal{E}_{\mathbf{C}}$  is a class of pointed  $\mathbf{C}$ -models, the triple  $(\mathcal{L}_{\mathbf{C}}, \mathcal{E}_{\mathbf{C}}, \Vdash)$  is a logic called the *molecular logic associated to  $\mathcal{E}_{\mathbf{C}}$  and  $\mathbf{C}$* .  $\square$

As one can easily notice, every atomic logic can be canonically mapped to an equi-expressive molecular logic: each atomic connective  $\odot$  of dimension signature  $(k_1, \dots, k_n, k)$  of the given atomic logic has to be transformed into the molecular connective of skeleton  $\odot(x_1^{k_1}, \dots, x_n^{k_n})$ . We recall that the  $x_i^k$  are in fact specific atomic connectives whose associated relations are the identity relations (see Example 3).

**Example 4** (Temporal logic). The *temporal language*  $\mathcal{L}_{\text{TL}}$  is defined inductively by the following grammar in BNF:

$$\mathcal{L}_{\text{TL}} : \varphi ::= \top \mid \perp \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid U(\varphi, \varphi) \mid S(\varphi, \varphi)$$

where  $p \in \text{ATM}_0$ . A *temporal model* is a tuple  $\mathfrak{M} = (W, \{<, P_1, \dots, P_n, \dots\})$  where:  $W$  is a non-empty set;  $< \subseteq W \times W$  is a binary relation over  $W$ ;  $P_1, \dots, P_n, \dots \subseteq W$  are unary relations over  $W$ .

We write  $w \in \mathfrak{M}$  for  $w \in W$  by abuse and the pair  $(\mathfrak{M}, w)$  is called a *pointed temporal model*. The class of all pointed temporal models is denoted  $\mathcal{C}_{\text{TL}}$ . We define the *satisfaction relation*  $\models_{\text{TL}} \subseteq \mathcal{C}_{\text{TL}} \times \mathcal{L}_{\text{TL}}$  by the following *truth conditions*. Below, we write  $(\mathfrak{M}, w) \models \varphi$  for  $((\mathfrak{M}, w), \varphi) \in \models_{\text{TL}}$ . For all temporal models  $\mathfrak{M} = (W, \{<, P_1, \dots, P_n, \dots\})$ , all  $w \in \mathfrak{M}$ , all  $\varphi, \psi \in \mathcal{L}_{\text{TL}}$  and all  $p_i \in \text{SKL}_0$ ,

$(\mathfrak{M}, w) \models \top$	always;
$(\mathfrak{M}, w) \models \perp$	never;
$(\mathfrak{M}, w) \models p_i$	iff $P_i(w)$ holds;
$(\mathfrak{M}, w) \models \neg\varphi$	iff it is not the case that $(\mathfrak{M}, w) \models \varphi$ ;
$(\mathfrak{M}, w) \models (\varphi \wedge \psi)$	iff $(\mathfrak{M}, w) \models \varphi$ and $(\mathfrak{M}, w) \models \psi$ ;
$(\mathfrak{M}, w) \models U(\varphi, \psi)$	iff there is $v \in W$ such that $w < v$ and $(\mathfrak{M}, v) \models \varphi$ and for all $u \in W$ such that $w < u < v$ , $(\mathfrak{M}, u) \models \psi$ ;
$(\mathfrak{M}, w) \models S(\varphi, \psi)$	iff there is $v \in W$ such that $v < w$ and $(\mathfrak{M}, v) \models \varphi$ and for all $u \in W$ such that $v < u < w$ , $(\mathfrak{M}, u) \models \psi$ .

The triple  $(\mathcal{L}_{\text{TL}}, \mathcal{C}_{\text{TL}}, \models_{\text{TL}})$  forms a logic, that we call *temporal logic*.

**Encoding of temporal logic as a molecular logic.** Let us consider the skeletons defined by the following first-order formulas:

$$\begin{aligned} \bullet_1(x) &\triangleq \exists yzz' (Py \wedge Qzz' \wedge Ryz z'x) \\ \bullet'_1(x) &\triangleq \exists yzz' (Py \wedge Qzz' \wedge Rxxz'y) \\ \bullet_2(x, x') &\triangleq \forall y (Py \vee \neg S y x x') \end{aligned}$$

$\odot_1, \odot'_1$  and  $\odot_2$  are atomic skeletons (this is independent from the definitions of R and S). The connectives of skeletons  $\otimes = \odot_1(x_1^1, \odot_2(x_2^1))$  and  $\otimes' = \odot'_1(x_1^1, \odot_2(x_2^1))$  are normal molecular connectives standing for the ‘Until’ and the ‘Since’ temporal operators  $U$  and  $S$ . Let  $\mathfrak{M}_1 = (W_1, \{<_1, P\})$  and  $\mathfrak{M}_2 = (W_2, \{<_2, P\})$  be two temporal models. We represent these temporal models by the C-models  $\mathfrak{M}_1^{U,S} = (W_1, \{R_1, S_1, P\})$  and  $\mathfrak{M}_2^{U,S} = (W_2, \{R_2, S_2, P\})$  respectively such that for all  $v_1, u_1, u'_1, w_1, w'_1 \in W_1$ ,

$$R_1 v_1 u_1 u'_1 w_1 \text{ iff } w_1 <_1 v_1, w_1 = u_1 \text{ and } v_1 = u'_1 \quad (1)$$

$$S_1 v_1 w_1 w'_1 \text{ iff } w_1 <_1 v_1 <_1 w'_1 \quad (2)$$

and likewise for  $R_2$  and  $S_2$  of  $\mathfrak{M}_2$ . One can show that for all  $\varphi \in \mathcal{L}_{\text{C}}$  and all  $w_1 \in W_1$ ,  $(\mathfrak{M}_1, w_1) \models \varphi$  iff  $(\mathfrak{M}_1^{U,S}, w_1) \Vdash \varphi$  (and likewise for  $\mathfrak{M}_2$  and  $\mathfrak{M}_2^{U,S}$ ).  $\square$

## 4 Boolean atomic and molecular logics

Atomic and molecular logics do not include Boolean connectives as primitive connectives. In fact, they can be defined in terms of specific atomic connectives, as follows.

**Definition 9** (Boolean connectives). The *Boolean connectives* called *conjunctions*, *disjunctions*, *negations* and *Boolean constants* (of dimension  $k$ ) are the atomic connectives denoted, respectively  $\text{BLN} \triangleq \{\wedge_k, \vee_k, \rightarrow_k, \top_k, \perp_k, \neg_k \mid k \in \mathbb{N}^*\}$  where

$$\begin{aligned} \wedge_k &\triangleq ((k, k, k, \exists, +, +, +, \text{Id}), 0) & \top_k &\triangleq ((k, \exists, +), 0) \\ \vee_k &\triangleq ((k, k, k, \forall, +, +, -, \text{Id}), 0) & \perp_k &\triangleq ((k, \forall, -), 0) \\ \neg_k &\triangleq ((k, k, \exists, -, +, \text{Id}), 0) & \rightarrow_k &\triangleq ((k, k, k, \forall, -, +, (3, 1, 2)), 0) \end{aligned}$$

In any C-model  $\mathfrak{M} = (W, \mathfrak{R})$  containing Boolean connectives, the associated relation of any  $\vee_k$ ,  $\wedge_k$  or  $\rightarrow_k$  is  $R_{\wedge_k} = R_{\vee_k} = R_{\rightarrow_k} \triangleq \{(\bar{w}, \bar{w}, \bar{w}) \mid \bar{w} \in W^k\}$ , the associated relation of  $\neg_k$  is  $R_{\neg_k} \triangleq \{(\bar{w}, \bar{w}) \mid \bar{w} \in W^k\}$  and the associated relation of any  $\top_k$  or  $\perp_k$  is  $R_{\perp_k} = R_{\top_k} \triangleq W^k$ . We will often omit the subscript  $k$  in  $\wedge_k$ ,  $\vee_k$ ,  $\rightarrow_k$ ,  $\top_k$ ,  $\perp_k$ ,  $\neg_k$  when it is clear from the context and simply write  $\wedge$ ,  $\vee$ ,  $\rightarrow$ ,  $\top$ ,  $\perp$ ,  $\neg$ . We define  $\text{BLN}^*$  to be the set of all atomic connectives which are residuated with some connective of  $\text{BLN}$ ,  $\text{BLN}^- \triangleq \{\odot \in \text{BLN}^* \mid \pm_j(\odot) = - \text{ for some } j \in \{1, 2\}\}$  those that have a negative tonicity in their argument(s) and  $\text{BLN}_k$  those of dimension  $k$ .

We say that a set of atomic connectives  $C$  is *complete for truth constants, conjunction and disjunction (resp. negation)* when it contains all truth constants, conjunctions and disjunctions  $\top_k, \perp_k, \wedge_k, \vee_k$  (resp. Boolean negation  $\neg_k$ ), for  $k$  ranging over all input types and output types of the atomic connectives of  $C$ . We say that a set  $C$  of atomic connectives is *Boolean* when it contains all conjunctions, disjunctions, material implications, constants as well as negations  $\wedge_k, \vee_k, \rightarrow_k, \top_k, \perp_k, \neg_k$ , for  $k$  ranging over all input dimensions and output dimensions of the connectives of  $C$ . The *Boolean completion* of a set of atomic connectives  $C$  is the smallest set of connectives including  $C$  which is Boolean. A *Boolean atomic logic* is an atomic logic such that its set of connectives is Boolean.  $\square$

**Proposition 1.** *Let  $C$  be a set of atomic connectives containing Boolean connectives. and let  $\mathfrak{M} = (W, \mathfrak{R})$  be a  $C$ -model. Then, for all  $k \in \mathbb{N}^*$ , all  $\varphi, \psi \in \mathcal{L}_C$ , if  $k(\varphi) = k(\psi) = k$ , then*

$$\begin{array}{ll} \llbracket \top_k \rrbracket^{\mathfrak{M}} &= W^k & \llbracket \varphi \wedge_k \psi \rrbracket^{\mathfrak{M}} &= \llbracket \varphi \rrbracket^{\mathfrak{M}} \cap \llbracket \psi \rrbracket^{\mathfrak{M}} \\ \llbracket \perp_k \rrbracket^{\mathfrak{M}} &= \emptyset & \llbracket \varphi \vee_k \psi \rrbracket^{\mathfrak{M}} &= \llbracket \varphi \rrbracket^{\mathfrak{M}} \cup \llbracket \psi \rrbracket^{\mathfrak{M}} \\ \llbracket \neg_k \varphi \rrbracket^{\mathfrak{M}} &= W^k - \llbracket \varphi \rrbracket^{\mathfrak{M}} & \llbracket \varphi \rightarrow_k \psi \rrbracket^{\mathfrak{M}} &= (W^k - \llbracket \varphi \rrbracket^{\mathfrak{M}}) \cup \llbracket \psi \rrbracket^{\mathfrak{M}} \end{array}$$

## 5 Atomic and molecular logics with information order

Many logics include in their semantics a preorder  $\sqsubseteq$  over states of the models, sometimes called an *information order* or an *order relation*, such as the family of relevance logics but also the Kripke semantics of intuitionistic logic. Intuitively,  $w \sqsubseteq v$  means that everything that is true at  $w$  is true at  $v$ . Hence, because of this intuitive interpretation, the valuation of propositional letters  $p$  by predicates  $P$  should consist of so-called ‘‘cones of situations’’ and should satisfy the following condition, sometimes called the ‘‘persistence’’ condition in intuitionistic logic:

$$\text{if } Pw \text{ and } w \sqsubseteq v \text{ then } Pv. \quad (\text{Heredit}y \text{ condition})$$

Some other conditions must be imposed on the accessibility relations associated to the connectives so that the Heredit}y condition transfers to all formulas of the language, which is a common *desideratum*, or only to a sub-language of the whole language.

As it turns out, because of its grounding in gaggle theory, these conditions can be automatically defined from the signature of the atomic connectives considered. This information order was part of Dunn’s original gaggle theory [18, p. 98-99] and can be very easily and naturally integrated within the framework of atomic and molecular logics.

**Definition 10** (Atomic and molecular model with information order). Let  $C$  be a set of atomic connectives. A *persistent set of connectives associated to  $C$*  is a set of atomic connectives  $D \subseteq C$  such that  $D \cap C_0 \neq \emptyset$  and  $D \cap \text{BLN}^- = \emptyset$ , that is,  $D$  contains some propositional letter(s) of  $C$  and no Boolean connective of negative tonicity.

Let  $\mathfrak{M} = (W, \mathfrak{R})$  be an atomic  $C$ -model. A  *$D$ -information order* over  $\mathfrak{M}$  is a preorder  $\sqsubseteq \subseteq \bigcup_{k \in \mathbb{N}^*} (W^k \times W^k)$  such that for all  $\odot \in D - \text{BLN}^*$  of skeleton  $(\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$ , all  $j \in \llbracket 1; n \rrbracket$

and all  $\bar{w}_1, \dots, \bar{w}_j, \bar{w}'_j, \dots, \bar{w}_n, \bar{w}, \bar{w}' \in \omega(\mathfrak{M}, \mathcal{C})$ ,

- if  $R_{\odot}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_j \dots \bar{w}_n \bar{w}$  and  $\bar{w}_j \sqsubseteq \bar{w}'_j$  then  $R_{\odot}^{\pm\sigma} \bar{w}_1 \dots \bar{w}'_j \dots \bar{w}_n \bar{w}$
- where  $\bar{w}_j \sqsubseteq \bar{w}'_j \triangleq \begin{cases} \bar{w}_j \sqsubseteq \bar{w}'_j & \text{if } \pm_j \mathcal{A} = + \\ \bar{w}'_j \sqsubseteq \bar{w}_j & \text{if } \pm_j \mathcal{A} = - \end{cases}$  (Tonicity postulates)
- if  $R_{\odot}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}$  and  $\bar{w} \sqsubseteq \bar{w}'$  then  $R_{\odot}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}'$ .

An atomic C-model (C-frame) together with a D-information order  $\sqsubseteq$  is called an *atomic C-model* (resp. *C-frame*) with D-information order. A *molecular model with D-information order* is a molecular model such that its associated C'-atomic model is with D-information order, where  $D \subseteq C'$  is such that  $D \cap C'_0 \neq \emptyset$  and  $D \cap \text{BLN}^- = \emptyset$ .

Note that any atomic or molecular model with D-information order satisfies the Heredity condition for the propositional letters  $p$  of D since this condition is an instance of the Tonicity postulates for the connectives  $p$ .

**Proposition 2.** *Let  $\mathcal{C}$  be a set of atomic (molecular) connectives and let  $\mathfrak{M}$  be an atomic (resp. molecular) C-model with D-information order  $\sqsubseteq$ . Then, for all  $\bar{w}, \bar{v} \in \omega(\mathfrak{M}, \mathcal{C})$  and all  $\varphi \in \mathcal{L}_D$ , if  $(\mathfrak{M}, \bar{w}) \Vdash \varphi$  and  $\bar{w} \sqsubseteq \bar{v}$  then  $(\mathfrak{M}, \bar{v}) \Vdash \varphi$ .*

*Proof.* By induction on  $\varphi$ . □

Information orders allow us to have a more fined-grained account over the class of (non-classical) logics. As one can easily see, if we decide to take  $\sqsubseteq$  as the equality  $=$ , then we just recover all our previous definitions of atomic and molecular models. So, in a sense, atomic and molecular models *with information order* are a generalization of our previous atomic and molecular models. By using information order, we have more freedom and accuracy over the definition of our semantics and therefore we can devise more refined and sometimes more complex (non-classical) logics.

**Definition 11** (Atomic and molecular logics with information order). *An atomic (molecular) logic with information order* is an atomic (resp. molecular) logic such that all its atomic (resp. molecular) models have an information order.

**Example 5.** On the one hand, intuitionistic and superintuitionistic/intermediate logics are canonical examples of atomic logics with information order. On the other hand, modal intuitionistic logic is an example of molecular logic with information order.

## 6 Generalities about display calculi and acyclic formulas

In the rest of the article, we will use the standard definitions of *logic*, *truth*, *validity*, *logical consequence*, *soundness*, *(strong) completeness*, etc (see [2, Section 7.1] for details for example). In this section, we introduce display calculi from an abstract and general perspective. All the definitions in this section are exactly (unless specified otherwise) the definitions of Ciabattoni and Ramanayake [12].

**Definition 12** (Display calculus, display property and display rules, [12]). Let  $L = (\mathcal{L}, \mathcal{C}, \Vdash)$  be a logic containing two types of connectives called *logical connectives* and *structural connectives* such that the set of structural connectives can be partitioned into two subsets called the *antecedent structural connectives* denoted  $[C^-]$  and the *succedent structural connectives* denoted  $[C^+]$ .

A *formula* is a well-formed expression of  $\mathcal{L}$  built up only from logical connectives and a *structure* is either a formula or a well-formed expression of  $\mathcal{L}$  built up inductively from other structures and structural connectives. Moreover, all well-formed expressions of  $\mathcal{L}$  are either formulas or structures. An *a-structure* (resp. *s-structure*) is a structure whose outermost structural connective  $[\odot]$  is such that  $[\odot] \in [\mathbf{C}^-]$  (resp.  $[\odot] \in [\mathbf{C}^+]$ ). The set of *a-structures* (resp. *s-structures*) is denoted  $\mathfrak{S}_{\text{ant}}$  (resp.  $\mathfrak{S}_{\text{suc}}$ ).

A *substructure* of a structure  $U$  or of a  $\mathcal{L}$ -consecution  $U \vdash V$  is a structure appearing in the scope of a structural connective of  $U$  or  $V$ . In that case, we denote it  $U[Z]$  and  $U \vdash V[Z]$ . We also denote by  $U[Z/Z']$  and  $U \vdash V[Z/Z']$  the structure and consecution obtained by substituting  $Z$  by  $Z'$  in  $U$  and  $U \vdash V$  respectively.

A sequent calculus  $\mathcal{P}$  for  $\mathcal{L}$  is said to have the *display property* if it contains a set of single-premise structural rules called the *display rules* such that

1. the rule upwards from conclusion to premise of a display rule is also a display rule;
2. for all  $U \vdash V$  derivable in  $\mathcal{P}$  and all substructures  $Z$  appearing in  $U \vdash V$  there is a structure  $W$  such that
  - either for all structures  $Z'$ , it holds that  $W \vdash Z'$  is derivable from  $U \vdash V[Z/Z']$  using the display rules;
  - or for all structures  $Z'$ , it holds that  $Z' \vdash W$  is derivable from  $U \vdash V[Z/Z']$  using the display rules.

We say that  $Z$  is *antecedant part* (resp. *consequent part*) in a sequent when it is equivalent to a sequent of the form  $Z \vdash U$  (resp.  $U \vdash Z$ ). A structure  $Z$  is *displayed* (in a sequent) if the sequent has the form  $Z \vdash U$  or  $U \vdash Z$ . A *display calculus* is a sequent calculus with the display property. A display calculus is said to be *proper* if it satisfies Belnap's conditions (C1) – (C8) (see [12] for a formulation of these conditions with our terminology). A calculus is a *typed proper display calculus* if it satisfies the conditions (C1) to (C5) and (C8) of [6] and two new conditions: (C6') and (C7'). Conditions (C6') and (C7') are the same as Belnap's (C6) and (C7) except that the substituted structure  $X$  and the part  $M$  where it is substituted should moreover have equal dimension.<sup>3</sup> By abuse and for simplicity, we will often call a *typed proper display calculus* simply 'proper display calculus'.

**Example 6.** In the setting of our atomic logics, the antecedent structural connectives  $[\mathbf{C}^-]$  are the structural connectives  $[\odot]$  such that  $\mathbb{A}(\odot) = \exists$  and the succedent structural connectives  $[\mathbf{C}^+]$  are the structural connectives  $[\odot]$  such that  $\mathbb{A}(\odot) = \forall$ . So, in this setting, the set of a-structures  $\mathfrak{S}_{\text{ant}}$  (resp. s-structures  $\mathfrak{S}_{\text{suc}}$ ) is the set of structures  $[\odot](X_1, \dots, X_n)$  such that  $\mathbb{A}(\odot) = \exists$  (resp.  $\mathbb{A}(\odot) = \forall$ ).

**Definition 13** (Structural and analytic inference rule). A *structural inference rule* for a language with logical and structural connectives is an inference rule whose representation only uses structural variables and structural connectives, no logical connective. An *analytic inference rule* is a structural inference rule satisfying Belnap's conditions (C1) – (C5) – (C6') – (C7') [6].

**Definition 14** (Amenable calculus, [12]). Suppose that  $\mathcal{P}$  is a proper display calculus for a language  $\mathcal{L}$  whose set of formulas of  $\mathcal{L}$  is denoted  $\text{For}\mathcal{L}$ . A proper display calculus satisfying the following conditions is said to be *amenable*.

- (1) (*interpretation functions*) There are functions  $l : \mathfrak{S}_{\text{ant}} \mapsto \text{For}\mathcal{L}$  and  $r : \mathfrak{S}_{\text{suc}} \mapsto \text{For}\mathcal{L}$  such that for all  $\varphi \in \text{For}\mathcal{L}$   $l(\varphi) = r(\varphi) = \varphi$  and for all  $X \in \mathfrak{S}_{\text{ant}}$  and all  $Y \in \mathfrak{S}_{\text{suc}}$ ,

<sup>3</sup>This notion of *typed proper display calculus* was not introduced by Ciabattoni & Ramanayake [12].

- (a)  $X \vdash l(X)$  and  $r(Y) \vdash Y$  are derivable in  $\mathcal{P}$ ;  
(b) if  $X \vdash Y$  is derivable in  $\mathcal{P}$ , then so is  $l(X) \vdash r(Y)$ .
- (2) (*logical constants*) It contains an a-structure constant and an s-structure constant that we will denote  $[\top]$  and  $[\perp]$  respectively (they are usually both denoted  $\mathbf{I}$ ). There are also logical constants  $\top, \perp \in \text{For}\mathcal{L}$  such that the following sequents are derivable for all  $X \in \mathfrak{S}_{\text{ant}}$  and all  $Y \in \mathfrak{S}_{\text{suc}}$ :

$$\perp \vdash Y \qquad X \vdash \top$$

- (3) (*logical connectives*) There are binary logical connectives  $\vee, \wedge \in \mathbf{C}$  such that the following are derivable for all  $\varphi, \psi \in \text{For}\mathcal{L}$  and all  $\odot \in \{\wedge, \vee\}$ :

- (i) commutativity:  $\varphi \odot \psi \vdash \psi \odot \varphi$   
(ii) associativity:  $\varphi \odot (\psi \odot \chi) \vdash (\varphi \odot \psi) \odot \chi$  and  $(\varphi \odot \psi) \odot \chi \vdash \varphi \odot (\psi \odot \chi)$

Also, for all  $X \in \mathfrak{S}_{\text{ant}}$  and all  $Y \in \mathfrak{S}_{\text{suc}}$ ,

- (a) $_{\vee}$   $\varphi \vdash Y$  and  $\psi \vdash Y$  implies  $\varphi \vee \psi \vdash Y$ ;  
(b) $_{\vee}$   $X \vdash \varphi$  implies  $X \vdash \varphi \vee \psi$ ;  
(a) $_{\wedge}$   $X \vdash \varphi$  and  $X \vdash \psi$  implies  $X \vdash \varphi \wedge \psi$ ;  
(b) $_{\wedge}$   $\varphi \vdash Y$  implies  $\varphi \wedge \psi \vdash Y$ .

*Remark 1.* The definition of an amenable calculus by Ciabattoni & Ramanayake [12, Definition 3.1] considers only the (Boolean) connectives  $\wedge, \vee, \top$  and  $\perp$  of dimension 1. Yet, all their results are straightforwardly transferable to our setting with multiple dimensions  $\wedge_k, \vee_k, \top_k$  and  $\perp_k$ .

**Definition 15** (*inv $\mathcal{P}$* ). Let  $\mathcal{P}$  be a display calculus  $\mathcal{P}$  for  $\mathcal{L}$  and let  $X \vdash Y$  be a  $\mathcal{L}$ -consecution. The set  $\text{inv}_{\mathcal{P}}(X \vdash Y)$  consists of all the sets of  $\mathcal{L}$ -consecutions obtained by applying upwards (*i.e.*, from conclusion to premise) some sequence of invertible logical rules in  $\mathcal{P}$  (and display rules in order to display the formula occurring in the  $\mathcal{L}$ -consecution) starting from  $X \vdash Y$ .

**Definition 16** (a-soluble and s-soluble, [12]). Let  $\mathcal{P}$  be a display calculus  $\mathcal{P}$  for  $\mathcal{L}$ . A formula  $\varphi \in \text{For}\mathcal{L}$  is *a-soluble* (resp. *s-soluble*) for  $\mathcal{P}$  if there is some  $\{U_i \vdash V_i\}_{i \in \Omega} \in \text{inv}_{\mathcal{P}}(\varphi \vdash [\perp])$  (resp.  $\in \text{inv}_{\mathcal{P}}([\top] \vdash \varphi)$ ) containing no logical connectives.

**Definition 17** ( $\mathcal{I}_0(\mathcal{P}), \mathcal{I}_1(\mathcal{P}), \mathcal{I}_2(\mathcal{P})$ , [12]). Let  $\mathcal{P}$  be an amenable calculus for  $\mathcal{L}$ . The classes  $\mathcal{I}_0(\mathcal{P}), \mathcal{I}_1(\mathcal{P}), \mathcal{I}_2(\mathcal{P}) \subseteq \text{For}\mathcal{L}$  are defined as follows:

- $\mathcal{I}_0(\mathcal{P}) = \text{ATM}_0$ ;

and for all  $\varphi \in \text{For}\mathcal{L}$ ,

- $\varphi \in \mathcal{I}_1(\mathcal{P})$  iff there is some  $\{U_i \vdash V_i\}_{i \in \Omega} \in \text{inv}_{\mathcal{P}}([\top] \vdash \varphi)$  such that  $\{U_i \vdash V_i\}_{i \in \Omega}$  does not contain logical connectives;
- $\varphi \in \mathcal{I}_2(\mathcal{P})$  iff there is some  $\{U_i \vdash V_i\}_{i \in \Omega} \in \text{inv}_{\mathcal{P}}([\top] \vdash \varphi)$  such that each antecedent part formula in  $U_i \vdash V_i$  is s-soluble and each consequent part formula in  $U_i \vdash V_i$  is a-soluble, for each  $i \in \Omega$ .

In the two latter cases, we say that  $\{U_i \vdash V_i\}_{i \in \Omega}$  witnesses  $\varphi$ .

Note that  $\mathcal{I}_0(\mathcal{P}) \subseteq \mathcal{I}_1(\mathcal{P}) \subseteq \mathcal{I}_2(\mathcal{P})$ .

**Definition 18** (Respect multiplicities, [12]). Let  $\mathcal{P}$  be a display calculus for  $\mathcal{L}$  and let  $S$  be a set of  $\mathcal{L}$ -consecutions. We denote by  $\text{ATM}_0(S)$  the set of propositional letters which occur in  $S$ . The set  $S$  is said to *respect multiplicities* with respect to a propositional letter  $p \in \text{ATM}_0(S)$  if  $S$  can be partitioned into one of the following forms using the display rules:

$$\{p \vdash U \mid p \notin U\} \cup \{V \vdash p \mid \text{every } p \text{ in } V \vdash p \text{ is cons. part}\} \cup \{s \in S \mid p \notin s\} \quad (3)$$

$$\{U \vdash p \mid p \notin U\} \cup \{p \vdash V \mid \text{every } p \text{ in } p \vdash V \text{ is ante. part}\} \cup \{s \in S \mid p \notin s\} \quad (4)$$

where  $p \notin U$ ,  $p \notin U \vdash V$  denote the fact that  $p$  does not occur in  $U$  or  $V$ .

**Definition 19** ( $S_p$ , [12]). Let  $\mathcal{P}$  be a display calculus for  $\mathcal{L}$  and let  $S$  be a set of  $\mathcal{L}$ -consecutions respecting multiplicities w.r.t.  $p$ . If it is not the case that  $p \vdash U \in S$  and  $V \vdash p \in S$  (up to display equivalence), then we set  $S_p$  as  $\{s \in S \mid p \notin s\}$ . Otherwise, define  $S_p$  as the union of  $\{s \in S \mid p \notin s\}$  and one of the following, depending on the display equivalent form of  $S$  as (3) or (4), respectively:

$$\{s \mid \text{obtain } s \text{ from } V \vdash p \in S \text{ by subst. each occ. of } p \text{ with a } U \text{ such that } p \vdash U \in S\}$$

$$\{s \mid \text{obtain } s \text{ from } p \vdash V \in S \text{ by subst. each occ. of } p \text{ with a } U \text{ such that } U \vdash p \in S\}$$

Note that if  $S$  respects multiplicities w.r.t.  $p$ , then  $p$  does not occur in  $S_p$ .

**Definition 20** (Acyclic set, [12]). Let  $\mathcal{P}$  be a display calculus for  $\mathcal{L}$ . A finite set  $S$  of  $\mathcal{L}$ -consecutions built only from structural connectives and atoms (and no logical connectives) is *acyclic* if

(i)  $\text{ATM}_0(S) = \emptyset$  or

(ii) there is  $p \in \text{ATM}_0(S)$  such that  $S$  respects multiplicities w.r.t.  $p$  and  $S_p$  is acyclic.

**Definition 21** (Acyclic formula, [12]). Let  $\mathcal{P}$  be a display calculus for  $\mathcal{L}$  and let  $\varphi \in \mathcal{I}_2(\mathcal{P})$ . If there is some set  $\{\rho_i\}_{i \in \Omega}$  of inference rules equivalent to  $\varphi$  obtained according to the algorithm of [12, Fig. 1, p. 18] such that the premises of each  $\rho_i$  ( $i \in \Omega$ ) are acyclic, then  $\varphi$  is called an *acyclic formula*.

**Definition 22** (Display calculus corresponding to a Hilbert calculus, [12]). Let  $\mathcal{P}$  be an amenable calculus for  $\mathcal{L}$  and let  $\mathcal{P}^H$  be a Hilbert calculus for  $\mathcal{L}$ . Then, we say that  $\mathcal{P}$  *corresponds* to  $\mathcal{P}^H$  if there is a function  $\mathbb{F}$  mapping  $\text{For}\mathcal{L}$ -consecutions to formulas of  $\text{For}\mathcal{L}$  such that for all  $\varphi \in \text{For}\mathcal{L}$ ,

- $\mathbb{F}([\top] \vdash \varphi) = \varphi$ ;
- for every instance  $\frac{X_1 \vdash Y_1 \quad \dots \quad X_n \vdash Y_n}{X_{n+1} \vdash Y_{n+1}}$  of a rule in  $\mathcal{P}$ , there is a derivation in  $\mathcal{P}^H$  of  $\mathbb{F}(l(X_{n+1}) \vdash r(Y_{n+1}))$ , assuming  $\mathbb{F}(l(X_1) \vdash r(Y_1)), \dots, \mathbb{F}(l(X_n) \vdash r(Y_n))$ ;
- for every instance of a rule  $\frac{\varphi_1 \quad \dots \quad \varphi_n}{\varphi_{n+1}}$  in  $\mathcal{P}^H$ , there is a derivation in  $\mathcal{P}$  of  $[\top] \vdash \varphi_{n+1}$ , assuming  $[\top] \vdash \varphi_1, \dots, [\top] \vdash \varphi_n$ .

**Definition 23** (Well-behaved calculus, [12]). An amenable calculus  $\mathcal{P}$  for  $\mathcal{L}$  is *well-behaved* if

- $\mathcal{P}$  corresponds to some Hilbert calculus  $\mathcal{P}^H$ ;



- $\mathcal{P}$  contains the following rules

$$\frac{\varphi \vdash U \quad \psi \vdash U}{\varphi \vee \psi \vdash U} \qquad \frac{U \vdash \varphi \quad U \vdash \psi}{U \vdash \varphi \wedge \psi}$$

Here,  $\wedge$  and  $\vee$  are the connectives in the definition of amenable calculus (not necessarily conjunction, disjunction);

- for every For $\mathcal{L}$ -consecution  $\varphi \vdash \psi$ , we have that  $\varphi \vdash \psi \in \text{inv}_{\mathcal{P}}([\top] \vdash \mathbb{F}(\varphi \vdash \psi))$ .

## 7 Group actions and residuations

In [2], a group action of the symmetric group over the set of gaggle connectives was introduced. We are going to redefine it in a simpler and more concise way. It will turn out to play a crucial role, in particular in the expression of the display rule.<sup>4</sup>

**Definition 24.** Let  $n \in \mathbb{N}^*$ ,  $\bullet = (k_1, \dots, k_{n+1}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma) \in \text{SKL}_n$  and  $\tau \in \mathfrak{S}_{n+1}$ . For all  $j \in \llbracket 0; n \rrbracket$ , we first define  $\Delta_j \triangleq \delta_j^\tau \pm_{\tau^{-1}(n+1)}$  where

$$\delta_j^\tau \triangleq \begin{cases} + & \text{if } j = \tau(n+1) \\ \delta_{n+1, \tau(n+1)} & \text{otherwise} \end{cases} \qquad \delta_{n+1, \tau(n+1)} \triangleq \begin{cases} + & \text{if } n+1 = \tau(n+1) \\ - & \text{otherwise} \end{cases}$$

and we also set  $\pm_{n+1} \triangleq +$  and  $\delta_{n+1}^\tau \triangleq +$ .<sup>5</sup> Then, we define the function  $a_n : \mathfrak{S}_{n+1} \times \text{SKL}_n \rightarrow \text{SKL}_n$  as follows:

$$a_n(\tau, \bullet) \triangleq (k_{\tau^{-1}(1)}, \dots, k_{\tau^{-1}(n+1)}, \Delta_0 \mathbb{A}, \Delta_1 \pm_{\tau^{-1}(1)}, \dots, \Delta_n \pm_{\tau^{-1}(n)}, \Delta_0 \pm, \tau \circ \sigma).$$

The function  $a_n$  induces a function  $\alpha_n : \text{ATM} \times \mathfrak{S}_{n+1} \rightarrow \text{ATM}$  on the set ATM of connectives of arity  $n$  defined by  $\alpha_n((\bullet, i), \tau) \triangleq (a_n(\tau, \bullet), i)$ . Likewise, we define the functions  $\beta_n : \mathbb{Z}/2\mathbb{Z} \times \text{ATM} \rightarrow \text{ATM}$  and  $\gamma_n : \mathbb{Z}/2\mathbb{Z} \times \text{ATM} \rightarrow \text{ATM}$  by

$$\beta_n(\pm, (\bullet, i)) \triangleq \begin{cases} (-\bullet, i) & \text{if } \pm = - \\ (\bullet, i) & \text{if } \pm = + \end{cases} \qquad \gamma_n(\pm, (\bullet, i)) \triangleq \begin{cases} (\sim \bullet, i) & \text{if } \pm = - \\ (\bullet, i) & \text{if } \pm = + \end{cases}$$

where  $-\bullet \triangleq (k_1, \dots, k_{n+1}, -\mathbb{A}, -\pm_1, \dots, -\pm_n, -\pm, \sigma)$

and  $\sim \bullet \triangleq (k_1, \dots, k_{n+1}, \mathbb{A}, \pm_1, \dots, \pm_n, -\pm, \sigma)$ .

$a_n(\tau, \bullet)$ ,  $\alpha_n(\tau, \odot)$  and  $\beta_n(\pm, \odot)$  are often denoted  $\tau \bullet$ ,  $\tau \odot$  and  $\pm \odot$  respectively.

*Remark 2.* Note that if  $n+1 = \tau(n+1)$  then

$$\tau \bullet = (k_{\tau^{-1}(1)}, \dots, k_{\tau^{-1}(n+1)}, \mathbb{A}, \pm_{\tau^{-1}(1)}, \dots, \pm_{\tau^{-1}(n)}, \pm, \tau \circ \sigma). \quad (\text{Res})$$

**Definition 25** (Common set of connectives). A set  $\mathcal{C}$  of atomic connectives is *common* when for all pairs of residuated connectives  $\odot, \odot' \in \mathcal{C}$ , we have that  $\mathcal{O}_{\alpha_n \bullet \beta_n}(\odot) = \mathcal{O}_{\alpha_n \bullet \beta_n}(\odot')$ , that is,  $\odot' = \tau_0 - \tau_1 \dots - \tau_m \odot$  for some  $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$ .

<sup>4</sup>The definition of  $a_n$  is inspired but slightly different from the group action introduced by Espejo-Boix [21, Definition 2]. We resort here to the extra components  $\pm_{n+1}$  and  $\delta_{n+1}^\tau$ , always equal to  $+$ .

<sup>5</sup>Setting  $\delta_{n+1}^\tau$  to  $+$  does not play any role in this definition but it might appear in the expressions and play a role if we apply the group action successively to several permutations  $\tau$ .

Common sets of atomic connectives correspond to a very large class of logics. As far as we know, all non-classical logics which are atomic logics are based on some common set of connectives.<sup>6</sup>

## 8 Display calculi for basic atomic logics

Often, it is not very clear how and why one should choose structural connectives in a calculus. In this section, we give some proposals on how one should choose them so as to enforce cut admissibility. Basically, they should be ‘displayable enough’ and chosen so as to ensure the display property of the sequent calculus.

**Definition 26** (Displayable enough set of connectives). A set of atomic connectives  $\mathbf{C}$  is *displayable enough* when for all  $\odot \in \mathbf{C}$  of arity  $n \in \mathbb{N}^*$  and all  $i \in \llbracket 1; n \rrbracket$ , there is  $\odot' \in \mathbf{C}$  such that  $\odot' = \tau_0 - \tau_1 \dots - \tau_m \odot$  for some  $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$  such that  $\tau_0 \circ \dots \circ \tau_m(i) = n + 1$ . It is *purely displayable* if moreover, for all  $i \in \llbracket 1; n + 1 \rrbracket$ , we have that  $\pm^i = +$ .<sup>7</sup>

Informally, a set of connectives is displayable enough when each argument of each of its connectives can be displayed as the sole antecedent or the sole consequent of a sequent. This concept was first introduced by Espejo-Boix [21, Theorem 5] and it will be formalized here by our Proposition 3. Note that for every common set of atomic connectives  $\mathbf{C}$  there is always a common displayable enough set of atomic connectives  $\mathbf{C}^+$  such that  $\mathbf{C} \subseteq \mathbf{C}^+$ .

**Definition 27** (Structures, consecutions). *Atomic structural connectives* are copies of the atomic connectives: for all sets of atomic connectives  $\mathbf{C}$ , its associated set of structural connectives is denoted  $[\mathbf{C}] \triangleq \{[\odot] \mid \odot \in \mathbf{C}\}$ . For all atomic connectives  $\odot$ , the *arity*, *signature*, *type signature*, *tonicity signature*, *quantification signature* of  $[\odot]$  are the same as  $\odot$ . Structural connectives are denoted  $[p], [p_1], [p_2], \dots$  and  $[\odot], [\odot_1], [\odot_2], \dots$ . For each  $k \in \mathbb{N}^*$ , we also introduce the (*Boolean*) *structural connective* associated to the Boolean connectives  $\wedge_k, \vee_k$ , denoted  $[\wedge_k], [\vee_k]$  and often simply  $[\wedge], [\vee]$  by abuse. We also denote  $*_k$ , and often simply  $*$  by abuse, the structural connective  $[\neg_k]$ .

Let  $(\mathbf{C}, \mathbf{C}^+)$  be a pair of sets of atomic connectives such that  $\mathbf{C} \subseteq \mathbf{C}^+$ . The *structural atomic language*  $[\mathcal{L}_{\mathbf{C}^+}]$  associated to the pair  $(\mathbf{C}, \mathbf{C}^+)$  is the smallest set that contains the atomic language  $\mathcal{L}_{\mathbf{C}}$  as well as  $[\mathbf{C}_0]$  and that is closed under the structural connectives of  $[\mathbf{C}^+] \cup \{*\}$  while respecting the dimension constraints. Its elements are called *structures* and their *dimensions* are defined like for formulas of  $\mathcal{L}_{\mathbf{C}}$ . A  $\mathcal{L}_{\mathbf{C}}$ -consecution (resp.  $[\mathcal{L}_{\mathbf{C}^+}]$ -consecution) is an expression of the form  $\varphi \vdash \psi$ ,  $\vdash \varphi$  or  $\varphi \vdash$  (resp.  $X \vdash Y$ ,  $\vdash X$  or  $Y \vdash$ ), where  $\varphi, \psi \in \mathcal{L}_{\mathbf{C}}$  (resp.  $X, Y \in [\mathcal{L}_{\mathbf{C}^+}]$ ) are of the same dimension. The set of all  $\mathcal{L}_{\mathbf{C}}$ -consecutions (resp.  $[\mathcal{L}_{\mathbf{C}^+}]$ -consecutions) is denoted  $\mathcal{S}_{\mathbf{C}}$  (resp.  $[\mathcal{S}_{\mathbf{C}^+}]$ ). Elements of  $\mathcal{L}_{\mathbf{C}}$  (resp.  $[\mathcal{L}_{\mathbf{C}^+}]$  and  $[\mathcal{S}_{\mathbf{C}^+}]$ ) are called *formulas* (resp. *structures* and *consecutions*); they are denoted  $\varphi, \psi, \alpha, \dots$  (resp.  $X, Y, U, V, \dots$  and  $X \vdash Y, U \vdash V, \dots$ ). Structures and consecutions are interpreted canonically over atomic  $\mathbf{C}$ -models exactly like the formulas to which they correspond. In particular, we have for all  $\mathbf{C}$ -models  $\mathfrak{M}$ , all structures  $X$  and all  $\bar{w} \in \omega(\mathfrak{M}, \mathbf{C})$  of the same dimension as  $X$  that  $\bar{w} \in \llbracket *X \rrbracket^{\mathfrak{M}}$  iff  $\bar{w} \notin \llbracket X \rrbracket^{\mathfrak{M}}$ .

**Proposition 3** (Display property). *If  $Z$  is a substructure of  $X$ , then  $tn(Z, X)$  is defined inductively as follows:*

- if  $X = Z$  then  $tn(Z, X) \triangleq +$ ;

<sup>6</sup>An exception might be Levesque’s logic of only knowing [32], because we might need to refer to the complement of the epistemic accessibility relation so as to be able to capture it as an atomic logic.

<sup>7</sup>We recall that  $\pm^i$  is defined in Fig. 5.

- *Structural rules*: for all  $\top, \wedge, \vee \in \mathbf{C}^+$ ,<sup>a</sup>

$$\frac{(X [\wedge] Y) \vdash U}{(Y [\wedge] X) \vdash U} \quad (\text{CI} \vdash) \quad \frac{X \vdash U}{(X [\wedge] Y) \vdash U} \quad (\text{K} \vdash) \quad \frac{(X [\wedge] X) \vdash U}{X \vdash U} \quad (\text{WI} \vdash)$$

$$\frac{([\top] [\wedge] X) \vdash U}{X \vdash U} \quad (\text{I} \vdash) \quad \frac{U \vdash \varphi \quad \varphi \vdash V}{U \vdash V} \quad \text{Cut}$$

If  $X$  is empty then  $(X [\wedge] [\top])$  and  $([\top] [\wedge] X)$  are  $[\top]$  and  $*X$  is empty.

- *Display rule*: for all  $\odot, \odot' \in \mathbf{C}^+$  such that  $\odot' = \tau_0 - \tau_1 \dots - \tau_m \odot$  for some  $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$ , denoting  $\rho \triangleq (\tau_0 \circ \tau_1 \circ \dots \circ \tau_m)^-$ ,

$$\frac{S([\odot], X_1, \dots, X_n, X_{n+1})}{S([\odot'], \pm^{\rho(1)} X_{\rho(1)}, \dots, \pm^{\rho(n)} X_{\rho(n)}, \pm^{\rho(n+1)} X_{\rho(n+1)})} \quad \text{DR}$$

where for all  $j \in \llbracket 1; n+1 \rrbracket$ ,  $\pm^j \triangleq \begin{cases} \pm_1^j \pm_2^j \dots \pm_m^j & \text{if } m \neq 0 \\ + & \text{if } m = 0 \end{cases}$  and for all  $i \in \llbracket 1; m \rrbracket$ ,  
 $\pm_i^j \triangleq \begin{cases} - & \text{if } \tau_i \circ \tau_{i+1} \circ \dots \circ \tau_m(j) = n+1 \\ + & \text{otherwise} \end{cases} \quad \pm X \triangleq \begin{cases} *X & \text{if } \pm = - \\ X & \text{if } \pm = + \end{cases}$

- *Axiom and introduction rules* : for all  $p \in \mathbf{C}_0$  and all  $\odot \in \mathbf{C}$ ,

$$\frac{p \vdash p}{S(X_1, \varphi_1) \quad \dots \quad S(X_n, \varphi_n)} \quad (\vdash \odot) \quad \frac{S([\odot], \varphi_1, \dots, \varphi_n, U)}{S(\odot, \varphi_1, \dots, \varphi_n, U)} \quad (\odot \vdash) \quad \text{Axiom}$$

where for all  $\odot \in \mathbf{C}$  of skeleton  $(\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$  such that:

- for all  $j \in \llbracket 1; n \rrbracket$ , we define  $S(X_j, \varphi_j) \triangleq \begin{cases} X_j \vdash \varphi_j & \text{if } \pm_j \mathbb{A} = - \\ \varphi_j \vdash X_j & \text{if } \pm_j \mathbb{A} = + \end{cases}$
  - for all  $\star \in \{\odot, [\odot]\}$ ,  $S(\star, X_1, \dots, X_n, X) \triangleq \begin{cases} \star(X_1, \dots, X_n) \vdash X & \text{if } \mathbb{A} = \exists \\ X \vdash \star(X_1, \dots, X_n) & \text{if } \mathbb{A} = \forall \end{cases}$
- such that the  $X_j$ s are non-empty and if  $\varphi_j$  is empty then  $\odot(\varphi_1, \dots, \varphi_n)$  is empty.

<sup>a</sup>We recall that  $\wedge$  and  $\vee$  are abusive notations for  $\top_k$ ,  $\wedge_k$  and  $\vee_k$ , for some  $k \in \mathbb{N}^*$ .

Figure 3: Display calculus  $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+}^{\text{BLN}}$

- if  $X = *Y$  and  $Z$  appears in  $Y$  then  $tn(Z, X) \triangleq -tn(Z, Y)$ ;
- if  $X = [\odot](X_1, \dots, X_n)$  and  $Z$  appears in  $X_j$  then  $tn(Z, X) \triangleq \pm_j(\odot)tn(Z, X_j)$ .

Let  $\mathbf{C}, \mathbf{C}^+$  be sets of atomic connectives such that  $\mathbf{C} \subseteq \mathbf{C}^+$  and  $\mathbf{C}^+$  is displayable enough. If  $Z$  is a substructure of  $X$  or  $Y$  and  $X \vdash Y$  is provable in  $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+}^{\text{BLN}}$ , then  $Z$  is an antecedent part (resp. consequent part) of  $X \vdash Y$  iff  $tn(Z, X) = +$  or  $tn(Z, Y) = -$  (resp.  $tn(Z, X) = -$  or

$\frac{}{[\top] \vdash \top} \quad (\top \vdash)$	$\frac{X \vdash \varphi \quad Y \vdash \psi}{(X [\wedge] Y) \vdash (\varphi \wedge \psi)} \quad (\wedge \vdash)$	$\frac{U \vdash (\varphi [\vee] \psi)}{U \vdash (\varphi \vee \psi)} \quad (\vee \vdash)$
$\frac{[\top] \vdash U}{\top \vdash U} \quad (\top \vdash)$	$\frac{(\varphi [\wedge] \psi) \vdash U}{(\varphi \wedge \psi) \vdash U} \quad (\wedge \vdash)$	$\frac{\varphi \vdash X \quad \psi \vdash Y}{(\varphi \vee \psi) \vdash (X [\vee] Y)} \quad (\vee \vdash)$
$\frac{}{\perp \vdash [\perp]} \quad (\perp \vdash)$	$\frac{\varphi \vdash X}{*X \vdash \neg \varphi} \quad (\neg \vdash)$	$\frac{U \vdash (\varphi [\supset] \psi)}{U \vdash (\varphi \supset \psi)} \quad (\supset \vdash)$
$\frac{U \vdash [\perp]}{U \vdash \perp} \quad (\perp \vdash)$	$\frac{* \varphi \vdash U}{\neg \varphi \vdash U} \quad (\neg \vdash)$	$\frac{X \vdash \varphi \quad \psi \vdash Y}{(\varphi \supset \psi) \vdash (X [\supset] Y)} \quad (\supset \vdash)$

Figure 4: Classical introduction rules as instances of the rules  $(\vdash \odot)$  and  $(\odot \vdash)$

$tn(Z, Y) = +$ .

**Theorem 2** (Soundness and strong completeness of  $\mathcal{P}_{C, C^+}^{\text{BLN}}$ , [4]). *Let  $(C, C^+)$  be a pair of common Boolean sets of atomic connectives such that  $C \subseteq C^+$  and  $C^+$  is displayable enough. The calculus  $\mathcal{P}_{C, C^+}^{\text{BLN}}$  of Fig. 3 is sound and complete for the basic atomic logic based on  $C$ . Moreover, it is a proper display calculus and enjoys cut elimination.*

As shown in Fig. 4, the classical introduction rules are all instances of rules  $(\vdash \odot)$  and  $(\odot \vdash)$ . The Axiom  $p \vdash p$  could be replaced by axioms and inference rules for propositional letters  $p$  which are special instances of the rules  $(\vdash \odot)$  and  $(\odot \vdash)$  of Fig. 3. Indeed, with  $\odot = p$ , we would have that  $n = 0$  and, replacing  $\odot$  with  $p$  in  $(\vdash \odot)$  and  $(\odot \vdash)$ , we would obtain the inference rules below. Note that  $(\vdash p)$  is in fact an axiom.

$$\frac{}{S([p], p)} (\vdash p) \qquad \frac{S([p], X)}{S(p, X)} (p \vdash)$$

where, if  $\otimes$  is  $p$  or  $[p]$ , then  $S(\otimes, X) \triangleq \begin{cases} \otimes \vdash X & \text{if } \mathcal{A} = \exists \\ X \vdash \otimes & \text{if } \mathcal{A} = \forall \end{cases}$ . Hence, for all  $p = (k, \mathcal{A}, \pm)$ , if  $\mathcal{A} = \exists$  then  $(\vdash p)$  and  $(p \vdash)$  would rewrite as follows:

$$\frac{}{[p] \vdash p} (\vdash p) \qquad \frac{[p] \vdash X}{p \vdash X} (p \vdash) \qquad \text{Axiom}^\#$$

and if  $\mathcal{A} = \forall$  then  $(\vdash p)$  and  $(p \vdash)$  would rewrite as follows:

$$\frac{}{p \vdash [p]} (\vdash p) \qquad \frac{X \vdash [p]}{X \vdash p} (p \vdash). \qquad \text{Axiom}^b$$

Note that in both cases, the standard axiom  $p \vdash p$  is derivable by applying  $(p \vdash)$  once again to  $[p] \vdash p$  or  $p \vdash [p]$ . Our rules  $(\vdash \top)$ ,  $(\top \vdash)$ ,  $(\perp \vdash)$  and  $(\vdash \perp)$  are in fact instances of these rules and are the same as those of Kracht [30] and Belnap [6]. Like in the calculus **DLM** of Kracht [30],

we impose some conditions on these propositional letters by means of the structural inference rule ( $\mathbf{I}\vdash$ ) so that these special atoms  $\top$  and  $\perp$  do behave as truth constants, as intended.

As for Rule DR, it guarantees the display property of the calculus, but it is also meant to ensure from a semantic point of view that residuated connectives are associated to the same relation. When we apply it to Boolean connectives, we can recover classical display rules. However, note that the set BLN of Boolean connectives is not displayable enough, and therefore  $\text{BLN}_0 = \{\wedge, \vee, \rightarrow, \top, \perp\}$  is not displayable enough either. In order to make it displayable, it can be completed by adding the residuated atomic connectives  $\subset \triangleq ((s_2, -, \sigma_3), 0) = (((1, 1, 1), \forall, +, -, -, (2, 3, 1)), 0)$  or  $\prec \triangleq ((s_5, +, \sigma_1), 0) = (((1, 1, 1), \exists, +, -, +, (1, 2, 3)), 0)$ , but many other choices are possible. Whatever displayable completion we choose, the following corollary will hold. It is a consequence of the soundness and completeness of  $\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}^{\text{BLN}}$ .

**Corollary 1.** *Let  $\mathcal{C}^+$  be any set of atomic connectives including  $\{\wedge, \vee\}$  which is displayable enough. Then, the following classical inference rules below (from [44, p. 29]) are all admissible in  $\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}^{\text{BLN}}$ .*

$$\frac{\frac{(X, Y) \vdash Z}{X \vdash (Z, *Y)}}{Y \vdash (*X, Z)} \qquad \frac{X \vdash (Y, Z)}{(X, *Z) \vdash Y} \qquad \frac{(X, *Z) \vdash Y}{(*Y, X) \vdash Z} \qquad \text{DR}^{\wedge \vee}$$

where  $(X, Y) \triangleq \begin{cases} (X \wedge Y) & \text{if } (X, Y) \text{ is antecedent part in the consecution} \\ (X \vee Y) & \text{if } (X, Y) \text{ is consequent part in the consecution.} \end{cases}$

**Definition 28** (Calculus  $\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}$ ). Let  $(\mathcal{C}, \mathcal{C}^+)$  be a pair of common sets of atomic connectives such that  $\mathcal{C}$  is *without Boolean connectives*,  $\mathcal{C} \subseteq \mathcal{C}^+$  and  $\mathcal{C}^+$  is displayable enough. The calculus  $\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}$  is the proof system consisting of  $\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}^{\text{BLN}}$  of Fig. 3 without the structural rules, which are replaced by the rules  $\text{DR}^\neg$  below.

$$\frac{X \vdash Y}{*Y \vdash *X} \qquad \frac{X \vdash *Y}{Y \vdash *X} \qquad \frac{*X \vdash Y}{*Y \vdash X} \qquad \text{DR}^\neg$$

One can easily show that the rules  $\text{DR}^\neg$  are derivable from the rules  $\text{DR}^{\wedge \vee}$  and ( $\mathbf{I}\vdash$ ) (if we assume that  $\mathcal{C}^+$  contains the Boolean constants  $\top$  and  $\perp$ ).

**Theorem 3.** *Let  $(\mathcal{C}, \mathcal{C}^+)$  be a pair of common sets of atomic connectives such that  $\mathcal{C}$  is without Boolean connectives,  $\mathcal{C} \subseteq \mathcal{C}^+$  and  $\mathcal{C}^+$  is displayable enough. The calculus  $\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}$  is sound and complete for the basic atomic logic based on  $\mathcal{C}$ .*

The kind of negation obeying these inference rules, and so the (structural) negation of  $\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}$  of Definition 28, is a ‘minimal’ negation in the sense of [19, 20] (see also the discussion on this topic on page 39).

## 9 Hilbert calculi for basic atomic logics

In this section on Hilbert calculi, we define the notion of provability (deducibility) from a set of formulas, *i.e.*  $\Sigma \vdash_{\mathcal{P}} \varphi$  as follows. Let  $\mathbb{L} = (\mathcal{L}, \mathcal{C}, \|\vdash)$  be an atomic logic and let  $\Sigma \subseteq \mathcal{L}$  (possibly empty and not necessarily closed under uniform substitution) and  $\varphi \in \mathcal{L}$  be of dimension  $k$ . Then, we say that  $\varphi$  is *provable from*  $\Sigma$  in a proof system  $\mathcal{P}$  for  $\mathbb{L}$ , written  $\Sigma \vdash_{\mathcal{P}} \varphi$ , when there is a proof of  $\varphi$  in the proof system  $\mathcal{P} + \Sigma$ .

We assume in what follows that  $\mathcal{C}$  is a class of pointed atomic models  $(\mathfrak{M}, \bar{w})$  but it could also be a class of pointed atomic frames  $(\mathfrak{F}, \bar{w})$ . We write  $(\mathfrak{M}, \bar{w}) \Vdash \Sigma$  when for all  $\psi \in \Sigma$ , we have  $(\mathfrak{M}, \bar{w}) \Vdash \psi$ . We say that  $\varphi$  is a *logical consequence* of  $\Sigma$ , written  $\Sigma \Vdash_{\perp} \varphi$ , when for all  $(\mathfrak{M}, \bar{w}) \in \mathcal{C}$ , if  $(\mathfrak{M}, \bar{w}) \Vdash \Sigma$  then  $(\mathfrak{M}, \bar{w}) \Vdash \varphi$ ;  $\varphi$  is *valid*, written  $\Vdash_{\perp} \varphi$ , when for all  $(\mathfrak{M}, \bar{w}) \in \mathcal{C}$ , we have  $(\mathfrak{M}, \bar{w}) \Vdash \varphi$ ;  $\varphi$  is *satisfiable* when there is a model  $(\mathfrak{M}, \bar{w}) \in \mathcal{C}$  such that  $(\mathfrak{M}, \bar{w}) \Vdash \varphi$ . An inference rule of the form “from  $\varphi_1, \dots, \varphi_n$  infer  $\varphi$ ” is *valid* when  $\varphi_1, \dots, \varphi_n$  being all valid implies that  $\varphi$  is valid. We say that  $\varphi$  is *globally valid* (*globally unsatisfiable*) when for all  $(\mathfrak{M}, \bar{w}) \in \mathcal{C}$  and for all  $\bar{w}' \in \omega(\mathfrak{M}, \mathcal{C})$  of the same dimension as  $\varphi$  (resp. for no  $\bar{w}' \in \omega(\mathfrak{M}, \mathcal{C})$ ), we have that  $(\mathfrak{M}, \bar{w}') \Vdash \varphi$ .<sup>8</sup>

The notion of *strong completeness* of a (Hilbert) calculus  $\mathcal{P}$  for a logic  $\mathbf{L}$  is defined as usual by  $\Sigma \Vdash_{\perp} \varphi$  implies  $\Sigma \vdash_{\mathcal{P}} \varphi$  and, conversely, a calculus  $\mathcal{P}$  is *sound* for a logic  $\mathbf{L}$  when  $\vdash_{\mathcal{P}} \varphi$  implies  $\Vdash_{\perp} \varphi$ .

**Theorem 4.** *Let  $\mathcal{C}$  be a common Boolean set of atomic connectives. The calculus  $\mathcal{P}_{\mathcal{C}}$  of Fig. 5 is sound and strongly complete for the basic atomic logic  $(\mathcal{L}_{\mathcal{C}}, \mathcal{C}_{\mathcal{C}}, \Vdash)$ .*

Our calculus  $\mathcal{P}_{\mathcal{C}}$  is well-defined and, to be more precise, the axiom A5 does not depend on the particular residuation equation  $\odot' = \tau_0 - \tau_1 \dots - \tau_m \odot$  that we pick, as proven by Proposition 4 of [4].

**Example 7** (Modal logic). If we take  $\mathcal{C} = \text{ATM}_0 \cup \text{BLN}_1 \cup \{\Box, \Diamond\}$  to be the Boolean set of atomic connectives of modal logic with  $\Diamond, \Box$  atomic connectives of skeletons  $((2, 1), +, t_1)$  and  $((2, 1), -, t_2)$  respectively then we have that  $(2 \ 1) - (2 \ 1) - \Box = \Diamond$  ( $\Box$  and  $\Diamond$  are dual of each other, see [2, Proposition 30]) and the  $\mathcal{C}$ -models  $\mathfrak{M} = (W, \mathfrak{R}) \in \mathcal{E}_{\mathcal{C}}$  are such that  $R_{\Diamond} = R_{\Box}$  and  $R_{\top} = R_{\perp} = W$ . The Hilbert calculus  $\mathcal{P}_{\mathcal{C}}$  that we obtain and which is sound and complete for (this) modal logic is spelled out in Figure 6. Note that this proof system is novel. It can be simplified and is in fact equivalent to the classical proof system of modal logic [8, Definition 1.39] (see [4, Appendix F] for details).  $\square$

## 10 Inductive atomic formulas

Universal and existential molecular connectives are essentially molecular connectives such that the quantification patterns of the quantification signatures of their successive atomic connectives are of the form  $\forall \dots \forall$  or  $\exists \dots \exists$  respectively. So, they essentially behave as ‘macroscopic’ atomic connectives of quantification signatures  $\forall$  or  $\exists$ .

First, just as we have tonicity signatures for atomic connectives, we can also define an adaptation of this notion for molecular connectives in general, which, we repeat, are some sort of ‘macroscopic’ atomic connectives.

**Definition 29** (Tonicity signature of a molecular connective). Let  $\otimes$  be a molecular connective and let  $\otimes'$  be a molecular sub-connective of  $\otimes$ . We define the *tonicity of  $\otimes'$  w.r.t.  $\otimes$* , denoted  $tn(\otimes', \otimes)$  inductively as follows. If  $\otimes = \otimes'$  then  $tn(\otimes', \otimes) = +$ . Otherwise, if  $\otimes = \odot(\otimes_1, \dots, \otimes_n)$  with  $\odot = (\bar{k}, \mathbb{E}, \pm_1, \dots, \pm_n, \pm, \sigma)$  and  $\otimes'$  appears in  $\otimes_j$  then  $tn(\otimes', \otimes) = \pm_j tn(\otimes', \otimes_j)$ . The *tonicity signature* of a molecular connective is the tuple  $(\pm_1, \dots, \pm_l)$  of the tonicities  $tn(x_i^k, \otimes)$  of the connectives labeling the leafs of the decomposition tree of  $\otimes$  of the form  $x_i^k$  (possibly with repetition). We say that the tonicity signature  $(\pm_1, \dots, \pm_l)$  of a molecular connective  $\otimes$  is *coherent* if all tonicities  $\pm_i$ s corresponding to the same  $x_j^k$  in the decomposition tree of  $\otimes$  are equal. In that case, its tonicity signature is also denoted  $(\pm_1, \dots, \pm_m)$  by abuse, where  $\pm_j$  is the unique tonicity of the corresponding  $x_j^k$  appearing in  $\otimes$ .  $\square$

<sup>8</sup>The distinction between validity/unsatisfiability and *global* validity/unsatisfiability plays a role in some logics which distinguish two kinds of states, such as relevance logic [43] where validity is defined w.r.t. *normal* worlds.

<ul style="list-style-type: none"> <li>• <i>Axiom schemas:</i></li> </ul>	
Any sound and complete axiomatization of propositional logic	CPC
For all $\odot \in \mathbf{C}$ such that $\mathcal{A}(\odot) = \exists$ ,	
if $\pm_j(\odot) = +$ then	
$\odot(\varphi_1, \dots, \varphi_j \vee \varphi'_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vee \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n)$	A1
if $\pm_j(\odot) = -$ then	
$\odot(\varphi_1, \dots, \varphi_j \wedge \varphi'_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vee \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n)$	A2
For all $\odot \in \mathbf{C}$ such that $\mathcal{A}(\odot) = \forall$ ,	
if $\pm_j(\odot) = +$ then	
$\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \wedge \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j \wedge \varphi'_j, \dots, \varphi_n)$	A3
if $\pm_j(\odot) = -$ then	
$\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \wedge \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j \vee \varphi'_j, \dots, \varphi_n)$	A4
For all $\odot, \odot' \in \mathbf{C}$ such that $\odot' = \tau_0 - \tau_1 \dots - \tau_m \odot$ for some $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$ :	
denoting $\rho = (\tau_0 \circ \tau_1 \circ \dots \circ \tau_m)^-$ ,	
$S\left(\odot', \pm^{\rho(1)}\varphi_{\rho(1)}, \dots, \pm^{n+1}\odot(\varphi_1, \dots, \varphi_n), \dots, \pm^{\rho(n+1)}\varphi_{\rho(n+1)}\right)$	A5
where $S(\odot, \psi_1, \dots, \psi_n, \psi) \triangleq \begin{cases} \odot(\psi_1, \dots, \psi_n) \rightarrow \psi & \text{if } \mathcal{A}(\odot) = \exists \\ \psi \rightarrow \odot(\psi_1, \dots, \psi_n) & \text{if } \mathcal{A}(\odot) = \forall \end{cases}$	
$\pm\varphi = \begin{cases} \neg\varphi & \text{if } \pm = - \\ \varphi & \text{if } \pm = + \end{cases}$ and for all $j \in \llbracket 1; n+1 \rrbracket$ , $\pm^j = \begin{cases} \pm_1^j \pm_2^j \dots \pm_m^j & \text{if } m \neq 0 \\ + & \text{if } m = 0 \end{cases}$	
where for all $i \in \llbracket 1; m \rrbracket$ , $\pm_i^j = \begin{cases} - & \text{if } \tau_i \circ \tau_{i+1} \circ \dots \circ \tau_m(j) = n+1 \\ + & \text{otherwise} \end{cases}$	
<ul style="list-style-type: none"> <li>• <i>Inference rules:</i></li> </ul>	
From $\varphi$ and $(\varphi \rightarrow \psi)$ , infer $\psi$	MP
For all $\odot \in \mathbf{C}$ such that $\mathcal{A}(\odot) = \exists$ ,	
if $\pm_j(\odot) = +$ then from $\neg\varphi_j$ , infer $\neg\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$	R1
if $\pm_j(\odot) = -$ then from $\varphi_j$ , infer $\neg\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$	R2
For all $\odot \in \mathbf{C}$ such that $\mathcal{A}(\odot) = \forall$ ,	
if $\pm_j(\odot) = +$ then from $\varphi_j$ , infer $\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$	R3
if $\pm_j(\odot) = -$ then from $\neg\varphi_j$ , infer $\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$	R4
For all $\odot \in \mathbf{C}$ ,	
if $\pm_j(\odot) = +$ then	
From $\varphi_j \rightarrow \varphi'_j$ , infer $\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n)$	R5
if $\pm_j(\odot) = -$ then	
From $\varphi_j \rightarrow \varphi'_j$ , infer $\odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$	R6
The formulas $\varphi, \psi, \varphi_1, \dots, \varphi_j, \varphi'_j, \dots, \varphi_n$ range over $\mathcal{L}_{\mathbf{C}}$ .	

Figure 5: Hilbert calculus  $\mathcal{P}_{\mathbf{C}}$

Any sound and complete axiomatization of propositional logic	(CPC)
$\Diamond(\varphi \vee \psi) \rightarrow \Diamond\varphi \vee \Diamond\psi$	(A1)
$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$	(A3)
$\Diamond\neg\varphi \rightarrow \neg\Box\varphi$	(A5)
$\neg\Diamond\varphi \rightarrow \Box\neg\varphi$	(A5)
From $\neg\varphi$ , infer $\neg\Diamond\varphi$	(R1)
From $\varphi$ , infer $\Box\varphi$	(R3)
From $\varphi \rightarrow \psi$ , infer $\Box\varphi \rightarrow \Box\psi$	(R5)
From $\varphi \rightarrow \psi$ , infer $\Diamond\varphi \rightarrow \Diamond\psi$	(R5)
From $\varphi$ and $\varphi \rightarrow \psi$ , infer $\psi$	(MP)
The formulas $\varphi, \psi$ range over $\mathcal{L}_C$ , $C$ are the connectives of Example 7.	

Figure 6: Hilbert calculus  $\mathcal{P}_C$  instantiated with the atomic connectives  $C$  of modal logic.

**Example 8.** In this section, the connective  $\supset$  is the implication of relevance logic (see Table 2) and  $\rightarrow$  is the (Boolean) material implication (see Definition 9). If  $\otimes = x_1^1 \supset \Box x_2^1$  then  $tn(\otimes, x_1^1) = -$  and  $tn(\otimes, x_2^1) = tn(\otimes, \Box x_2^1) = +$ . Therefore, if  $\otimes' = x_1^1 \supset \Box x_1^1$  then the tonicity signature of  $\otimes'$  is not coherent.

Note the very close similarity between this definition and the definition of  $tn(Z, X)$  in Proposition 3.

**Definition 30** (Basic universal and existential molecular connective). Let  $C$  be a set of atomic connectives. A molecular connective  $\otimes$  based on  $C$  is *basic universal* (resp. *existential*) when the tonicity signature of  $\otimes$  is coherent and the following hold:

- $\mathcal{A}(\otimes) = \forall$  (resp.  $\mathcal{A}(\otimes) = \exists$ );
- for each node of its decomposition tree labeled with  $\odot = (\bar{k}, \mathcal{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$  and each of its  $j$ th children labeled with some  $\odot_j \in \text{MOL}$  such that the subtree generated by this  $j$ th children contains at least one  $x_j^k$ , we have that  $\mathcal{A}(\odot_j) = \pm_j \mathcal{A}$ .

**Example 9.** On the one hand, the molecular connective  $\odot(p, x_1^k)$  is a basic universal (resp. existential) molecular connective if  $\mathcal{A}(\odot) = \forall$  (resp.  $\mathcal{A}(\odot) = \exists$ ). Likewise,  $(x_1^1 \supset \Box x_2^1)$  and  $\otimes(\Diamond x_1^1, p)$  are universal and existential molecular connectives respectively. On the other hand, the molecular connectives  $\Box\Diamond^- x_1^1$  and  $(\Box x_1^1 \supset \Box x_2^1)$  are neither universal nor existential molecular connectives.

**Proposition 4** ([5]). *Rules R3-R4 of Fig. 5 are globally valid for all coherent molecular connectives. Moreover, axioms A1-A4 and rules R2-R3 are also globally valid for basic existential and universal molecular connectives.*

**Definition 31** (Universal and existential molecular connective). Let  $L \triangleq (\mathcal{L}_C, \mathcal{E}_C, \Vdash)$  be a Boolean atomic logic based on a set of atomic connectives  $C$ . A molecular connective  $\otimes$  based on  $C$  is *universal* (resp. *existential*) in  $L$  when the tonicity signature of  $\otimes$  is coherent, the rules R5-R6 of Figure 5 are globally valid in  $L$  for  $\otimes$  and the axioms A3-A4 (resp. A1-A2) and the rules R3-R4 (resp. R1-R2) are globally valid in  $L$  for  $\otimes$ .

Note that every basic universal or existential molecular connective is universal or existential respectively because of Proposition 4. The following proposition shows roughly the converse,



more precisely that a universal or existential molecular connective can always be turned into an equivalent basic one. Hence, our more general and semantic-based Definition 31 of universal and existential molecular connectives is somehow equivalent to its ‘basic’ syntactic-based Definition 30.

**Proposition 5** ([5]). *For every molecular logic, there is an equally expressive molecular logic where all universal and existential molecular connectives are equivalent to atomic connectives of universal and existential quantification signature respectively.*<sup>9</sup>

**Definition 32** (Positive and negative formula). Let  $\mathbf{L} \triangleq (\mathcal{L}_{\mathbf{C}}, \mathcal{E}_{\mathbf{C}}, \Vdash)$  be an atomic logic based on a set of atomic connectives  $\mathbf{C}$  and let  $\varphi \in \mathcal{L}_{\mathbf{C}}$ . The formula  $\varphi$  can be written as a formula of the form  $\otimes(p_1, \dots, p_n)$  where  $p_1, \dots, p_n$  are all the propositional letters that occur in  $\varphi$  (possibly with repetitions) and  $\otimes$  is a molecular connective (not necessarily belonging to  $\mathbf{C}$ ). We say that  $\varphi$  is *positive* (resp. *negative*) if for all  $j \in \llbracket 1; n \rrbracket$ ,  $tn(\otimes, j)(p_j) = +$  (resp.  $tn(\otimes, j)(p_j) = -$ ).  $\square$

**Example 10.** The formulas  $p, \Box p$  and  $p \otimes p$  are positive and  $\neg p, \Box \neg p$  and  $\Diamond \neg p$  are negative. The formulas  $p \supset p$  and  $\Box p \wedge \Box \neg p$  are neither positive nor negative.  $\square$

**Definition 33** (Essentially universal and existential formulas). Let  $\mathbf{L} \triangleq (\mathcal{L}_{\mathbf{C}}, \mathcal{E}_{\mathbf{C}}, \Vdash)$  be an atomic logic based on a set of atomic connectives  $\mathbf{C}$ . An *essentially universal formula* (resp. *essentially existential formula*) of  $\mathbf{L}$  is either a negative (resp. positive) formula or a formula of  $\mathcal{L}_{\mathbf{C}}$  equivalent to a formula of the form  $\otimes(\varphi_1, \dots, \varphi_{i-1}, p, \varphi_{i+1}, \dots, \varphi_n)$  where  $\otimes$  is a universal molecular connective (resp. existential molecular connective) such that for all  $j \in \llbracket 1; n \rrbracket \setminus \{i\}$ ,

- if  $tn(\otimes, j) = -$  (resp.  $tn(\otimes, j) = +$ ) then  $\varphi_j$  is a positive formula or a globally valid formula of  $\mathbf{L}$
- if  $tn(\otimes, j) = +$  (resp.  $tn(\otimes, j) = -$ ) then  $\varphi_j$  is a negative formula or a globally unsatisfiable formula of  $\mathbf{L}$
- $p \in \mathbf{C}_0$  is a propositional letter in an arbitrary position  $i \in \llbracket 1; n \rrbracket$  but such that  $tn(\otimes, i) = +$  (resp.  $tn(\otimes, i) = -$ ).

In that case,  $p$  is called the *head* of the essentially universal (resp. existential) formula and the formula is said to be *headed*.  $\square$

**Example 11.** Propositional letters  $p$  are essentially universal formulas because they are equivalent to formulas of the form  $x_i^k p$  (since  $x_i^k$  can be seen as a universal molecular connective). Moreover, on the one hand,  $((\Box p \supset p) \supset q)$  is a formula of the modal logic KT which is essentially universal because  $(\Box p \supset p)$  is valid in KT, its head is  $q$ . On the other hand,  $((p \supset \Box p) \supset q)$  is neither an essentially universal formula nor an essentially existential formula of the modal logic KT because  $p \supset \Box p$  is neither valid nor positive.  $\square$

**Definition 34** (Regular formula). Let  $\mathbf{L} \triangleq (\mathcal{L}_{\mathbf{C}}, \mathcal{E}_{\mathbf{C}}, \Vdash)$  be an atomic logic based on a set of atomic connectives  $\mathbf{C}$ . A *regular formula*  $\varphi$  of  $\mathbf{L}$  is a formula  $\varphi \in \mathcal{L}_{\mathbf{C}}$  of the form  $\varphi = \otimes(\varphi_1, \dots, \varphi_n)$  where  $\otimes$  is a universal molecular connective of  $\mathcal{L}_{\mathbf{C}}$  and such that for all  $j \in \llbracket 1; n \rrbracket$ ,  $\varphi_j$  is an essentially universal formula if  $tn(\otimes, j) = -$  or an essentially existential formula if  $tn(\otimes, j) = +$ . The headed formulas  $\varphi_1, \dots, \varphi_n$  are called the *main components* of  $\varphi$  and the *heads* of  $\varphi$  are the heads of  $\varphi_1, \dots, \varphi_n$  (if they exist).  $\square$

<sup>9</sup>We use [3, Definition 1] for the definition of *equally expressive* logics.

**Example 12.** We recall that the tonicity signature of  $\supset$  is  $(-, +)$ . The formula  $((\Box p \supset p) \supset q) \supset \Box q$  is a regular formula of the modal logic **KT** because the subformula  $((\Box p \supset p) \supset q)$  is essentially universal (see Example 11) and  $\Box q$  is essentially existential (it is positive). The formula  $((p \supset \Box p) \supset q) \supset \Box q$  is not a regular formula of the modal logic **KT** because the subformula  $((p \supset \Box p) \supset q)$  is not an essentially universal formula (see Example 11).  $\square$

There might indeed be no head of  $\varphi$  if the  $\varphi_i$ s are positive or negative formulas.

**Definition 35** (Essential and inessential atom). An occurrence of a propositional letter in a regular formula  $\varphi$  is *essential in  $\varphi$*  if it is the head of a main component of the formula, otherwise it is *inessential in  $\varphi$* . A propositional letter in a regular formula  $\varphi$  is *essential in  $\varphi$*  if it has at least one essential occurrence in it, otherwise it is *inessential in  $\varphi$* .  $\square$

**Example 13.** The propositional letters  $p$  and  $r$  are essential in the modal polyadic formula  $[\alpha](\neg p, \neg[\alpha](r, \neg(\beta)q))$  while  $q$  is inessential there. The propositional letters  $p$  and  $q$  are essential in the modal polyadic formula  $[\iota_3](\neg p, \neg[\alpha](\iota_2(\alpha, \alpha)))(\neg p, q), \langle \alpha \rangle[\alpha][\alpha]q$ .  $\square$

**Definition 36** (Dependency digraph). Given a regular formula  $\varphi = \otimes(\varphi_1, \dots, \varphi_n)$  with main components  $\{\varphi_1, \dots, \varphi_k\}$ , the *dependency digraph* of  $\varphi$  is a digraph  $G = (V_\varphi, E_\varphi)$  where  $V_\varphi = \{p_1, \dots, p_n\}$  is the set of heads of  $\varphi$  and we set  $p_i E_\varphi p_j$  iff  $p_i$  occurs as an inessential propositional letter in a formula from  $\{\varphi_1, \dots, \varphi_k\}$  with a head  $p_j$ . A digraph is *acyclic* if it does not contain oriented cycles.  $\square$

**Definition 37** (Inductive atomic formula). Let  $L \triangleq (\mathcal{L}_C, \mathcal{E}_C, \Vdash)$  be an atomic logic based on a set of atomic connectives  $C$ . An *inductive atomic formula of  $L$*  is a regular formula  $\varphi$  of  $L$  with an acyclic dependency digraph. If  $L$  is a basic atomic logic then  $\varphi$  is simply called an inductive atomic formula.  $\square$

**Example 14.** Like in [24], the formula of modal logic  $p \wedge \Box(\Diamond p \supset \Box q) \supset \Diamond \Box \Box q$  which, written in the purely modal polyadic language, is  $[\iota_3](\neg p, \neg[\alpha](\iota_2(\alpha, \alpha)))(\neg p, q), \langle \alpha \rangle[\alpha][\alpha]q$  is an inductive atomic formula. The formula  $((\Box p \supset p) \supset q) \supset \Box q$  is an inductive atomic formula of the modal logic **KT** but not an inductive (atomic) formula, also in the sense of [24], because  $\Box p \supset p$  is neither a positive nor a valid formula. The Löb formula  $(\Box(\Box p \supset p) \supset \Box p)$  and the McKinsey formula  $(\Box \Diamond p \supset \Diamond \Box p)$  as well as the axiom **K** are not Sahlqvist formulas, in fact not even inductive atomic formulas. The Dirk Gently's Principle  $(p \supset q) \vee (q \supset p)$  (**DGP**) of the intermediate logic **LC** and the formula  $(p \wedge q) \supset p$  are not inductive atomic formulas. Finally, the formula  $\zeta \triangleq ((p \supset p) \supset q) \supset q$  from the relevance logic **E** has a local first-order correspondent but is not an inductive (atomic) formula. This was already observed by Conradie & Goranko [13].  $\square$

## 11 Acyclic formulas are inductive

**Definition 38** (Amenable pair of atomic connectives). A pair  $(C, C^+)$  of sets of atomic connectives is *amenable* when  $C$  and  $C^+$  are Boolean and common,  $C \subseteq C^+$  and for all  $\odot \in C^+$  of arity  $n > 0$  and permutation signature  $\sigma$ , there is a residuated connective  $\odot' \in C$  of permutation signature  $\sigma'$  such that  $\sigma' \circ \sigma^{-1}(n+1) = n+1$ .

If  $C^+$  is a set of connectives which includes all the atomic connectives of all the orbits of  $C$ , then  $(C, C^+)$  is amenable. So, every set of atomic connectives can be completed by another set of atomic connectives  $C^+$  so that  $(C, C^+)$  is amenable. In the above definition, we give in fact requirements on the pair of logical and structural connectives  $(C, C^+)$  so that the following proposition holds. These requirements are not minimal, they could be weakened (by removing

the assumption that  $C$  is Boolean and, in particular, complete for negation) in order to match exactly with Definition 14 of Ciabattoni & Ramanayake, but the definition would then be much more complex.

**Proposition 6.** *Let  $(C, C^+)$  be an amenable pair of sets of atomic connectives. Then  $\mathcal{P}_{C, C^+}^{BLN}$  is amenable (in the sense of Ciabattoni & Ramanayake, Definition 14).*

**Proposition 7.** *Let  $(C, C^+)$  be an amenable pair of sets of atomic connectives. A formula  $\varphi \in \mathcal{L}_C$  is s-soluble (resp. a-soluble) for  $\mathcal{P}_{C, C^+}^{BLN}$  iff it can be equivalently rewritten as  $\otimes(p_1, \dots, p_n)$  where  $\otimes$  is a basic universal molecular connective (resp. basic existential molecular connective) and  $p_1, \dots, p_n$  are propositional letters.*

**Proposition 8.** *Let  $C$  be a common Boolean set of atomic connectives and let  $\varphi \in \mathcal{L}_C$ . Then, for all sets of atomic connectives  $C^+$  such that  $(C, C^+)$  is amenable, the following holds:*

- $\varphi \in \mathcal{I}_1(\mathcal{P}_{C, C^+}^{BLN})$  iff  $\varphi$  can be equivalently rewritten as  $\otimes(p_1, \dots, p_n)$  where  $\otimes$  is a universal molecular connective and  $p_1, \dots, p_n \in C_0$ ;
- $\varphi \in \mathcal{I}_2(\mathcal{P}_{C, C^+}^{BLN})$  iff  $\varphi$  can be equivalently rewritten as  $\otimes(\varphi_1, \dots, \varphi_n)$  where  $\otimes$  is a basic universal molecular connective and each  $\varphi_j$  is
  - either an atom
  - or of the form  $\otimes_j(p_1, \dots, p_m)$  with  $\otimes_j$  basic universal molecular connective (that is, it is s-soluble) if  $tn(\otimes, j) = -$
  - or of the form  $\otimes'_j(p'_1, \dots, p'_m)$  with  $\otimes'_j$  basic existential molecular connective (that is, it is a-soluble) if  $tn(\otimes, j) = +$ ,

for some  $p_1, \dots, p_m, p'_1, \dots, p'_m \in C_0$ .

Hence, every  $\varphi \in \mathcal{I}_2(\mathcal{P}_{C, C^+}^{BLN})$  is a regular formula, but the converse does not necessarily hold. Since the definitions of  $\mathcal{I}_1(\mathcal{P}_{C, C^+}^{BLN})$  and  $\mathcal{I}_2(\mathcal{P}_{C, C^+}^{BLN})$  actually do not depend on  $C^+$ , they will also be denoted  $\mathcal{I}_1(\mathcal{P}_C)$  and  $\mathcal{I}_2(\mathcal{P}_C)$ .

In other words,  $\varphi \in \mathcal{I}_2(\mathcal{P}_C)$  if  $\varphi$  begins with a universal quantification and is followed by an existential quantification: the alternation depth of the quantification signatures does not exceed 1.

**Theorem 5.** *Let  $C$  be a common Boolean set of atomic connectives. For all  $\varphi \in \mathcal{I}_2(\mathcal{P}_C)$ ,  $\varphi$  is an inductive atomic formula iff  $\varphi$  is acyclic.*

**Corollary 2.** *Let  $C$  be a common Boolean set of atomic connectives. Let  $\Sigma$  be a set of acyclic formulas of  $\mathcal{I}_2(\mathcal{P}_C)$ . The calculus  $\mathcal{P}_C + \Sigma$  is sound and strongly complete for the Boolean atomic logic  $(\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ , where  $\mathcal{E}_C$  is the class of pointed  $C$ -frames defined by the first-order correspondents of the inductive atomic formulas of  $\Sigma$ .*

*Proof.* It follows from [5, Theorem 6] and the fact that acyclic formulas are inductive atomic formulas according to Theorem 5.  $\square$

## 12 Main results

Roughly, the following theorem states that a calculus is properly displayable iff it is sound and complete w.r.t. a class of frames defined by acyclic/inductive formulas. The assumption that  $C^+$

be displayable enough in this theorem is not a real precondition since, as we said after Definition 26, every set of atomic connectives  $C$  can be extended into a displayable enough set of atomic connectives  $C^+$ .

**Theorem 6.** *Let  $(C, C^+)$  be a pair of common sets of atomic connectives such that  $C \subseteq C^+$  and  $C^+$  is displayable enough. Let  $\rho_0$  be a set of structural inference rules for  $[\mathcal{L}_{C^+}]$ . The calculus  $\mathcal{P}_{C, C^+} + \rho_0$  is equivalent to a proper display calculus of the form  $\mathcal{P}_{C, C^+} + \rho$  iff it is sound and complete for an atomic logic based on  $C$  whose class of pointed  $C$ -frames (with or without information order) is definable by the local first-order correspondents of a set  $\Sigma$  of acyclic formulas of  $\mathcal{I}_2(\mathcal{P}_{C^{BLN}})$  where  $C^{BLN}$  is the Boolean completion of  $C$ . Moreover the acyclic formulas of  $\Sigma$  are effectively computable from the analytic structural rules of  $\rho$ , and vice versa, as specified in Fig. 7.*

The following corollary is a direct consequence of the previous theorem.

**Corollary 3.** *Let  $(C, C^+)$  be a pair of common sets of atomic connectives such that  $C \subseteq C^+$  and  $C^+$  is displayable enough. Every proper display calculus extending  $\mathcal{P}_{C, C^+}$  is sound and complete for a logic  $(\mathcal{L}_C, \mathcal{E}_C, \Vdash)$  whose class of pointed  $C$ -frames  $\mathcal{E}_C$  is definable by a set of acyclic formulas of  $\mathcal{I}_2(\mathcal{P}_{C^{BLN}})$  where  $C^{BLN}$  is the Boolean completion of  $C$ .*

Hence, the modal logic  $KL \triangleq K + \{\Box(\Box p \rightarrow p) \rightarrow \Box p\}$  is not displayable by an extension of  $\mathcal{P}_{C, C^+}$  (for some amenable pair  $(C, C^+)$  of common sets of atomic connectives) because it is not sound and complete w.r.t. any class of (Kripke) frames. Likewise, the modal logic  $KM \triangleq K + \{\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi\}$  is not displayable by an extension of  $\mathcal{P}_{C, C^+}$  (for some amenable pair  $(C, C^+)$  of common sets of atomic connectives) because the McKinsey formula does not correspond to a first-order condition on (Kripke) frames. Below is a consequence of the admissibility of the Cut for displayable logics.

**Theorem 7.** *Let  $C_1, C_2$  be two common sets of atomic connectives such that  $C_1 \subseteq C_2$ . If the class of pointed  $C_2$ -frames  $\mathcal{E}_{C_2}$  of the logic  $(\mathcal{L}_{C_2}, \mathcal{E}_{C_2}, \Vdash)$  is definable by a set of acyclic formulas of  $\mathcal{I}_2(\mathcal{P}_{C_1^{BLN}})$  (where  $C_1^{BLN}$  is the Boolean completion of  $C_1$ ) then  $(\mathcal{L}_{C_2}, \mathcal{E}_{C_2}, \Vdash)$  is a conservative extension of  $(\mathcal{L}_{C_1}, \mathcal{E}_{C_2}, \Vdash)$ .*

In particular, Theorem 7 holds if  $C_2$  is the Boolean completion of  $C_1$ . It also entails that every atomic logic axiomatized by inductive atomic formulas is conservatively extended by atomic connectives that do not satisfy any specific constraint or that belong to an orbit of one of the connectives of the original atomic logic. As immediate corollary of Theorem 7 we get the following theorems. The only difference with respect to Corollary 2 and [5, Theorems 6&7] is that the atomic logics are not necessarily Boolean.

**Theorem 8.** *Let  $C$  be a common set of atomic connectives. Let  $\Sigma$  be a set of acyclic formulas of  $\mathcal{I}_2(\mathcal{P}_C)$ . The calculus  $\mathcal{P}_C + \Sigma$  is sound and strongly complete for the atomic logic  $(\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ , where  $\mathcal{E}_C$  is the class of pointed  $C$ -frames defined by the first-order correspondents of the inductive atomic formulas of  $\Sigma$ .*

**Theorem 9.** *Let  $L \triangleq (\mathcal{L}_C, \mathcal{E}_C, \Vdash)$  be an atomic logic based on a common set  $C$  of atomic connectives and a set  $\mathcal{E}_C$  of pointed  $C$ -frames with or without  $D$ -information order. Let  $\Sigma \subseteq \mathcal{L}_C$  be a set of inductive atomic formulas of  $L$  such that  $\mathcal{E}_C$  is the class of pointed  $C$ -frames defined by the first-order correspondents of the inductive atomic formulas of  $\Sigma$ . Then, the calculus  $\mathcal{P}_C + \Sigma +$  (Uniform  $D$ -substitution) is sound and strongly complete for  $L$ , where (Uniform  $D$ -substitution)*

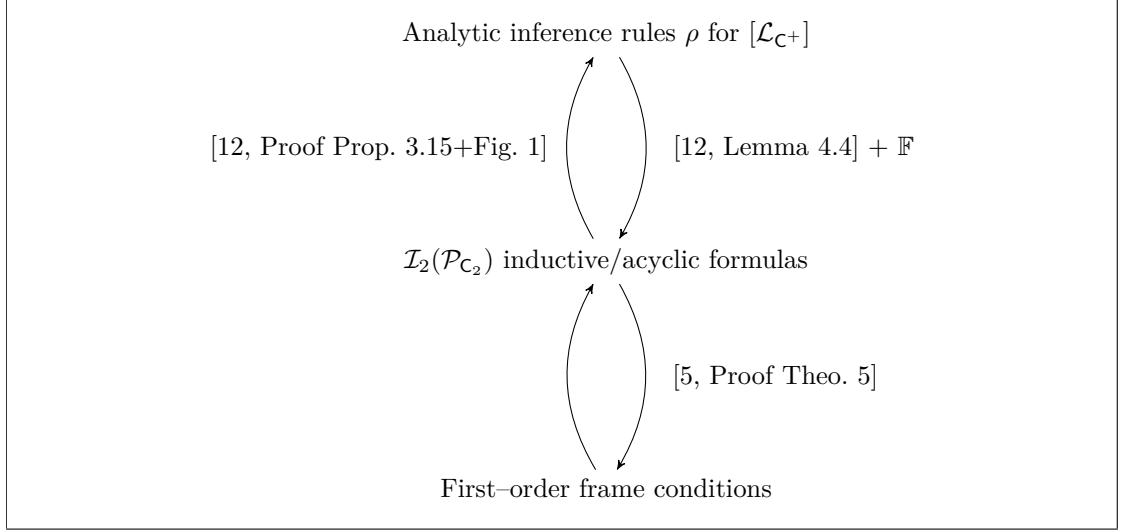


Figure 7: Translations from analytic structural inference rules to first-order frame conditions, and vice versa

is the following rule of uniform substitution:

For all  $\chi \in \Sigma$  and all prop. letters  $p_1, \dots, p_k \in D$  and  $\alpha_1, \dots, \alpha_k \in \mathcal{L}_D$   
and all prop. letters  $q_1, \dots, q_l \in C - D$  and  $\beta_1, \dots, \beta_l \in \mathcal{L}_C$ :  
From  $\chi$ , infer  $\chi[p_1/\alpha_1, \dots, p_k/\alpha_k, q_1/\beta_1, \dots, q_l/\beta_l]$  (Uniform D-substitution)

where  $\chi[p_1/\alpha_1, \dots, p_k/\alpha_k, q_1/\beta_1, \dots, q_l/\beta_l]$  is the formula resulting from the uniform substitution in  $\chi$  of  $p_1$  by  $\alpha_1$  and ... and  $p_k$  by  $\alpha_k$  and  $q_1$  by  $\beta_1$  and ... and  $q_l$  by  $\beta_l$ .

*Remark 3.* In the formulation of the Rule (Uniform D-substitution), we could replace in  $\chi$  the propositional letters  $q_1, \dots, q_l$  by formulas schemas  $\psi_1, \dots, \psi_l$  and reformulate this rule as a uniform substitution rule over an axiom schema instead of a single formula  $\chi$ . This would be particularly appropriate if  $C_0 \cap (C - D) = \emptyset$ .

### 13 Examples of correspondences

In Fig. 9, we provide a number of first-order frame correspondents of analytic inference rules. All the connectives are of dimension  $(1, 1)$  or  $(1, 1, 1)$  and by abuse we confuse the atomic connectives with their skeletons. In these figures, we use the following abbreviations and notations: for all  $i, j \in \{1, 2, 3\}$ ,

- $[\otimes_i]$  and  $[\otimes_j]$  range over  $\{[\otimes_1], [\otimes_2], [\otimes_3]\}$  and  $[\oplus_i]$  and  $[\oplus_j]$  range over  $\{[\oplus_1], [\oplus_2], [\oplus_3]\}$  (see Fig. 8);
- $[\prec_i]$  ranges over  $\{[\prec_1], [\prec_2], [\prec_3]\}$  and  $[\succ_i]$  ranges over  $\{[\succ_1], [\succ_2], [\succ_3]\}$  (see Fig. 8);
- $[C_i]$  ranges over  $\{[C_1], [C_2], [C_3]\}$  and  $[D_i]$  ranges over  $\{[D_1], [D_2], [D_3]\}$  (see Fig. 8);
- $[\sim^i]$  and  $[\sim^j]$  range over  $\{[\sim^1], [\sim^2], [\sim^3], [\sim^4]\}$  (see Fig. 8).

Connective	Skeleton	Connective	Skeleton	Connective	Skeleton
$\mathbf{t}$	$(1, \exists, +)$	$\otimes_3$	$(s_1, +, \sigma_1)$	$\supset_3$	$(s_3, -, \sigma_1)$
$\mathbf{t}'$	$(1, \forall, +)$	$\otimes_2$	$(s_1, +, \sigma_3)$	$\supset_2$	$(s_3, -, \sigma_3)$
$\mathbf{f}$	$(1, \forall, -)$	$\otimes_1$	$(s_1, +, \sigma_5)$	$\supset_1$	$(s_3, -, \sigma_5)$
$\mathbf{f}'$	$(1, \exists, -)$	$\oplus_3$	$(s_4, -, \sigma_1)$	$\subset_3$	$(s_2, -, \sigma_1)$
$\sim^1$	$(t_4, +, \tau_1)$	$\oplus_2$	$(s_4, -, \sigma_3)$	$\subset_2$	$(s_2, -, \sigma_3)$
$\sim^2$	$(t_4, +, \tau_2)$	$\oplus_1$	$(s_4, -, \sigma_5)$	$\subset_1$	$(s_2, -, \sigma_5)$
$\sim^3$	$(t_3, -, \tau_1)$	$\succ_3$	$(s_6, +, \sigma_1)$		
$\sim^4$	$(t_3, -, \tau_2)$	$\succ_2$	$(s_6, +, \sigma_3)$		
$\diamond^-$	$(t_1, +, \tau_1)$	$\succ_1$	$(s_6, +, \sigma_5)$		
$\diamond$	$(t_1, +, \tau_2)$	$\prec_3$	$(s_5, +, \sigma_1)$		
$\square^-$	$(t_2, -, \tau_1)$	$\prec_2$	$(s_5, +, \sigma_3)$		
$\square$	$(t_2, -, \tau_2)$	$\prec_1$	$(s_5, +, \sigma_5)$		

Figure 8: Some modal and substructural connectives of atomic logics

Structural connectives are generally denoted  $[\odot_i] = [(t_i, \pm, \sigma_i)]$ . In these first-order correspondents, we use the notation  $R \triangleq R_{\odot_i}^{\pm\sigma_i}$  if  $R_{\odot_i}^{\pm\sigma_i}$  is a ternary relation and  $S \triangleq R_{\odot_i}^{\pm\sigma_i}$  if  $R_{\odot_i}^{\pm\sigma_i}$  is a binary relation, as well as the following notations:

$$R(xy)zw \triangleq \exists u(Rxyu \wedge Ruzw) \qquad R_x(yz)w \triangleq \exists u(Ryzu \wedge Rxuw)$$

The abbreviations  $R(xy)zw$  and  $R_x(yz)w$  correspond to the abbreviations of [40]. In these first-order correspondents, the nullary connectives (atoms)  $\mathbf{t}$  and  $\mathbf{f}$  are associated in the frames to the unary predicates denoted  $T$  and  $F$  respectively:  $R_{\mathbf{t}} = T$  and  $R_{\mathbf{f}} = F$ .

The structural connective “;” of the literature [40] corresponds in our setting to the structural connective  $[\otimes_3]$ . Likewise, the usual structural connective  $\bullet$  corresponds in our setting to the structural connective  $[\diamond^-]$  (resp.  $[\square]$ ) when  $\bullet$  is in a-part (resp. s-part).

*Remark 4.* We often present the rules in such a way that the transformations are performed on the antecedent side. We could obtain equivalent rules by taking the contraposition of each rule, as we did with rules  $\mathbf{N}_6^l$  and  $\mathbf{N}_7^l$ : the former performs the changes on the antecedent side of the consecutions and the latter on the consequent side, yet they correspond at the semantic level to the same first-order condition. In fact, we could take any display equivalent of a given rule, it would yield the same semantic condition at the level of frames. Display rules are just present in a display calculus in order to associate residuated connectives to the same relation, they do not impose any specific conditions on these relations.

### 13.1 From analytic inference rules to first-order frame conditions

In this section, we execute the algorithms of Fig. 7 (top to bottom) on some classical structural rules to compute the first-order frame conditions to which they correspond. The interpretation functions  $l$  and  $r$  are defined in a canonical way.

$$\text{Rule } \mathbf{K}_1^i : \frac{X \vdash Z}{X [\otimes_i] Y \vdash Z}$$

1. Algorithm [12, Lemma 4.4] +  $\mathbb{F}$ :

Inference rule	Local first-order correspondent $\chi(x)$	
$\frac{((X [\otimes_i] Y) [\otimes_i] Z) \vdash U}{(X [\otimes_i] (Y [\otimes_i] Z)) \vdash U}$	$\forall yzw (Rx(yz)w \rightarrow R(xy)zw)$	$B^{e,i}$
$\frac{(Y [\otimes_i] X) \vdash U}{(X [\otimes_i] Y) \vdash U}$	$\forall yz (Ryzx \rightarrow Rzyx)$	$CI^i$
$\frac{(X [\otimes_i] X) \vdash U}{X \vdash U}$	$Rxxx$	$WI^i$
$\frac{X \vdash U}{(X [\otimes_i] Y) \vdash U}$	$\forall yz (Ryzx \rightarrow y \sqsubseteq x)$	$K_1^i$
$\frac{X \vdash U}{(Y [\otimes_i] X) \vdash U}$	$\forall yz (Ryzx \rightarrow z \sqsubseteq x)$	$K_2^i$
$\frac{[\sim^2] X \vdash U}{[\sim^1] X \vdash U}$	$\forall y (Sxy \rightarrow Syx)$	$N_1^j$
$\frac{X \vdash U}{[\sim^2] [\sim^4] X \vdash U}$	$\forall yz (Sxy \wedge Syz \rightarrow x \sqsubseteq z)$	$N_2^j$
$\frac{[\sim^2] [\sim^4] X \vdash U}{X \vdash U}$	$\exists yz (Sxy \wedge Syz \wedge x \sqsubseteq z)$	$N_3^j$
$\frac{[\sim^j] X \vdash U}{*X \vdash U}$	$Sxx$	$N_4^j$
$\frac{*X \vdash U}{[\sim^j] X \vdash U}$	$\forall y (Syx \rightarrow x = y)$	$N_5^j$
$\frac{U \vdash (X [\supset_i] Y)}{U \vdash ([\sim^j] X [\oplus_i] Y)}$	$\forall yzw (Ryzx \wedge Swz \rightarrow Rywx)$	$N_6^j$
$\frac{(X [\prec_i] Y) \vdash U}{(X [\otimes_i] [\sim^j] Y) \vdash U}$	$\forall yzw (Ryzx \wedge Swz \rightarrow Rywx)$	$N_7^j$
$\frac{[\sim^j] X \vdash U}{[\sim^i] X \vdash U}$	$\forall y (R_{\sim^j}^{\pm\tau_j} yx \rightarrow R_{\sim^i}^{\pm\tau_i} yx)$	$BI^{i,j}$
$\frac{[\sim^i] [\mathbf{f}] \vdash U}{[\mathbf{t}] \vdash U}$	$Tx \rightarrow \exists y (Syx \wedge \neg Fy)$	$Q^i$
$\frac{([\mathbf{t}] [\otimes_i] X) \vdash U}{X \vdash U}$	$\exists y (Ryxx \wedge Ty)$	$t_2^i$
$\frac{X \vdash U}{([\mathbf{t}] [\otimes_i] X) \vdash U}$	$\forall yz (Tx \wedge Rxyz \rightarrow y \sqsubseteq z)$	$t_1^i$

Figure 9: First-order correspondents of ‘classical’ inference rules

$$(l(X [\otimes_i] Y) \vdash r(Z))[Z/l(X)][X/p][Y/q]$$

$$p \otimes_i q \vdash p$$

$$p \otimes_i q \rightarrow p$$

2. Algorithm [5, Proof Theo. 5]:

$$\forall PQ [\exists yz (Ryzx \wedge Py \wedge Qz) \rightarrow Px]$$

$$\forall PQ \forall yz [Ryzx \wedge Py \wedge Qz \rightarrow Px]$$

As minimal valuation, we thus take  $\sigma(P) \triangleq \lambda t. y \sqsubseteq t$  and  $\sigma(Q) \triangleq \lambda t. z \sqsubseteq t$ .

$$\forall yz (Ryzx \rightarrow y \sqsubseteq x)$$

If we considered atomic frames *without* information order, we would take as minimal valuations  $\sigma(P) \triangleq \lambda t. y = t$  and  $\sigma(Q) \triangleq \lambda t. z = t$  and we would obtain:

$$\forall yz (Ryzx \rightarrow y = x).$$

$$\text{Rule CI}^i : \frac{Y [\otimes_i] X \vdash Z}{X [\otimes_i] Y \vdash Z}$$

1. Algorithm [12, Lemma 4.4] +  $\mathbb{F}$ :

$$(l(X [\otimes_i] Y) \vdash r(Z)) [Z/l(Y [\otimes_i] X)][X/p][Y/q]$$

$$p \otimes_i q \vdash q \otimes_i p$$

$$p \otimes_i q \rightarrow q \otimes_i p$$

2. Algorithm [5, Proof Theo. 5]:

$$\forall PQ [\exists yz (Ryzx \wedge Py \wedge Qz) \rightarrow \exists y'z' (Ry'z'x \wedge Qy' \wedge Pz')]$$

$$\forall PQ \forall yz [Ryzx \wedge Py \wedge Qz \rightarrow \exists y'z' (Ry'z'x \wedge Qy' \wedge Pz')]$$

As minimal valuation, we thus take  $\sigma(P) \triangleq \lambda t. y \sqsubseteq t$  and  $\sigma(Q) \triangleq \lambda t. z \sqsubseteq t$ .

$$\forall yz [Ryzx \rightarrow \exists y'z' (Ry'z'x \wedge z \sqsubseteq y' \wedge y \sqsubseteq z')]$$

$\forall yz [Ryzx \rightarrow Rzyx]$  because of the Tonicity postulates.

$$\text{Rule WI}^i : \frac{X [\otimes_i] X \vdash Y}{X \vdash Y}$$

1. Algorithm [12, Lemma 4.4] +  $\mathbb{F}$ :

$$(l(X) \vdash r(Y)) [Y/l(X [\otimes_i] X)][X/p]$$

$$p \vdash p \otimes_i p$$

$$p \rightarrow p \otimes_i p$$

2. Algorithm [5, Proof Theo. 5]:

$$\forall P [Px \rightarrow \exists yz (Ryzx \wedge Py \wedge Pz)]$$

As minimal valuation we thus take  $\sigma(P) \triangleq \lambda t. x \sqsubseteq t$ .

$$\exists yz (Ryzx \wedge x \sqsubseteq y \wedge x \sqsubseteq z)$$

$Rxxx$  because of the Tonicity postulates.



### 13.2 From inductive/acyclic formulas to analytic inference rules

The algorithm of Ciabattini & Ramanayake [12, Proof Prop. 3.15+Fig. 1] has been run on the axioms of relevance logic A8 - A16 of Fig. 7 of [5] (corresponding to the classical axioms A8 - A16 of [37]) resulting in the analytic inference rules of Fig. 14. All these axioms are  $\mathcal{I}_2(\mathcal{P}^B)$  acyclic/inductive atomic formulas. We illustrate the four steps of this algorithm with the basic axiom A8 of relevance logic below.<sup>10</sup>

**Axiom A8** :  $(p \supset \sim q) \rightarrow (q \supset \sim p)$ .

1.  $\vdash (p \supset \sim q) \rightarrow (q \supset \sim p)$   
 $\vdash (p \supset \sim q) [\rightarrow] (q \supset \sim p)$   
 $(p \supset \sim q) \vdash (q \supset \sim p)$   
 $(p \supset \sim q) \vdash (q [\supset] \sim p)$   
 $(p \supset \sim q) ; q \vdash \sim p$   
 $(p \supset \sim q) ; q \vdash [\sim] p$   
 $p \supset \sim q \vdash q [\supset] [\sim] p$
2.  $\frac{X_1 \vdash p \supset \sim q}{X_1 \vdash q [\supset] [\sim] p}$   
 $\frac{X_1 \vdash p \supset \sim q \quad X_2 \vdash q}{X_2 \vdash [\sim] p [\supset] X_1}$  because  $\frac{X_1 \vdash \psi [\supset] [\sim] p}{X_1 ; q \vdash [\sim] p}$  DR  
 $\frac{X_1 \vdash p \supset \sim q \quad X_2 \vdash q}{X_3 \vdash [\sim] (X_1 ; X_2)}$  because  $\frac{X_2 \vdash [\sim] p [\supset] X_1}{(X_1 ; X_2) \vdash [\sim] p}$  DR  
 $\frac{X_2 \vdash [\sim] p [\supset] X_1}{p \vdash [\sim] (X_1 ; X_2)}$  DR
3.  $\frac{X_1 \vdash p [\supset] [\sim] q \quad X_2 \vdash q \quad X_3 \vdash p}{X_3 \vdash [\sim] (X_1 ; X_2)}$
4. (a) elimination of  $p$ :  
 $\frac{p \vdash [\sim] q [\supset] X_1 \quad X_2 \vdash q \quad X_3 \vdash p}{X_3 \vdash [\sim] (X_1 ; X_2)}$   
 $\frac{X_3 \vdash [\sim] q [\supset] X_1 \quad X_2 \vdash \psi}{X_3 \vdash [\sim] (X_1 ; X_2)}$
- (b) elimination of  $q$ :  
 $\frac{X_1 [\otimes] X_3 \vdash [\sim] q \quad X_2 \vdash q}{X_3 \vdash [\sim] (X_1 ; X_2)}$   
 $\frac{q \vdash [\sim] (X_1 [\otimes] X_3) \quad X_2 \vdash q}{X_3 \vdash [\sim] (X_1 [\otimes] X_2)}$   
 $\frac{X_2 \vdash [\sim] (X_1 [\otimes] X_3)}{X_3 \vdash [\sim] (X_1 [\otimes] X_2)}$

<sup>10</sup>As already noted by Greco et al. [25, p. 62], there is a “striking” similarity between this algorithm and the (first phase of the) ALBA algorithm of [25], the main difference being the use of nominals  $\mathbf{i}$  and co-nominals  $\mathbf{m}$  instead of structure variables  $X_1, X_2, \dots$

$$\begin{array}{c}
\text{(c) (optional) simplifications:} \\
\frac{X_1 [\otimes] X_2 \vdash [\sim] (X_1 [\otimes] X_3)}{X_3 \vdash [\sim] X_2} \\
\frac{X_1 [\otimes] X_2 \vdash [\sim] X_3}{X_1 [\otimes] X_3 \vdash [\sim] X_2}.
\end{array}$$

## 14 Case study: propositional logic

In this section, we show how we can recover the classical conjunction  $\wedge$ , disjunction  $\vee$  and material implication  $\rightarrow$  as well as the truth constants  $\top$  and  $\perp$  and the Boolean negation  $\neg$  by adding specific inference rules to the display calculus  $\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}$  introduced in Definition 28. These rules are in fact refinements of Gentzen's structural rules for classical logic. In this section, all the connectives are of dimension  $(1, 1)$  or  $(1, 1, 1)$  and by abuse we confuse the connectives with their skeletons. Moreover, we consider frames *without* information order.

### 14.1 A semantic reconstruction

**Lemma 1.** *Let  $\mathcal{C}$  be a set of atomic connectives. Every  $\mathcal{C}$ -frame  $\mathfrak{F}$  without information order validating the first-order conditions  $\{C^f, W^f, K_1^i\}$  is such that for all  $x, y, z \in \mathfrak{F}$ ,*

$$R_{\otimes_i}^{\pm\sigma_i} xyz \text{ iff } x = y = z. \quad (5)$$

*Every  $\mathcal{C}$ -frame  $\mathfrak{F}$  without information order validating the first-order conditions  $\{N_4^i, N_5^i\}$  is such that for all  $x, y, z \in \mathfrak{F}$ ,*

$$R_{\sim_i}^{\pm\sigma_i} xy \text{ iff } x = y. \quad (6)$$

*Proof.* Since the frames are without information order, the corresponding FO formulas are obtained by replacing the preorder  $\sqsubseteq$  by the equality  $=$ . Then, it suffices to combine the first-order conditions corresponding to the respective analytic inference rules.  $\square$

**Theorem 10.** *Let  $\mathcal{C}, \mathcal{C}^+$  be common sets of atomic connectives such that  $\mathcal{C} \subseteq \mathcal{C}^+$  and  $\mathcal{C}^+$  is displayable enough.*

- *Every proper display calculus extending  $\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}$  with analytic structural rules including  $N_4^i, N_5^i$  for some  $i \in \llbracket 1; 3 \rrbracket$  is sound and complete for a logic based on a class of  $\mathcal{C}$ -frames without information order in which  $\sim_j$  is the Boolean negation  $\neg$  if it belongs to  $\mathcal{C}$ .*
- *Every proper display calculus extending  $\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}$  with analytic structural rules including  $K_1^i, W^f, C^f$  for some  $i \in \llbracket 1; 3 \rrbracket$  is sound and complete for a logic based on a class of  $\mathcal{C}$ -frames without information order in which  $\otimes_i$  and  $\oplus_i$  are the classical conjunction  $\wedge$  and disjunction  $\vee$  respectively if they belong to  $\mathcal{C}$ .*
- *Every proper display calculus extending  $\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}$  with analytic structural rules including  $K_1^i, W^f, C^f, \mathfrak{t}_2$  for some  $i \in \llbracket 1; 3 \rrbracket$  is sound and complete for a logic based on a class of  $\mathcal{C}$ -frames without information order in which  $\mathfrak{t}$  (resp.  $\mathfrak{f}$ ) is the verum constant  $\top$  (resp. the falsum constant  $\perp$ ) if it belongs to  $\mathcal{C}$ .*
- *Every proper display calculus extending  $\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}$  with analytic structural rules including  $K_1^i, W^f, C^f, \mathfrak{t}_2, Q^i, N_7^i$  or  $K_1^i, W^f, C^f, \mathfrak{t}_2, Q^i, N_6^i$  for some  $i \in \llbracket 1; 3 \rrbracket$  is sound and complete for a logic based on a class of  $\mathcal{C}$ -frames without information order in which the atomic connectives  $\sim_i$  are the Boolean negation and  $\mathfrak{t}$  (resp.  $\mathfrak{f}$ ) is the verum constant  $\top$  (resp. the falsum constant  $\perp$ ) if they belong to  $\mathcal{C}$ .*

<ul style="list-style-type: none"> <li>• <i>Structural rules:</i></li> </ul>			
$\frac{(X [\wedge] Y) \vdash U}{(Y [\wedge] X) \vdash U} \text{ (CI } \vdash)$	$\frac{X \vdash U}{(X [\wedge] Y) \vdash U} \text{ (K } \vdash)$	$\frac{(X [\wedge] X) \vdash U}{X \vdash U} \text{ (WI } \vdash)$	$\frac{X \vdash Y}{*Y \vdash *X}$
<ul style="list-style-type: none"> <li>• <i>Display rules:</i></li> </ul>			
$\frac{(X [\wedge] Y) \vdash Z}{X \vdash (Z [\vee] *Y)}$	$\frac{X \vdash (Y [\vee] Z)}{(X [\wedge] *Z) \vdash Y}$	$\frac{X \vdash *Y}{Y \vdash *X}$	$\frac{*X \vdash Y}{*Y \vdash X}$
<ul style="list-style-type: none"> <li>• <i>Introduction rules:</i> <math>(\vdash \wedge), (\wedge \vdash), (\vdash \vee), (\vee \vdash), (\vdash \neg), (\neg \vdash)</math> of Fig. 4 and the Axiom.</li> </ul>			

Figure 10: The display calculus CPC for propositional logic

*Proof.* It is a direct consequence of Theorem 6 and Lemma 1. We just combine the local first-order conditions corresponding to the various inference rules and we obtain the expected results. We only prove the proposition for a calculus including the rules  $K_1^i, WI^i, CI^i, t_2^i, Q^i, N_7^i$  for an arbitrary  $i \in \llbracket 1; 3 \rrbracket$ . Because of Lemma 1, the ternary relation  $R \triangleq R_{\otimes_i}^{\pm\sigma_i}$  is the identity relation. Moreover, by conditions  $\{t_2^i, Q^i\}$ , we obtain that the relation  $S \triangleq R_{\sim^i}^{\pm\sigma_i}$  is *serial*, that is, for all  $x$ , there is  $y$  such that  $Sxy$ . Therefore, thanks to  $N_7^i$ , we finally obtain that for all  $x, y$ , we have that  $Sxy$  if, and only if,  $x = y$ :  $S$  is the identity relation. Because of the truth conditions for  $\sim_j$ , we obtain that  $\sim^i$  is the Boolean negation. The proof that  $\mathbf{t}$  and  $\mathbf{f}$  correspond to the truth constant  $\top$  and  $\perp$  is similar and follows from our proof that  $S$  and  $R$  are both the identity relations.  $\square$

**Definition 39.** Let  $(C, C^+)$  be an amenable pair of atomic connectives such that  $C = \{\wedge, \vee, \neg\} \triangleq \{((s_1, +, \sigma_1), 1), ((s_4, -, \sigma), 1), ((t_4, +, \tau_2), 1)\}$ ,  $C \subseteq C^+$  and  $C^+$  is displayable enough. We define the proper display calculus  $PL^0$  as follows, with  $\otimes_i = \wedge$ ,  $\oplus_i = \vee$  and  $\sim^j = \neg$ :

$$PL^0 \triangleq \mathcal{P}_{C, C^+} + \{CI^i, WI^i, K_1^i \mid i \in \llbracket 1; 3 \rrbracket\} + \{N_4^i, N_5^i\}. \quad (PL^0)$$

**Theorem 11.**  $PL^0$  proves the same theorems of  $\mathcal{S}_C$  as the display calculus CPC of Fig. 10.

*Proof.*  $PL^0$  is sound and complete w.r.t. propositional logic by Theorem 10. The display calculus of Fig. 10 is equivalent to the propositional part of  $\mathcal{P}_{C, C^+}^{BLN}$  because of Corollary 1: any display inference in  $\mathcal{P}_{C, C^+}^{BLN}$  involving the binary connectives  $[\wedge]$  or  $[\vee]$  can be replaced by a display inference of  $DR^{\wedge\vee}$ . So, the result follows from the soundness and completeness of the calculus  $\mathcal{P}_{C, C^+}^{BLN}$  for basic Boolean atomic logics of Theorem 2.  $\square$

## 14.2 A syntactic reconstruction

The simplification of the semantics of the connectives when one adds structural rules to the calculus has a counterpart on the syntactic side, that is proof-theoretically. We are going to see in the sequel that the addition of the classical structural rules to the display calculus for basic atomic logics  $\mathcal{P}_{C, C^+}$  in  $PL^0$  allows us to greatly simplify the structural part and the display rules of this calculus. Here, we consider *all* structural connectives belonging to the orbit of  $\mathcal{O}_{\alpha_2 * \beta_2}(\wedge)$

and  $\mathcal{O}_{\alpha_1 * \beta_1}(\neg)$ . That is, we define  $C^+ \triangleq \mathcal{O}_{\alpha_2 * \beta_2}(\wedge) \cup \mathcal{O}_{\alpha_1 * \beta_1}(\neg)$ . Our arguments would transfer even if we considered a smaller set of structural connectives  $C^+$ , yet assuming that it is displayable enough. There are respectively 48 binary atomic connectives in  $\mathcal{O}_{\alpha_2 * \beta_2}(\wedge)$  and 8 unary atomic connectives in  $\mathcal{O}_{\alpha_1 * \beta_1}(\neg)$ . The set of atomic connectives  $C^+$  is obviously displayable enough. Many rules thus become redundant or useless as well as many structural connectives. We are going to show in particular that we can reduce the number of binary structural connectives from 48 to 1 and the number of unary structural connectives from 8 to 1 and recover the classical display calculus for propositional logic of Fig. 10.

**Simplification of the structural binary connectives.** Below are some of the instances of the display rule DR of  $PL^0$  for our 48 binary structural connectives:

$$\begin{array}{cccc}
\frac{(X [\otimes] Y) \vdash Z}{X \vdash (Z [c'] Y)} & \frac{X \vdash (Y [\oplus] Z)}{(X [c'] Z) \vdash Y} & \frac{(X [\uparrow_3] Y) \vdash Z}{(Z [\uparrow'_1] Y) \vdash X} & \frac{X \vdash (Y [\downarrow_3] Z)}{Y \vdash (X [\downarrow'_1] Z)} \\
\frac{Y \vdash (Z [c] X)}{(X [c'] Y) \vdash Z} & \frac{(X [c] Y) \vdash Z}{X \vdash (Z [\oplus] Y)} & \frac{(Z [\uparrow_2] X) \vdash Y}{(Y [\uparrow'_3] X) \vdash Z} & \frac{Z \vdash (X [\downarrow_2] Y)}{X \vdash (Z [\downarrow'_3] Y)} \\
\frac{X \vdash (Y [\supset] Z)}{Y \vdash (X [\supset'] Z)} & \frac{(Z [c] X) \vdash Y}{(Y [c'] X) \vdash Z} & \frac{(Y [\uparrow_1] Z) \vdash X}{(X [\uparrow'_2] Z) \vdash Y} & \frac{Y \vdash (Z [\downarrow_1] X)}{Z \vdash (Y [\downarrow'_2] X)}
\end{array} \quad (\text{DR})$$

where, if  $c$  is the cycle  $c = (1 \ 2 \ 3 \ 4 \ 5 \ 6)$ , the tuple of atomic connectives  $(\otimes, c', c, \oplus, \supset, \supset')$  is one of the following, for some  $i \in \{1, 3, 5\}$ ,

$$((s_1, +, \sigma_i), (s_2, -, \sigma_{c(i)}), (s_2, -, \sigma_{c^2(i)}), (s_1, +, \sigma_{c^3(i)}), (s_3, -, \sigma_{c^4(i)}), (s_3, -, \sigma_{c^5(i)}))$$

and  $(\oplus, c', c, \uparrow, \uparrow', \uparrow')$  is one of the following, for some  $i \in \{1, 3, 5\}$ ,

$$((s_4, -, \sigma_i), (s_5, +, \sigma_{c(i)}), (s_5, +, \sigma_{c^2(i)}), (s_4, -, \sigma_{c^3(i)}), (s_6, +, \sigma_{c^4(i)}), (s_6, +, \sigma_{c^5(i)}))$$

$$(\uparrow_3, \uparrow'_1, \uparrow_2, \uparrow'_3, \uparrow_1, \uparrow'_2) = ((s_7, +, \sigma_1), (s_7, +, \sigma_2), (s_7, +, \sigma_3), (s_7, +, \sigma_4), (s_7, +, \sigma_5), (s_7, +, \sigma_6))$$

$$(\downarrow_3, \downarrow'_1, \downarrow_2, \downarrow'_3, \downarrow_1, \downarrow'_2) = ((s_8, -, \sigma_1), (s_8, -, \sigma_2), (s_8, -, \sigma_3), (s_8, -, \sigma_4), (s_8, -, \sigma_5), (s_8, -, \sigma_6))$$

We are going to see by proving a series of lemmas that these display rules can be greatly simplified and that the number of structural connectives can also be drastically reduced from 48 to 1 using the structural rules in combination with the display rules. Our first lemma states that the structural rules are also derivable when  $[\otimes_3]$  is replaced by  $[\otimes_1]$  or  $[\otimes_2]$ .

**Lemma 2.** *The following rules are derivable in  $PL^0$  for all  $i, j \in \llbracket 1; 3 \rrbracket$ :*

$$\frac{X [\otimes_j] Y \vdash U}{X [\otimes_i] Y \vdash U} \qquad \frac{Z \vdash X [\oplus_j] Y}{Z \vdash X [\oplus_i] Y} \quad \mathbf{T1}_{i[\otimes_i]j}$$



$$\begin{array}{cccc}
\frac{\frac{[\sim^2] X \vdash Y}{[\sim^1] Y \vdash X}}{X \vdash [\sim^4] Y} & \frac{X \vdash [\sim^4] Y}{Y \vdash [\sim^3] X} & \frac{X \vdash [\Box] Y}{[\Diamond^-] X \vdash Y} & \frac{X \vdash [\Box^-] Y}{[\Diamond] X \vdash Y} \\
\frac{X \vdash [\sim^4] Y}{[\sim^2] * Y \vdash X} & \frac{X \vdash [\sim^3] Y}{[\sim^1] * Y \vdash * X} & \frac{X \vdash [\Box] Y}{X \vdash [\sim^4] * Y} & \frac{X \vdash [\Box^-] * Y}{X \vdash [\sim^3] * Y}
\end{array}$$

where  $(\sim^2, \sim^1, \Box, \Diamond^-, \Box^-, \Diamond, \sim^4, \sim^3)$  is

$$((t_4, +, \tau_2), (t_4, +, \tau_1), (t_2, -, \tau_2), (t_1, +, \tau_1), (t_2, -, \tau_1), (t_1, +, \tau_2), (t_3, -, \tau_2), (t_3, -, \tau_1)).$$

**Lemma 3.** *The following rules are derivable in  $PL^0$ :*

$$\begin{array}{cccc}
\frac{[\sim^2] X \vdash Y}{[\sim^1] X \vdash Y} & \frac{X \vdash [\sim^4] Y}{X \vdash [\sim^3] Y} & \frac{[\Diamond^-] X \vdash Y}{[\Diamond] X \vdash Y} & \frac{X \vdash [\Box^-] Y}{X \vdash [\Box] Y} \\
\frac{[\sim^1] X \vdash Y}{*X \vdash Y} & \frac{X \vdash [\sim^3] Y}{X \vdash *Y} & \frac{[\Diamond] X \vdash Y}{X \vdash Y} & \frac{X \vdash [\Box] Y}{X \vdash Y}
\end{array}$$

*Proof.* The first block of derivations follows easily from  $N_4^j$  and  $N_5^j$ . The second follows from the instances of DR above (the second line) and the first block. For the rest, we only prove it for  $[\Box]$  and  $[\Diamond]$ , the other proofs being similar.

$$\begin{array}{ccc}
\frac{\frac{X \vdash [\Box] Y}{X \vdash [\sim^4] * Y} \text{ DR}}{X \vdash **Y} \text{ Sec. Block} & & \frac{\frac{\frac{[\Diamond] X \vdash Y}{X \vdash [\Box^-] Y} \text{ DR}}{X \vdash [\sim^3] * Y} \text{ DR}}{[\sim^1] **Y \vdash * X} \text{ DR}}{\frac{***Y \vdash * X}{X \vdash Y} \text{ DR}^\neg} \text{ DR}^\neg
\end{array}$$

□

Thus, the structural connectives  $[\sim^2], [\sim^1], [\sim^4], [\sim^3]$  can be replaced by the structural connective  $*$  in the proofs of  $PL^0$  and the structural connectives  $[\Diamond], [\Diamond^-], [\Box], [\Box^-]$  can be removed or replaced by  $**$  in the proofs of  $PL^0$ . In doing so, we obtain rules which are equivalent to the third and fourth block of display rules of Fig. 10. Compiling all these simplifications of notations for unary as well as binary connectives, we obtain the display calculus of Fig. 10.

The theorem below follows from our previous explanations, lemmas and theorems.

**Theorem 12.** *Every proof in  $PL^0$  can be translated into a proof in CPC, and vice versa. Hence,  $PL^0$  proves the same theorems of  $S_C$  as the display calculus CPC of Fig. 10.*

*Remark 5.* The rule of associativity  $B^{c:i}$  is derivable in  $PL^0$ , we do not need to introduce this principle as primitive. The proof can be found in [2, p. 56].

## 15 Case study: FDE and relevance logics

### 15.1 First Degree Entailment (FDE)

FDE is recalled in [5]. We propose here an encoding into an atomic logic which is slightly different from the one in [5]. We introduce the atomic connectives  $C_{FDE} = ATM_0 \cup \{\wedge, \vee\} \cup \{\sim^1, \sim^2, \sim^3, \sim^4\}$

<ul style="list-style-type: none"> <li>• <i>Structural rules:</i></li> </ul>			
$\frac{(X [\wedge] Y) \vdash U}{(Y [\wedge] X) \vdash U} \text{ (CI} \vdash)$	$\frac{X \vdash U}{(X [\wedge] Y) \vdash U} \text{ (K} \vdash)$	$\frac{(X [\wedge] X) \vdash U}{X \vdash U} \text{ (WI} \vdash)$	
$\frac{X \vdash Y}{*Y \vdash *X}$	$\frac{X \vdash Y}{[\sim]Y \vdash [\sim]X} \text{ Ctr}$		
<ul style="list-style-type: none"> <li>• <i>Display rules DR:</i></li> </ul>			
$\frac{X \vdash *Y}{Y \vdash *X}$	$\frac{*X \vdash Y}{*Y \vdash X}$	$\frac{(X [\wedge] Y) \vdash Z}{X \vdash (Z [\vee] *Y)}$	$\frac{X \vdash (Y [\vee] Z)}{(X [\wedge] *Z) \vdash Y}$
$\frac{X \vdash *Y}{Y \vdash *X}$	$\frac{*X \vdash Y}{*Y \vdash X}$	$\frac{(X [\wedge] Y) \vdash Z}{Y \vdash (*X [\vee] Z)}$	$\frac{X \vdash (Y [\vee] Z)}{(*Y [\wedge] X) \vdash Z}$
$\frac{X \vdash [\sim]Y}{Y \vdash [\sim]X}$	$\frac{[\sim]X \vdash Y}{[\sim]Y \vdash X}$	$\frac{[\sim]X \vdash Y}{*Y \vdash [\sim]*X}$	$\frac{X \vdash [\sim]Y}{[\sim]*Y \vdash *X}$
<ul style="list-style-type: none"> <li>• <i>Introduction rules:</i> (<math>\vdash \wedge</math>), (<math>\wedge \vdash</math>), (<math>\vdash \vee</math>), (<math>\vee \vdash</math>) of Fig. 4, the Axiom, as well as</li> </ul>			
$\frac{\varphi \vdash X}{[\sim]X \vdash \sim \varphi} \text{ (}\vdash\sim^{12}\text{)}$	$\frac{[\sim]\varphi \vdash U}{\sim \varphi \vdash U} \text{ (}\sim^{12}\vdash\text{)}$		
$\frac{X \vdash \varphi}{\sim \varphi \vdash [\sim]X} \text{ (}\vdash\sim^{34}\text{)}$	$\frac{U \vdash [\sim]\varphi}{U \vdash \sim \varphi} \text{ (}\sim^{34}\vdash\text{)}$		

Figure 11: The display calculus  $\mathcal{P}^{\text{FDE}}$ , sound and complete for FDE

where

$$\begin{aligned} \sim^1 &\triangleq ((1, 1, \exists, -, +, \text{Id}), 1) & \sim^2 &\triangleq ((1, 1, \exists, -, +, (2\ 1)), 1) \\ \sim^3 &\triangleq ((1, 1, \forall, -, -, \text{Id}), 1) & \sim^4 &\triangleq ((1, 1, \forall, -, -, (2\ 1)), 1) \end{aligned}$$

The set of atomic connectives  $\{\sim^1, \sim^2, \sim^3, \sim^4\}$  is displayable enough. But the set of Boolean connectives  $\{\wedge, \vee\}$  is not displayable enough. So, one could argue that we should consider a displayable enough completion  $\mathcal{C}_{\text{FDE}}^+$  of  $\mathcal{C}_{\text{FDE}}$ . For that, we could add the connective  $\rightarrow$  and  $\subset = ((s_2, -, \sigma_3), 0) = (((1, 1, 1), \forall, +, -, -, (2, 3, 1)), 0)$  to  $\mathcal{C}_{\text{FDE}}$  for example. However, we do not need to do so because rules  $\text{DR}^{\wedge, \vee}$  are valid and already ensure the display property for  $\wedge$  and  $\vee$ . Indeed, one could show as we did in Section 14.2 that some extra atomic connectives (such as  $\subset$  and  $\rightarrow$ ) added to  $\mathcal{C}_{\text{FDE}}$  so that  $\mathcal{C}_{\text{FDE}}^+$  is displayable enough would be redundant, because of the presence of the structural rules. In other words, one can show that any derivation including these extra structural connectives (such as  $\subset$  and  $\rightarrow$ ) could be transformed into a derivation without them.

**Proposition 9.** *The display calculus  $\mathcal{P}^{\text{FDE}}$  of Fig. 11 is sound and complete for FDE.*

*Proof.* The calculus  $\mathcal{P}^{\text{FDE}}$  is obtained from  $\mathcal{P}_{\mathcal{C}_{\text{FDE}}, \mathcal{C}_{\text{FDE}}^+} + \{\mathbf{N}_1^j, \mathbf{N}_2^j, \mathbf{N}_3^j\}$ . Indeed, the first-order frame conditions corresponding to  $\{\mathbf{N}_1^j, \mathbf{N}_2^j, \mathbf{N}_3^j\}$  impose on their associated relation  $R_{\sim^i}$  to be the

graph of an involutive function. Then, this calculus is transformed by identifying the connectives  $\sim^1, \sim^2, \sim^3, \sim^4$  into a single connective  $\sim$ . The four display rules for the relevant negation  $\sim$ , which are all instances of DR, are obtained from the following equations:  $\sim^2 = (1\ 2)\ \sim^1$  and  $\sim^4 = (1\ 2)\ \sim^3$  and  $\sim^3 = (1\ 2) - (1\ 2) - \text{Id}\ \sim^1$  and  $\sim^1 = (1\ 2) - (1\ 2) - \text{Id}\ \sim^3$ . One can then easily show that all the rules of  $\mathcal{P}_{\mathcal{C}_{\text{FDE}}, \mathcal{C}_{\text{FDE}}^+} + \{\mathbf{N}_1^j, \mathbf{N}_2^j, \mathbf{N}_3^j\}$  are provably equivalent in  $\mathcal{P}_{\mathcal{C}_{\text{FDE}}, \mathcal{C}_{\text{FDE}}^+}$  to the rules involving  $[\sim]$  appearing in the calculus of Fig. 11.  $\square$

In [5] we use different axioms to obtain a Hilbert axiomatization of FDE; these axioms boil down as well to identify the connectives  $\sim^1, \sim^2, \sim^3, \sim^4$ . The kind of negation obeying these axioms or inference rules is a ‘‘De Morgan’’ negation in the sense of [19, 20]. On the other hand, note that the Boolean negation only appears as the structural connective  $*$  and not as a logical connective.

## 15.2 Relevance logics

In this section, we consider atomic frames  $\mathfrak{F}$  with information order (see Section 5). The family of relevance logics is recalled in [5, Section 11], as well as its encoding into atomic logics. We take up this encoding here. We introduce the atomic connectives  $\mathbf{C}_{\mathbf{B}} = \text{ATM}_0 \cup \{\mathbf{t}, \sim, \wedge, \vee, \supset, \otimes\}$  where

$$\begin{aligned} \otimes &\triangleq (((1, 1, 1), \exists, +, +, +, (1, 2, 3)), 1) & \mathbf{t} &\triangleq ((1, \exists, +), 1) \\ \supset &\triangleq (((1, 1, 1), \forall, -, +, -, (3, 1, 2)), 1) \end{aligned}$$

The persistent set of connectives associated to  $\mathbf{C}_{\mathbf{B}}$  is the same:  $\mathbf{D}_{\mathbf{B}} = \mathbf{C}_{\mathbf{B}}$ .

However,  $\mathbf{C}_{\mathbf{B}}$  is not displayable enough: the residuated connectives  $\otimes$  and  $\supset$  do not allow us to display every argument. We need to add another atomic connective to make it displayable enough: we add the co-implication  $\subset \triangleq (((1, 1, 1), \forall, +, -, -, (3, 2, 1)), 1)$  (see Fig. 1). We could have chosen another atomic connective, such as for example  $\setminus$  or  $\oplus_3$  (corresponding to the ‘‘;’’ of [39] when it is in consequent part). We obtain the displayable enough set of atomic connectives  $\mathbf{C}_{\mathbf{B}}^+ \triangleq \mathbf{C}_{\mathbf{B}} \cup \{\subset\}$ . As is common in the literature, we denote by  $;$  the structural connective  $;$   $\triangleq [\otimes]$  (when it is in consequent part, ‘‘;’’ corresponds to  $[\oplus_3]$ ).

The analytic inference rules of Fig. 14 have been obtained by applying the algorithm of [12, Proof Prop. 3.15+Fig. 1] to the inductive atomic formulas of [5, Fig. 7]. Then, applying Theorem 6 to this set of analytic inference rules, we obtain the following result.

**Proposition 10.** *Let  $\varphi \in \mathcal{L}_{\mathbf{C}_{\mathbf{B}}}$ . The formula  $\varphi$  is valid in the relevance logic  $\mathbf{B}$  iff  $\mathbf{t} \vdash \varphi$  is provable in the display calculus  $\mathcal{P}^{\mathbf{B}}$  of Fig. 13. Let  $\Sigma$  be a set of inference rules of Fig. 14. The formula  $\varphi$  is valid in the relevance logic  $\mathbf{B} + \Sigma$  iff  $\mathbf{t} \vdash \varphi$  is provable in  $\mathcal{P}^{\mathbf{B}} + \Sigma$ .*

*Proof.* It is very similar to the proofs of Theorems 9 and 10 in [5].  $\square$

## 15.3 On the conservativity of Boolean extensions of relevance logics

It is well-known that many relevance logics are conservatively extended with Boolean negation [23, 38]. Our Theorems 7 and 5 yield further results in that direction: every relevance logic extending the basic relevance logic  $\mathbf{B}$  with  $\mathcal{I}_2(\mathcal{P}_{\mathbf{C}_{\mathbf{B}}})$  acyclic/inductive atomic formulas can be conservatively extended with Boolean negation. As it turns out, all the standard axioms A8-A16 of relevance logics recalled in [5] are inductive atomic formulas (we even computed their first-order frame correspondents in the companion article). Hence, all the relevance logics of Fig. 12 are conservative extensions of the same logics without the Boolean negation. This was already



DW	=	B+A8	T	=	TW + A11+ A14
DWX	=	DW+A13		[=	TWX+A11+A14]
TW	=	DW +A9+A10	RW	=	TW + A12
TWX	=	TW + A13	R	=	RW+A11 [=T+A12]
	[=	DWX +A9+A10]	RWK	=	RW+A15
			RM	=	R+A16

Figure 12: Classical relevance logics.

<ul style="list-style-type: none"> <li>• The display calculus <math>\mathcal{P}^{FDE}</math> of Fig. 11:</li> <li>• Structural rules and display rules DR:</li> </ul>		
$\frac{([\mathbf{t}]; X) \vdash U}{X \vdash U} \mathbf{t}_2^i$	$\frac{X \vdash U}{([\mathbf{t}]; X) \vdash U} \mathbf{t}_1^i$	$\frac{(X; Y) \vdash Z}{X \vdash (Y [\supset] Z)} \quad \frac{X \vdash (Y [\supset] Z)}{Y \vdash (Z [\subset] X)}$
<ul style="list-style-type: none"> <li>• Introduction rules:</li> </ul>		
$\frac{}{[\mathbf{t}] \vdash \mathbf{t}} (\vdash \top)$	$\frac{U \vdash (\varphi [\supset] \psi)}{U \vdash (\varphi \supset \psi)} (\vdash \supset)$	$\frac{X \vdash \varphi \quad Y \vdash \psi}{(X; Y) \vdash (\varphi \otimes \psi)} (\vdash \otimes)$
$\frac{[\mathbf{t}] \vdash U}{\mathbf{t} \vdash U} (\top \vdash)$	$\frac{X \vdash \varphi \quad \psi \vdash Y}{(\varphi \supset \psi) \vdash (X [\supset] Y)} (\supset \vdash)$	$\frac{(\varphi; \psi) \vdash U}{(\varphi \otimes \psi) \vdash U} (\otimes \vdash)$

Figure 13: The display calculus  $\mathcal{P}^B$  for relevance logic B

proven in the literature as we said. A notable exception concerns the relevance logic E, whose Boolean extension is not a conservative extension, as proven by Mares [33]. This result is coherent with our Theorem 7 since the logic E is axiomatized with the axiom  $\zeta \triangleq ((p \supset p) \supset q) \supset q$  of Example 14 which is not an inductive atomic formula (as already observed by Conradie & Goranko [13]). Further (negative) results about the conservativity and non-conservativity of Boolean extensions of relevance logics have been obtained by Øgaard [36, 35, 34]. He showed using some proof-theoretical and algebraic methods that in fact many relevance logics are not conservatively extended with Boolean negation, in particular when these relevance logics include a modality or the Ackermann constant  $\mathbf{t}$ . All his results are consistent with our Theorem 7. Our theorem sheds new light on the reasons why some relevance logics are conservatively extended with Boolean negation and not others: this is simply due to the shape of the axioms axiomatizing the respective relevance logics, whether they are inductive atomic formulas or not. Our Theorem 7 yields in that respect many new conservativity results that solves (partly) one of his open problems [34, p. 383] because our relevance logics with the Ackermann truth constant  $\mathbf{t}$  are ‘ $\mathbf{t}$ -distinctive’ according to his terminology (it follows from the third item of Proposition 10 in [5]).

Inference rule	Local first-order correspondent $\chi(x)$	
$\frac{(U ; X) \vdash [\sim]Y}{(U ; Y) \vdash [\sim]X}$	$\forall yz (Rxyz \rightarrow Rxz^*y^*)$	R8 (A8)
$\frac{(X ; (Y ; Z)) \vdash U}{((Y ; X) ; Z) \vdash U}$	$\forall yzw (R(xy)zw \rightarrow Ry(xz)w)$	B' (A9)
$\frac{(X ; (Y ; Z)) \vdash U}{((X ; Y) ; Z) \vdash U}$	$\forall yzw (R(xy)zw \rightarrow Rx(yz)w)$	B (A10)
$\frac{((X ; Y) ; Y) \vdash U}{(X ; Y) \vdash U}$	$\forall yz (Rxyz \rightarrow R(xy)yz)$	W (A11)
$\frac{(X ; Y) \vdash U}{(Y ; X) \vdash U}$	$\forall yz (Rxyz \rightarrow Ryxz)$	CI (A12)
$\frac{X \vdash [\mathbf{t}] \quad Y \vdash Z}{(*Z ; X) \vdash [\sim]Y}$	$Tx \rightarrow x^* \sqsubseteq x$	R13 (A13)
$\frac{Y \vdash [\sim](X ; Y)}{Y \vdash [\sim]X}$	$(Tx \rightarrow x^* \sqsubseteq x) \wedge (\neg Tx \rightarrow Rxx^*x)$	R14 (A14)
$\frac{X \vdash U}{(X ; Y) \vdash U}$	$Rxyz \rightarrow x \sqsubseteq z$	K (A15)
$\frac{X \vdash Z \quad Y \vdash Z}{(X ; Y) \vdash Z}$	$Rxyz \rightarrow (x \sqsubseteq z \vee y \sqsubseteq z)$	R16 (A16)
where $R(xy)zw \triangleq \exists u (Rxyu \wedge Ruzw)$ $Rx(yz)w \triangleq \exists u (Ryzu \wedge Rxuw)$		

Figure 14: Analytic inference rules for relevance logics and their first-order correspondents.

## 16 Case study: intuitionistic and superintuitionistic logics

In this section, we consider atomic frames and models *with* information order (see Section 5). The method that we are going to present in order to deal with intuitionistic and superintuitionistic logics within our framework is based on the approach developed in [1, Section 8]. A similar encoding had been proposed by Bimbó & Dunn [7, p. 291-295].

### 16.1 Intuitionistic logic

The language of intuitionistic logic is usually based on the the following set of logical connectives:  $\mathbf{C}_{\text{IPC}} \triangleq \text{ATM}_0 \cup \{\top, \perp, \wedge, \vee, \supset\}$ . We present the Kripke-style semantics of intuitionistic logic. An *intuitionistic Kripke frame* is a pair  $(W, \sqsubseteq)$  where  $W$  is a non-empty set (of possible worlds) and  $\sqsubseteq \subseteq W \times W$  is a reflexive and transitive binary relation over  $W$ . An *intuitionistic Kripke model* is a triple  $\mathcal{M} \triangleq (W, \sqsubseteq, V)$  where  $(W, \sqsubseteq)$  is an intuitionistic Kripke frame and  $V : \text{ATM}_0 \rightarrow 2^W$  is a mapping obeying the following condition: for all  $w, v \in W$ , for all  $p \in \text{ATM}_0$ ,

$$\text{if } w \in V(p) \text{ and } w \sqsubseteq v \text{ then } v \in V(p). \quad (\text{Hereditiy condition})$$

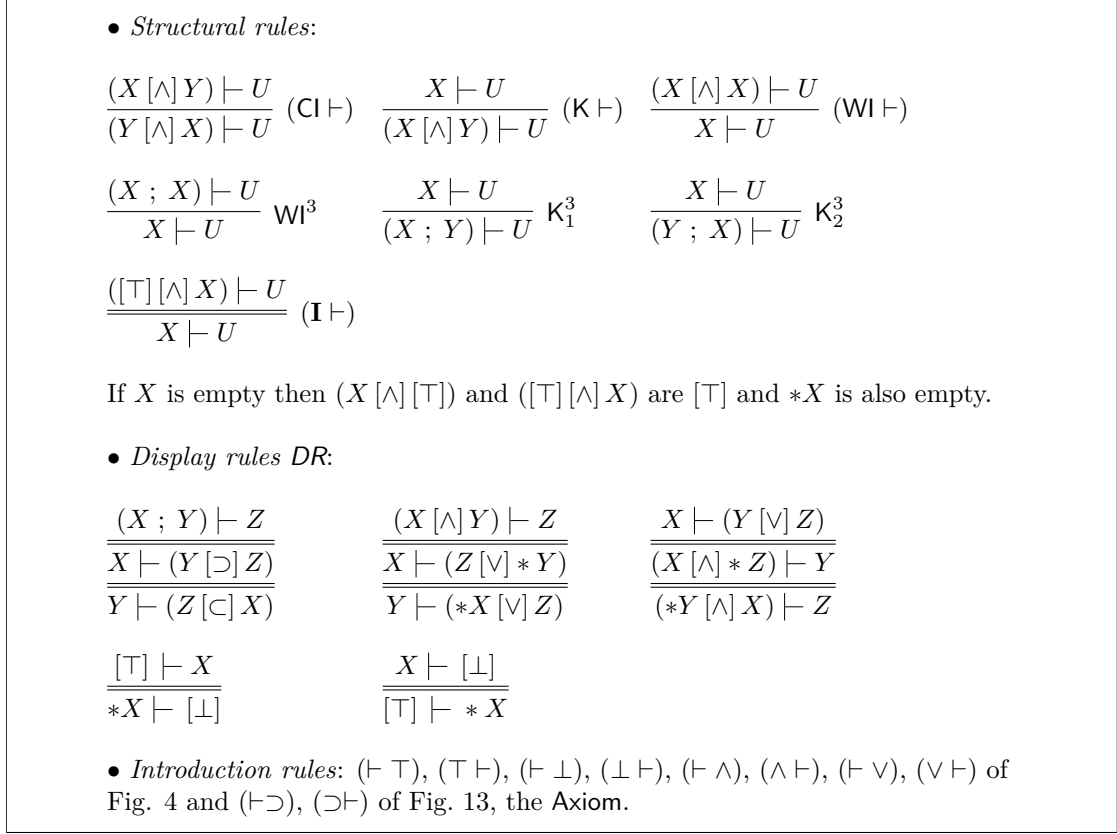


Figure 15: The display calculus  $\mathcal{P}^{\text{IPC}}$  sound and complete for intuitionistic logic

We abusively write  $w \in \mathcal{M}$  for  $w \in W$  and  $(\mathcal{M}, w)$  is called a *pointed intuitionistic Kripke model*. The class of all pointed intuitionistic Kripke models (frames) is denoted  $\mathcal{E}_{\text{IPC}}$  (resp.  $\mathcal{F}_{\text{IPC}}$ ). The *intuitionistic evaluation relation*  $\models \subseteq \mathcal{E}_{\text{IPC}} \times \mathcal{L}_{\text{IPC}}$  is defined inductively as follows. Let  $(\mathcal{M}, w)$  be a pointed intuitionistic Kripke model and  $\varphi \in \mathcal{L}_{\text{IPC}}$ . The truth conditions for  $\top$ ,  $\perp$  and  $\wedge, \vee$  are defined as usual like in Proposition 1. The truth condition for the connective  $\supset$  is defined as follows:

$$(\mathcal{M}, w) \models \varphi \supset \psi \text{ iff for all } v \in \mathcal{M} \text{ such that } w \sqsubseteq v, \text{ if } (\mathcal{M}, v) \models \varphi \text{ then } (\mathcal{M}, v) \models \psi$$

Hence, the triple  $\text{IPC} \triangleq (\mathcal{L}_{\text{IPC}}, \mathcal{E}_{\text{IPC}}, \models)$  is a logic, called *intuitionistic logic*.

**Encoding of intuitionistic logic as an atomic logic.** The atomic connectives are also denoted  $\mathbf{C}_{\text{IPC}} \triangleq \text{ATM}_0 \cup \{\top, \perp, \wedge, \vee, \supset\}$  where  $\top, \perp, \wedge, \vee$  are the usual Boolean connectives and  $\supset \triangleq ((s_3, -, \sigma_5), 1)$  (it has the same skeleton as  $\supset_1$  and the relevant implication). The persistent set of connectives associated to  $\mathbf{C}_{\text{IPC}}$  is the same:  $\mathbf{D}_{\text{IPC}} = \mathbf{C}_{\text{IPC}}$ . We recall that the truth condition for the atomic connective  $\supset$  is the following:

$$(\mathfrak{M}, w) \Vdash \varphi \supset \psi \text{ iff for all } u, v \in \mathfrak{M} \text{ such that } R_{\supset} wuv, \text{ if } (\mathfrak{M}, u) \Vdash \varphi \text{ then } (\mathfrak{M}, v) \Vdash \psi$$

This set of atomic connectives  $\mathbf{C}_{\text{IPC}}$  is not displayable enough. So, let us consider the set of atomic connectives  $\mathbf{C}_{\text{IPC}}^+ = \mathbf{C}_{\text{IPC}} \cup \{\otimes, \subset\}$  where  $\otimes = \otimes_3 \triangleq ((s_1, +, \sigma_1), 1)$  and  $\subset = \subset_2 \triangleq$

$((s_2, -, \sigma_3), 1)$ . This set  $\mathbf{C}_{\text{IPC}}^+$  is displayable enough and even purely displayable. Note that we could have chosen other atomic connectives than  $\{\otimes_3, \sqsubset\}$  to extend  $\mathbf{C}_{\text{IPC}}$  so as to obtain a displayable enough set of connectives. We choose them because it is then easier to state the appropriate structural rules (although they could also be stated differently, see Remark 4). The idea underlying the encoding of intuitionistic logic into an atomic logic is to impose conditions on the ternary relation by means of specific structural rules so that for all  $u, v, w$ ,

$$R_{\supset}uvw \text{ iff } u \sqsubseteq w \text{ and } v \sqsubseteq w \quad (7)$$

This ensures that the semantics for  $\supset$  with this specific ternary relation will coincide with the semantics of the intuitionistic implication. These conditions are obtained by choosing adequately the structural rules and in particular by including in the calculus the Rules  $\mathbf{K}_1^i$  and  $\mathbf{K}_2^i$  for  $i = 3$ . The proof of the following proposition is very similar to the corresponding one in [1, Section 8].

**Proposition 11.** *The display calculus  $\mathcal{P}^{\text{IPC}}$  of Fig. 15 is sound and complete for intuitionistic logic.*

This result is corroborated by the fact that  $\mathbf{FNL}_{wc}$  (the Full non-associative Lambek Calculus with contraction and weakening) coincides with intuitionistic logic [9, 22].

## 16.2 Intermediate and superintuitionistic logics

A superintuitionistic logic is a propositional logic extending intuitionistic logic. Intermediate logics are consistent superintuitionistic logics [10]. So, they are logics between intuitionistic logic and classical propositional logic. There exists a continuum of different intermediate logics [28] and just as many such logics exhibit the disjunction property.<sup>11</sup> Intermediate logics form a complete lattice with intuitionistic logic as the least element and classical logic as the greatest element.<sup>12</sup>

As is well known, classical logic is obtained from intuitionistic logic by adding the principle of excluded middle PEM. It turns out that if we add the corresponding first-order condition to the condition (7) characterizing the relation associated to  $\supset$  in intuitionistic logic, this amounts to have a ternary relation  $R_{\supset}$  satisfying the following condition:

$$R_{\supset}uvw \text{ iff } u \equiv v \text{ and } v \sqsubseteq w \quad (8)$$

where  $u \equiv v$  is a shorthand for  $u \sqsubseteq v$  and  $v \sqsubseteq u$ . Assuming condition (8), the intuitionistic implication  $\supset$  then collapses with the material implication  $\rightarrow$  of classical propositional logic. Indeed, one can easily prove that for all  $\varphi, \psi \in \mathcal{L}_{\text{CIPC}}$ , the formula  $(\varphi \supset \psi) \leftrightarrow (\varphi \rightarrow \psi)$  is valid in any  $\mathbf{C}_{\text{IPC}}$ -frame satisfying Condition (8).

We can weaken PEM in many ways and thus obtain a plethora of logics called intermediate logics. For example, the logic KC, also called Jankov's logic [27] is the logic obtained from intuitionistic logic by adding Axiom wPEM (weak PEM). If we assume condition (7), then this condition is equivalent to the following:

$$\text{if } u \sqsubseteq v \text{ and } u \sqsubseteq w \text{ then there is } x \text{ such that } v \sqsubseteq x \text{ and } w \sqsubseteq x \quad (\text{Confluence})$$

This is a condition that characterizes the frames of KC [29, Table 1].<sup>13</sup>

<sup>11</sup>The disjunction property for a logic  $\mathbf{L}$  is the following: if  $\varphi \vee \psi$  is provable/valid in  $\mathbf{L}$  then either  $\varphi$  is provable/valid in  $\mathbf{L}$  or  $\psi$  is provable/valid in  $\mathbf{L}$ .

<sup>12</sup>A *complete lattice* is a partially ordered set  $(L, \leq)$  such that every subset  $A$  of  $L$  has both a greatest lower bound (the *infimum*, also called the *meet*) and a least upper bound (the *supremum*, also called the *join*) in  $(L, \leq)$ . The *meet* is denoted  $\bigwedge A$ , and the *join* is denoted  $\bigvee A$ .

<sup>13</sup>This condition is also sometimes replaced by other formulations, such as: “KC is complete w.r.t. finite rooted frames with unique top points” [17, p. 64] or KC is the logic “characterized by the class of directed [intuitionistic Kripke] frames” [16, Proposition 3.2]. These formulations are equivalent because of the properties of S4.2 [11].

<b>Inductive atomic axiom</b>	<b>Inference rule</b>	<b>Local FO correspondent</b>	$\chi(x)$
$p \vee \neg_h p$	$\frac{X \vdash Y}{*Y \vdash (X \supset [\perp])}$	$\forall yz (Rxyz \rightarrow y \sqsubseteq x)$	PEM
$\neg_h p \vee \neg_h \neg_h p$	$\frac{Y \vdash (X \supset [\perp])}{*(X \supset [\perp]) \vdash (Y \supset [\perp])}$	$\forall x_1 x_2 y_1 y_2 (Rxx_1 x_2 \wedge Rxy_1 y_2 \rightarrow \exists w Ry_1 x_1 w)$	wPEM

where  $\neg_h \varphi \triangleq (\varphi \supset \perp)$  is the intuitionistic negation.

Figure 16: Axioms and analytic inference rules for intuitionistic and intermediate logics

**Proposition 12.** *The display calculus  $\mathcal{P}^{IPC} + wPEM$  is sound and complete for the intermediate logic  $KC$ .*

## 17 Related works and conclusion

**Related works.** Restall [39, 40] proved some correspondence theory results, connecting display logic structural rules with Hilbert axioms and corresponding frame conditions. However, he did so for a much smaller class of (relevance) formulas than our class of inductive atomic formulas and only for relevance logics with the Routley-Meyer semantics. He also did not characterize in any way the class of (relevance) logics admitting properly displayable calculi.

Greco et al. [25] adapted and generalized the framework of Goranko & Vakarelov [24] dealing with the correspondence theory of polyadic modal logics to their DLE-logics, just as we did it with our atomic logics. In particular, they introduced the notion of inductive inequality which is the counterpart in their setting to the notion of Goranko & Vakarelov's inductive formula. It is therefore not surprising that we come up with notions and results which are very similar to their notions.<sup>14</sup> Unlike theirs, our notions are genuine instances of the notions introduced by Goranko & Vakarelov [24] and operate and apply directly at the level of the Kripke-style relational semantics, like [24], because we set a formal connection between our atomic logics and modal polyadic logics [5, Proposition 8]. It is in fact the soundness and completeness of our calculi w.r.t. a Kripke-style relational semantics which allows us to import directly the results and notions of Goranko & Vakarelov [24] in our framework. This is different from what has been done

<sup>14</sup>To be more precise, the tonicity of molecular connectives corresponds to their *sign* inherited by the leaves in the *signed generation tree* [25, Definition 14]; essentially universal molecular formulas are concatenations of normal operators that behave 'like boxes' and correspond to their positive *PIA formulas* or negative *skeleton* [25, Definition 15]; essentially existential molecular formulas are compositions of normal operators that behave 'like diamonds' and correspond to their negative *PIA formulas* or positive *skeleton* [25, Definition 15]; the universal molecular connective part is the *skeleton* part of [25, Definition 15], and the several essentially existential and universal formulas attached to it are the maximal *PIA formulas* containing the *critical* occurrences (these occurrences are called essential here and in [24]). Every branch of a regular formula is *good* [25, Definition 15] since the lower part of the formula is all PIA, and the upper is all skeleton. All in all, inductive atomic formulas of  $\mathcal{I}_2(\mathcal{P}_C)$  correspond to *analytic inductive formulas* [25, Definition 55]. Moreover, the *order types* for propositional letters [25, p. 18] different from 1 can all be replaced equivalently by variables of order type 1 thanks to the Boolean negation, which is present in our framework.

by Greco et al. [25] where such a formal connection is absent. They do not connect their work to the work of Goranko & Vakarelov as tightly as we do because they simply do not provide a Kripke semantics to their DLE-logics. Instead, they only provide evidence [14, Section 3] that, as their name suggests, inductive inequalities, which extend analytic inductive inequalities, are the distributive counterparts of and ‘project over’ the inductive formulas of Goranko and Vakarelov in the classical setting. Another difference with our work is that the ordering  $\Omega$  on the propositional variables which is obtained in our approach constructively by a topological sort of the dependency digraph associated to a regular formula (see the proof of Theorem 5) is not determined by Greco et al. [25] and is only assumed to exist for inequalities to be inductive (together with an order type). This said, Greco et al. [25, Appendix D] prove that their analytic inductive inequalities coincide with acyclic formulas of  $\mathcal{I}_2(\text{DL})$ . This result is similar in spirit to our Theorem 5. Finally, since they do not resort to a Kripke-style relational semantics, they do not consider models with information order. This renders the study of intuitionistic and intermediate logics rather difficult in their setting.

As we already pointed out in Section 15.3, our Theorem 7 generalizes and sheds new light on a number of results related to the conservativity of relevance logics when extended with Boolean negation (in particular those of [23, 38, 33, 36, 35, 34]). This said, the conditions for obtaining conservative extensions of logics in this Theorem are only sufficient conditions, we do not know whether they are also necessary.

**Concluding remarks.** In this article, we have connected the work of Goranko & Vakarelov [24] to the work of Ciabattoni & Ramanayake [12] by applying the latter to our atomic logics and showing that acyclic formulas for our basic display calculi are in fact inductive in the sense of the former (our Theorem 5) and therefore canonical. We limited our investigations to atomic logics. Our results could be adapted to molecular logics, but for connectives which are universal or existential (because of Proposition 5). However, much work remains to be done so as to obtain similar general results for molecular logics.

We dealt with a number of case studies in Sections 14, 15 and 16: propositional logic, FDE and relevance logics, intuitionistic and intermediate logics, respectively. As the reader will have surely noticed, many other non-classical logics could be revisited within the framework of atomic and molecular logics. Sound and complete calculi could then be obtained automatically from their mere reformulation in this framework. This may lead to the development of novel theorem provers, for intermediate logics for example [26]. As for intermediate logics, they have often been studied thanks to the Gödel-Tarski translation into modal logic using some back-and-forth translation of results between intermediate logics and their modal companions [10]. Our work offers a novel semantics and a novel approach to the study of intermediate logics, but this is the case for many other non-classical logics.

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## A Proofs of Propositions 3, 6, 7, 8

The proof of Proposition 3 can be found in [4].

**Proposition 6.** *Let  $(C, C^+)$  be an amenable pair of sets of atomic connectives. Then  $\mathcal{P}_{C, C^+}$  is amenable (in the sense of Ciabattoni & Ramanayake, Definition 14).*

*Proof.* By spelling out in detail the definition of an amenable pair of sets of atomic connectives, for all  $\odot \in C^+$  of arity  $n > 0$ , there are  $\odot' \in C$  of dimension signature  $(k'_1, \dots, k'_{n+1})$  and  $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$  such that  $\odot' = \tau_0 - \dots - \tau_m \odot$  and  $\tau_0 \circ \dots \circ \tau_m(n+1) = n+1$ . We denote  $\bar{\tau} \triangleq \tau_0 \circ \dots \circ \tau_m$ .

We first prove Condition (1) of Definition 14. We define the interpretation function  $f$ , which is equal to both the interpretations functions  $l$  and  $r$  of Definition 14, as follows:

$$f : \quad \mathfrak{S}_{\text{ant}} \cup \mathfrak{S}_{\text{suc}} \rightarrow \mathcal{L}_C$$

$$[\odot](X_1, \dots, X_n) \mapsto \odot' \left( \pm_{\bar{\tau}^{-}(1)} f(X_{\bar{\tau}^{-}(1)}), \dots, \pm_{\bar{\tau}^{-}(n)} f(X_{\bar{\tau}^{-}(n)}) \right)$$

where  $\odot' \in C$  is the atomic connective of dimension signature  $(k'_1, \dots, k'_{n+1})$  associated to  $\odot$  in Definition 38 and for all  $i \in \{1, \dots, n\}$  we have that

$$\pm_i \triangleq \begin{cases} \neg^{k'_i} & \text{if } \pm^i = - \\ \text{empty} & \text{otherwise.} \end{cases}$$

Then, by induction, if  $[\odot](X_1, \dots, X_n) \in \mathfrak{S}_{\text{ant}}$ , that is  $\mathcal{A}(\odot) = \exists$ , then  $\mathcal{A}(\odot') = \exists$  too and one can prove that

$$[\odot] \left( \pm_{\bar{\tau}^{-}(1)} X_{\bar{\tau}^{-}(1)}, \dots, \pm_{\bar{\tau}^{-}(n)} X_{\bar{\tau}^{-}(n)} \right) \vdash \odot' \left( \pm_{\bar{\tau}^{-}(1)} f(X_{\bar{\tau}^{-}(1)}), \dots, \pm_{\bar{\tau}^{-}(n)} f(X_{\bar{\tau}^{-}(n)}) \right) \quad (9)$$

because one applies  $(\vdash \odot)$  to  $S(\pm^1, \pm_1 f(X_1)), \dots, S(\pm^n X_n, \pm_n f(X_n))$ , which all hold by Induction Hypothesis. Now, we apply Rule DR to Axiom 9 and we obtain, because  $\bar{\tau}(n+1) = n+1$ , that  $[\odot](X_1, \dots, X_n) \vdash \odot'(\pm_{\bar{\tau}-(1)} f(X_{\bar{\tau}-(1)}), \dots, \pm_{\bar{\tau}-(n)} f(X_{\bar{\tau}-(n)}))$ , i.e.  $[\odot](X_1, \dots, X_n) \vdash f([\odot](X_1, \dots, X_n))$ .

A similar reasoning proves that, if  $[\odot](X_1, \dots, X_n) \in \mathfrak{S}_{\text{SUC}}$ , that is  $\mathcal{A}(\odot) = \forall$ , then  $f([\odot](X_1, \dots, X_n)) \vdash [\odot](X_1, \dots, X_n)$  and likewise, if  $X \vdash Y$  is derivable in  $\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}^{\text{BLN}}$ , then so is  $f(X) \vdash f(Y)$ , using again display Rule DR but this time also  $(\odot \vdash)$ .

The proofs of Conditions (2) and (3) of Definition 14 follow straightforwardly from the fact that we assumed that  $\mathcal{C}$  is complete for truth constants, disjunctions and conjunctions.  $\square$

**Proposition 7.** *Let  $(\mathcal{C}, \mathcal{C}^+)$  be an amenable pair of sets of atomic connectives. A formula  $\varphi \in \mathcal{L}_{\mathcal{C}}$  is s-soluble (resp. a-soluble) for  $\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}^{\text{BLN}}$  iff it can be equivalently rewritten as  $\otimes(p_1, \dots, p_n)$  where  $\otimes$  is a basic universal molecular connective (resp. basic existential molecular connective) and  $p_1, \dots, p_n$  are propositional letters.*

*Proof.* We prove it for the s-soluble case, the proof for the a-soluble case is similar. On the one hand, if  $\varphi$  can be equivalently rewritten as  $\otimes(p_1, \dots, p_n)$  with  $\otimes$  a basic universal atomic connective, then we can iteratively apply Rule  $(\odot \vdash)$  bottom up to  $\mathbf{I} \vdash \otimes(p_1, \dots, p_n)$ . In doing so, we will transform all logical connectives into structural connectives. On the other hand, if  $\otimes$  is not a universal connective, then by applying the same process bottom up, we will reach a consecution of the form  $S([\odot], X_1, \dots, X_n, \odot(\varphi_1, \dots, \varphi_n))$  where we cannot apply Rule  $(\odot \vdash)$  or the converse of Rule  $(\text{WI} \vdash)$  anymore, but only Rule  $(\vdash \odot)$  or  $(\text{K} \vdash)$  (if  $[\odot]$  is  $[\wedge]$  and  $X_1$  is different from  $X_2$ ) which are not invertible. In that case,  $\varphi$  is not s-soluble.  $\square$

**Proposition 8.** *Let  $\mathcal{C}$  be a common Boolean set of atomic connectives and let  $\varphi \in \mathcal{L}_{\mathcal{C}}$ . Then, for all sets of atomic connectives  $\mathcal{C}^+$  such that  $(\mathcal{C}, \mathcal{C}^+)$  is amenable, the following holds:*

- $\varphi \in \mathcal{I}_1(\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}^{\text{BLN}})$  iff  $\varphi$  can be equivalently rewritten as  $\otimes(p_1, \dots, p_n)$  where  $\otimes$  is a universal molecular connective and  $p_1, \dots, p_n \in \mathcal{C}_0$ ;
- $\varphi \in \mathcal{I}_2(\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}^{\text{BLN}})$  iff  $\varphi$  can be equivalently rewritten as  $\otimes(\varphi_1, \dots, \varphi_n)$  where  $\otimes$  is a basic universal molecular connective and each  $\varphi_j$  is
  - either an atom
  - or of the form  $\otimes_j(p_1, \dots, p_m)$  with  $\otimes_j$  basic universal molecular connective (that is, it is s-soluble) if  $\text{tn}(\otimes, j) = -$
  - or of the form  $\otimes'_j(p'_1, \dots, p'_m)$  with  $\otimes'_j$  basic existential molecular connective (that is, it is a-soluble) if  $\text{tn}(\otimes, j) = +$ ,

for some  $p_1, \dots, p_m, p'_1, \dots, p'_m \in \mathcal{C}_0$ .

Hence, every  $\varphi \in \mathcal{I}_2(\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}^{\text{BLN}})$  is a regular formula, but the converse does not necessarily hold. Since the definitions of  $\mathcal{I}_1(\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}^{\text{BLN}})$  and  $\mathcal{I}_2(\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}^{\text{BLN}})$  actually do not depend on  $\mathcal{C}^+$ , they will also be denoted  $\mathcal{I}_1(\mathcal{P}_{\mathcal{C}})$  and  $\mathcal{I}_2(\mathcal{P}_{\mathcal{C}})$ .

*Proof.* We only prove the case for  $\mathcal{I}_2(\mathcal{P}_{\mathcal{C}})$ , the case  $\mathcal{I}_1(\mathcal{P}_{\mathcal{C}, \mathcal{C}^+}^{\text{BLN}})$  follows easily. We first prove the right to left direction. If  $\otimes$  is a basic universal molecular connective, then  $[\top] \vdash \otimes(\varphi_1, \dots, \varphi_n)$  can be transformed applying the invertible Rule  $(\odot \vdash)$  bottom up into its counterpart structural connective  $[\top] \vdash [\otimes](\varphi_1, \dots, \varphi_n)$ . Then, with our assumptions on  $\varphi_1, \dots, \varphi_n$ , we have that all  $\varphi_j = \otimes_j(p_1, \dots, p_m)$  are either s-soluble (if  $\varphi_j$  is antecedent part in  $[\top] \vdash [\otimes](\varphi_1, \dots, \varphi_n)$ ) or a-soluble (if  $\varphi_j$  is consequent part in  $[\top] \vdash [\otimes](\varphi_1, \dots, \varphi_n)$ ), because of Propositions 3 and 7.

So,  $\varphi \in \mathcal{I}_2(\mathcal{P}_C)$ . For the left to right direction, if  $\otimes$  is not a basic universal molecular connective, then it cannot be transformed applying the invertible Rule ( $\odot \vdash$ ) bottom up into its counterpart structural connective and therefore  $\varphi \notin \mathcal{I}_2(\mathcal{P}_C)$ . Likewise, if one of the  $\varphi_j$  is not of the expected form, then by Proposition 7 it is not s-soluble (or a-soluble) and therefore  $\varphi \notin \mathcal{I}_2(\mathcal{P}_C)$ .  $\square$

## B Proofs of Theorem 5

**Lemma 4.** *Let  $(C, C^+)$  be an amenable pair of atomic connectives. Let  $\varphi \in \mathcal{I}_2(\mathcal{P}_C)$  be a regular atomic formula, let  $\mathcal{S}$  be a set of premises (consecutions) of a rule belonging to the set of semistructural rules equivalent to  $\varphi$  obtained according to Ciabattoni & Ramanayake [12, Proposition 3.15] and let  $(V_\varphi, E_\varphi)$  be the dependency digraph of  $\varphi$ . If for all  $p$  in  $\mathcal{S}$  there is no  $q \in V_\varphi$  such that  $qE_\varphi p$ , then  $\mathcal{S}$  respects multiplicities w.r.t.  $p$ .*

*Proof.* Let  $p$  be in  $\mathcal{S}$  and assume that there is no  $q \in V_\varphi$  such that  $qE_\varphi p$ . Then, the consecutions of  $\mathcal{S}$  that contain  $p$  are of the form  $M \vdash \otimes(p)$  with  $\otimes$  (structural) universal atomic connective or  $\otimes'(p) \vdash N$  with  $\otimes'$  (structural) existential atomic connective. Then, using the display rules, these consecutions are equivalent to consecutions of the form  $U \vdash p$  and  $p \vdash V$  with  $p$  not occurring in  $U$  and  $V$ . Hence,  $\mathcal{S}$  respects multiplicities w.r.t.  $p$ .  $\square$

**Theorem 5.** *Let  $C$  be a Boolean set of atomic connectives. For all  $\varphi \in \mathcal{I}_2(\mathcal{P}_C)$ ,  $\varphi$  is an inductive atomic formula iff  $\varphi$  is acyclic.*

*Proof.* Assume that  $\varphi \in \mathcal{I}_2(\mathcal{P}_C)$  is inductive. We are going to prove that it is acyclic. We start with an initial set  $\mathcal{S}$  of consecutions corresponding to the premises of the semistructural rules equivalent to  $\varphi$  (obtained according to Ciabattoni & Ramanayake [12, Proposition 3.15]). We have to find an order on the heads  $p$  of  $V_\varphi$  on which we apply the transformation of Ciabattoni & Ramanayake [12, Definition 3.21]. This order is provided by the topological sort [15, p. 612] of the dependency digraph  $(V_\varphi, E_\varphi)$ , which is a directed acyclic graph. Then, we iteratively apply Lemma 4 in combination with [12, Definition 3.21]. This way, we can ensure to remove all atoms  $p$  from the initial set of consecutions  $\mathcal{S}$ .

Conversely, if  $\varphi \in \mathcal{I}_2(\mathcal{P}_C)$  is not inductive then the dependency digraph of  $\varphi$ ,  $(V_\varphi, E_\varphi)$ , contains an oriented cycle  $(p_1, \dots, p_k)$ . We are going to show that in any order in which we apply the Step 4 of the transformation of Ciabattoni & Ramanayake [12, Definition 3.21] on the atoms  $p_1, \dots, p_k$  we reach a set of consecutions containing a consecution with a propositional letter appearing both as antecedent part and consequent part. This entails that any final set of consecutions obtained by this transformation does not respect multiplicities and therefore that  $\varphi$  is not acyclic because this outcome does not depend on the order on which we apply the cut-closure elimination of Ciabattoni & Ramanayake [12, Definition 3.21] on  $p_1, \dots, p_k$ .

To prove that claim, we first adapt the notion of dependency digraph of a formula to define a notion of dependency digraph associated to a set of consecutions  $\mathcal{S}$ . These sets of consecutions  $\mathcal{S}$  correspond in fact to each set of premises of the semi-structural rules equivalent to  $[\top] \vdash \varphi$  (following the procedure defined by Ciabattoni & Ramanayake [12, Definition 3.15]). One can show that a propositional letter  $p$  belongs to  $V_\varphi$ , the set of vertices of the dependency digraph of  $\varphi$  (*i.e.* the set of heads of  $\varphi$ ), iff  $p$  occurs in the corresponding set  $\mathcal{S}$  of consecutions in the scope of a universal (existential) structural atomic connective  $[\odot]$  at place  $j$  with  $\pm_j(\odot) = +$  (resp.  $\pm_j(\odot) = -$ ). Likewise, one sets  $p_i E_\varphi p_j$  in the dependency digraph of  $\varphi$  (*i.e.*  $p_i$  occurs as an inessential propositional letter in a formula from  $\{\varphi_1, \dots, \varphi_k\}$  with a head  $p_j$ ) iff  $p_j$  occurs in the corresponding set  $\mathcal{S}$  of consecutions in the scope of a universal (existential) structural atomic connective  $[\odot]$  at place  $j$  with  $\pm_j(\odot) = +$  (resp.  $\pm_j(\odot) = -$ ) and  $p_i$  occurs in the same consecution and the same universal (existential) structural atomic connective  $[\odot]$  at place  $i$  with

$\pm_i(\odot) = -$  (resp.  $\pm_i(\odot) = +$ ). Following these observations, we define the *dependency digraph* of a set of consecutions  $\mathcal{S}$  as the graph  $(V_{\mathcal{S}}, E_{\mathcal{S}})$  where  $V_{\mathcal{S}}$  is the set of atoms appearing in  $\mathcal{S}$  in the scope of a universal (or existential) structural atomic connective  $[\odot]$  at place  $j$  with  $\pm_j(\odot) = +$  (resp.  $\pm_j(\odot) = -$ ) and we set  $p_i E_{\mathcal{S}} p_j$  iff  $p_j$  occurs in  $\mathcal{S}$  in the scope of a universal (or existential) structural atomic connective  $[\odot]$  at place  $j$  with  $\pm_j(\odot) = +$  (resp.  $\pm_j(\odot) = -$ ) and  $p_i$  occurs in the same consecution and the same universal (or existential) structural atomic connective  $[\odot]$  at place  $i$  with  $\pm_i(\odot) = -$  (resp.  $\pm_i(\odot) = +$ ). Therefore, there is an oriented cycle in  $(V_{\varphi}, E_{\varphi})$  iff there is an oriented cycle in  $(V_{\mathcal{S}}, E_{\mathcal{S}})$  for some set of consecutions  $\mathcal{S}$  corresponding to a set of premises of the semi-structural rules equivalent to  $[\top] \vdash \varphi$  (following the procedure defined by Ciabattoni & Ramanayake [12, Definition 3.15]).

Now, applying the transformation of Ciabattoni & Ramanayake [12, Definition 3.21] to a set of consecutions  $\mathcal{S}$  respecting the multiplicity with regard to  $p$  yields a new set of consecutions  $\mathcal{S}_p$  whose dependency digraph does not contain the propositional letter  $p$  anymore. In particular, if we have a pattern  $p_i E_{\mathcal{S}} p, p E_{\mathcal{S}} p_j$  in the dependency digraph associated to  $\mathcal{S}$  then we obtain a dependency digraph associated to  $\mathcal{S}_p$  which is the same except that this pattern is replaced by the pattern  $p_i E_{\mathcal{S}_p} p_j$ . We illustrate this transformation by the following example (quantification signatures and tonicity of connectives are mentioned as superscripts):

$$\begin{array}{ccc}
M \vdash [\overset{\forall}{\odot}](\bar{p}_i, \bar{p}) & & M' \vdash [\overset{\forall}{\odot}](\bar{p}, \bar{p}_j) \\
[(2 \overset{\exists}{\exists})\odot](p_i^+, M^+) \vdash p & & p \vdash [(1 \overset{\forall}{\exists})\odot](M', p_j^+) \\
& & [(2 \overset{\exists}{\exists})\odot](p_i^+, M^+) \vdash [(1 \overset{\forall}{\exists})\odot](M', p_j^+) \\
& & M' \vdash [\overset{\forall}{\odot}'] \left( [(2 \overset{\exists}{\exists})\odot](p_i^+, M^+), p_j^+ \right)
\end{array}$$

Therefore, if  $(V_{\mathcal{S}}, E_{\mathcal{S}})$  contains an oriented cycle with the pattern  $p_i E_{\mathcal{S}} p, p E_{\mathcal{S}} p_j$  then  $(V_{\mathcal{S}_p}, E_{\mathcal{S}_p})$  contains an oriented cycle of length smaller by one. If we iterate this process on the atoms  $p_1, \dots, p_k$  of the initial cycle, we will end up with a dependency digraph which contains a reflexive state  $p E_{\mathcal{S}_k} p$ . Hence, we have a consecution  $S \in \mathcal{S}_k$  which contains both an antecedent part and a consequent part occurrence of  $p$ . Thus,  $\mathcal{S}_k$  cannot respect multiplicities.  $\square$

## C Proofs of Theorems 6 and 7

**Lemma 5.** *Let  $(C, C^+)$  be an amenable pair of atomic connectives such that  $C^+$  is displayable enough. Then, the display calculus  $\mathcal{P}_{C, C^+}^{\text{BLN}}$  is well-behaved for the Hilbert calculus  $\mathcal{P}_C$ .*

*Proof.* The function  $\mathbb{F}$  is defined by  $\mathbb{F}(\varphi \vdash \psi) \triangleq \varphi \rightarrow \psi$ . The proof that the display calculus  $\mathcal{P}_{C, C^+}^{\text{BLN}}$  corresponds to the Hilbert calculus  $\mathcal{P}_C$  is the proof that can be found in [4, Theorem 4] to prove the completeness of  $\mathcal{P}_{C, C^+}^{\text{BLN}}$ . The proof of the second item (the additive versions of the introduction rules for  $\wedge$  and  $\vee$  are derivable) can be found in [2, Proposition 46]. The proof that  $\varphi \vdash \psi \in \text{inv}_{\mathcal{P}}([\top] \vdash \mathbb{F}(\varphi \vdash \psi))$  follows from the following derivation. Note that  $(K \vdash)$  is

invertible for this instance because of Rules  $(\mathbf{I}\vdash)$  and  $(\mathbf{CI}\vdash)$ .

$$\frac{\frac{\varphi \vdash \psi}{[\top] [\wedge] \varphi \vdash \psi} (\mathbf{K}\vdash), (\mathbf{CI}\vdash), (\mathbf{I}\vdash)}{[\top] \vdash \varphi [\rightarrow] \psi} \text{DR}$$

$$\frac{[\top] \vdash \varphi [\rightarrow] \psi}{[\top] \vdash \varphi \rightarrow \psi} (\odot\vdash) \quad \square$$

**Theorem 6.** *Let  $(\mathbf{C}, \mathbf{C}^+)$  be a pair of common sets of atomic connectives such that  $\mathbf{C} \subseteq \mathbf{C}^+$  and  $\mathbf{C}^+$  is displayable enough. Let  $\rho_0$  be a set of structural inference rules for  $[\mathcal{L}_{\mathbf{C}^+}]$ . The calculus  $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+} + \rho_0$  is equivalent to a proper display calculus of the form  $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+} + \rho$  iff it is sound and complete for an atomic logic based on  $\mathbf{C}$  whose class of pointed  $\mathbf{C}$ -frames (with or without information order) is definable by the local first-order correspondents of a set  $\Sigma$  of acyclic formulas of  $\mathcal{I}_2(\mathcal{P}_{\mathbf{C}^{\text{BLN}}})$  where  $\mathbf{C}^{\text{BLN}}$  is the Boolean completion of  $\mathbf{C}$ . Moreover the acyclic formulas of  $\Sigma$  are effectively computable from the analytic structural rules of  $\rho$ , and vice versa, as specified in Fig. 7.*

*Proof.* For the left to right direction, assume that  $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+} + \rho_0$  is equivalent to a proper display calculus. Let  $\mathbf{C}_2$  be the Boolean completion of  $\mathbf{C}$  and let  $(\mathbf{C}_2, \mathbf{C}_2^+)$  be an amenable pair of common sets of atomic connectives such that  $\mathbf{C}^+ \subseteq \mathbf{C}_2^+$ . Then,  $\mathcal{P}_{\mathbf{C}, \mathbf{C}_2^+} + \rho_0$  is still equivalent to a proper display calculus because  $\mathbf{C}_2^+$  is only a superset of  $\mathbf{C}^+$ . Moreover,  $\mathcal{P}_{\mathbf{C}_2, \mathbf{C}_2^+}^{\text{BLN}}$  is a well-behaved calculus for  $\mathcal{P}_{\mathbf{C}_2}$  by Lemma 5. Now,  $\mathcal{P}_{\mathbf{C}_2, \mathbf{C}_2^+}^{\text{BLN}} + \rho_0$  is equivalent to a proper display calculus because  $\mathcal{P}_{\mathbf{C}, \mathbf{C}_2^+} + \rho_0$  is. Then, it corresponds to a Hilbert calculus  $\mathcal{P}_{\mathbf{C}_2} + \Sigma$  for a set  $\Sigma$  of acyclic formulas of  $\mathcal{I}_2(\mathcal{P}_{\mathbf{C}_2})$  by [12, Theorem 4.6]. Now, by Theorem 5,  $\Sigma$  is also a set of inductive atomic formulas of  $\mathcal{L}_{\mathbf{C}}$ . Therefore, by the canonicity of inductive formulas [5, Theorem 6],  $\mathcal{P}_{\mathbf{C}_2} + \Sigma$  is sound and complete for an atomic logic  $(\mathcal{L}_{\mathbf{C}}, \mathcal{E}_{\mathbf{C}}, \Vdash)$  whose class of pointed  $\mathbf{C}$ -frames is definable by the local first-order correspondents of the set  $\Sigma$ . Now, every cut-free proof of a  $\mathcal{L}_{\mathbf{C}}$ -consecution in  $\mathcal{P}_{\mathbf{C}_2, \mathbf{C}_2^+}^{\text{BLN}} + \rho_0$  can be transformed into a proof of the same  $\mathcal{L}_{\mathbf{C}}$ -consecution in  $\mathcal{P}_{\mathbf{C}, \mathbf{C}_2^+} + \rho_0$  by following the same transformation as in the proof of [4, Theorem 5] (if some Boolean connectives belong to  $\mathbf{C}$  then their corresponding structural rules already belong to  $\rho_0$  anyway). So,  $\mathcal{P}_{\mathbf{C}, \mathbf{C}_2^+} + \rho_0$  is also sound and complete for an atomic logic  $(\mathcal{L}_{\mathbf{C}}, \mathcal{E}_{\mathbf{C}}, \Vdash)$  whose class of pointed  $\mathbf{C}$ -frames is definable by the local first-order correspondents of the set  $\Sigma$ . Finally, since  $\mathbf{C}^+ \subseteq \mathbf{C}_2^+$  and  $\mathbf{C}^+$  is already displayable enough and  $\rho_0$  is a set of structural inference rules for  $[\mathcal{L}_{\mathbf{C}^+}]$ ,  $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+} + \rho_0$  is sound and complete for an atomic logic  $(\mathcal{L}_{\mathbf{C}}, \mathcal{E}_{\mathbf{C}}, \Vdash)$  whose class of pointed  $\mathbf{C}$ -frames is definable by the local first-order correspondents of the set  $\Sigma$ : every proof in  $\mathcal{P}_{\mathbf{C}, \mathbf{C}_2^+} + \rho_0$  can be transformed into a proof in  $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+} + \rho_0$ .

For the right to left direction, assume that  $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+} + \rho_0$  is sound and complete for an atomic logic  $(\mathcal{L}_{\mathbf{C}}, \mathcal{E}_{\mathbf{C}}, \Vdash)$  whose class of pointed  $\mathbf{C}$ -frames  $\mathcal{E}_{\mathbf{C}}$  is definable by the local first-order correspondents of a set  $\Sigma$  of acyclic formulas of  $\mathcal{I}_2(\mathcal{P}_{\mathbf{C}_2})$  where  $\mathbf{C}_2$  is the Boolean completion of  $\mathbf{C}$ . Then, by [12, Theorem 4.6], there is a set of structural inference rules  $\rho$  and a displayable enough set of atomic connectives  $\mathbf{C}_2^+ \supseteq \mathbf{C}_2$  such that  $\mathcal{P}_{\mathbf{C}_2, \mathbf{C}_2^+}^{\text{BLN}} + \rho$  is a proper display calculus corresponding to  $\mathcal{P}_{\mathbf{C}_2} + \Sigma$ . By Corollary 2,  $\mathcal{P}_{\mathbf{C}_2} + \Sigma$  is sound and complete for  $(\mathcal{L}_{\mathbf{C}}, \mathcal{E}_{\mathbf{C}}, \Vdash)$ . Therefore,  $\mathcal{P}_{\mathbf{C}_2, \mathbf{C}_2^+}^{\text{BLN}} + \rho$  is sound and complete for  $(\mathcal{L}_{\mathbf{C}}, \mathcal{E}_{\mathbf{C}}, \Vdash)$ , like  $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+} + \rho_0$ . Now, every cut-free proof of a  $\mathcal{L}_{\mathbf{C}}$ -consecution in  $\mathcal{P}_{\mathbf{C}_2, \mathbf{C}_2^+}^{\text{BLN}} + \rho$  can be transformed into a proof of the same  $\mathcal{L}_{\mathbf{C}}$ -consecution in  $\mathcal{P}_{\mathbf{C}, \mathbf{C}_2^+} + \rho$  by following the same transformation as in the proof of [4, Theorem 5]. Thus,  $\mathcal{P}_{\mathbf{C}, \mathbf{C}_2^+} + \rho$  is sound and complete for  $(\mathcal{L}_{\mathbf{C}}, \mathcal{E}_{\mathbf{C}}, \Vdash)$ , like  $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+} + \rho_0$ , and they are based on the same language. Since  $\mathcal{P}_{\mathbf{C}, \mathbf{C}_2^+} + \rho$  is a proper display calculus, like its extension  $\mathcal{P}_{\mathbf{C}_2, \mathbf{C}_2^+}^{\text{BLN}} + \rho$ , we have that  $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+} + \rho_0$  is equivalent to a proper display calculus.  $\square$

**Theorem 7.** *Let  $C_1, C_2$  be two common sets of atomic connectives such that  $C_1 \subseteq C_2$ . If the class of pointed  $C_2$ -frames  $\mathcal{E}_{C_2}$  of the logic  $(\mathcal{L}_{C_2}, \mathcal{E}_{C_2}, \Vdash)$  is definable by a set of acyclic formulas of  $\mathcal{I}_2(\mathcal{P}_{C_1^{BLN}})$  (where  $C_1^{BLN}$  is the Boolean completion of  $C_1$ ) then  $(\mathcal{L}_{C_2}, \mathcal{E}_{C_2}, \Vdash)$  is a conservative extension of  $(\mathcal{L}_{C_1}, \mathcal{E}_{C_2}, \Vdash)$ .*

*Proof.* Let  $C_2^+$  be a common set of atomic connectives which is a displayable enough extension of both  $C_1$  and  $C_2$ . It follows from the assumptions and Theorem 6 that there is a proper display calculus  $\mathcal{P}_{C_2, C_2^+} + \rho$  with analytic inference rules  $\rho$  in  $[\mathcal{L}_{C_1}]$  which is sound and complete for  $(\mathcal{L}_{C_2}, \mathcal{E}_{C_2}, \Vdash)$ . Now, for all  $\mathcal{L}_{C_1}$ -consecutions, every proof of this consecution in  $\mathcal{P}_{C_2, C_2^+} + \rho$  is a proof in  $\mathcal{P}_{C_1, C_1^+} + \rho$ , and vice versa, because the calculi have the subformula property and enjoy cut elimination. So,  $\mathcal{P}_{C_2, C_2^+} + \rho$  is a conservative extension of  $\mathcal{P}_{C_1, C_1^+} + \rho$  and  $\mathcal{P}_{C_1, C_1^+} + \rho$  is sound and complete for  $(\mathcal{L}_{C_1}, \mathcal{E}_{C_2}, \Vdash)$ . The result then follows by soundness and completeness.  $\square$