A Characterization of Properly Displayable Atomic and Molecular Logics

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Abstract

We apply the general results of Ciabattoni & Ramanayake about display calculi to atomic and molecular logics. We obtain an instantiation of their "$\mathcal{I}_2$ acyclic" formulas in our framework. Then, we show that our versions of $\mathcal{I}_2$ acyclic formulas are inductive formulas in the sense of Goranko & Vakarelov, again adapted to our framework. This allows us to provide a characterization of displayable calculi extending basic atomic and molecular logics in terms of these formulas: a calculus is properly displayable iff it is sound and complete w.r.t. a class of models defined by $\mathcal{I}_2$ acyclic formulas. For molecular logics, connectives should be of a specific format for that theorem to apply. Then, we give examples of correspondences between structural inference rules and first-order frame properties. We also detail some of the computations that enable to transform one to the other. Afterwards, we apply our characterization theorem and we obtain proper display calculi for propositional logic. In doing so, we show how classical and standard inference rules reappear as simplifications of more general and abstract rules that stem from the display calculi of our atomic logics.

1 Introduction

Despite their complicated automated proof search, display calculi introduced by Belnap [7] have the advantage that a proof of cut-elimination can be easily obtained by inspecting the shape of the inference rules of such calculi and also that adding so-called analytic inference rules to any display calculus preserves the cut-elimination property. Determining to which formulas these analytic inference rules correspond has been studied in the case of tense logics by Kracht [22] and in the general case by Ciabattoni & Ramanayake [11]. The latter introduced a general framework for dealing with display calculi and they showed that these analytic inference rules somehow correspond to so–called "$\mathcal{I}_2$ acyclic" formulas which coincide in the framework of Palmigiano et al. with so-called "analytic inductive inequalities" [20, Appendix D]. Ciabattoni & Ramanayake also obtain a general characterization of properly displayable calculi extending any base display calculus in terms of these formulas. Their general theorem implies in particular the result of Kracht [22] for tense logics.

In that article, we are going to apply their general results to our atomic and molecular logics. We will not only instantiate their general result but also extend it. Indeed, their characterization is 'syntactic' somehow. We will show that our $\mathcal{I}_2$ acyclic formulas are in fact inductive in the sense of Goranko & Vakarelov [17] (a result similar to [20, Proposition 82]). This implies that they are also canonical and we can therefore obtain a characterization which is somehow 'semantic'. To be more precise, we will prove that a calculus is properly displayable iff it is sound and complete w.r.t. a class of models defined by acyclic formulas. Moreover, we will provide algorithms to translate analytic inference rules into first–order frame conditions. For molecular logics, connectives should be in a specific format for that theorem to apply.

Our characterization theorem of the properly displayable atomic and molecular logics is quite general. It applies to an infinite number of logics and in particular to one of the most studied of all, propositional logic. It is therefore natural to study the outcomes that it would produce from a proof–theoretical perspective if we take propositional logic as input for that theorem. Indeed, propositional logic is an extension of our basic atomic logic and we can wonder what happens if we apply our theorem with the Gentzen structural rules.

We will illustrate the various correspondence algorithms that play a role in our characterization theorem with well–known structural inference rules. We will also see that adding the classical Gentzen structural rules to our base display calculus $\mathsf{GGL}^0_{\mathsf{C}}$ of Figure 4 (where $\mathsf{C} = \{\wedge, \vee, \neg\}$) simplifies greatly the display rules and the 48 structural connectives which are associated to the binary connectives $\wedge$, $\vee$ and the 16 structural connectives associated to the unary connective $\neg$. They almost all collapse and can all be expressed alone by means of the usual structural comma , and the operation $*$. Many rules thus become redundant or useless as well as many structural connectives. The addition of Gentzen structural rules to our base display calculus $\mathsf{GGL}^0_{\mathsf{C}}$ allows us to recover a classical display calculus for propositional logic which is very close to the usual Gentzen sequent calculus for propositional logic.

Organization of the article. We start in Section 2 by recalling some basics of group theory that will be used in the sequel. In Sections 3, 4 and 5, we recall atomic and molecular logics as well as their Boolean versions. In Section 6, we recall and slightly adapt the general framework for dealing with display calculi introduced by Ciabattoni & Ramanayake [11]. In Section 9, we introduce inductive molecular formulas which are natural adaptation of the inductive formula of Goranko & Vakarelov [17]. In Section 10, we show that acyclic formulas are inductive. Finally, in Section 11, we state our characterization theorem. Then, in Section 12 we provide examples of correspondence translations from structural inference rules to first–order frame conditions to illustrate it. Finally, in Section 13, we apply our theorem to the structural inference rules of classical propositional logic and reconstruct a display calculus for it. We discuss related works in Section 14 and in particular the framework of Palmigiano & Al and conclude.

Note. The article is self–contained. It is the third part of a series of articles on the proof and correspondence theory of atomic and molecular logics, starting with [5, 4]. All the proofs are in the appendix or in the companion articles.

2 Notions of group theory

We first recall some basics of group theory (see for instance [27] for more details).

Permutations and cycles. If X is a non-empty set, a *permutation* of X is a bijection $\sigma : X \to X$. We denote the set of all permutations of X by $\mathfrak{S}_X$. In the important special case when $X = \{1, \ldots, n\}$, we write $\mathfrak{S}_n$ instead of $\mathfrak{S}_X$. Note that $|\mathfrak{S}_n| = n!$, where $|Y|$ denotes the number of elements in a set Y. A permutation σ on the set $\{1, \ldots, n\}$ such that $\sigma(1) = x_1, \sigma(2) =$

$x_2, \ldots, \sigma(n) = x_n$ is denoted $(x_1, x_2, \ldots, x_n)$. For example, $(1,3,2)$ is the permutation σ such that $\sigma(1) = 1, \sigma(2) = 3$ and $\sigma(3) = 2$.

If $x \in X$ and $\sigma \in \mathfrak{S}_X$, then σ *fixes* x if $\sigma(x) = x$ and σ *moves* x if $\sigma(x) \neq x$. Let $j_1, \ldots, j_r$ be distincts integers between 1 and n. If $\sigma \in \mathfrak{S}_n$ fixes the remaining $n - r$ integers and if $\sigma(j_1) = j_2, \sigma(j_2) = j_3, \ldots, \sigma(j_{r-1}) = j_r, \sigma(j_r) = j_1$ then σ is an r*–cycle*; one also says that σ is a cycle of *length* r. Denote σ by $(j_1\ j_2\ \ldots\ j_r)$. A 2–cycle which merely interchanges a pair of elements is called a *transposition*.

Two permutations $\sigma, \tau \in \mathfrak{S}_X$ are *disjoint* if every x moved by one is fixed by the other. A family of permutations $\sigma_1, \sigma_2, \ldots, \sigma_n$ is *disjoint* if each pair of them is disjoint. Every permutation $\sigma \in \mathfrak{S}_n$ is either a cycle or a product of disjoint cycles. Moreover, this factorization is unique except for the order in which the factors occur.

Groups. A *group* $(G, \circ)$ is a non–empty set G equipped with an associative operation $\circ : G \times G \to G$ and containing an element denoted Id_G called the *neutral element* such that: $\mathrm{Id}_G \circ a = a = a \circ \mathrm{Id}_G$ for all $a \in G$; for every $a \in G$, there is an element $b \in G$ such that $a \circ b = \mathrm{Id}_G = b \circ a$. This element b is unique and called the *inverse* of a, denoted a^{-1}. The set $\mathfrak{S}_n$ with the composition operation is a group called the *symmetric group on n letters*.

A non–empty subset S of a group G is a *subgroup* of G if $s \in S$ implies $s^{-1} \in S$ and $s, t \in S$ imply $s \circ t \in S$. In that case, S is also a group in its own right.

If X is a subset of a group G, then the smallest subgroup of G containing X, denoted by $\langle X \rangle$, is called the *subgroup generated by* X. For example, $\mathfrak{S}_n = \langle (1\,2), (2\,3), \ldots, (i\ i+1), \ldots, (n-1\ n) \rangle = \langle (n\,1), (n\,2), \ldots, (n\ n-1) \rangle = \langle (n-1\ n), (1\ 2\ \ldots\ n) \rangle$. $\mathfrak{S}_n$ is also generated by $(1\,2)$ and 3–cycles. For $n \geq 3$, the *alternating group* $\mathfrak{A}_n$ is the subgroup of $\mathfrak{S}_n$ generated by the n–cycles of $\mathfrak{S}_n$.

In fact, if X is non–empty, then $\langle X \rangle$ is the set of all the words on X, that is, elements of G of the form $x_1^{\pm_1} x_2^{\pm_2} \ldots x_n^{\pm_n}$ where $x_1, \ldots, x_n \in X$ and $\pm_1, \ldots, \pm_n$ are either -1 or empty.

Free groups and free products. If X is a subset of a group F, then F is a *free group* with *basis* X if, for every group G and every function $f : X \to G$, there exists a unique homomorphism $\varphi : F \to G$ extending f. One can prove that a free group with basis X always exists and that X generates F. We therefore use the notation $F = \langle X \rangle$ also for free groups.

If G and H are groups, the *free product* of G and H is a group P and homomorphisms j_G and j_H such that, for every group Q and all homomorphisms $f_G : G \to Q$ and $f_H : H \to Q$, there exists a unique homomorphism $\varphi : P \to Q$ with $\varphi j_G = f_G$ and $\varphi j_H = f_H$. Such a group always exists and it is unique modulo isomorphism, we denote it $G * H$. This definition can be generalized canonically to the case of a finite number of groups $G_1, \ldots, G_n$, yielding the free product $G_1 * \ldots * G_n$.

Group actions. If X is a set and G a group, an *action of* G *on* X is a function $\alpha : G \times X \to X$ given by $(g, x) \mapsto gx$ such that: $\mathrm{Id}x = x$ for all $x \in X$; $(g_1 g_2)x = g_1(g_2 x)$ for all $x \in X$ and all $g_1, g_2 \in G$. If $x \in X$ and α an action of a group G on X, then the *orbit* of x under α is $\mathcal{O}_\alpha(x) \triangleq \{\alpha(g, x) \mid g \in G\}$. The orbits form a partition of X.

Fact 2.1. *If α is an action of G on a set X and H is a subgroup of G, then the restriction of α to H, denoted α_H, is also an action of H on the set X.*

Definition 2.2. Let G and H be two groups. If α and β are actions of G and H on a set X, then the *free action* $\alpha * \beta$ is the mapping $\alpha * \beta : G * H \times X \to X$ given by $\alpha * \beta(g, x) \triangleq \alpha(g_1, \beta(h_1, \ldots, \alpha(g_n, \beta(h_n, x))))$, where $g = g_1 h_1 \ldots g_n h_n$ is the factorization of g in the free group $G * H$.

This definition can be generalized canonically to the case of a finite number of actions $\alpha_1, \ldots, \alpha_n$, yielding the mapping $\alpha_1 * \ldots * \alpha_n$.

Proposition 2.3 ([5])**.** *If $\alpha_1, \ldots, \alpha_n$ are actions of $G_1, \ldots, G_n$ on a set X respectively, then the mapping $\alpha_1 * \ldots * \alpha_n$ is an action of the (free) group $G_1 * \ldots * G_n$ on X.*

3 Atomic logics

The truth conditions of the connectives of atomic logics are defined by first-order formulas of the form $\forall x_1 \ldots x_n (\pm_1 \mathsf{P}_1 x_1 \vee \ldots \vee \pm_n \mathsf{P}_n x_n \vee \pm \mathsf{R} x_1 \ldots x_n x)$ or $\exists x_1 \ldots x_n (\pm_1 \mathsf{P}_1 x_1 \wedge \ldots \wedge \pm_n \mathsf{P}_n x_n \wedge \pm \mathsf{R} x_1 \ldots x_n x)$ where $\pm_i$ is either empty or $\neg$. We will represent the structure of these formulas by means of so–called *skeletons* whose various arguments capture the different features that allow us to redefine them completely.

Definition 3.1 (Atomic skeletons)**.** The sets of *atomic skeletons* $\mathbb{P}$ and $\mathbb{C}$ are defined as follows:

$$\begin{aligned}\mathbb{P} &\triangleq \mathfrak{S}_1 \times \{+,-\} \times \{\forall, \exists\} \times \mathbb{N}^* \\ \mathbb{C} &\triangleq \mathbb{P} \cup \bigcup_{n \in \mathbb{N}^*} \left\{ \mathfrak{S}_{n+1} \times \{+,-\} \times \{\forall, \exists\} \times \mathbb{N}^* \times {\mathbb{N}^*}^n \times \{+,-\}^n \right\}.\end{aligned}$$

$\mathbb{P}$ is called the set of *atom skeletons* and $\mathbb{C}$ is called the set of *connective skeletons*. They can be represented by tuples $(\sigma, \pm, Æ, \overline{k}, \vec{\pm}_j)$ or $(\sigma, \pm, Æ, k)$ if it is a propositional letter skeleton, where $Æ \in \{\forall, \exists\}$ is called the *quantification signature* of the skeleton, $\overline{k} = (k, k_1, \ldots, k_n) \in {\mathbb{N}^*}^{n+1}$ is called the *type signature* of the skeleton and $\vec{\pm}_j = (\pm_1, \ldots, \pm_n) \in \{+,-\}^n$ is called the *tonicity signature* of the skeleton; $(Æ, \overline{k}, \vec{\pm}_j)$ is called the *signature* of the skeleton. The *arity* of a skeleton $\circledast \in \mathbb{C}$ is n, its *input types* are $k_1, \ldots, k_n$ and its *output type* is k. The set of n–ary connective skeletons, for $n > 0$, is denoted $\mathbb{C}_n$.

We define the mapping $\cdot : \{+,-\} \times \{+,-\} \to \{+,-\}$ by $\cdot(+,-) = \cdot(-,+) = -$ and $\cdot(-,-) = \cdot(+,+) = +$ ($(\{+,-\}, \cdot)$ is therefore isomorphic to the group $\mathbb{Z}/2\mathbb{Z}$). For better readability, for all $\pm, \pm' \in \{+,-\}$, we write $\pm\pm'$ for $\cdot(\pm, \pm')$. Informally, $\forall$ is associated with $+$ and $\exists$ is associated with $-$. We formalize this association with the function $\pm : \{\forall, \exists\} \to \{+,-\}$ defined by $\pm(\forall) \triangleq +, \pm(\exists) \triangleq -$ and the inverse function $Æ : \{+,-\} \to \{\forall, \exists\}$ defined by $Æ(+) \triangleq \forall, Æ(-) \triangleq \exists$. Also, we define the function $+ : \{\forall, \exists\} \to \{\forall, \exists\}$ by $+(\forall) \triangleq \forall$ and $+(\exists) \triangleq \exists$ and the function $- : \{\forall, \exists\} \to \{\forall, \exists\}$ by $-(\forall) \triangleq \exists$ and $-(\exists) \triangleq \forall$. For better readability, we write $+\forall, +\exists, -\forall, -\exists$ instead of $+(\forall), +(\exists), -(\forall), -(\exists)$.

Definition 3.2 (Action of the symmetric group)**.** Let $n \in \mathbb{N}^*$. We define the function $\alpha_n : \mathfrak{S}_{n+1} \times \mathbb{C}_n \to \mathbb{C}_n, (\tau, \star) \mapsto \tau\star$ inductively as follows. Let $\star = \big(\sigma, \pm, Æ, \overline{k}, (\pm_1, \ldots, \pm_n)\big) \in \mathbb{C}_n$ and let $c \in \mathfrak{S}_{n+1}$.

- If c is the transposition $r_j = (j\ n+1)$, then

$$r_j\star \triangleq \big((j\ n+1) \circ \sigma, - \pm_j \pm, - \pm_j Æ, \overline{k}, (- \pm_j \pm_1, \ldots, \pm_j, \ldots, - \pm_j \pm_n)\big). \quad (1)$$

  The connective r_j is called the *residual of $\star$ w.r.t. its j^{th} argument.*

- If c is the cycle $(j_1\ j_2\ \ldots\ j_k\ n+1)$, then $c\star \triangleq r_{j_1}(r_{j_2} \ldots (r_{j_k}\star))$, where $r_j \triangleq (j\ n+1)$ for all j.

- If c is a cycle fixing $n+1$, then

$$c\star \triangleq \left(c \circ \sigma, \pm, Æ, \overline{k}, \left(\pm_{c(1)}, \pm_{c(2)}, \ldots, \pm_{c(n)}\right)\right). \tag{2}$$

Finally, if τ is an arbitrary permutation of $\mathfrak{S}_{n+1}$, it can be factorized into a product of disjoint cycles $\tau = c_1 c_2 \ldots c_k$ and this factorization is unique (modulo its order) [27]. So, we define $\tau\star \triangleq c_1\left(c_2 \ldots (c_k \star)\right)$.

Our definition is based on cycles and not on transpositions because the decomposition of any permutation into disjoint cycles is unique modulo its order, unlike its decomposition into transpositions. The mapping α_n is well-defined [3].

Proposition 3.3 ([5])**.** *For all $n \in \mathbb{N}^*$, the mapping $\alpha_n : \mathfrak{S}_{n+1} \times \mathbb{C}_n \to \mathbb{C}_n$ is a group action of $\mathfrak{S}_{n+1}$ on $\mathbb{C}_n$.*

Definition 3.4 (Atomic connectives)**.** An *(atomic) connective* or *propositional letter* is a symbol to which is associated a connective skeleton or a propositional letter skeleton respectively. Its arity, signature, quantification signature, type signature, tonicity signature, input and output types are the same as its skeleton. By abuse, we sometimes identify a connective with its skeleton. If C is a set of atomic connectives, its set of atoms is denoted $\mathbb{P}(\mathsf{C})$. If $\circledast$ is a connective, its skeleton is denoted $\star$.

Every set of connectives C is partitioned into a set of *orbits* $\mathcal{O}$ such that for all connectives $\circledast, \circledast'$ belonging to the same orbit $\mathcal{O}$ of skeletons $\star = (\sigma, \pm, \vec{k}, \vec{\pm_j})$ and $\star' = (\sigma', \pm', \vec{k}', \vec{\pm_j}')$, there are $\tau_1, \ldots, \tau_n \in \mathfrak{S}_{n+1}$ such that $\tau_1 - \ldots \tau_{n-1} - \tau_n \star = \star'$. In that case, we also write that $\tau_1 - \ldots \tau_{n-1} - \tau_n \circledast = \circledast'$ and $\circledast'$ is sometimes defined by this relation.

Propositional letters are denoted p, p_1, p_2, *etc.*, connectives are denoted $\circledast, \circledast_1, \circledast_2$, *etc.* and skeletons are denoted $\star, \star_1, \star_2$, *etc.*

Definition 3.5 (Atomic language)**.** Let C be a set of atomic connectives. The *(typed) atomic language* $\mathcal{L}_{\mathsf{C}}$ associated to C is the smallest set that contains the propositional letters of C and that is closed under the atomic connectives of C while respecting the type constraints. That is,

- $\mathbb{P}(\mathsf{C}) \subseteq \mathcal{L}_{\mathsf{C}}$ and the *type* of an element of $\mathbb{P}$ is its output type k;
- for all $\circledast \in \mathsf{C}$ of arity $n > 0$ and of type signature $(k, k_1, \ldots, k_n)$ and for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$ of types $k_1, \ldots, k_n$ respectively, we have that $\circledast(\varphi_1, \ldots, \varphi_n) \in \mathcal{L}_{\mathsf{C}}$ and $\circledast(\varphi_1, \ldots, \varphi_n)$ is of *type* k.

Elements of $\mathcal{L}_{\mathsf{C}}$ are called *formulas* and are denoted $\varphi, \psi, \alpha, \ldots$ The *type of a formula* $\varphi \in \mathcal{L}_{\mathsf{C}}$ is denoted $k(\varphi)$. The set of all formulas of type k of $\mathcal{L}_{\mathsf{C}}$ is denoted $\mathcal{L}^k_{\mathsf{C}}$.

A set of atomic connectives C is *plain* if for all $\circledast \in \mathsf{C}$ of skeleton $\star = (\sigma, \pm, Æ, (k, k_1, \ldots, k_n), (\pm_1, \ldots, \pm_n))$ there are atoms $p_1, \ldots, p_n \in \mathbb{P}$ of types $k_1, \ldots, k_n$ respectively. *In the sequel, we assume that all sets of connectives C are plain.*

Our assumption that all sets of connectives C considered are plain makes sense. Indeed, we want all connectives of C to appear in some formula of $\mathcal{L}_{\mathsf{C}}$. If C was not plain then there would be a connective of C (with input type k) which would be necessarily composed with another connective of C (of output type k), if we want such a connective to appear in a formula of $\mathcal{L}_{\mathsf{C}}$. Yet, in that case, we should instead view C as a set of *molecular* connectives (introduced in the next section).

Definition 3.6 (C–model)**.** Let C be a set of atomic connectives. A *C–model* is a tuple $M = (W, \mathcal{R})$ where W is a non-empty set and $\mathcal{R}$ is a set of relations over W such that each n–ary connective $\circledast \in \mathsf{C}$ which is not a Boolean connective of type signature $(k, k_1, \ldots, k_n)$ is associated to a $k_1 + \ldots + k_n + k$–ary relation $R_\circledast \in \mathcal{R}$ such that for all connectives $\circledast_1, \circledast_2 \in \mathsf{C}$ which belong to the same orbit $\mathcal{O}$, we have that $R_{\circledast_1} = R_{\circledast_2}$.

An *assignment* is a tuple $(w_1, \ldots, w_k) \in W^k$ for some $k \in \mathbb{N}^*$, generally denoted $\overline{w}$. A *pointed C–model* $(M, \overline{w})$ is a C–model M together with an assignment $\overline{w}$. In that case, we say that $(M, \overline{w})$ *is of type* k. The class of all pointed C–models is denoted $\mathcal{M}_\mathsf{C}$.

Definition 3.7 (Atomic logics)**.** Let C be a set of atomic connectives and let $M = (W, \mathcal{R})$ be a C–model. We define the *interpretation function of $\mathcal{L}_\mathsf{C}$ in M*, denoted $[\![\cdot]\!]^M : \mathcal{L}_\mathsf{C} \to \bigcup_{k \in \mathbb{N}^*} W^k$, inductively as follows: for all propositional letters $p \in \mathsf{C}$ of type k, all connectives $\circledast \in \mathsf{C}$ of skeleton $(\sigma, \pm, Æ, (k, k_1, \ldots, k_n), (\pm_1, \ldots, \pm_n))$ of arity $n > 0$ and for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_\mathsf{C}$,

$$
\begin{aligned}
[\![p]\!]^M &\triangleq \begin{cases} R_p & \text{if } \pm = + \\ W^k - R_p & \text{if } \pm = - \end{cases} \\
[\![\circledast(\varphi_1, \ldots, \varphi_n)]\!]^M &\triangleq f_\circledast([\![\varphi_1]\!]^M, \ldots, [\![\varphi_n]\!]^M)
\end{aligned}
$$

where the function $f_\circledast$ is defined as follows: for all $W_1 \in \mathcal{P}(W^{k_1}), \ldots, W_n \in \mathcal{P}(W^{k_n})$, $f_\circledast(W_1, \ldots, W_n) \triangleq \left\{\overline{w} \in W^k \mid \mathcal{C}^\circledast(W_1, \ldots, W_n, \overline{w})\right\}$ where $\mathcal{C}^\circledast(W_1, \ldots, W_n, \overline{w})$ is called the *truth condition* of $\circledast$ and is defined as follows:

- if $Æ = \forall$: "$\forall \overline{w}_1 \in W^{k_1} \ldots \overline{w}_n \in W^{k_n} \, (\overline{w}_1 \pitchfork_1 W_1 \vee \ldots \vee \overline{w}_n \pitchfork_n W_n \vee R_\circledast^{\pm\sigma} \overline{w}_1 \ldots \overline{w}_n \overline{w})$";
- if $Æ = \exists$: "$\exists \overline{w}_1 \in W^{k_1} \ldots \overline{w}_n \in W^{k_n} \, (\overline{w}_1 \pitchfork_1 W_1 \wedge \ldots \wedge \overline{w}_n \pitchfork_n W_n \wedge R_\circledast^{\pm\sigma} \overline{w}_1 \ldots \overline{w}_n \overline{w})$";

where, for all $j \in [\![1; n]\!]$, $\overline{w}_j \pitchfork_j W_j \triangleq \begin{cases} \overline{w}_j \in W_j & \text{if } \pm_j = + \\ \overline{w}_j \notin W_j & \text{if } \pm_j = - \end{cases}$ and $R_\circledast^{\pm\sigma} \overline{w}_1 \ldots \overline{w}_{n+1}$ holds iff $\pm R_\circledast \overline{w}_{\sigma^-(1)} \ldots \overline{w}_{\sigma^-(n+1)}$ with the notations $+R_\circledast \triangleq R_\circledast$ and $-R_\circledast \triangleq W^{k+k_1+\ldots+k_n} - R_\circledast$.

If $\mathcal{E}_\mathsf{C}$ is a class of pointed C–models, the *satisfaction relation* $\Vdash \subseteq \mathcal{E}_\mathsf{C} \times \mathcal{L}_\mathsf{C}$ is defined as follows: for all $\varphi \in \mathcal{L}_\mathsf{C}$ and all $(M, \overline{w}) \in \mathcal{E}_\mathsf{C}$, $((M, \overline{w}), \varphi) \in \Vdash$ iff $\overline{w} \in [\![\varphi]\!]^M$. We usually write $(M, \overline{w}) \Vdash \varphi$ instead of $((M, \overline{w}), \varphi) \in \Vdash$ and we say that φ is *true* in $(M, \overline{w})$. The triple $(\mathcal{L}_\mathsf{C}, \mathcal{E}_\mathsf{C}, \Vdash)$ is a logic called the *atomic logic associated to $\mathcal{E}_\mathsf{C}$ and C*. The logics of the form $(\mathcal{L}_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$ are called *basic atomic logics*.

Example 3.8 (Modal logic)**.** Modal logic is an example of atomic logic, where $\mathsf{C} = \{p, \top, \bot, \wedge, \vee, \Diamond, \Box\}$ is such that

- $\top, \bot$ are connectives of skeletons $(\mathrm{Id}, +, \exists, 1)$ and $(\mathrm{Id}, -, \forall, 1)$ respectively;
- $\wedge, \vee, \Diamond, \Box$ are connectives of skeletons $(\sigma_1, +, s_1)$, $(\sigma_1, -, s_4)$, $(\tau_2, +, t_1)$ and $(\tau_2, -, t_2)$ respectively;
- C-models $M = (W, \mathcal{R}) \in \mathcal{E}_\mathsf{C}$ are such that $R_\wedge = R_\vee = \{(w, w, w) \mid w \in W\}$, $R_\Diamond = R_\Box$ and $R_\top = R_\bot = W$.

With these conditions on the C–models of $\mathcal{E}_\mathsf{C}$, for all $(M, w) \in \mathcal{E}_\mathsf{C}$,

$$
\begin{array}{lll}
w \in [\![\Diamond_j \varphi]\!]^M & \text{iff} & \exists v (v \in [\![\varphi]\!]^M \wedge R_{\Diamond_j} wv) \\
w \in [\![\Box_j \varphi]\!]^M & \text{iff} & \forall v (v \in [\![\varphi]\!]^M \vee -R_{\Box_j} wv) \\
w \in [\![\wedge(\varphi, \psi)]\!]^M & \text{iff} & \exists vu (v \in [\![\varphi]\!]^M \wedge u \in [\![\psi]\!]^M \wedge R_\wedge vuw) \\
 & \text{iff} & w \in [\![\varphi]\!]^M \wedge w \in [\![\psi]\!]^M \\
w \in [\![\vee(\varphi, \psi)]\!]^M & \text{iff} & \forall vu (v \in [\![\varphi]\!]^M \vee u \in [\![\psi]\!]^M \vee -R_\vee vuw) \\
 & \text{iff} & w \in [\![\varphi]\!]^M \vee w \in [\![\psi]\!]^M
\end{array}
$$

Permutations of $\mathfrak{S}_2$	unary signatures
$\tau_1 = (1,2)$	$t_1 = (\exists, (1,1), +)$
$\tau_2 = (2,1)$	$t_2 = (\forall, (1,1), +)$
	$t_3 = (\forall, (1,1), -)$
	$t_4 = (\exists, (1,1), -)$
Permutations of $\mathfrak{S}_3$	**binary signatures**
$\sigma_1 = (1,2,3)$	$s_1 = (\exists, (1,1,1), (+,+))$
$\sigma_2 = (3,2,1)$	$s_2 = (\forall, (1,1,1), (+,-))$
$\sigma_3 = (3,1,2)$	$s_3 = (\forall, (1,1,1), (-,+))$
$\sigma_4 = (2,1,3)$	$s_4 = (\forall, (1,1,1), (+,+))$
$\sigma_5 = (2,3,1)$	$s_5 = (\exists, (1,1,1), (+,-))$
$\sigma_6 = (1,3,2)$	$s_6 = (\exists, (1,1,1), (-,+))$
	$s_7 = (\exists, (1,1,1), (-,-))$
	$s_8 = (\forall, (1,1,1), (-,-))$

Figure 1: Permutations of $\mathfrak{S}_2$ and $\mathfrak{S}_3$ and 'orbits' of unary and binary signatures

The Boolean conjunction and disjunction $\wedge$ and $\vee$ are defined using the connectives of C by means of special relations $R_\wedge$ and $R_\vee$. However, they could obviously be defined directly. Other examples are given in Figure 2 and in the companion articles [5, 4] and [6]. All the possible truth conditions of unary and binary atomic connectives of type signature $(1,1,\dots,1)$ are in [3].

4 Molecular Logics

Molecular logics are basically logics whose primitive connectives are compositions of atomic connectives in which it is possible to repeat the same argument at different places in the connective. That is why we call them 'molecular', just as molecules are compositions of atoms in chemistry.

Definition 4.1 (Molecular skeleton and connective)**.** The class $\mathbb{C}^*$ of *molecular skeletons* is the smallest set such that:

- $\mathbb{P} \subseteq \mathbb{C}^*$ and $\mathbb{C}^*$ contains as well, for each $k, l \in \mathbb{N}^*$, a symbol id_k^l of *type signature* (k,k), *output type* k and *arity* 1;
- for all $\circledast \in \mathbb{C}$ of type signature $(k, k_1^0, \dots, k_n^0)$ and all $c_1, \dots, c_n \in \mathbb{C}^*$ of output types or types (if they are propositional letters) $k_1^0, \dots, k_n^0$ respectively, $c \triangleq \circledast(c_1, \dots, c_n)$ is a molecular skeleton of $\mathbb{C}^*$ of *output type* k.

If $c \in \mathbb{C}^*$, we define its *decomposition tree* as follows. If $c = p \in \mathbb{P}$ or $c = id_k^l$, then its decomposition tree T_c is the tree consisting of a single node labeled with p or id_k^l respectively. If $c = \circledast(c_1, \dots, c_n) \in \mathbb{C}^*$ then its decomposition tree T_c is the tree defined inductively as follows: the root of T_c is c and it is labeled with $\circledast$ and one sets edges between that root and the roots $c_1, \dots, c_n$ of the decomposition trees $T_{c_1}, \dots, T_{c_n}$ respectively.

If $c \triangleq \circledast(c_1, \dots, c_n)$ is a molecular skeleton with output type k and $k_1, \dots, k_m$ are the ks of the different id_k^ls which appear in $c_1, \dots, c_n$ (in an order which follows the first appearance of the id_k^ls in the inorder traversal of the decomposition trees of $c_1, \dots, c_n$), then the *type signature*

Atomic connective	Truth condition	Non–classical con. in the literature
	The conjunction orbit	
$\varphi\ (\sigma_1,+,s_1)\ \psi$	$\exists vu\,(v \in \llbracket\varphi\rrbracket \wedge u \in \llbracket\psi\rrbracket \wedge Rvuw)$	$\varphi \circ \psi$ [23], $\varphi \otimes_3 \psi$ [2]
$\varphi\ (\sigma_2,-,s_2)\ \psi$	$\forall vu\,(v \in \llbracket\varphi\rrbracket \vee u \notin \llbracket\psi\rrbracket \vee -Rwuv)$	
$\varphi\ (\sigma_3,-,s_2)\ \psi$	$\forall vu\,(v \in \llbracket\varphi\rrbracket \vee u \notin \llbracket\psi\rrbracket \vee -Ruwv)$	$/$ [23], $\varphi \subset_2 \psi$ [2]
$\varphi\ (\sigma_4,+,s_1)\ \psi$ $= \psi\ (\sigma_1,+,s_1)\ \varphi$	$\exists vu\,(v \in \llbracket\varphi\rrbracket \wedge u \in \llbracket\psi\rrbracket \wedge Ruvw)$	
$\varphi\ (\sigma_5,-,s_3)\ \psi$ $= \psi\ (\sigma_2,-,s_2)\ \varphi$	$\forall vu\,(v \notin \llbracket\varphi\rrbracket \vee u \in \llbracket\psi\rrbracket \vee -Rwvu)$	$\backslash$ [23], $\varphi \supset_1 \psi$ [2]
$\varphi\ (\sigma_6,-,s_3)\ \psi$ $= \psi\ (\sigma_3,-,s_2)\ \varphi$	$\forall vu\,(v \notin \llbracket\varphi\rrbracket \vee u \in \llbracket\psi\rrbracket \vee -Rvwu)$	
	The but–not orbit	
$\varphi\ (\sigma_1,+,s_5)\ \psi$	$\exists vu\,(v \in \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Rvuw)$	$\varphi \prec_3 \psi$ [2]
$\varphi\ (\sigma_2,-,s_4)\ \psi$	$\forall vu\,(v \in \llbracket\varphi\rrbracket \vee u \in \llbracket\psi\rrbracket \vee -Rwuv)$	
$\varphi\ (\sigma_3,+,s_6)\ \psi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \in \llbracket\psi\rrbracket \wedge Ruwv)$	$\varphi \succ_2 \psi$ [2]
$\varphi\ (\sigma_4,+,s_6)\ \psi$ $= \psi\ (\sigma_1,+,s_5)\ \varphi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \in \llbracket\psi\rrbracket \wedge Ruvw)$	$\varphi \obslash \psi$ [21, 25]
$\varphi\ (\sigma_5,-,s_4)\ \psi$ $= \psi\ (\sigma_2,-,s_4)\ \varphi$	$\forall vu\,(v \in \llbracket\varphi\rrbracket \vee u \in \llbracket\psi\rrbracket \vee -Rwvu)$	$\varphi \oplus \psi$ [21, 25] $\varphi \oplus_1 \psi$ [2]
$\varphi\ (\sigma_6,+,s_5)\ \psi$ $= \psi\ (\sigma_3,+,s_6)\ \varphi$	$\exists vu\,(v \in \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Rvwu)$	$\varphi \oslash \psi$ [21, 25]
	The stroke orbit	
$\varphi\ (\sigma_1,+,s_7)\ \psi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Rvuw)$	$\varphi \mid_3 \psi$ [1, 18]
$\varphi\ (\sigma_2,+,s_7)\ \psi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Rwuv)$	
$\varphi\ (\sigma_3,+,s_7)\ \psi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Ruwv)$	
$\varphi\ (\sigma_4,+,s_7)\ \psi$ $= \psi\ (\sigma_1,+,s_7)\ \varphi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Ruvw)$	
$\varphi\ (\sigma_5,+,s_7)\ \psi$ $= \psi\ (\sigma_2,+,s_7)\ \varphi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Rwvu)$	$\varphi \mid_1 \psi$ [1, 18]
$\varphi\ (\sigma_6,+,s_7)\ \psi$ $= \psi\ (\sigma_3,+,s_7)\ \varphi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Rvwu)$	$\varphi \mid_2 \psi$ [1, 18]

Figure 2: Some binary connectives of atomic logics of type $(1,1,1)$

of c is $(k, k_1, \ldots, k_m)$ and its *arity* is m. We also define the *quantification signature* $Æ(c)$ of $c = \circledast(c_1, \ldots, c_n)$ by $Æ(c) \triangleq Æ(\circledast)$.

A *molecular connective* is a symbol to which is associated a molecular skeleton. Its arity, type signature, output type, quantification signature and decomposition tree are the same as its skeleton.

The set of *atomic connectives associated to a set* C *of molecular connectives* is the set of labels different from id_k^l of the decomposition trees of the molecular connectives of C.

One needs to introduce the connective id_k^l to deal with molecular connectives whose skeletons are for example of the form $\circledast(p, id_k^l)$ where $p \in \mathbb{P}$ or molecular connectives in which the same argument(s) appear at different places, like for example in $\circledast(id_k^1, \ldots, id_k^1)$ which is of arity 1.

Definition 4.2 (Molecular language)**.** Let C be a set of molecular connectives. The *(typed) molecular language* $\mathcal{L}_{\mathsf{C}}$ associated to C is the smallest set that contains the propositional letters and that is closed under the molecular connectives while respecting the type constraints. That is,

- the propositional letters of C belong to $\mathcal{L}_{\mathsf{C}}$;

- for all $\circledast \in \mathsf{C}$ of type signature $(k, k_1, \ldots, k_m)$ and for all $\varphi_1, \ldots, \varphi_m \in \mathcal{L}_\mathsf{C}$ of types $k_1, \ldots, k_m$ respectively, we have that $\circledast(\varphi_1, \ldots, \varphi_m) \in \mathcal{L}_\mathsf{C}$ and $\circledast(\varphi_1, \ldots, \varphi_m)$ is of *type* k.

Elements of $\mathcal{L}_\mathsf{C}$ are called *molecular formulas* and are denoted $\varphi, \psi, \alpha, \ldots$ The *type of a formula* $\varphi \in \mathcal{L}_C$ is denoted $k(\varphi)$. We use the same abbreviations as for the atomic language.

Definition 4.3 (Molecular logic)**.** If C is a set of molecular connectives, then a *C–model* M is a C'–model M where C' is the set of atomic connectives associated to C. The truth conditions for molecular connectives are defined naturally from the truth conditions of atomic connectives. We define the *interpretation function of* $\mathcal{L}_C$ *in* M, denoted $[\![\cdot]\!]^M : \mathcal{L}_\mathsf{C} \to \bigcup_{k \in \mathbb{N}^*} W^k$, inductively as follows: for all propositional letters $p \in \mathsf{C}$ of skeleton $(\sigma, \pm, Æ, k)$, all molecular connectives $\circledast(c_1, \ldots, c_n) \in \mathsf{C}^*$ of arity $m > 0$ and all $k, l \in \mathbb{N}^*$, for all $\varphi, \varphi_1, \ldots, \varphi_m \in \mathcal{L}_\mathsf{C}$,

$$
\begin{aligned}
[\![p]\!]^M &\triangleq \pm R_p \\
[\![id_k^l(\varphi)]\!]^M &\triangleq [\![\varphi]\!]^M \\
[\![\circledast(c_1, \ldots, c_n)\,(\varphi_1, \ldots, \varphi_m)]\!]^M &\triangleq f_\circledast\left([\![c_1(\varphi_1^1, \ldots, \varphi_{i_1}^1)]\!]^M, \ldots, [\![c_n(\varphi_1^n, \ldots, \varphi_{i_n}^n)]\!]^M\right)
\end{aligned}
$$

where for all $j \in \{1, \ldots, n\}$, the formulas $\varphi_1^j, \ldots, \varphi_{i_j}^j$ are those $\varphi_1, \ldots, \varphi_m$ for which there is a corresponding id_k^l in c_j (the φ_i^js appear in the same order as their corresponding id_k^ls in c_j).

If $\mathcal{E}_\mathsf{C}$ is a class of pointed C–models, the triple $(\mathcal{L}_\mathsf{C}, \mathcal{E}_\mathsf{C}, \Vdash)$ is a logic called the *molecular logic associated to* $\mathcal{E}_\mathsf{C}$ *and* C.

As one can easily notice, every atomic logic can be canonically mapped to an equi-expressive molecular logic: each atomic connective $\circledast$ of type signature $(k, k_1, \ldots, k_n)$ of the given atomic logic has to be transformed into the molecular connective of skeleton $\circledast(id_{k_1}^1, \ldots, id_{k_n}^n)$. Note that the id_k^l are in fact specific atomic connectives whose associated relations are the identity relations.

Example 4.4. Modal logic with a neighborhood semantics, temporal logic and modal intuitionistic logic are examples of molecular logics, but in fact all logics such that the truth conditions of their connectives are expressible in terms of first-order formulas are molecular logics. See [6] for more details.

5 Boolean Atomic and Molecular Logics

Atomic and molecular logics do not include Boolean connectives as primitive connectives. In fact, they can be defined in terms of specific atomic connectives, as follows.

Definition 5.1 (Boolean connectives)**.** The *Boolean connectives* called *conjunctions*, *disjunctions*, *negations* and *Boolean constants* (of type k) are the atomic connectives denoted, respectively:

$$\mathbb{B} \triangleq \{\wedge_k, \vee_k, \neg_k, \top_k, \bot_k \mid k \in \mathbb{N}^*\}$$

The skeleton of $\wedge_k$ is $(1, +, \exists, (k, k, k), (+, +))$, the skeleton of $\vee_k$ is $(1, -, \forall, (k, k, k), (+, +))$, the skeleton of $\neg_k$ is $(1, +, \exists, (k, k), -)$, the skeleton of $\top_k$ is $(1, +, \exists, k)$ and the skeleton of $\bot_k$ is $(1, -, \forall, k)$.

In any C-model $M = (W, \mathcal{R})$ containing Boolean connectives, the associated relation of any $\vee_k$ or $\wedge_k$ is $R_{\wedge_k} = R_{\vee_k} \triangleq \{(\overline{w}, \overline{w}, \overline{w}) \mid \overline{w} \in W^k\}$, the associated relation of any $\neg_k$ is $R_{\neg_k} \triangleq \{(\overline{w}, \overline{w}) \mid \overline{w} \in W^k\}$ and the associated relation of any $\top_k$ or $\bot_k$ is $R_{\bot_k} = R_{\top_k} \triangleq W^k$.

We say that a set of atomic or molecular connectives C is *Boolean* when it contains all conjunctions, disjunctions, constants as well as negations $\wedge_k, \vee_k, \top_k, \bot_k, \neg_k$, for k ranging over all input types and output types of the connectives of C. The *Boolean completion* of a set of atomic or molecular connectives C is the smallest set of connectives including C which is Boolean. A *Boolean atomic or molecular logic* is an atomic or molecular logic such that its set of connectives is Boolean.

Proposition 5.2 ([5]). *Let C be a set of atomic connectives containing Boolean connectives. and let $M = (W, \mathcal{R})$ be a C-model. Then, for all $k \in \mathbb{N}^*$, all $\varphi, \psi \in \mathcal{L}_\mathsf{C}$, if $k(\varphi) = k(\psi) = k$, then*

$$\begin{array}{rcl} [\![\top_k]\!]^M & \triangleq & W^k \\ [\![\bot_k]\!]^M & \triangleq & \emptyset \\ [\![\neg_k \varphi]\!]^M & \triangleq & W^k - [\![\varphi]\!]^M \\ [\![(\varphi \wedge_k \psi)]\!]^M & \triangleq & [\![\varphi]\!]^M \cap [\![\psi]\!]^M \\ [\![(\varphi \vee_k \psi)]\!]^M & \triangleq & [\![\varphi]\!]^M \cup [\![\psi]\!]^M. \end{array}$$

It turns out that Boolean negation can also be simulated systematically at the level of atomic connectives by applying a transformation on them. The Boolean negation of a formula then boils down to taking the Boolean negation of the outermost connective of the formula. This transformation is defined as follows.

Definition 5.3 (Boolean negation). Let $\circledast$ be a n–ary connective of skeleton $(\sigma, \pm, \text{Æ}, \overline{k}, \pm_1, \ldots, \pm_n)$. The *Boolean negation of* $\circledast$ is the connective $-\circledast$ of skeleton $(\sigma, -\pm, -\text{Æ}, \overline{k}, -\pm_1, \ldots, -\pm_n)$ where $-\text{Æ} \triangleq \exists$ if $\text{Æ} = \forall$ and $-\text{Æ} \triangleq \forall$ otherwise, which is associated in any C–model to the same relation as $\circledast$. If $\varphi = \circledast(\varphi_1, \ldots, \varphi_n)$ is an atomic formula, the *Boolean negation of* φ is the formula $-\varphi \triangleq - \circledast (\varphi_1, \ldots, \varphi_n)$.

Proposition 5.4 ([5]). *Let C be a set of atomic connectives such that $-\circledast \in \mathsf{C}$ for all $\circledast \in \mathsf{C}$. Let $\varphi \in \mathcal{L}_\mathsf{C}$ of type k and let $M = (W, \mathcal{R})$ be a C–model. Then, for all $\overline{w} \in W^k$, $\overline{w} \in [\![-\varphi]\!]^M$ iff $\overline{w} \notin [\![\varphi]\!]^M$.*

6 Acyclic formulas and generalities about display calculi

In this section and in the rest of the article, we will use the same definitions as in [5, Section 6] regarding the notions of *logic*, *truth*, *validity*, *logical consequence*, *soundness*, *(strong) completeness*, *etc.*

In this section, we introduce display calculi from an abstract and general perspective following the presentation and definitions of Ciabattoni and Ramanayake [11] with a slight modification in Definition 6.4.

Definition 6.1 (Display calculus, display property and display rules, [11]). Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic containing two types of connectives called *logical connectives* and *structural connectives* such that the set of structural connectives can be partitioned into two subsets called the *antecedent structural connectives* denoted $\left[\mathsf{C}^-\right]$ and the *succedent structural connectives* denoted $\left[\mathsf{C}^+\right]$.

A *formula* is a well–formed expression of $\mathcal{L}$ built up only from logical connectives and a *structure* is either a formula or a well–formed expression of $\mathcal{L}$ built up inductively from other structures and structural connectives. Moreover, all well–formed expressions of $\mathcal{L}$ are either formulas or structures. An *a-structure* (resp. *s-structure*) is a structure whose outermost structural connective $[\circledast]$ is such that $[\circledast] \in [\mathsf{C}^-]$ (resp. $[\circledast] \in [\mathsf{C}^+]$). The set of *a-structures* (resp. *s-structures*) is denoted $\mathfrak{S}_{\text{ant}}$ (resp. $\mathfrak{S}_{\text{suc}}$).

A *substructure* of a structure U or of a $\mathcal{L}$–consecution $U \models V$ is a structure appearing in the scope of a structural connective of U or V. In that case, we denote it $U[Z]$ and $U \models V[Z]$. We also denote by $U[Z/Z']$ and $U \models V[Z/Z']$ the structure and consecution obtained by substituting Z by Z' in U and $U \models V$ respectively.

A sequent calculus P for $\mathcal{L}$ is said to have the *display property* if it contains a set of single–premise structural rules called the *display rules* such that

1. the rule upwards from conclusion to premise of a display rule is also a display rule;
2. for all $U \models V$ derivable in P and all substructures Z appearing in $U \models V$ there is a structure W such that
   - either for all structures Z', it holds that $W \models Z'$ is derivable from $U \models V[Z/Z']$ using the display rules;
   - or for all structures Z', it holds that $Z' \models W$ is derivable from $U \models V[Z/Z']$ using the display rules.

A structure Z is *displayed* (in a sequent) if the sequent has the form $Z \models U$ or $U \models Z$. In the former (resp. latter) case, the occurrence Z is said to be *a-part* (*s-part*) in the sequent. A *display calculus* is a sequent calculus with the display property. A display calculus is said to be *proper* if it satisfies Belnap's conditions $(C1)-(C8)$ (see [11] for a formulation of these conditions with our terminology).

Definition 6.2 (Structural and analytic inference rule)**.** A *structural inference rule* for a language with logical and structural connectives is an inference rule whose representation only uses structural variables and structural connectives, no logical connective. An *analytic* inference rule is a structural inference rule satisfying Belnap's conditions $(C1)-(C7)$ [7].

Definition 6.3 (Amenable calculus, [11])**.** Suppose that P is a proper display calculus for a language $\mathcal{L}$ whose set of formulas of $\mathcal{L}$ is denoted For$\mathcal{L}$. A proper display calculus satisfying the following conditions is said to be *amenable*.

(1) (*interpretation functions*) There are functions $l : \mathfrak{S}_{\text{ant}} \mapsto \text{For}\mathcal{L}$ and $r : \mathfrak{S}_{\text{suc}} \mapsto \text{For}\mathcal{L}$ such that for all $\varphi \in \text{For}\mathcal{L}$ $l(\varphi) = r(\varphi) = \varphi$ and for all $X \in \mathfrak{S}_{\text{ant}}$ and all $Y \in \mathfrak{S}_{\text{suc}}$,

   (a) $X \models l(X)$ and $r(Y) \models Y$ are derivable in P;

   (b) if $X \models Y$ is derivable in P, then so is $l(X) \models r(Y)$.

(2) (*logical constants*) It contains an a–structure constant and an s–structure constant (usually both denoted $\mathbf{I}$). There are also logical constants $c_a, c_s \in \text{For}\mathcal{L}$ such that the following sequents are derivable for all $X \in \mathfrak{S}_{\text{ant}}$ and all $Y \in \mathfrak{S}_{\text{suc}}$:

$$c_a \models Y \qquad\qquad X \models c_s$$

(3) (*logical connectives*) There are binary logical connectives $\vee, \wedge \in \mathsf{C}$ such that the following are derivable for all $\varphi, \psi \in \text{For}\mathcal{L}$ and all $\circledast \in \{\wedge, \vee\}$:

   (i) commutativity: $\varphi \circledast \psi \models \psi \circledast \varphi$

   (ii) associativity: $\varphi \circledast (\psi \circledast \chi) \models (\varphi \circledast \psi) \circledast \chi$ and $(\varphi \circledast \psi) \circledast \chi \models \varphi \circledast (\psi \circledast \chi)$

Also, for all $X \in \mathfrak{S}_{\mathrm{ant}}$ and all $Y \in \mathfrak{S}_{\mathrm{suc}}$,

(a)$_\vee$ $\varphi \models Y$ and $\psi \models Y$ implies $\varphi \vee \psi \models Y$;

(b)$_\vee$ $X \models \varphi$ implies $X \models \varphi \vee \psi$;

(a)$_\wedge$ $X \models \varphi$ and $X \models \psi$ implies $X \models \varphi \wedge \psi$;

(b)$_\wedge$ $\varphi \models Y$ implies $\varphi \wedge \psi \models Y$.

We need to slightly modify the following definition of Ciabattoni & Ramanayake [11] because we allow structural connectives of the form $[p]$ where p is a propositional letter. We forbid to use the rules $(p \ \vdash)$ and $(\vdash \ p)$ in the computation of *inv* so that no $[p]$ appears when we execute the algorithms.

Definition 6.4 (*inv*)**.** Let P be a display calculus P for $\mathcal{L}$ and let $X \models Y$ be a $\mathcal{L}$–consecution. The set $inv(X \models Y)$ consists of all the sets of $\mathcal{L}$–consecutions obtained by applying upwards (*i.e.*, from conclusion to premise) some sequence of invertible logical rules in P (and display rules in order to display the formula occurring in the $\mathcal{L}$–consecution) starting from $X \models Y$, except the invertible rules $(p \ \vdash)$ (if the skeleton of p is $(1, \pm, \forall, k)$) and $(\vdash \ p)$ (if the skeleton of p is $(1, \pm, \exists, k)$).

Definition 6.5 (a-soluble and s-soluble, [11])**.** Let P be a display calculus P for $\mathcal{L}$. A formula $\varphi \in$ For$\mathcal{L}$ is *a-soluble* (resp. *s-soluble*) if there is some $\{U_i \models V_i\}_{i\in\Omega} \in inv(\varphi \models \mathbf{I})$ (resp. $\in inv(\mathbf{I} \models \varphi)$) containing no logical connectives.

Definition 6.6 ($\mathcal{I}_0(\mathsf{P}), \mathcal{I}_1(\mathsf{P}), \mathcal{I}_2(\mathsf{P})$, [11])**.** Let P be an amenable calculus for $\mathcal{L}$. The classes $\mathcal{I}_0(\mathsf{P}), \mathcal{I}_1(\mathsf{P}), \mathcal{I}_2(\mathsf{P}) \subseteq$ For$\mathcal{L}$ are defined as follows:

- $\mathcal{I}_0(\mathsf{P}) = \mathbb{P}$;

and for all $\varphi \in$ For$\mathcal{L}$,

- $\varphi \in \mathcal{I}_1(\mathsf{P})$ iff there is some $\{U_i \models V_i\}_{i\in\Omega} \in inv(\mathbf{I} \models \varphi)$ such that $\{U_i \models V_i\}_{i\in\Omega}$ does not contain logical connectives;
- $\varphi \in \mathcal{I}_1(\mathsf{P})$ iff there is some $\{U_i \models V_i\}_{i\in\Omega} \in inv(\mathbf{I} \models \varphi)$ such that each a-part formula in $U_i \models V_i$ is s-soluble and each s-part formula in $U_i \models V_i$ is a-soluble, for each $i \in \Omega$.

In the two latter cases, we say that $\{U_i \models V_i\}_{i\in\Omega}$ witnesses φ.

Note that $\mathcal{I}_0(\mathsf{P}) \subseteq \mathcal{I}_1(\mathsf{P}) \subseteq \mathcal{I}_2(\mathsf{P})$.

Definition 6.7 (Respect multiplicities, [11])**.** Let P be a display calculus for $\mathcal{L}$ and let S be a set of $\mathcal{L}$–consecutions. We denote by $\mathbb{P}(S)$ the set of atoms which occur in S. The set S is said to *respect multiplicities* with respect to an propositional letter $p \in \mathbb{V}(S)$ if S can be partitioned into one of the following forms using the display rules:

$$\{p \models U \mid p \notin U\} \cup \{V \models p \mid \text{every } p \text{ in } V \models p \text{ is s-part}\} \cup \{s \in S \mid p \notin s\} \tag{3}$$

$$\{U \models p \mid p \notin U\} \cup \{p \models V \mid \text{every } p \text{ in } p \models V \text{ is a-part}\} \cup \{s \in S \mid p \notin s\} \tag{4}$$

where $p \notin U$, $p \notin U \models V$ denotes the fact that p does not occur in U or V.

Definition 6.8 (S_p, [11])**.** Let P be a display calculus for $\mathcal{L}$ and let S be a set of $\mathcal{L}$–consecutions respecting multiplicities w.r.t. p. If it is not the case that $p \vdash U \in S$ and $V \vdash p \in S$ (up to display equivalence), then we set S_p as $\{s \in S \mid p \notin S\}$. Otherwise, define S_p as the union of $\{s \in S \mid p \notin s\}$ and one of the following, depending on the display equivalent form of S as (3) or (4), respectively:

$$\{s \mid \text{obtain } s \text{ from } V \vdash p \in S \text{ by substituting each occurrence of } p \text{ with a } U \text{ such that } p \vdash U \in S\}$$
$$\{s \mid \text{obtain } s \text{ from } p \vdash V \in S \text{ by substituting each occurrence of } p \text{ with a } U \text{ such that } U \vdash p \in S\}$$

Note that if S respect multiplicities w.r.t. p, then p does not occur in S_p.

Definition 6.9 (Acyclic set, [11])**.** Let P be a display calculus for $\mathcal{L}$. A finite set S of $\mathcal{L}$–consecutions built only from structural connectives and atoms (and no logical connectives) is *acyclic* if

(i) $\mathbb{P}(S) = \varnothing$ or

(ii) there is $p \in \mathbb{P}(S)$ such that S respects multiplicities w.r.t. p and S_p is acyclic.

Definition 6.10 (Acyclic formula, [11])**.** Let P be a display calculus for $\mathcal{L}$ and let $\varphi \in \mathcal{I}_2(\mathsf{P})$. If there is some set $\{\rho_i\}_{i \in \Omega}$ of inference rules equivalent to φ obtained according to the algorithm of [11, Fig. 1, p. 18] such that the premises of each ρ_i ($i \in \Omega$) are acyclic, then φ is called an acyclic formula.

Definition 6.11 (Display calculus corresponding to a Hilbert calculus, [11])**.** Let P be an amenable calculus for $\mathcal{L}$ and let $\mathsf{P}^{\mathcal{H}}$ be a Hilbert calculus for $\mathcal{L}$. Then, we say that P *corresponds* to $\mathsf{P}^{\mathcal{H}}$ if there is a function $\mathbb{F}$ mapping For$\mathcal{L}$–consecutions to formulas of For$\mathcal{L}$ such that for all $\varphi \in \mathrm{For}\mathcal{L}$,

- $\mathbb{F}(\mathbf{I} \vdash \varphi) = \varphi$;
- for every instance $\dfrac{X_1 \vdash Y_1 \quad \dots \quad X_n \vdash Y_n}{X_{n+1} \vdash Y_{n+1}}$ of a rule in P, there is a derivation in $\mathsf{P}^{\mathcal{H}}$ of $\mathbb{F}(l(X_{n+1}) \vdash r(Y_{n+1}))$, assuming $\mathbb{F}(l(X_1) \vdash r(Y_1)), \dots, \mathbb{F}(l(X_n) \vdash r(Y_n))$;
- for every instance of a rule $\dfrac{\varphi_1 \quad \dots \quad \varphi_n}{\varphi_{n+1}}$ in $\mathsf{P}^{\mathcal{H}}$, there is a derivation in P of $\mathbf{I} \vdash \varphi_{n+1}$, assuming $\mathbf{I} \vdash \varphi_1, \dots, \mathbf{I} \vdash \varphi_n$.

Definition 6.12 (Well-behaved calculus, [11])**.** An amenable calculus P for $\mathcal{L}$ is *well-behaved* if

- P corresponds to some Hilbert calculus $\mathsf{P}^{\mathcal{H}}$;
- P contains the following rules

$$\frac{\varphi \vdash U \quad \psi \vdash U}{\varphi \vee \psi \vdash U} \qquad\qquad \frac{U \vdash \varphi \quad U \vdash \psi}{U \vdash \varphi \wedge \psi}$$

  Here, $\wedge$ and $\vee$ are the connectives in the definition of amenable calculus (not necessarily conjunction, disjunction);
- for every For$\mathcal{L}$–consecution $\varphi \vdash \psi$, we have that $\varphi \vdash \psi \in inv(\mathbf{I} \vdash \mathbb{F}(\varphi \vdash \psi))$.

7 Display calculi for atomic and molecular logics

7.1 Structures and consecutions for atomic logics

In order to provide a sound and complete calculus for an atomic logic based on a set of connectives $\mathsf{C} \subseteq \mathbb{C}$, we will need to resort to the connectives of C which are in the orbits of the free action $\alpha_n * \beta_n$ (for appropriate ns). We introduce these extra connectives in the language as *structural* connectives.

Definition 7.1 (Structural connectives)**.** *(Atomic) structural connectives* are copies of the connectives: for all sets of atomic connectives C, its associated set of structural connectives is denoted $[\mathsf{C}] \triangleq \{[\circledast] \mid \circledast \in \mathsf{C}\}$. For all atomic connectives $\circledast$, the *arity*, *signature*, *type signature*, *tonicity signature*, *quantification signature* of $[\circledast]$ are the same as $\circledast$. Structural connectives are denoted $[p], [p_1], [p_2], \ldots$ and $[\circledast], [\circledast_1], [\circledast_2], \ldots$ For each $k \in \mathbb{N}^*$, we also introduce the *(Boolean) structural connective* associated to the Boolean connectives $\wedge_k, \vee_k$, denoted $\ ,_k\ $ and often simply $\ ,\ $ by abuse and the *structural constant* associated to the constants $\top_k$ and $\bot_k$, denoted $\mathbf{I}_k$ and often simply $\mathbf{I}$ by abuse.

Definition 7.2 (Structural atomic language and consecutions)**.** Let C be a set of atomic connectives. The *structural atomic language* $[\mathcal{L}_\mathsf{C}]$ is the smallest set that contains the atomic language $\mathcal{L}_\mathsf{C}$, the structures $*\varphi$ for all $\varphi \in \mathcal{L}_\mathsf{C}$ as well as $[\mathbb{P}(\mathsf{C})]$ and that is closed under the structural connectives of $[\mathsf{C}] \cup \{\ ,_k \mid k \in \mathbb{N}^*\}$ while respecting the type constraints. Its elements are called *structures* and their *types* are defined like for formulas of $\mathcal{L}_\mathsf{C}$.

A $\mathcal{L}_C$*–consecution* (resp. $[\mathcal{L}_C]$*–consecution*) is an expression of the form $\varphi \vdash \psi$ (resp. $X \vdash Y$), where $\varphi, \psi \in \mathcal{L}^k_\mathsf{C}$ are of the same type, for some $k \in \mathbb{N}^*$ (resp. $X, Y \in [\mathcal{L}_\mathsf{C}]$ are of the same type, for some $k \in \mathbb{N}^*$). The set of all $\mathcal{L}_\mathsf{C}$–consecutions (resp. $[\mathcal{L}_\mathsf{C}]$–consecutions) is denoted $\mathcal{S}_\mathsf{C}$ (resp. $[\mathcal{S}_\mathsf{C}]$).

Elements of $\mathcal{L}_\mathsf{C}$ (resp. $[\mathcal{L}_\mathsf{C}]$ and $[\mathcal{S}_\mathsf{C}]$) are called *formulas* (resp. *structures* and *consecutions*); they are denoted $\varphi, \psi, \alpha, \ldots$ (resp. $X, Y, A, B, \ldots$ and $X \vdash Y, A \vdash B, \ldots$). □

Definition 7.3 (Boolean negation)**.** Let $X \in [\mathcal{L}]$ be a structure. The *Boolean negation of* X, denoted $*X$, is defined inductively as follows ($-\circledast$ was defined in Definition 5.3):

$$*X \triangleq \begin{cases} [-\circledast](X_1, \ldots, X_n) & \text{if } X = [\circledast](X_1, \ldots, X_n) \\ (*X_1\ ,\ *X_2) & \text{if } X = (X_1\ ,\ X_2) \\ \mathbf{I} & \text{if } X = \mathbf{I} \\ \varphi & \text{if } X = *\varphi \\ *\varphi & \text{if } X = \varphi \in \mathcal{L} \end{cases}$$

Note that from that definition, for all structures $X \in [\mathcal{L}]$, it follows that $** X = X$.

Definition 7.4 (Formula associated to a structure)**.** Let C be a set of atomic connectives. We define inductively the function τ_0 and τ_1 from structures of $[\mathcal{L}_\mathsf{C}]$ to formulas of $\mathcal{L}_\mathsf{C}$ as follows: for all $i \in \{0,1\}$, all $\circledast \in \mathsf{C}$ of skeleton $(\sigma, \pm, Æ, \overline{k}, (\pm_1, \ldots, \pm_n))$,

$$\begin{aligned} \tau_0(\mathbf{I}) &\triangleq \top \\ \tau_1(\mathbf{I}) &\triangleq \bot \\ \tau_i(\varphi) &\triangleq \varphi \\ \tau_i(*\varphi) &\triangleq \neg\varphi \\ \tau_0(X\ ,_k\ Y) &\triangleq (\tau_0(X) \wedge_k \tau_0(Y)) \\ \tau_1(X\ ,_k\ Y) &\triangleq (\tau_1(X) \vee_k \tau_1(Y)) \\ \tau_i([\circledast](X_1, \ldots, X_n)) &\triangleq \circledast(\tau_{i_1}(X_1), \ldots, \tau_{i_n}(X_n)) \end{aligned}$$

where for all $j \in [\![1;n]\!]$, $\tau_{i_j}(X_j) \triangleq \begin{cases} \tau_i(X_j) & \text{if } \pm_j = + \\ \tau_{1-i}(X_j) & \text{if } \pm_j = - \end{cases}$.

Then, we define the function τ from $[\mathcal{L}_{\mathsf{C}}]$–consecutions of $[\mathcal{S}_{\mathsf{C}}]$ to $\mathcal{L}_{\mathsf{C}}$–consecutions of $\mathcal{S}_{\mathsf{C}}$ as follows:

$$\tau(X \models Y) \quad \triangleq \quad \tau_0(X) \models \tau_1(Y)$$

Instead of a single structural connective $\;,\;$, we could introduce two Boolean structural connectives $[\wedge]$, $[\vee]$ as a copy of the Boolean connectives $\wedge, \vee$, like for the other atomic connectives $\circledast$. This would not be usual but in line with our approach. This would greatly simplify the definition of the function τ since the interpretation of the structural connectives would then not be context-dependent as here. In particular one would not need two functions τ_0 and τ_1 but only one. We proceed as follows on the one hand in order to stay in line with current practice and on the other hand because it simplifies the subsequent calculus $\mathsf{GGL}_{\mathsf{C}}$ of Figure 3: we use one structural connective (,) instead of two ($[\wedge]$ and $[\vee]$). This said, it would be easily possible to adapt and rewrite the calculus $\mathsf{GGL}_{\mathsf{C}}$ with these two structural connectives $[\wedge]$ and $[\vee]$: the structural connective $\;,\;$ would need to be replaced by $[\wedge]$ in the premise of (dr_2) and in $(\mathsf{B}\vdash), (\mathsf{Cl}\vdash), (\mathsf{K}\vdash), (\wedge\vdash), (\mathbf{I}\vdash)$ and by $[\vee]$ in the conclusion of (dr_2) and in $(\vdash\mathsf{B}), (\vdash\mathsf{Cl}), (\vdash\mathsf{K}), (\vdash\vee)$ (see below). The same comment applies to $\mathbf{I}$: we could introduce two structural connectives $[\top]$ and $[\bot]$ instead of a single one and obtain a more uniform calculus.

Definition 7.5 (Interpretation of atomic structures and consecutions)**.** Let C be a set of atomic connectives and let $M = (W, \mathcal{R})$ be a C–model. We extend the interpretation function $[\![\cdot]\!]^M$ of $\mathcal{L}_{\mathsf{C}}$ in M to $\mathcal{L}_{\mathsf{C}}$–consecutions of $\mathcal{S}_{\mathsf{C}}$ as follows: for all $\varphi, \psi \in \mathcal{L}_{\mathsf{C}}$ and all $w \in W$, we have that $w \in [\![\varphi \models \psi]\!]^M$ iff if $w \in [\![\varphi]\!]^M$ then $w \in [\![\psi]\!]^M$, we have that $w \in [\![\ \models \psi]\!]^M$ iff $w \in [\![\psi]\!]^M$ and we have that $w \in [\![\varphi \models\ ]\!]^M$ iff $w \notin [\![\varphi]\!]^M$. We then extend in a natural way the interpretation function $[\![\cdot]\!]^M$ of $\mathcal{L}_{\mathsf{C}}$ in M to $[\mathcal{L}_{\mathsf{C}}]$–consecutions of $[\mathcal{S}_{\mathsf{C}}]$ as follows: for all $X \in \mathcal{L}_{\mathsf{C}}$, all $X \models Y \in [\mathcal{S}_{\mathsf{C}}]$ and all $w \in W$, we have that $w \in [\![X \models Y]\!]^M$ if, and only if, $w \in [\![\tau(X \models Y)]\!]^M$. If $\mathcal{E}_{\mathsf{C}}$ is a class of C–models, then the satisfaction relation $\Vdash\ \subseteq \mathcal{E}_{\mathsf{C}} \times [\mathcal{S}_{\mathsf{C}}]$ is defined like for formulas of $\mathcal{L}$.

7.2 Display calculi for Boolean atomic logics

Our calculus is defined relatively to an orbit of connectives. This means that if we have a basic atomic logic defined on the basis of some connectives C and if we want to obtain a sound and complete calculus for that logic, we need to consider in the proof system the following associated set of connectives. This is because in the completeness proof, we need to apply the abstract law of residuation (of Definition 3.2) for any arguments j and consider the Boolean negation for each connective.

$$\mathcal{O}(\mathsf{C}) \triangleq (\mathsf{C} \cap \mathbb{B}) \cup \bigcup_{\circledast \in \mathsf{C} - \mathbb{B}} \{\tau_1 - \ldots - \tau_m \circledast \mid \circledast \text{ is of arity } n \text{ and } \tau_1, \ldots, \tau_m \in \mathfrak{S}_{n+1}\} \tag{5}$$

Definition 7.6. Let C be a Boolean set of atomic connectives. We denote by $\mathsf{GGL}_{\mathsf{C}}$ the calculus of Figure 3 where the introduction rules $(\vdash\circledast)$ and $(\circledast\vdash)$ are defined for the connectives $\circledast$ of C, where the rule (dr_1) is defined for the elements τ of an arbitrary set of generators of $\mathfrak{S}_{n+1}$ (for each n ranging over the arities of the connectives of C) and where the structural connectives $[\circledast]$ range over $[\mathcal{O}(\mathsf{C})]$.

Theorem 7.7 (Soundness and strong completeness, [5])**.** *Let C be a Boolean set of atomic connectives. The calculus $\mathsf{GGL}_{\mathsf{C}}$ is sound and strongly complete for the basic atomic logic $(\mathcal{S}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$.*

Structural rules:

$$\frac{(X\ ,\ Y) \vdash U}{(Y\ ,\ X) \vdash U}\ (\mathsf{Cl} \vdash) \qquad \frac{X \vdash U}{(X\ ,\ Y) \vdash U}\ (\mathsf{K} \vdash)$$

$$\frac{(X\ ,\ X) \vdash U}{X \vdash U}\ (\mathsf{Wl} \vdash) \qquad \frac{\mathbf{I} \vdash U}{\vdash U}\ (\mathbf{I} \vdash)$$

$$\frac{U \vdash \varphi \quad \varphi \vdash V}{U \vdash V}\ cut$$

Display rules:

$$\frac{S\left([\circledast], X_1, \dots, X_n, X_{n+1}\right)}{S\left([\tau\circledast], X_{\tau(1)}, \dots, X_{\tau(n)}, X_{\tau(n+1)}\right)}\ (\mathsf{dr}_1) \qquad \frac{(X\ ,\ Y) \vdash Z}{X \vdash (Z\ ,\ *Y)}\ (\mathsf{dr}_2)$$

Introduction rules:

$$\frac{}{\bot \vdash \mathbf{I}}\ (\vdash \bot) \qquad \frac{U \vdash \mathbf{I}}{U \vdash \bot}\ (\bot \vdash)$$

$$\frac{}{\mathbf{I} \vdash \top}\ (\vdash \top) \qquad \frac{\mathbf{I} \vdash U}{\top \vdash U}\ (\top \vdash)$$

$$\frac{U \vdash\ *\varphi}{U \vdash \neg\varphi}\ (\vdash \neg) \qquad \frac{*\varphi \vdash U}{\neg\varphi \vdash U}\ (\neg \vdash)$$

$$\frac{U \vdash \varphi \quad U \vdash \psi}{U \vdash (\varphi \wedge \psi)}\ (\vdash \wedge) \qquad \frac{\varphi \vdash U}{(\varphi \wedge \psi) \vdash U}\ (\wedge \vdash)$$

$$\frac{U \vdash \varphi}{U \vdash (\varphi \vee \psi)}\ (\vdash \vee) \qquad \frac{\varphi \vdash U \quad \psi \vdash U}{(\varphi \vee \psi) \vdash U}\ (\vee \vdash)$$

$$\frac{U_1 \vdash V_1 \quad \dots \quad U_n \vdash V_n}{S\left([\circledast], X_1, \dots, X_n, \circledast(\varphi_1, \dots, \varphi_n)\right)}\ (\vdash \circledast) \qquad \frac{S\left([\circledast], \varphi_1, \dots, \varphi_n, U\right)}{S\left(\circledast, \varphi_1, \dots, \varphi_n, U\right)}\ (\circledast \vdash)$$

In rules $(\vdash \circledast)$ and $(\circledast \vdash)$, for all $\circledast \in \mathsf{C}$ of skeleton $\star = (\sigma, \pm, \text{Æ}, \overline{k}, (\pm_1, \dots, \pm_n))$:

- for all $j \in [\![1; n]\!]$, we set $U_j \vdash V_j \triangleq \begin{cases} X_j \vdash \varphi_j & \text{if } \pm_j \pm (\text{Æ}) = - \\ \varphi_j \vdash X_j & \text{if } \pm_j \pm (\text{Æ}) = + \end{cases}$

  such that, in rule $(\vdash \circledast)$, for all j X_j is not empty and if φ_j is empty for some j then $\circledast(\varphi_1, \dots, \varphi_n)$ is also empty.

- for all $c \in \{\circledast, [\circledast]\}$, $S(c, X_1, \dots, X_n, X) \triangleq \begin{cases} c(X_1, \dots, X_n) \vdash X & \text{if Æ} = \exists \\ X \vdash c(X_1, \dots, X_n) & \text{if Æ} = \forall \end{cases}$.

If X is empty then $*X$ is empty and $(X\ ,\ Y)$ and $(Y\ ,\ X)$ are equal to Y.

Figure 3: Calculus GGL_C

The axioms and inference rules for atoms p are special instances of the rules $(\vdash \circledast)$ and $(\circledast \vdash)$ of Figure 3. With $\circledast = p$, we have that $n = 0$ and, replacing $\circledast$ with p in $(\vdash \circledast)$, we obtain the inference rules below. Note that $(\vdash p)$ is in fact an axiom.

$$\frac{}{S([p], p)}\ (\vdash p) \qquad\qquad \frac{S([p], X)}{S(p, X)}\ (p \vdash)$$

where, if $\circledast$ is p or $[p]$, then $S(\circledast, X) \triangleq \begin{cases} \circledast \models X & \text{if } Æ = \exists \\ X \models \circledast & \text{if } Æ = \forall \end{cases}$.

Hence, for all $p = (\mathrm{Id}, \pm, Æ, k)$, if $Æ = \exists$ then $(\vdash p)$ and $(p \vdash)$ rewrite as follows:

$$\frac{}{[p] \models p}\ (\vdash p) \qquad\qquad \frac{[p] \models X}{p \models X}\ (p \vdash) \tag{6}$$

and if $Æ = \forall$ then $(\vdash p)$ and $(p \vdash)$ rewrite as follows:

$$\frac{}{p \models [p]}\ (\vdash p) \qquad\qquad \frac{X \models [p]}{X \models p}\ (p \vdash) \tag{7}$$

Note that in both cases, the standard axiom $p \models p$ is derivable by applying $(p \vdash)$ once again to $[p] \models p$ or $p \models [p]$. If $[p]$ is replaced by $\mathbf{I}$ and p by $\top$ in the first pair and if $[p]$ is replaced by $\mathbf{I}$ and p by $\bot$ in the second pair then we obtain respectively the operational rules $(\models \top)$, $(\top \models)$, $(\bot \models)$ and $(\models \bot)$ of Kracht [22] and Belnap [7]. This is meaningful since truth constants can be seen as special propositional letters, those that are always true or always false. Then, one needs, like in the calculus **DLM** of Kracht [22], to impose some conditions on these atoms by means of the structural inference rules $(\mathbf{I} \vdash)$ and $(Q \vdash)$ so that these special atoms $\top$ and $\bot$ do behave as truth constants, as intended. The reading of $\mathbf{I}$, either as $\top$ or as $\bot$, could be separated by means of two structural constants, like for the structural connective $\,,\,$. Alternatively, one can easily prove that adding the following axioms to our calculus $\mathsf{GGL_C}$ is enough to capture the standard truth constants $\top$ and $\bot$:

$$\frac{}{\bot \models}\ (\bot \vdash) \qquad\qquad \frac{}{\models \top}\ (\vdash \top)$$

Atomic logics have four different propositional letter skeletons of type 1: $(\mathrm{Id}, +, \forall, 1)$, $(\mathrm{Id}, +, \exists, 1)$, $(\mathrm{Id}, -, \forall, 1)$, $(\mathrm{Id}, -, \exists, 1)$. Their eight introduction rules are the same as the eight rules $(\mathbf{1}), (\bot), (\top), (\mathbf{0})$ of Belnap's display calculus for linear logic [8, p. 19]. Hence, with appropriate structural rules, our four propositional letter skeletons of type 1 can stand for the four propositional constants of linear logic.

7.3 Display calculi for atomic logics

Until now, our display calculi are sound and complete for logics including the Boolean connectives. However, we would like to obtain calculi for plain atomic logics, without Boolean connectives. Indeed, we consider the latter to be more primitive than Boolean atomic logics because even the Boolean connectives can be seen as particular atomic connectives, interpreted over special relations (identity relations, see Example 3.8). These special relations are obtained at the proof-theroretical level by imposing the validity of Gentzen's structural rules. So, in this section, we are going to define sound and complete calculi for (plain) atomic logics, without Boolean connectives.

Display rules:

$$\frac{S\left([\circledast], X_1, \ldots, X_n, X_{n+1}\right)}{S\left(\left[\tau \circledast\right], X_{\tau(1)}, \ldots, X_{\tau(n)}, X_{\tau(n+1)}\right)} \left(\mathsf{dr}_1\right) \qquad \frac{X \vdash Y}{*Y \vdash \; * X} \left(\mathsf{dr}_2'\right)$$

Introduction rules:

$$\frac{U_1 \vdash V_1 \quad \ldots \quad U_n \vdash V_n}{S\left([\circledast], X_1, \ldots, X_n, \circledast\left(\varphi_1, \ldots, \varphi_n\right)\right)} (\vdash \circledast) \qquad \frac{S\left([\circledast], \varphi_1, \ldots, \varphi_n, U\right)}{S\left(\circledast, \varphi_1, \ldots, \varphi_n, U\right)} (\circledast \vdash)$$

In rules $(\vdash \circledast)$ and $(\circledast \vdash)$, for all $\circledast = (\sigma, \pm, (Æ, (\pm_1, \ldots, \pm_n))) \in \mathsf{C}$:

- for all $j \in [\![1; n]\!]$, we set $U_j \vdash V_j \triangleq \begin{cases} X_j \vdash \varphi_j & \text{if } \pm_j \pm (Æ) = - \\ \varphi_j \vdash X_j & \text{if } \pm_j \pm (Æ) = + \end{cases}$

  such that, in rule $(\vdash \circledast)$, for all j X_j is not empty and with the convention that if φ_j is empty for some j then $\circledast(\varphi_1, \ldots, \varphi_n)$ is also empty.
- for all $\star \in \{\circledast, [\circledast]\}$, $S(\star, X_1, \ldots, X_n, X) \triangleq \begin{cases} \star(X_1, \ldots, X_n) \vdash X & \text{if } Æ = \exists \\ X \vdash \; \star(X_1, \ldots, X_n) & \text{if } Æ = \forall. \end{cases}$

Figure 4: Calculus $\mathsf{GGL}^0_{\mathsf{C}}$

Definition 7.8. Let C be a set of atomic connectives without Boolean connectives. We denote by $\mathsf{GGL}^0_{\mathsf{C}}$ the calculus of Figure 4 where the introduction rules $(\vdash \circledast)$ and $(\circledast \vdash)$ are defined for the connectives $\circledast$ of C, where the rule (dr_1) is defined for the elements τ of an arbitrary set of generators of $\mathfrak{S}_{n+1}$ (for each n ranging over the arities of the connectives of C) and where the structural connectives $[\circledast]$ range over $[\mathcal{O}(\mathsf{C})]$.

The calculus $\mathsf{GGL}^0_{\mathsf{C}}$ is in fact the calculus of Figure 3 where the structural rules and the introduction rules for the Boolean connectives have been removed and where the display rule dr_2 has been replaced by the display rule $\left(\mathsf{dr}_2'\right)$.

Theorem 7.9 (Soundness and strong completeness, [5])**.** *Let C be a set of atomic connectives without Boolean connectives. The calculus $\mathsf{GGL}^0_{\mathsf{C}}$ is sound and strongly complete for the basic atomic logic $(\mathcal{S}^0_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$.*

7.4 Display calculi for molecular logics

In this section, we provide display calculi for molecular logics whose connectives are so-called 'universal' or 'existential'. Universal and existential molecular connectives are essentially molecular connectives such that the quantification patterns of the quantification signatures of their successive atomic connectives are of the form $\forall \ldots \forall$ or $\exists \ldots \exists$ respectively. They essentially behave as 'macroscopic' atomic connectives of quantification signatures $\forall$ or $\exists$.

Definition 7.10 (Universal and existential molecular connective)**.** A *universal (resp. existential) molecular skeleton* is a molecular skeleton c different from any id^l_k for any $k, l \in \mathbb{N}^*$ such that $Æ(c) = \forall$ (resp. $Æ(c) = \exists$) and such that for each node of its decomposition tree labeled with $\circledast = (\sigma, \pm, Æ, \overline{k}, (\pm_1, \ldots, \pm_n))$ and each of its jth children labeled with some $\circledast_j \in \mathbb{C}$ such that the subtree generated by this j^{th} children contains at least one id_k^l, we have that $Æ(\circledast_j) = \pm_j Æ$. A *universal (resp. existential) molecular connective* is a molecular connective with a universal (resp. existential) skeleton.

Example 7.11. On the one hand, the molecular connective $\circledast(p, id_k^l)$ is a universal (resp. existential) molecular connective if $Æ(\circledast) = \forall$ (resp. $Æ(\circledast) = \exists$). Likewise, $\supset (id_1^1, \Box id_1^2)$ and $\otimes(\Diamond id_1^1, p)$ are universal and existential molecular connectives respectively. On the other hand, the molecular connectives $\Box \Diamond^- id_1^1$ and $\supset (\Box id_1^1, \Box id_1^2)$ are neither universal nor existential molecular connectives.

Definition 7.12. Let C be a set of molecular connectives which are either universal or existential. The calculi $\mathsf{GGL}_\mathsf{C}^{0,*}$ and $\mathsf{GGL}_\mathsf{C}^{*}$ are the display calculi such that:

- the introduction rule for each molecular connective of C is the composition of the introduction rules of the atomic connectives which compose it;
- the display rules are those which are associated to each atomic connective that appears in a molecular connective of C;
- the structural rules of $\mathsf{GGL}_\mathsf{C}^{*}$ are the same as those of GGL_C ($\mathsf{GGL}_\mathsf{C}^{0,*}$ does not contain any).

Note that if C are atomic connectives then $\mathsf{GGL}_\mathsf{C}^{0,*}$ and $\mathsf{GGL}_\mathsf{C}^{*}$ are $\mathsf{GGL}_\mathsf{C}^{0}$ and GGL_C respectively.

Theorem 7.13 (Soundness and strong completeness, [5])**.** *Let C be a set of molecular connectives which are all either universal or existential. If C is without Boolean connectives then the calculus $\mathsf{GGL}_\mathsf{C}^{0,*}$ is sound and strongly complete for the molecular logic $(\mathcal{S}_\mathsf{C}^0, \mathcal{M}_\mathsf{C}, \Vdash)$. If C contains Boolean connectives then the calculus $\mathsf{GGL}_\mathsf{C}^{*}$ is sound and strongly complete for the molecular logic $(\mathcal{S}_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$.*

8 Hilbert calculi for atomic logics

In this section on Hilbert calculi, we define the notion of provability (deducibility) from a set of formulas, *i.e.* $\Gamma \vdash_\mathsf{P} \varphi$, differently, like for modal logic [10, Definition 4.4]. If $\mathsf{L} = (\mathcal{L}, E, \models)$ is a Boolean atomic logic and we have that $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$ of type k, then we say that φ is *provable from* Γ in a proof system P for $\mathcal{L}$, written $\Gamma \vdash_\mathsf{P} \varphi$, when $\vdash_\mathsf{P} \varphi$ or there are $n \in \mathbb{N}^*$ and $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_\mathsf{P} (\varphi_1 \wedge_k \ldots \wedge_k \varphi_n) \rightarrow_k \varphi$ (we use the abbreviation $\varphi \rightarrow_k \psi \triangleq (\neg_k \varphi \vee_k \psi)$).

Definition 8.1. Let C be a set of atomic connectives complete for conjunctions and disjunctions. We denote by $\mathsf{GGL}_\mathsf{C}^{\mathcal{H}}$ the calculus of Figure 5 restricted to the axioms and inference rules which mention the atomic connectives of C.

Theorem 8.2 (Soundness and strong completeness, [5])**.** *Let C be a set of atomic connectives complete for Boolean connectives such that $\mathcal{O}(\mathsf{C}) = \mathsf{C}$. The calculus $\mathsf{GGL}_\mathsf{C}^{\mathcal{H}}$ is sound and strongly complete for the Boolean basic atomic logic $(\mathcal{L}_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$.*

Proposition 8.3. *Let C be a set of atomic connectives complete for conjunctions and disjunctions and such that $\mathcal{O}(\mathsf{C}) = \mathsf{C}$. Then, the display calculus GGL_C corresponds to the Hilbert calculus $\mathsf{GGL}_\mathsf{C}^{\mathcal{H}}$.*

Axiom schemas:

$\top, \neg\bot$ (A$_0$)
$(\varphi \to (\varphi \wedge \varphi))$ (A$_1$)
$((\varphi \wedge \psi) \to \varphi)$ (A$_2$)
$((\varphi \to \psi) \to (\neg(\psi \wedge \chi) \to \neg(\chi \wedge \varphi)))$ (A$_3$)

$\neg \circledast (\varphi_1, \ldots, \varphi_n) \leftrightarrow - \circledast (\varphi_1, \ldots, \varphi_n)$ (A$_4$)

For all $\circledast$ of skeleton $(\sigma, \pm, \overline{k}, (\exists, (\pm_1, \ldots, \pm_j, \ldots, \pm_n)))$:
if $\pm_j = +$ then
$(\circledast(\varphi_1, \ldots, \varphi_j \vee \varphi'_j, \ldots, \varphi_n) \to (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi'_j, \ldots, \varphi_n)))$ (A$_5$)
if $\pm_j = -$ then
$(\circledast(\varphi_1, \ldots, \varphi_j \wedge \varphi'_j, \ldots, \varphi_n) \to (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi'_j, \ldots, \varphi_n)))$ (A$_6$)

For all $\circledast$ such that $Æ(\circledast) = \exists : \circledast(\varphi_1, \ldots, \tau_j \circledast (\varphi_1, \ldots, \varphi_n), \ldots, \varphi_n) \to \varphi_j$ (A$_7$)
For all $\circledast$ such that $Æ(\circledast) = \forall : \varphi_j \to \circledast(\varphi_1, \ldots, \tau_j \circledast (\varphi_1, \ldots, \varphi_n), \ldots, \varphi_n)$ (A$_8$)

Inference rules:

from φ and $(\varphi \to \psi)$, infer ψ (MP)

For all $\circledast$ of skeleton $(\sigma, \pm, \overline{k}, (\forall, (\pm_1, \ldots, \pm_j, \ldots, \pm_n)))$:
if $\pm_j = +$ then from φ_j, infer $\circledast (\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n)$ (R$_1$)
if $\pm_j = -$ then from $\neg\varphi_j$, infer $\circledast (\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n)$ (R$_2$)

For all $\circledast$ of skeleton $(\sigma, \pm, \overline{k}, (\exists, (\pm_1, \ldots, \pm_j, \ldots, \pm_n)))$:
if $\pm_j = +$ then from $\varphi_j \to \psi_j$, infer $\circledast (\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \to \circledast(\varphi_1, \ldots, \psi_j, \ldots, \varphi_n)$ (R$_3$)
if $\pm_j = -$ then from $\varphi_j \to \psi_j$, infer $\circledast (\varphi_1, \ldots, \psi_j, \ldots, \varphi_n) \to \circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n)$ (R$_4$)

Figure 5: Calculus $\mathsf{GGL}^{\mathcal{H}}$

9 Inductive atomic formulas

Universal and existential molecular connectives are essentially molecular connectives such that the quantification patterns of the quantification signatures of their successive atomic connectives are of the form $\forall \ldots \forall$ or $\exists \ldots \exists$ respectively. They essentially behave as 'macroscopic' atomic connectives of quantification signatures $\forall$ or $\exists$.

Definition 9.1 (Universal and existential molecular connective)**.** A *universal (resp. existential) molecular skeleton* is a molecular skeleton c different from any id^l_k for any $k, l \in \mathbb{N}^*$ such that $Æ(c) = \forall$ (resp. $Æ(c) = \exists$) and such that for each node of its decomposition tree labeled with $\circledast = (\sigma, \pm, Æ, \overline{k}, (\pm_1, \ldots, \pm_n))$ and each of its jth children labeled with some $\circledast_j \in \mathbb{C}$ such that the subtree generated by this j^{th} children contains at least one id^l_k, we have that $Æ(\circledast_j) = \pm_j Æ$. A *universal (resp. existential) molecular connective* is a molecular connective with a universal (resp. existential) skeleton.

Example 9.2. On the one hand, the molecular connective $\circledast(p, id^l_k)$ is a universal (resp. existential) molecular connective if $Æ(\circledast) = \forall$ (resp. $Æ(\circledast) = \exists$). Likewise, $\supset_1 (id^1_1, \Box id^2_1)$ and $\otimes_3(\Diamond id^1_1, p)$ are universal and existential molecular connectives respectively. On the other hand, the molecular connectives $\Box \Diamond id^1_1$ and $\supset_1 (\Box id^1_1, \Box id^2_1)$ are neither universal nor existential molecular connectives.

Definition 9.3 (Tonicity)**.** The *tonicity w.r.t. the jth argument* of a molecular connective c, denoted $tn(c, j)$, is defined inductively as follows:

- if $c = \circledast$ is an atomic connective of skeleton $\star = (\sigma, \pm, Æ, \overline{k}, (\pm_1, \dots, \pm_n)) \in \mathbb{C}$, then $tn(c,j) = \pm_j$;
- if $c = \neg c'$ then $tn(c,j) = -tn(c',j)$;
- if $c = \wedge(c_1, c_2)$ or $c = \vee(c_1, c_2)$ then $tn(c,j) = tn(c_k, j_k)$, where c_k is the molecular connective c_1 or c_2 that takes the jth argument of c also as argument at place j_k;
- if $c = \circledast(c_1, \dots, c_n)$ with $\star = (\sigma, \pm, Æ, \overline{k}, (\pm_1, \dots, \pm_n))$ then $tn(c,j) = \pm_k tn(c_k, j_k)$ where c_k is the molecular connective of the decomposition tree of c which takes the jth argument of c as argument at place j_k.

Definition 9.4 (Positive and negative formula)**.** Let C be a set of molecular connectives and let $\varphi \in \mathcal{L}^*_{\mathsf{C}}$. The formula φ can be written as a formula of the form $c(p_1, \dots, p_n)$ where $p_1, \dots, p_n$ are all the propositional letters that occur in φ (possibly with repetitions) and c is a molecular connective (not necessarily belonging to C). We say that φ is *positive* (resp. *negative*) if for all $j \in [\![1;n]\!]$, $tn(c,j)(p_j) = +$ (resp. $tn(c,j)(p_j) = -$).

Definition 9.5 (Essentially universal and existential formulas)**.** An *essentially universal formula* (resp. *essentially existential formula*) is either a negative (resp. positive) atomic formula or a formula of a molecular language of the form $c(\varphi_1, \dots, \varphi_{i-1}, p, \varphi_{i+1}, \dots, \varphi_n)$ where c is a universal molecular connective (resp. existential molecular connective) and for all $j \in [\![1;n]\!] \backslash \{i\}$, φ_j is either a positive formula if $tn(c,j) = -$ (resp. $tn(c,j) = +$) or a negative formula if $tn(c,j) = +$ (resp. $tn(c,j) = -$) and $p \in \mathbb{P}$ is a propositional letter in an arbitrary position $i \in [\![1;n]\!]$ but such that $tn(c,i) = +$ (resp. $tn(c,i) = -$). In that case, p is called the *head* of the essentially universal (resp. existential) formula and the formula is said to be *headed*.

Definition 9.6 (Regular formula)**.** A *regular formula* φ is a formula of a molecular language of the form $\varphi = c(\varphi_1, \dots \varphi_n)$ where c is a universal molecular connective and such that for all $j \in [\![1;n]\!]$, φ_j is an essentially universal formula if $tn(c,j) = -$ or an essentially existential formula if $tn(c,j) = +$. The headed formulas $\varphi_1, \dots, \varphi_n$ are called the *main components of* φ and the *heads of* φ are the heads of $\varphi_1, \dots, \varphi_n$ (if they exist).

There might indeed be no head of φ if the φ_is are positive or negative formulas.

Example 9.7. The modal formula $(\Box p \to (\Box q \to \Diamond r))$ can be seen as a regular formula $c(\varphi_1, \varphi_2, \varphi_3)$ where c is the universal molecular connective associated to $c(x,y,z) \triangleq (x \to (y \to z))$, φ_1 is the essentially universal formula $\Box p$, φ_2 is the essentially universal formula $\Box q$ and φ_3 is the essentially existential formula $\Diamond r$. Note that φ_3 is in fact a positive formula and does not contain any head.

Definition 9.8 (Essential and inessential atom)**.** An occurrence of a propositional letter in a regular formula φ is *essential in* φ if it is the head of a main component of the formula, otherwise it is *inessential in* φ. A propositional letter in a regular formula φ is *essential in* φ if it has at least one essential occurrence in it, otherwise it is *inessential in* φ.

Definition 9.9 (Dependency digraph)**.** Given a regular formula $\varphi = c(\varphi_1, \dots, \varphi_n)$ with main components $\{\varphi_1, \dots, \varphi_k\}$, the *dependency digraph* of φ is a digraph $G = (V_\varphi, E_\varphi)$ where $V_\varphi = \{p_1, \dots, p_n\}$ is the set of heads of φ and we set $p_i E_\varphi p_j$ iff p_i occurs as an inessential propositional letter in a formula from $\{\varphi_1, \dots, \varphi_k\}$ with a head p_j. A digraph is *acyclic* if it does not contain oriented cycles.

Definition 9.10 (Inductive atomic formula)**.** An *inductive atomic formula* is a regular formula with an acyclic dependency digraph.

10 Acyclic formulas are inductive

Proposition 10.1. *Let C be a Boolean set of atomic connectives such that $\mathcal{O}(C) = C$. Then GGL_C is* amenable.

Remark 1. The definition of an amenable calculus by Ciabattoni & Ramanayake [11, Definition 3.1] considers only the Boolean connectives $\wedge$, $\vee$, $\top$ and $\bot$ of type 1. Yet, all their results are straightforwardly transferable to our setting with multiple types $\wedge_k$ $\vee_k$, $\top_k$ and $\bot_k$.

Proposition 10.2. *Let C be a set of atomic connectives. A formula $\varphi \in \mathcal{L}_C$ is* s-soluble *(resp.* a-soluble*) iff it can be equivalently rewritten as $c(p_1, \ldots, p_n)$ where c is a universal atomic connective (resp. existential atomic connective) and $p_1, \ldots, p_n$ are atoms.*

Proposition 10.3. *Let C be a Boolean set of atomic connectives. For all $\varphi \in \mathcal{L}_C$,*

- *$\varphi \in \mathcal{I}_1(GGL_C)$ iff φ can be equivalently rewritten as $c(p_1, \ldots, p_n)$ where c is a universal molecular connective and $p_1, \ldots, p_m \in \mathbb{P}(C)$;*
- *$\varphi \in \mathcal{I}_2(GGL_C)$ iff φ can be equivalently rewritten as $c(\varphi_1, \ldots, \varphi_n)$ where c is a universal molecular connective and each φ_j is either an atom or of the form $c_j(p_1, \ldots, p_m)$ with c_j universal (that is, it is s-soluble) if $tn(c, j) = -$ or of the form $c'_j(p'_1, \ldots, p'_m)$ with c'_j existential (that is, it is a-soluble) if $tn(c, j) = +$, for some $p_1, \ldots, p_m, p'_1, \ldots, p'_m \in \mathbb{P}(C)$.*

Hence, every $\varphi \in \mathcal{I}_2(GGL_C)$ is a regular formula, but the converse does not necessarily hold.

In other words, $\varphi \in \mathcal{I}_2(\mathsf{GGL}_C)$ if φ begins with a universal quantification and the alternation depth of the quantification signatures does not exceed 1.

Theorem 10.4. *Let C be a set of atomic connectives. For all $\varphi \in \mathcal{I}_2(GGL_C)$, φ is an inductive atomic formula iff φ is acyclic.*

Corollary 10.5. *Let C be a Boolean set of atomic connectives and let S be a set of acyclic formulas of $\mathcal{I}_2(GGL_C)$. The calculus $GGL_C^{\mathcal{H}} + S$ is sound and strongly complete for the Boolean atomic logics $(\mathcal{L}_C, \mathcal{E}_C, \Vdash)$, where $\mathcal{E}_C$ is the class of pointed C–frames defined by the first–order correspondents of the inductive atomic formulas of S.*

11 A characterization of displayable molecular logics

Roughly, the following theorem states that a calculus is properly displayable iff it is sound and complete w.r.t. a class of models defined by acyclic formulas.

Theorem 11.1. *Let C be a set of molecular connectives which are universal or existential and let ρ_0 be a set of structural inference rules for $[\mathcal{L}_C]$. The calculus $GGL_C^{0,*} + \rho_0$ is equivalent to a proper display calculus $GGL_C^{0,*} + \rho$ iff it is sound and complete for a molecular logic $(\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ whose class of C–frames is definable by the local first–order correspondents of a set Δ of acyclic formulas of $\mathcal{I}_2(GGL_{C'})$ where C' is the Boolean completion of $\mathcal{O}(C'')$, for C'' the set of associated atomic connectives of C. Moreover the acyclic formulas of Δ are effectively computable from the analytic structural rules of ρ, and vice versa, as specified in Figure 6.*

Corollary 11.2. *Let C be a set of atomic connectives. Every proper display calculus extending GGL_C^0 is sound and strongly complete for a logic $(\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ whose class of C–frames $\mathcal{E}_C$ is definable by a set of acyclic formulas of $\mathcal{I}_2(GGL_{C'})$ where C' is the Boolean completion of $\mathcal{O}(C)$. Moreover the acyclic formulas of Δ are effectively computable from the analytic structural rules of the calculus.*

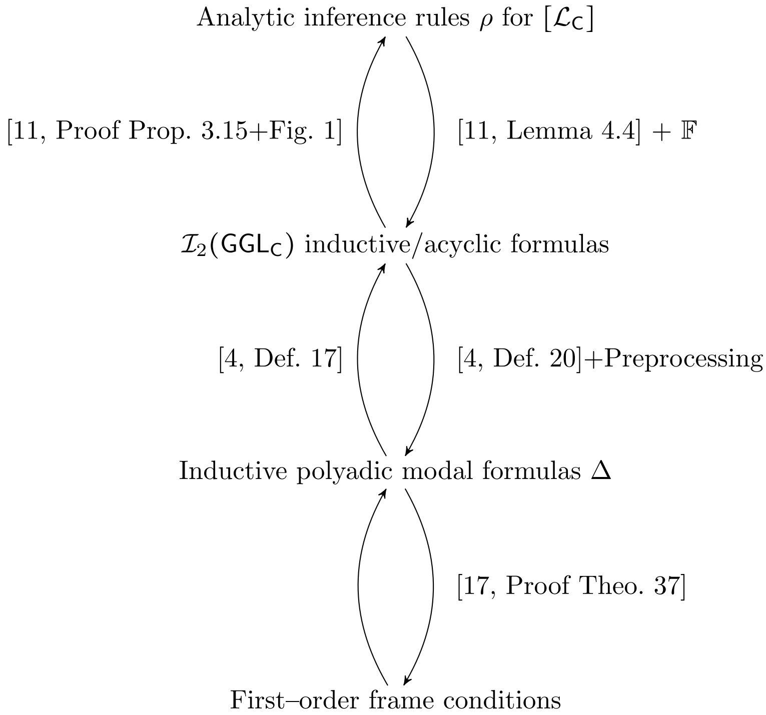

Figure 6: Translations from analytic structural inference rules to first-order frame conditions, and vice versa

Hence, the modal logic $\mathsf{KL} \triangleq \mathsf{K} + \{\Box(\Box p \to p) \to \Box p)\}$ is not displayable by an extension of $\mathsf{GGL}_{\mathsf{C}}$ (for some C) because it is not sound and strongly complete w.r.t. any class of (Kripke) frames. Likewise, the modal logic $\mathsf{KM} \triangleq \mathsf{K} + \{\Box \Diamond \varphi \to \Diamond \Box \varphi\}$ is not displayable by an extension of $\mathsf{GGL}_{\mathsf{C}}$ (for some C) because the McKinsey formula does not correspond to a first–order condition on (Kripke) frames.

12 Examples of correspondences

In Figure 8, we provide a number of first–order frame correspondents of analytic inference rules. All the connectives are of type $(1,1)$ or $(1,1,1)$ and by abuse we confuse the atomic connectives with their skeletons. In these figures, we use the following abbreviations and notations: for all $i, j \in \{1,2,3\}$,

- $[\otimes_i]$ and $[\otimes_j]$ range over $\{[\otimes_3],[\otimes_2],[\otimes_1]\}$ and $[\oplus_i]$ and $[\oplus_j]$ range over $\{[\oplus_3],[\oplus_2],[\oplus_1]\}$ (see Figure 7);
- $[\prec_i]$ ranges over $\{[\prec_1],[\prec_2],[\prec_3]\}$ and $[\succ_i]$ ranges over $\{[\succ_1],[\succ_2],[\succ_3]\}$(see Figure 7);
- $[\subset_i]$ ranges over $\{[\subset_3],[\subset_2],[\subset_1]\}$ and $[\supset_i]$ ranges over $\{[\supset_3],[\supset_2],[\supset_1]\}$ (see Figure 7);
- $[\sim_i]$ and $[\sim_j]$ range over $\{[\sim_1],[\sim_2]\}$ and $[-_i]$ and $[-_j]$ range over $\{[-_1],[-_2]\}$ (see Figure 7).

Connective	Atomic skeleton	Connective	Atomic skeleton
t	$(Id,+,-)$	$\otimes_3$	$(\sigma_1,+,s_1)$
t'	$(Id,+,+)$	$\otimes_2$	$(\sigma_3,+,s_1)$
f	$(Id,-,+)$	$\otimes_1$	$(\sigma_5,+,s_1)$
f'	$(Id,-,-)$	$\oplus_3$	$(\sigma_1,-,s_4)$
$\sim_1$	$(\tau_2,+,t_4)$	$\oplus_2$	$(\sigma_3,-,s_4)$
$\sim_2$	$(\tau_1,t_4,+)$	$\oplus_1$	$(\sigma_5,s_4,-)$
$-_1$	$(\tau_2,t_3,+)$	$\succ_3$	$(\sigma_1,s_6,+)$
$-_2$	$(\tau_1,t_3,+)$	$\succ_2$	$(\sigma_3,s_6,+)$
$\Diamond$	$(\tau_2,t_1,+)$	$\succ_1$	$(\sigma_5,s_6,+)$
$\Diamond^-$	$(\tau_1,t_1,+)$	$\prec_3$	$(\sigma_1,s_5,+)$
$\Box^-$	$(\tau_1,t_2,-)$	$\prec_2$	$(\sigma_3,s_5,+)$
$\Box$	$(\tau_2,t_2,-)$	$\prec_1$	$(\sigma_5,s_5,+)$
		$\supset_3$	$(\sigma_1,s_3,-)$
		$\supset_2$	$(\sigma_3,s_3,-)$
		$\supset_1$	$(\sigma_5,s_3,-)$
		$\subset_3$	$(\sigma_1,s_2,-)$
		$\subset_2$	$(\sigma_3,s_2,-)$
		$\subset_1$	$(\sigma_5,s_2,-)$

Figure 7: Some modal and substructural connectives of atomic logics

Structural connectives are generally denoted $[\otimes_i] = [(\sigma_i, t_i, \pm)]$ (or $[\otimes_j] = [(\sigma_j, t_j, \pm)]$). In these first–order correspondents, we use the notation $R \triangleq R^{\pm\sigma_i}_{\otimes_i}$ if $R^{\pm\sigma_i}_{\otimes_i}$ is a ternary relation and $S \triangleq R^{\pm\sigma_i}_{\otimes_i}$ if $R^{\pm\sigma_i}_{\otimes_i}$ is a binary relation, as well as the following notations:

$$R(xy)zw \triangleq \exists u(Rxyu \wedge Ruzw) \qquad\qquad Rx(yz)w \triangleq \exists u(Ryzu \wedge Rxuw)$$

In these first–order correspondents, the nullary connectives (atoms) t and f are associated in the frames to the unary predicates denoted T and F respectively.

The abbreviations $R(xy)zw$ and $Rx(yz)w$ correspond to the abbreviations of [26]. The structural connective ";" of the literature [26] corresponds in our setting to the structural connective $[\otimes_3]$. Likewise, the usual structural connective $\bullet$ corresponds in our setting to the structural connective $[\Diamond^-]$ (resp. $[\Box]$) when $\bullet$ is in a–part (resp. s–part).

In the rest of this section, we execute the algorithms of Figure 6 (top to bottom) on some classical structural inference rules to compute the first–order frame conditions to which they correspond. Each item in the following subsections corresponds to one of the three algorithms mentioned. The interpretation functions l and r are τ_0 and τ_1 of Definition 7.4 respectively. For the third and fourth items, we also use notations introduced in [4].

12.1 Rule K_{i}

- $\dfrac{X \vdash Z}{X\,[\otimes_{\mathrm{i}}]\,Y \vdash Z}$
- $(l(X\,[\otimes_{\mathrm{i}}]\,Y) \vdash r(Z))[Z/l(X)][X/p][Y/q]$

  $p \otimes_{\mathrm{i}} q \vdash p$

  $p \otimes_{\mathrm{i}} q \to p$

- $[i_2(\alpha_{\otimes_i}, i_0)](\neg p, \neg q, p)$
- $\forall y_1y_2y_3 \forall PQ \left(R_{i_2(\alpha_{\otimes_i}, i_0)} y_1y_2y_3x \to \neg P(y_1) \vee \neg Q(y_2) \vee P(y_3)\right)$
  $\forall PQ \forall y_1y_2y_3 \left(R_{i_2(\alpha_{\otimes_i}, i_0)} y_1y_2y_3x \to \neg P(y_1) \vee \neg Q(y_2) \vee P(y_3)\right)$
  $\forall PQ \forall y_1y_2y_3 \left(R_{i_2(\alpha_{\otimes_i}, i_0)} y_1y_2y_3x \wedge P(y_1) \wedge Q(y_2) \to P(y_3)\right)$
  As minimal valuations, we take $\sigma(P) := \lambda u.y_1 = u$ and $\sigma(Q) := \lambda u.y_2 = u$ (we use the notations of [10]).
  $\forall y_1y_2y_3 \left(R_{i_2(\alpha_{\otimes_i}, i_0)} y_1y_2y_3x \to y_1 = y_3\right)$
  $\forall y_1y_2y_3 \left(R_{i_2} uy_3x \wedge Ry_1y_2u \to y_1 = y_3\right)$
  $\forall y_1y_2 \left(Ry_1y_2x \to y_1 = x\right)$.

12.2 Rule CI_{i}

- $\dfrac{Y\,[\otimes_i]\,X \vdash Z}{X\,[\otimes_i]\,Y \vdash Z}$
- $(l(X\,[\otimes_i]\,Y) \vdash r(Z))\,[Z/l(Y\,[\otimes_i]\,X)][X/p][Y/q]$
  $p \otimes_i q \vdash q \otimes_i p$
  $p \otimes_i q \to q \otimes_i p$
- $[\underbrace{i_2(\alpha_{\otimes_i}(i_0, i_0), i_0)}_{\alpha}](\neg p, \neg q, \neg[\alpha_{\otimes_i}](\neg q, \neg p))$
- $\forall PQ \forall y_1y_2y_3\,[R_\alpha y_1y_2y_3x \to \neg P(y_1) \vee \neg Q(y_2) \vee \exists y_3'y_3''\,(Ry_3'y_3''y_3 \wedge Q(y_3') \wedge P(y_3''))]$
  As minimal valuations, we take $\sigma(P) : \lambda u.u = y_1$ and $\sigma(Q) : \lambda u.u = y_2$.
  $\forall uy_1y_2y_3\,[R_{i_2} xuy_3 \wedge Ry_1y_2u \to \exists y_3'y_3''\,(Ry_3'y_3''y_3 \wedge y_3' = y_2 \wedge y_3'' = y_1)]$
  $\forall y_1y_2\,(Ry_1y_2x \to Ry_2y_1x)$.

12.3 Rule WI_{i}

- $\dfrac{X\,[\otimes_i]\,X \vdash Y}{X \vdash Y}$
- $(l(X) \vdash r(Y))\,[Y/l(X\,[\otimes_i]\,X)][X/p]$
  $p \vdash p \otimes_i p$
  $p \to p \otimes_i p$
- $[i_2](\neg p, \neg[\alpha_{\otimes_i}](\neg p, \neg p))$
- $\forall P \forall y_1y_2\,[R_{i_2} xy_1y_2 \to \neg P(y_1) \vee \exists y_2'y_2''\,(Ry_2'y_2'y_2' \wedge P(y_2') \wedge P(y_2''))]$
  As minimal valuation we take $\sigma(P) : \lambda u.u = y_1$.
  $\forall y_1y_2\,[x = y_1 = y_2 \to \exists y_2'y_2''\,(Ry_2'y_2''y_2 \wedge y_2' = y_1 \wedge y_2'' = y_1)]$
  $Rxxx$.

Analytic Inference Rule	Local First-order Correspondent $\chi(x)$	
$\dfrac{(X\,[\otimes_{\mathrm{i}}]\,Y)\,[\otimes_{\mathrm{i}}]\,Z \vdash U}{X\,[\otimes_{\mathrm{i}}]\,(Y\,[\otimes_{\mathrm{i}}]\,Z) \vdash U}$	$\forall yzw\,(Rx(yz)w \to R(xy)zw)$	$\mathsf{B}^{c}_{\mathrm{i}}$
$\dfrac{Y\,[\otimes_{\mathrm{i}}]\,X \vdash Z}{X\,[\otimes_{\mathrm{i}}]\,Y \vdash Z}$	$\forall yz\,(Ryzx \to Rzyx)$	CI_{i}
$\dfrac{X\,[\otimes_{\mathrm{i}}]\,X \vdash Y}{X \vdash Y}$	$Rxxx$	WI_{i}
$\dfrac{X \vdash Z}{X\,[\otimes_{\mathrm{i}}]\,Y \vdash Z}$	$\forall yz\,(Ryzx \to x = y)$	K_{i}
$\dfrac{X\,[\prec_{\mathrm{i}}]\,Y \vdash Z}{X\,[\otimes_{\mathrm{i}}]\,[\sim_{\mathrm{j}}]\,Y \vdash Z}$	$\forall yzw\,(Ryzx \wedge Swz \to Rywx)$	$\mathsf{N}^{1}_{\mathrm{i}}$
$\dfrac{X\,[\succ_{\mathrm{i}}]\,Y \vdash Z}{[\sim_{\mathrm{j}}]\,X\,[\otimes_{\mathrm{i}}]\,Y \vdash Z}$	$\forall yzw\,(Ryzx \wedge Swy \to Rwzx)$	$\mathsf{N}^{2}_{\mathrm{i}}$
$\dfrac{[\sim_{\mathrm{j}}]\,X \vdash U}{*X \vdash U}$	Sxx	$\mathsf{N}^{3}_{\mathrm{i}}$
$\dfrac{*X \vdash U}{[\sim_{\mathrm{j}}]\,X \vdash U}$	$\forall y\,(Syx \to x = y)$	$\mathsf{N}^{4}_{\mathrm{i}}$
$\dfrac{[\sim_{\mathrm{j}}]\,X \vdash Y}{[\sim_{\mathrm{i}}]\,X \vdash Y}$	$\forall y\left(R^{\pm\sigma_i}_{\otimes_i} yx \to R^{\pm\sigma_j}_{\otimes_j} yx\right)$	$\mathsf{BI}_{\mathrm{i\ j}}$
$\dfrac{[\sim_{\mathrm{i}}]\,[f] \vdash U}{[t] \vdash U}$	$Tx \to \exists y(Syx \wedge \neg Fy)$	Q_{i}
$\dfrac{[t]\,[\otimes_{\mathrm{i}}]\,X \vdash U}{X \vdash U}$	$\exists y\,(Ryxx \wedge Ty)$	$\mathbf{I}^{1}_{\mathrm{i}}$
$\dfrac{[t] \vdash U}{\vdash U}$	Tx	$\mathbf{I}^{2}_{\mathrm{i}}$
$\dfrac{U \vdash [f]}{U \vdash}$	$\neg Fx$	$\mathbf{I}^{3}_{\mathrm{i}}$
$\dfrac{U \vdash [\square]\,X}{U \vdash X}$	Sxx	$\mathsf{T}_{\bullet}$
$\dfrac{U \vdash [\square]\,X}{U \vdash [\square]\,[\square]\,X}$	$\forall yz\,(Sxy \wedge Syz \to Sxz)$	$\mathsf{4}_{\bullet}$

Figure 8: First–order correspondents of 'classical' inference rules

12.4 Rule $\mathsf{N}^1_{\mathrm{i}}$

- $\dfrac{X\,[\prec_{\mathrm{i}}]\,Y \vdash Z}{X\,[\otimes_{\mathrm{i}}]\,[\sim_{\mathrm{j}}]\,Y \vdash Z}$

- $(l(X\,[\otimes_{\mathrm{i}}]\,[\sim_{\mathrm{j}}]\,Y) \vdash r(Z))\,[Z/l(X\,[\prec_{\mathrm{i}}]\,Y)][X/p][Y/q]$
  $(p \otimes_{\mathrm{i}} \sim_j q) \vdash (p \prec_{\mathrm{i}} q)$
  $(p \otimes_{\mathrm{i}} \sim_j q) \to (p \prec_{\mathrm{i}} q)$

- $[\underbrace{i_2(\alpha_{\otimes_{\mathrm{i}}}(i_0, i_0), i_0)}_{\alpha}](\neg p, [\alpha_{\sim_j}]q, \neg[\alpha_{\prec_{\mathrm{i}}}](\neg p, q))$
  The dependency digraph of this inductive formula φ is $pE_\varphi q$.

- $\forall PQ \forall y_1 y_2 y_3 (R_\alpha y_1 y_2 y_3 x \to \neg P(y_1) \vee \forall y_2'(R_{\alpha_{\sim_j}} y_2' y_2 \to Q(y_2')) \vee \neg \forall y_3' y_3''(R_{\alpha_{\prec_{\mathrm{i}}}} y_3' y_3'' y_3 \to \neg P(y_3') \vee Q(y_3'')))$
  As minimal valuations, we take $\sigma(P) \coloneqq \lambda u.u = y_1$ and $\sigma(Q) \coloneqq \lambda u.R_{\alpha_{\prec_{\mathrm{i}}}} y_3 y_1 u$.
  $\forall y_1 y_2 y_3 y_4 \left(R_\alpha y_1 y_2 y_3 y_4 x \to \forall y_2' \left(R_{\alpha_{\sim_j}} y_2' y_2 \to R y_1 y_2' y_3\right)\right)$
  $\forall u v y_1 y_2 y_3 \left[R_{i_2} x u v \wedge R y_1 y_2 u \wedge R_{i_2} y_3 y_4 v \to \forall y_2' \left(R_{\alpha_{\sim_j}} y_2' y_2 \to R y_1 y_2' y_3\right)\right]$
  $\forall y_1 y_2 \left(R y_1 y_2 x \to \forall y_2' \left(R_{\alpha_{\sim_j}} y_2' y_2 \to R y_1 y_2' x\right)\right)$
  $\forall y_1 y_2 y_2' \left(R y_1 y_2 x \wedge R_{\alpha_{\sim_j}} y_2' y_2 \to R y_1 y_2' x\right)$.

12.5 Rule $\mathbf{I}^1_{\mathrm{i}}$

- $\dfrac{[t]\,[\otimes_{\mathrm{i}}]\,X \vdash U}{X \vdash U}$

- $(l(X) \vdash r(U))\,[U/l([t]\,[\otimes_{\mathrm{i}}]\,X)][X/p]$
  $p \vdash t \otimes_{\mathrm{i}} p$
  $p \to t \otimes_{\mathrm{i}} p$

- $[i_2]\,(\neg p, \neg[\alpha_{\otimes_{\mathrm{i}}}](\neg t, \neg p))$

- $\forall P \forall u y_1\,(R_{i_2} y_1 u x \to \neg P(y_1) \vee \exists z_1 z_2\,(R z_1 z_2 u \wedge T(z_1) \wedge P(z_2)))$
  As minimal valuation, we take $\sigma(P) : \lambda u.u = y_1$.
  $\forall u y_1 y_2\,(x = y_1 = u \to \exists z_1 z_2\,(R z_1 z_2 u \wedge T(z_1) \wedge z_2 = y_1))$
  $\exists z_1\,(R z_1 x x \wedge T(z_1))$.

13 A reconstruction of propositional logic

In this section, we show how we can recover the classical conjunction $\wedge$ and disjunction $\vee$ as well as the truth constants $\top$ and $\bot$ and the Boolean negation $\neg$ by adding specific inference rules to the display calculus $\mathsf{GGL}^0_{\mathsf{C}}$ of Figure 4. These rules are in fact refinements of Gentzen's structural rules for classical logic. In this section, all the connectives are of type $(1,1)$ or $(1,1,1)$ and by abuse we confuse the connectives with their skeletons.

13.1 A semantic reconstruction

Lemma 13.1. *Let C be a set of atomic connectives. Every C–frame F satisfying the first-order conditions $\{CI_i, WI_i, K_i\}$ is such that for all $x, y, z \in F$,*

$$R^{\pm\sigma_i}_{\otimes_i} xyz \text{ iff } x = y = z. \tag{8}$$

Every C–frame F satisfying the first-order conditions $\{N^3_i, N^4_i\}$ is such that for all $x, y, z \in F$,

$$R^{\pm\sigma_i}_{\otimes_i} xy \text{ iff } x = y. \tag{9}$$

Proof: It suffices to combine the local first–order conditions corresponding to the respective analytic inference rules. □

Theorem 13.2. *Let C be a set of atomic connectives.*

- *If C contains an atom denoted t (resp. f) then every proper display calculus extending GGL^0_C with analytic structural rules including $\boldsymbol{I}^2_i$ (resp. $\boldsymbol{I}^3_i$) is sound and strongly complete for a logic in which t (resp. f) is the verum truth constant $\top$ (resp. the falsum truth constant $\bot$).*
- *If C contains a unary atomic connective then every proper display calculus extending GGL^0_C with analytic structural rules including N^3_i, N^4_i for some $i \in [\![1;3]\!]$ is sound and strongly complete for a logic in which $\sim_j$ and $-_i$ (if they belong to C) are the Boolean negation $\neg$.*
- *If C contains a binary atomic connective then every proper display calculus extending GGL^0_C with analytic structural rules including K_i, WI_i, CI_i for some $i \in [\![1;3]\!]$ is sound and strongly complete for a logic in which $\otimes_i$ and $\oplus_i$ (if they belong to C) are the classical conjunction $\wedge$ and disjunction $\vee$ respectively.*
- *If C contains a binary atomic connective and an atom denoted t (resp. f) then every proper display calculus extending GGL^0_C with analytic structural rules including K_i, WI_i, CI_i,$\boldsymbol{I}^1_i$ for some $i \in [\![1;3]\!]$ is sound and strongly complete for a logic in which t (resp. f) is the verum constant $\top$ (resp. the falsum constant $\bot$).*
- *If C contains a binary and a unary atomic connective then every proper display calculus extending GGL^0_C with analytic structural rules including K_i, WI_i, CI_i,$\boldsymbol{I}^1_i$,Q_i,N^1_i or K_i, WI_i, CI_i,$\boldsymbol{I}^1_i$,Q_i,N^2_i for some $i \in [\![1;3]\!]$ is sound and strongly complete for a logic in which the atomic connective $\sim_i$ and $-_i$ (if they belong to C) are both the Boolean negation.*

Proof: It is a direct consequence of Theorem 11.1 and Lemma 13.1. We just combine the local first–order conditions corresponding to the various inference rules and we obtain the expected results. We only prove the proposition for a calculus including the rules K_i, WI_i, CI_i,$\mathbf{I}^1_i$,Q_i,N^1_i for an arbitrary $i \in [\![1;3]\!]$. Because of Lemma 13.1, the ternary relation $R^{\pm\sigma_i}_{\otimes_i}$ is the identity relation. Moreover, by conditions $\{\mathbf{I}^1_i, \mathsf{Q}_i\}$, we obtain that the relation S is *serial*, that is, for all x, there is y such that Sxy. Therefore, we finally obtain that for all x, y, we have that Sxy if, and only if, $x = y$: S is the identity relation. □

Definition 13.3. Let $\mathsf{C} \triangleq \{\wedge, \vee, \neg\} \triangleq \{(\sigma_1, +, s_1), (\sigma, -, \sigma_4), (\tau_2, +, t_4)\}$. We define the proper display calculus PL^0 as follows, with $j = 1$:

$$\mathsf{GGL}^0_\mathsf{C} + \{\mathsf{CI}_i, \mathsf{WI}_i, \mathsf{K}_i \mid i \in [\![1;3]\!]\} + \{\mathsf{N}^3_i, \mathsf{N}^4_i\} \qquad \mathsf{PL}^0$$

Structural rules:

$$\frac{(X \mathbin{,} Y) \vdash U}{(Y \mathbin{,} X) \vdash U}\ (\mathsf{CI} \vdash) \qquad \frac{X \vdash U}{(X \mathbin{,} Y) \vdash U}\ (\mathsf{K} \vdash) \qquad \frac{(X \mathbin{,} X) \vdash U}{X \vdash U}\ (\mathsf{WI} \vdash)$$

Display rule:

$$\begin{array}{c} (X \mathbin{,} Y) \vdash Z \\ \hline\hline X \vdash (Z \mathbin{,} *Y) \end{array}\ (\mathsf{dr}_2)$$

Introduction rules:

$$\frac{X \vdash \varphi \quad Y \vdash \psi}{(X \mathbin{,} Y) \vdash (\varphi \wedge \psi)}\ (\vdash \wedge) \qquad \frac{(\varphi \mathbin{,} \psi) \vdash U}{(\varphi \wedge \psi) \vdash U}\ (\wedge \vdash)$$

$$\frac{U \vdash \varphi}{U \vdash (\varphi \vee \psi)}\ (\vdash \vee) \qquad \frac{\varphi \vdash U \quad \psi \vdash U}{(\varphi \vee \psi) \vdash U}\ (\vee \vdash)$$

$$\frac{U \vdash *\varphi}{U \vdash \neg\varphi}\ (\vdash \neg) \qquad \frac{*\varphi \vdash U}{\neg\varphi \vdash U}\ (\neg \vdash)$$

Figure 9: The display calculus PL for propositional logic

Theorem 13.4. *PL^0 proves the same theorems of $\mathcal{S}_\mathsf{C}$ as the display calculus PL of Figure 9.*

Proof: It follows from the soundness and completeness of the calculus GGL_C for basic Boolean atomic logics of Theorem 7.7. The display calculus of Figure 9 is indeed the propositional part of GGL_C. □

13.2 A syntactic reconstruction

The simplification of the semantics of the connectives when one adds structural rules to the calculus has a counterpart on the syntactic side, that is proof–theoretically. We are going to see in the sequel that the addition of the classical structural rules to the display calculus for basic atomic logics $\mathsf{GGL}^0_\mathsf{C}$ in PL^0 allows us to greatly simplify the structural part and the display rules of this calculus. Many rules thus become redundant or useless as well as many structural connectives. We are going to show in particular that we can reduce the number of structural connectives from 48 to 1 and recover the classical display calculus for propositional logic of Figure 9.

Simplification of the structural binary connectives. Even if C contains only three logical connectives, the display calculus PL^0 contains many more structural connectives, namely the 48 structural connectives of $[\mathcal{O}(\mathsf{C})]$ [3, Proposition 32]. Below are the instances of the display rule

(dr_1) of PL^0 for these binary structural connectives:

$$
\begin{array}{c}
(X\,[\otimes]\,Y) \vdash Z \\ \hline\hline
X \vdash (Z\,[\subset']\,Y) \\ \hline\hline
Y \vdash (Z\,[\subset]\,X) \\ \hline\hline
(Y\,[\otimes']\,X) \vdash Z \\ \hline\hline
X \vdash (Y\,[\supset]\,Z) \\ \hline\hline
Y \vdash (X\,[\supset']\,Z)
\end{array}
\qquad
\begin{array}{c}
X \vdash (Y\,[\oplus]\,Z) \\ \hline\hline
(X\,[\prec']\,Z) \vdash Y \\ \hline\hline
(X\,[\prec]\,Y) \vdash Z \\ \hline\hline
X \vdash (Z\,[\oplus']\,Y) \\ \hline\hline
(Z\,[\succ]\,X) \vdash Y \\ \hline\hline
(Y\,[\succ']\,X \vdash Z)
\end{array}
\qquad
\begin{array}{c}
(X\,[\uparrow_3]\,Y) \vdash Z \\ \hline\hline
(Z\,[\uparrow'_1]\,Y) \vdash X \\ \hline\hline
(Z\,[\uparrow_2]\,X) \vdash Y \\ \hline\hline
(Y\,[\uparrow'_3]\,X) \vdash Z \\ \hline\hline
(Y\,[\uparrow_1]\,Z) \vdash X \\ \hline\hline
(X\,[\uparrow'_2]\,Z) \vdash Y
\end{array}
\qquad
\begin{array}{c}
X \vdash (Y\,[\downarrow_3]\,Z) \\ \hline\hline
Y \vdash (X\,[\downarrow'_1]\,Z) \\ \hline\hline
Z \vdash (X\,[\downarrow_2]\,Y) \\ \hline\hline
X \vdash (Z\,[\downarrow'_3]\,Y) \\ \hline\hline
Y \vdash (Z\,[\downarrow_1]\,X) \\ \hline\hline
Z \vdash (Y\,[\downarrow'_2]\,X)
\end{array}
\qquad (\mathsf{dr}_1)
$$

where, if c is the cycle $c = (1\ 2\ 3\ 4\ 5\ 6)$, the tuple of atomic connectives $(\otimes, \subset', \subset, \otimes', \supset, \supset')$ ranges over

$$\left\{((\sigma_i,+,s_1),(\sigma_{c(i)},-,s_2),(\sigma_{c^2(i)},-,s_2),(\sigma_{c^3(i)},+,s_1),(\sigma_{c^4(i)},-,s_3),(\sigma_{c^5(i)},-,s_3)) \mid i \in \{1,3,5\}\right\}$$

and $(\oplus, \prec', \prec, \vee', \succ, \succ')$ ranges over

$$\left\{((\sigma_i,-,s_4),(\sigma_{c(i)},+,s_5),(\sigma_{c^2(i)},+,s_5),(\sigma_{c^3(i)},-,s_4),(\sigma_{c^4(i)},+,s_6),(\sigma_{c^5(i)},+,s_6)) \mid i \in \{1,3,5\}\right\}$$

and

$$
\begin{aligned}
(\uparrow_3, \uparrow'_1, \uparrow_2, \uparrow'_3, \uparrow_1, \uparrow'_2) &\triangleq ((\sigma_1,+,s_7),(\sigma_2,+,s_7),(\sigma_3,+,s_7),(\sigma_4,+,s_7),(\sigma_5,+,s_7),(\sigma_6,+,s_7)) \\
(\downarrow_3, \downarrow'_1, \downarrow_2, \downarrow'_3, \downarrow_1, \downarrow'_2) &\triangleq ((\sigma_1,-,s_8),(\sigma_2,-,s_8),(\sigma_3,-,s_8),(\sigma_4,-,s_8),(\sigma_5,-,s_8),(\sigma_6,-,s_8))
\end{aligned}
$$

In fact, we have here just enumerated the 48 connectives of $\mathcal{O}(\mathsf{C})$. The other display rule of PL^0 is the following:

$$\begin{array}{c} X \vdash Y \\ \hline\hline *Y \vdash\ *X \end{array}\ (\mathsf{dr}'_2)$$

We are going to see by proving a series of propositions that these display rules can be greatly simplified and that the number of structural connectives can also be drastically reduced from 48 to 1 using the strucural rules in combination with the display rules. Our first proposition states that the structural rules are also derivable when $[\otimes_3]$ is replaced by $[\otimes_1]$ or $[\otimes_2]$.

Lemma 13.5. *The following rules are derivable in PL^0 for all $i, j \in [\![1;3]\!]$:*

$$
\begin{array}{c} X\,[\otimes_j]\,Y \vdash U \\ \hline\hline X\,[\otimes_i]\,Y \vdash U \end{array}
\qquad\qquad
\begin{array}{c} Z \vdash X\,[\oplus_j]\,Y \\ \hline\hline Z \vdash X\,[\oplus_i]\,Y \end{array}
\qquad\qquad \mathsf{TI}_{i[\otimes_i]j}
$$

Hence, we can identify all the structural connectives $[\otimes_i]$ for all $i \in [\![1;3]\!]$.

Lemma 13.6. *The following rules are derivable in PL^0: for all $i \in \{1,2,3\}$,*

$$
\begin{array}{c} (X\,[\prec_i]\,Y) \vdash Z \\ \hline\hline (X\,[\otimes_i] * Y) \vdash Z \end{array}
\qquad
\begin{array}{c} (X\,[\succ_i]\,Y) \vdash Z \\ \hline\hline (*X\,[\otimes_i]\,Y) \vdash Z \end{array}
\qquad
\begin{array}{c} X \vdash (Y\,[\subset_i]\,Z) \\ \hline\hline X \vdash (Y\,[\oplus_i] * Z) \end{array}
\qquad
\begin{array}{c} X \vdash (Y\,[\supset_i]\,Z) \\ \hline\hline X \vdash (*Y\,[\oplus_i]\,Z) \end{array}
$$

$$
\begin{array}{c} (X\,[\prec'_i]\,Y) \vdash Z \\ \hline\hline (X\,[\otimes_i] * Y) \vdash Z \end{array}
\qquad
\begin{array}{c} (X\,[\succ'_i]\,Y) \vdash Z \\ \hline\hline (*X\,[\otimes_i]\,Y) \vdash Z \end{array}
\qquad
\begin{array}{c} X \vdash (Y\,[\subset'_i]\,Z) \\ \hline\hline X \vdash (Y\,[\oplus_i] * Z) \end{array}
\qquad
\begin{array}{c} X \vdash (Y\,[\supset'_i]\,Z) \\ \hline\hline X \vdash (*Y\,[\oplus_i]\,Z) \end{array}
$$

$$
\begin{array}{c} (X\,[\uparrow_i]\,Y) \vdash Z \\ \hline\hline (*X\,[\otimes_i] * Y) \vdash Z \end{array}
\qquad
\begin{array}{c} X \vdash (Y\,[\downarrow_i]\,Z) \\ \hline\hline X \vdash (*Y\,[\oplus_i] * Z) \end{array}
$$

Lemmas 13.6 and 13.5 entail that we can replace in the proofs of PL^0 all binary structural connectives by a combination of a single binary structural connective " , " (whose interpretation will be context dependent whether it is a-part or s-part) and the operator $*$. Doing so, we obtain the following set of display rules that are equivalent to the various instances of (dr_1):

$$
\begin{array}{c}
(X \,,\, Y) \vdash Z \\ \hline\hline
X \vdash (Z \,,\, *Y) \\ \hline\hline
Y \vdash (Z \,,\, *X) \\ \hline\hline
(Y \,,\, X) \vdash Z \\ \hline\hline
X \vdash (*Y \,,\, Z) \\ \hline\hline
Y \vdash (*X \,,\, Z)
\end{array}
\qquad
\begin{array}{c}
X \vdash (Y \,,\, Z) \\ \hline\hline
(X \,,\, *Z) \vdash Y \\ \hline\hline
(X \,,\, *Y) \vdash Z \\ \hline\hline
X \vdash (Z \,,\, Y) \\ \hline\hline
(*Z \,,\, X) \vdash Y \\ \hline\hline
(*Y \,,\, X \vdash Z)
\end{array}
\qquad
\begin{array}{c}
(*X \,,\, *Y) \vdash Z \\ \hline\hline
(*Z \,,\, *Y) \vdash X \\ \hline\hline
(*Z \,,\, *X) \vdash Y \\ \hline\hline
(*Y \,,\, *X) \vdash Z \\ \hline\hline
(*Y \,,\, *Z) \vdash X \\ \hline\hline
(*X \,,\, *Z) \vdash Y
\end{array}
\qquad
\begin{array}{c}
(*X \,,\, *Y) \vdash Z \\ \hline\hline
(*Z \,,\, *Y) \vdash X \\ \hline\hline
(*Z \,,\, *X) \vdash Y \\ \hline\hline
(*Y \,,\, *X) \vdash Z \\ \hline\hline
(*Y \,,\, *Z) \vdash X \\ \hline\hline
(*X \,,\, *Z) \vdash Y
\end{array}
\qquad (\mathsf{dr}_1)
$$

The rules can be further simplified using the structural rule of permutation. Moreover, if we use the same conventions regarding the empty structures like in our calculus $\mathsf{GGL_C}$ of Figure 6, namely if X is empty then $*X$ is empty and $(X \,,\, Y)$ and $(Y \,,\, X)$ are equal to Y then (dr_1) and (dr'_2) altogether are equivalent to the following single display rule:

$$
\begin{array}{c}
(X \,,\, Y) \vdash Z \\ \hline\hline
X \vdash (Z \,,\, *Y)
\end{array}
\ (\mathsf{dr}_2)
$$

Simplification of the unary structural connectives. We can perform similar simplifications with the unary connectives. Below are the display rules for the unary structural connectives:

$$
\begin{array}{c}
[\sim_1]X \vdash Y \\ \hline\hline
[\sim_2]Y \vdash X
\end{array}
\qquad
\begin{array}{c}
X \vdash [\Box]Y \\ \hline\hline
[\Diamond^-]X \vdash Y
\end{array}
\qquad
\begin{array}{c}
X \vdash [\Box^-]Y \\ \hline\hline
[\Diamond]X \vdash Y
\end{array}
\qquad
\begin{array}{c}
X \vdash [\bot_1]Y \\ \hline\hline
Y \vdash [\bot_2]X
\end{array}
$$

where $(\sim_1, \sim_2, \Box, \Diamond^-, \Box^-, \Diamond, \bot_1, \bot_2)$ is

$$((\tau_2,+,t_4),(\tau_1,+,t_4),(\tau_2,-,t_2),(\tau_1,+,t_1),(\tau_1,-,t_2),(\tau_2,+,t_1),(\tau_2,-,t_3),(\tau_1,-,t_3)).$$

Note that $\sim_1$ is in fact our $\neg$.

Lemma 13.7. *The following rules are derivable in PL^0:*

$$
\begin{array}{c}
[\sim_1]X \vdash Y \\ \hline\hline
[\sim_2]X \vdash Y \\ \hline\hline
*X \vdash Y
\end{array}
\qquad
\begin{array}{c}
X \vdash [\bot_1]Y \\ \hline\hline
X \vdash [\bot_2]Y \\ \hline\hline
X \vdash *Y
\end{array}
\qquad
\begin{array}{c}
[\Diamond^-]X \vdash Y \\ \hline\hline
[\Diamond]X \vdash Y \\ \hline\hline
X \vdash Y
\end{array}
\qquad
\begin{array}{c}
X \vdash [\Box^-]Y \\ \hline\hline
X \vdash [\Box]Y \\ \hline\hline
X \vdash Y
\end{array}
$$

Thus, the structural connectives $[\sim_1], [\sim_2], [\bot_1], [\bot_2]$ can be replaced by the operator $*$ in the proofs of PL^0 and the structural connectives $[\Diamond], [\Diamond^-], [\Box], [\Box^-]$ can be removed or replaced by $**$ in the proofs of PL^0.

Lemma 13.8. *In PL^0, the introduction rules for $\neg = \sim_1$ are equivalent to the following rules:*

$$
\frac{*\varphi \vdash U}{\neg\varphi \vdash U}
\qquad\qquad
\frac{U \vdash *\varphi}{U \vdash \neg\varphi}
$$

We therefore introduce a single negation connective $\neg$ which behaves like $-_1$ and $\sim_1$. Likewise, the structural connectives $[\sim_1], [-_1]$ and all the other unary structural connectives of $[\mathcal{O}(\mathsf{C})]$ can be replaced by the single operator $*$ because this replacement does not produce more or less theorems regarding $-_1$ and $\sim_1$. We recall that our operator $*$ is slightly different from the usual structural connective of display logics (see a comment in [5] or [3, p.38-39]). With all these simplifications of notations, we obtain the display calculus of Figure 9. We have therefore reconstructed the propositional part of our display calculus GGL_C of Figure 6.

Theorem 13.9. *Every proof in PL^0 can be translated into a proof in PL, and vice versa. Hence, PL^0 proves the same theorems of $\mathcal{S}_\mathsf{C}$ as the display calculus PL of Figure 9.*

Remark 2. Note that the rule of associativity B^c_i is derivable in PL^0, we do not need to introduce this principle as primitive. Here is the proof:

$$
\begin{prooftree}
\AxiomC{$((X , Y) , Z) \vdash U$}
\RightLabel{$(\mathsf{dr}_2) , (\vdash \mathsf{Cl})$}
\UnaryInfC{$Z \vdash (*(X , Y) , U)$}
\RightLabel{$(\mathsf{K} \vdash) , (\mathsf{Cl} \vdash)$}
\UnaryInfC{$(Y , Z) \vdash (*(X , Y) , U)$}
\RightLabel{$(\mathsf{K} \vdash) , (\mathsf{Cl} \vdash)$}
\UnaryInfC{$(X , (Y , Z)) \vdash (*(X , Y) , U)$}
\RightLabel{$(\vdash \mathsf{Cl}) , (\mathsf{dr}_2) , (\mathsf{Cl} \vdash)$}
\UnaryInfC{$((X , Y) , (X , (Y , Z))) \vdash U$}
\RightLabel{$(\mathsf{dr}_2) , (\vdash \mathsf{Cl})$}
\UnaryInfC{$(X , Y) \vdash (*(X , (Y , Z)) , U)$}
\RightLabel{$(\mathsf{Cl} \vdash) , (\mathsf{dr}_2) , (\vdash \mathsf{Cl})$}
\UnaryInfC{$Y \vdash (*X , (*(X , (Y , Z)) , U))$}
\RightLabel{$(\mathsf{K} \vdash)$}
\UnaryInfC{$(Y , Z) \vdash (*X , (*(X , (Y , Z)) , U)$}
\RightLabel{$(\vdash \mathsf{Cl}) , (\mathsf{dr}_2) , (\mathsf{Cl} \vdash)$}
\UnaryInfC{$(X , (Y , Z)) \vdash (*(X , (Y , Z)) , U)$}
\RightLabel{$(\vdash \mathsf{Cl}) , (\mathsf{dr}_2)$}
\UnaryInfC{$((X , (Y , Z)) , (X , (Y , Z)) \vdash U$}
\RightLabel{$(\mathsf{Wl} \vdash)$}
\UnaryInfC{$(X , (Y , Z)) \vdash U$}
\end{prooftree}
$$

14 Related work and conclusion

The DLE-logics introduced by Greco et al.. [20] are similar to our basic atomic logics. Their families $\mathcal{F}$ and $\mathcal{G}$ correspond in our framework to connectives of "quantification signatures" $\exists$ and $\forall$ respectively. Likewise, their order types correspond in our framework to "tonicity signatures". Hence, several of their notions correspond to notions introduced by Dunn's gaggle theory [14, 15].

The main difference between their and our work is that we prove the completeness of our calculi w.r.t. a Kripke-style relational semantics. We also introduce a generalized form of residuation based on the symmetric group which is novel. Unlike them, we originally introduce the Boolean negation as a primitive connective, even if one can dispose of it after proving cut elimination. An important difference between Greco & Al.'s DLE-logics and our atomic and molecular logics lies in our introduction and use of types and in the fact that we consider compositions of atomic connectives as primitive connectives. These generalizations are motivated at length in [6]. Basically, some logics/protologics cannot be represented without the use of types, such as temporal logic [6, Example 8], arrow logic, many-dimensional logics [24] and first-order logic. This use of types is crucial to represent these logics and it is also instrumental in showing that any protologic is as expressive as a molecular logic, which constitutes the main result of [6]. It complexifies the soundness and completeness proof of the present article compared to the soundness and completeness proof of [3] for gaggle logics, which are actually atomic logics of type $(1, 1, \ldots, 1)$. This said, maybe the main differences with the work of Palmigiano & Al. remains the fact that we are able to define automatically from the connectives of a given atomic logic (or specific molecular logics) sound and strongly complete display and Hilbert calculi in a generic fashion together with their *Kripke-style relational semantics* for which they are sound and complete. In particular, they do not provide a Kripke-style relational semantics to their DLE-logics, only an algebraic one which more or less mimics the axioms and inference rules of their DLE-logics. Our proofs of soundness and completeness w.r.t. the Kripke-style relational semantics resorts to the results of Dunn's gaggle theory and are not straightforward.

Greco et al. [20] also adapted and generalized the framework of Goranko & Vakarelov [17] dealing with the correspondence theory of polyadic modal logics to their DLE-logics, just as we did it with our atomic and molecular logics. In particular, they introduced the notion of inductive inequality which is the counterpart in their setting to the notion of Goranko & Vakarelov's inductive formula. It is therefore not surprising that we come up with notions and results which are very similar to their notions. To be more precise, the tonicity of molecular connectives corresponds to their *sign* inherited by the leaves in the *signed generation tree* [19, Definition 14]; essentially universal molecular formulas are concatenations of normal operators that behave 'like boxes' and correspond to their positive *PIA formulas* or negative *skeleton* [19, Definition 15]; essentially existential molecular formulas are compositions of normal operators that behave 'like diamonds' and correspond to their negative *PIA formulas* or positive *skeleton* [19, Definition 15]; the universal molecular connective part is the *skeleton* part of [19, Definition 15], and the several essentially existential and universal formulas attached to it are the maximal *PIA formulas* containing the *critical* occurrences (these occurrences are called essential here and in [17]). Every branch of a molecular regular formula is *good* [19, Definition 15] since the lower part of the formula is all PIA, and the upper is all skeleton. All in all, molecular inductive formulas of $\mathcal{I}_2(\mathsf{GGL}_\mathsf{C})$ correspond to analytic inductive formulas [19, Definition 55]. Moreover, the *order types* for propositional letters [19, p. 18] different from 1 can all be replaced equivalently by variables of order type 1 thanks to the Boolean negation, which is present in our framework.

One of the main differences is that our notions are genuine instances of the notions introduced by Goranko & Vakarelov [17] and operate and apply directly at the level of the Kripke-style relational semantics, like [17], because we set a formal connection between our atomic logics and modal polyadic logics [4, Theorem 5.16], unlike what has been done by Greco et al. [20] where such a formal connection is absent. It is in fact the soundness and completeness of our calculi w.r.t. a Kripke-style relational semantics which allows to import directly the results and notions of Goranko & Vakarelov [17] in our framework. Instead, evidence is given in [12, Section 3] to the effect that, as their name suggests, inductive inequalities, which extend analytic inductive inequalities, are the distributive counterparts of and 'project over' the inductive formulas of Goranko and Vakarelov in the classical setting. Another difference with our work is that the ordering Ω on the propositional variables which is obtained in our approach constructively by a topological sort of the dependency digraph associated to a regular molecular formula (see the proof of Theorem 10.4) is not determined by Greco et al. [19] and is only assumed to exist for inequalities to be inductive (together with an order type). Overall, our approach is in any case different: we connect the work of Goranko & Vakarelov [17] to the work of Ciabattoni & Ramanayake [11] by applying the latter to our atomic and molecular logics and showing that acyclic formulas for our basic display calculi are in fact inductive (our Theorem 10.4) and therefore canonical. This said, Greco et al. [20, Appendix D] prove that their analytic inductive inequalities coincide with acyclic formulas of $\mathcal{I}_2(\mathsf{DL})$. This result is similar in spirit to our Theorem 10.4.

Atomic logics are logics of residuation to which types are added. Residuated logics have been extensively studied in the algebraic approach to logic [16]. However, it still remains to propose and adapt these algebraic approaches and semantics to our atomic and molecular logics and to show how a proof of completeness for atomic and molecular logics w.r.t. to our Kripke-style relational semantics can be obtained, as well as the other results in our series of articles. In that respect, the duality theory relating the algebraic and our Kripke-style relational semantics remains to be developed for atomic and molecular logics, in the spirit of the one for modal logic for example [10, Section 5] or for other non-classical logics like in Bimbo & Dunn [9].

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A Proofs of Propositions 10.1, 10.2, 10.3

Proposition A.1. *Let C be a Boolean set of atomic connectives such that $\mathcal{O}(\mathsf{C}) = \mathsf{C}$. Then $\mathsf{GGL}_{\mathsf{C}}$ is* amenable.

Proof: The interpretation functions l and r are τ_0 and τ_1 of Definition 7.4 respectively and we set $\mathbb{F}(X \vdash Y) \triangleq l(X) \to r(Y)$ where $\varphi \to \psi \triangleq (\neg\varphi \vee \psi)$. It suffices to check the definition. □

Proposition A.2. *Let C be a set of atomic connectives. A formula $\varphi \in \mathcal{L}_{\mathsf{C}}$ is* s-soluble *(resp.* a-soluble*) iff it can be equivalently rewritten as $c(p_1, \dots, p_n)$ where c is a universal atomic connective (resp. existential atomic connective) and $p_1, \dots, p_n$ are atoms.*

Proof: We prove it for the s-soluble case, the proof for the a-soluble case is similar. On the one hand, if φ can be equivalently rewritten as $c(p_1, \ldots, p_n)$ with c a universal atomic connective, then we can iteratively apply rule $(\circledast \vdash)$ bottom up to $\mathbf{I} \mid\!\!- c(p_1, \ldots, p_n)$. In doing so, we will transform all logical connectives into structural connectives. On the other hand, if c is not a universal connective, then by applying the same process bottom up, we will reach a consecution of the form $S([\circledast], X_1, \ldots, X_n, \circledast(\varphi_1, \ldots, \varphi_n))$ where we cannot apply rule $(\circledast \vdash)$ anymore, but only rule $(\vdash \circledast)$ which is not invertible. In that case, φ is not s-soluble. □

Proposition A.3. *Let C be a Boolean set of atomic connectives. For all $\varphi \in \mathcal{L}_C$,*

- *$\varphi \in \mathcal{I}_1(GGL_C)$ iff φ can be equivalently rewritten as $c(p_1, \ldots, p_n)$ where c is a universal molecular connective and $p_1, \ldots, p_m \in \mathbb{P}(C)$;*
- *$\varphi \in \mathcal{I}_2(GGL_C)$ iff φ can be equivalently rewritten as $c(\varphi_1, \ldots, \varphi_n)$ where c is a universal molecular connective and each φ_j is either an atom or of the form $c_j(p_1, \ldots, p_m)$ with c_j universal (that is, it is s-soluble) if $tn(c, j) = -$ or of the form $c'_j(p'_1, \ldots, p'_m)$ with c'_j existential (that is, it is a-soluble) if $tn(c, j) = +$, for some $p_1, \ldots, p_m, p'_1, \ldots, p'_m \in \mathbb{P}(C)$.*

Hence, every $\varphi \in \mathcal{I}_2(GGL_C)$ is a regular formula, but the converse does not necessarily hold.

Proof: We only prove the case for $\mathcal{I}_2(\mathsf{GGL_C})$, the case $\mathcal{I}_1(\mathsf{GGL_C})$ follows easily. On the one hand, if c is a universal atomic connective, then it can be transformed applying the invertible rule $(\circledast \vdash)$ bottom up into its counterpart structural connective. Then, with our assumptions, we have that all $\varphi_j = c_j(p_1, \ldots, p_m)$ are either s-soluble (if φ_j is protoantecedent part) or a-soluble (if φ_j is protoconsequent part), because of Proposition 10.2. So, $\varphi \in \mathcal{I}_2(\mathsf{GGL_C})$. On the other hand, if c is not a universal atomic connective, then it cannot be transformed applying the invertible rule $(\circledast \vdash)$ bottom up into its counterpart structural connective and therefore $\varphi \notin \mathcal{I}_2(\mathsf{GGL_C})$. Likewise, if one of the φ_j is not of the expected form, then by Proposition 10.2 it is not s-soluble (or a-soluble) and therefore $\varphi \notin \mathcal{I}_2(\mathsf{GGL_C})$. □

B Proofs of Theorem 10.4 and Corollary 10.5

Lemma B.1. *Let $\varphi \in \mathcal{I}_2(GGL_C)$ be a regular atomic formula, let $\mathcal{S}$ be a set of premises (consecutions) of a rule belonging to the set of semistructural rules equivalent to φ obtained according to Ciabattoni & Ramanayake [11, Proposition 3.15] and let (V_φ, E_φ) be the dependency digraph of φ. If for all p in $\mathcal{S}$ there is no $q \in V_\varphi$ such that $qE_\varphi p$, then $\mathcal{S}$ respects multiplicities w.r.t. p.*

Proof: Let p in $\mathcal{S}$ and assume that there is no $q \in V_\varphi$ such that $qE_\varphi p$. Then, the consecutions of $\mathcal{S}$ that contain p are of the form $M \mid\!\!- c(p)$ with c (structural) universal atomic connective or $c'(p) \mid\!\!- N$ with c' (structural) existential atomic connective. Then, using the display rules, these consecutions are equivalent to consecutions of the form $U \mid\!\!- p$ and $p \mid\!\!- V$ with p not occurring in U and V. Hence, $\mathcal{S}$ respects multiplicities w.r.t. p. □

Theorem B.2. *Let C be a set of atomic connectives. For all $\varphi \in \mathcal{I}_2(GGL_C)$, φ is a molecular inductive formula iff φ is acyclic.*

Proof: Assume that $\varphi \in \mathcal{I}_2(\mathsf{GGL_C})$ is inductive. We are going to prove that it is acyclic. We start with an initial set $\mathcal{S}$ of consecutions corresponding to the premises of the semistructural rules equivalent to φ (obtained according to Ciabattoni & Ramanayake [11, Proposition 3.15]). We have to find an order on the heads p of V_φ on which we apply the transformation of Ciabattoni & Ramanayake [11, Definition 3.21]. This order is provided by the topological sort [13, p. 612] of the dependency digraph (V_φ, E_φ), which is a directed acyclic graph. Then, we iteratively apply Lemma B.1 in combination with [11, Definition 3.21]. This way, we can ensure to remove all atoms p from the initial set of consecutions $\mathcal{S}$.

Conversely, if $\varphi \in \mathcal{I}_2(\mathsf{GGL_C})$ is not inductive then the dependency digraph of φ, (V_φ, E_φ), contains an oriented cycle $(p_1, \ldots, p_k)$. We are going to show that in any order in which we apply the Step 4 of the transformation of Ciabattoni & Ramanayake [11, Definition 3.21] on the atoms $p_1, \ldots, p_k$ we reach a set of consecutions containing a consecution with a propositional letter appearing both as a-part and s-part. This entails that any final set of consecutions obtained by this transformation does not respect multiplicities and therefore that φ is not acyclic because this outcome does not depend on the order on which we apply the cut-closure elimination of Ciabattoni & Ramanayake [11, Definition 3.21] on $p_1, \ldots, p_k$.

To prove that claim, we first adapt the notion of dependency digraph of a formula to define a notion of dependency digraph associated to a set of consecutions $\mathcal{S}$. These sets of consecutions $\mathcal{S}$ correspond in fact to each set of premises of the semi–structural rules equivalent to $\mathbf{I} \vdash \varphi$ (following the procedure defined by Ciabattoni & Ramanayake [11, Definition 3.15]). One can show that a propositional letter p belongs to V_φ, the set of vertices of the dependency digraph of φ (*i.e.* the set of heads of φ), iff p occurs in the corresponding set $\mathcal{S}$ of consecutions in the scope of a universal (existential) structural atomic connective $[\circledast]$ at place j with $tn(\circledast, j) = +$ (resp. $tn(\circledast, j) = -$). Likewise, one sets $p_i E_\varphi p_j$ in the dependency digraph of φ (*i.e.* p_i occurs as an inessential propositional letter in a formula from $\{\varphi_1, \ldots, \varphi_k\}$ with a head p_j) iff p_j occurs in the corresponding set $\mathcal{S}$ of consecutions in the scope of a universal (existential) structural atomic connective $[\circledast]$ at place j with $tn(\circledast, j) = +$ (resp. $tn(\circledast, j) = -$) and p_i occurs in the same consecution and the same universal (existential) structural atomic connective $[\circledast]$ at place i with $tn(\circledast, i) = -$ (resp. $tn(\circledast, i) = +$). Following these observations, we define the *dependency digraph* of a set of consecutions $\mathcal{S}$ as the graph $(V_\mathcal{S}, E_\mathcal{S})$ where $V_\mathcal{S}$ is the set of atoms appearing in $\mathcal{S}$ in the scope of a universal (existential) structural atomic connective $[\circledast]$ at place j with $tn(\circledast, j) = +$ (resp. $tn(\circledast, j) = -$) and we set $p_i E_\mathcal{S} p_j$ iff p_j occurs in $\mathcal{S}$ in the scope of a universal (existential) structural atomic connective $[\circledast]$ at place j with $tn(\circledast, j) = +$ (resp. $tn(\circledast, j) = -$) and p_i occurs in the same consecution and the same universal (existential) structural atomic connective $[\circledast]$ at place i with $tn(\circledast, i) = -$ (resp. $tn(\circledast, i) = +$). Therefore, there is an oriented cycle in (V_φ, E_φ) iff there is an oriented cycle in $(V_\mathcal{S}, E_\mathcal{S})$ for some set of consecutions $\mathcal{S}$ corresponding to a set of premises of the semi–structural rules equivalent to $\mathbf{I} \vdash \varphi$ (following the procedure defined by Ciabattoni & Ramanayake [11, Definition 3.15]).

Now, applying the transformation of Ciabattoni & Ramanayake [11, Definition 3.21] to a set of consecutions $\mathcal{S}$ respecting the multiplicity with regard to p yields a new set of consecutions $\mathcal{S}_p$ whose dependency digraph does not contain the propositional letter p anymore. In particular, if we have a pattern $p_i E_\mathcal{S} p$, $p E_\mathcal{S} p_j$ in the dependency digraph associated to $\mathcal{S}$ then we obtain a dependency digraph associated to $\mathcal{S}_p$ which is the same except that this pattern is replaced by the pattern $p_i E_{\mathcal{S}_p} p_j$. We illustrate this transformation by the following example (quantification signatures and tonicity of connectives are mentioned as superscripts):

$$\dfrac{\dfrac{M \vdash [\overset{\forall}{\otimes}](\overset{-}{p_i},\overset{+}{p})}{[r_2\overset{\exists}{\otimes}](\overset{+}{p_i},\overset{+}{M}) \vdash p} \qquad \dfrac{M' \vdash [\overset{\forall}{\otimes}{}'](\overset{-}{p},\overset{+}{p_j})}{p \vdash [r_1\overset{\forall}{\otimes}{}'](\overset{-}{M'},\overset{+}{p_j})}}{\dfrac{[r_2\overset{\exists}{\otimes}](\overset{+}{p_i},\overset{+}{M}) \vdash [r_1\overset{\forall}{\otimes}{}'](\overset{-}{M'},\overset{+}{p_j})}{M' \vdash [\overset{\forall}{\otimes}{}']\left(\overset{-}{[r_2\overset{\exists}{\otimes}](\overset{+}{p_i},\overset{+}{M})},\overset{+}{p_j}\right)}}$$

Therefore, if $(V_{\mathcal{S}}, E_{\mathcal{S}})$ contains an oriented cycle with the pattern $p_i E_{\mathcal{S}} p$, $p E_{\mathcal{S}} p_j$ then $(V_{\mathcal{S}_p}, E_{\mathcal{S}_p})$ contains an oriented cycle of length smaller by one. If we iterate this process on the atoms $p_1, \ldots, p_k$ of the initial cycle, we will end up with a dependency digraph which contains a reflexive state $p E_{\mathcal{S}_k} p$. Hence, we have a consecution $S \in \mathcal{S}_k$ which contains both an a-part and a s-part occurrence of p. Thus, $\mathcal{S}_k$ cannot respect multiplicities. □

Corollary B.3. *Let C be a Boolean set of atomic connectives and let S be a set of acyclic formulas of $\mathcal{I}_2(\mathsf{GGL}_C)$. The calculus $\mathsf{GGL}_C^{\mathcal{H}} + S$ is sound and strongly complete for the Boolean atomic logics $(\mathcal{L}_C, \mathcal{E}_C, \Vdash)$, where $\mathcal{E}_C$ is the class of pointed C–frames defined by the first–order correspondents of the inductive atomic formulas of S.*

Proof: It follows from Theorem 7.9 and the fact that acyclic formulas are inductive atomic formulas according to Theorem 10.4. □

C Proofs of Theorem 11.1 and Corollary 11.2

Lemma C.1. *Let C be a Boolean set of atomic connectives such that $\mathcal{O}(C) = C$. Then, the display calculus GGL_C is well-behaved for the Hilbert calculus $\mathsf{GGL}_C^{\mathcal{H}}$.*

Proof: No difficulty, it suffices to check the definitions. □

Lemma C.2. *Let C be a set of atomic connectives and let C' be an extension of C with other atomic or Boolean connectives. Then, for all sets of analytic inference rules ρ, $\mathsf{GGL}_{C'}^0 + \rho$ is a conservative extension of $\mathsf{GGL}_C^0 + \rho$.*

Proof: It is a standard argument, it follows from the fact that the calculi have the subformula property and are without the cut rule (that they admit). □

Theorem C.3. *Let C be a set of molecular connectives which are universal or existential and let ρ_0 be a set of structural inference rules for $[\mathcal{L}_C]$. The calculus $\mathsf{GGL}_C^{0,*} + \rho_0$ is equivalent to a proper display calculus $\mathsf{GGL}_C^{0,*} + \rho$ iff it is sound and complete for a molecular logic $(\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ whose class of C–frames is definable by the local first–order correspondents of a set Δ of acyclic formulas of $\mathcal{I}(\mathsf{GGL}_{C'})$ where C' is the Boolean completion of $\mathcal{O}(C'')$, for C'' the set of associated atomic connectives of C. Moreover the acyclic formulas of Δ are effectively computable from the analytic structural rules of ρ, and vice versa, as specified in Figure 6.*

Proof: We first prove it for the case where C is a set of atomic connectives. In that case, $\mathsf{GGL}^{0,*}_{\mathsf{C}}$ is $\mathsf{GGL}^{0}_{\mathsf{C}}$.

For the left to right direction, assume that $\mathsf{GGL}^{0}_{\mathsf{C}}+\rho_0$ is equivalent to a proper display calculus. Then, $\mathsf{GGL}_{\mathsf{C}'}+\rho_0$, where C' is the completion of $\mathcal{O}(\mathsf{C})$ with conjunctions and disjunctions, is also equivalent to a proper display calculus. Moreover $\mathsf{GGL}_{\mathsf{C}'}$ is a well-behaved calculus for $\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}'}$ by Lemma C.1. Then, since $\mathsf{GGL}_{\mathsf{C}'}$ is equivalent to a proper display calculus, it corresponds to a Hilbert calculus $\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}'}+\Delta$ for a set Δ of acyclic formulas of $\mathcal{I}_2(\mathsf{GGL}_{\mathsf{C}'})$ by [11, Theorem 4.6]. Now, by Theorem 10.4, Δ is also a set of inductive formulas of $\mathcal{L}_{\mathsf{C}'}$. Therefore, by the canonicity of inductive formulas [17], $\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}'}+\Delta$ is sound and complete for an atomic logic $(\mathcal{L}_{\mathsf{C}},\mathcal{E}_{\mathsf{C}},\Vdash)$ whose class of C–frames is definable by the local first–order correspondents of the set Δ. So, since $\mathsf{GGL}_{\mathsf{C}'}+\rho_0$ is a conservative extension of $\mathsf{GGL}^{0}_{\mathsf{C}}+\rho_0$ by Lemma C.2, $\mathsf{GGL}^{0}_{\mathsf{C}}+\rho_0$ is also sound and complete for an atomic logic $(\mathcal{L}_{\mathsf{C}},\mathcal{E}_{\mathsf{C}},\Vdash)$ whose class of C–frames is definable by the local first–order correspondents of the set Δ.

For the right to left direction, assume that $\mathsf{GGL}^{0}_{\mathsf{C}}+\rho_0$ is sound and complete for an atomic logic $(\mathcal{L}_{\mathsf{C}},\mathcal{E}_{\mathsf{C}},\Vdash)$ whose class of C–frames is definable by the local first–order correspondents of a set Δ of acyclic formulas of $\mathcal{I}(\mathsf{GGL}_{\mathsf{C}'})$ where C' is the completion of $\mathcal{O}(\mathsf{C})$ with conjunctions and disjunctions. Then, by [11, Theorem 4.6], there is a set of structural inference rules ρ such that $\mathsf{GGL}_{\mathsf{C}'}+\rho$ is a proper display calculus corresponding to $\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}'}+\Delta$. By Corollary 10.5, $\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}'}+\Delta$ is sound and complete for $(\mathcal{L}_{\mathsf{C}},\mathcal{E}_{\mathsf{C}},\Vdash)$. Therefore, $\mathsf{GGL}_{\mathsf{C}'}+\rho$ is sound and complete for $(\mathcal{L}_{\mathsf{C}},\mathcal{E}_{\mathsf{C}},\Vdash)$, like $\mathsf{GGL}^{0}_{\mathsf{C}}+\rho_0$. Now, by Lemma C.2, $\mathsf{GGL}_{\mathsf{C}'}+\rho$ is a conservative extension of $\mathsf{GGL}^{0}_{\mathsf{C}}+\rho$. Thus, both $\mathsf{GGL}^{0}_{\mathsf{C}}+\rho_0$ and $\mathsf{GGL}^{0}_{\mathsf{C}}+\rho$ are sound and complete for $(\mathcal{L}_{\mathsf{C}},\mathcal{E}_{\mathsf{C}},\Vdash)$ and they are based on the same language. Since $\mathsf{GGL}^{0}_{\mathsf{C}}+\rho$ is a proper display calculus, like its extension $\mathsf{GGL}_{\mathsf{C}'}+\rho$, we have that $\mathsf{GGL}^{0}_{\mathsf{C}}+\rho_0$ is equivalent to a proper display calculus.

Now, if C is a set of *molecular* connectives and C' is its set of associated atomic connectives, every proper display calculus $\mathsf{GGL}^{0}_{\mathsf{C}'}+\rho$ proves the same $[\mathcal{L}_{\mathsf{C}}]$–consecutions as $\mathsf{GGL}^{0,*}_{\mathsf{C}}+\rho$, as proved in the proof of [5, Theorem 6]. The result for the case of molecular connectives follows. □

Corollary C.4. *Let C be a set of atomic connectives. Every proper display calculus extending $\mathsf{GGL}^{0}_{\mathsf{C}}$ is sound and strongly complete for a logic $(\mathcal{L}_{\mathsf{C}},\mathcal{E}_{\mathsf{C}},\Vdash)$ whose class of C–frames $\mathcal{E}_{\mathsf{C}}$ is definable by a set of acyclic formulas of $\mathcal{I}_2(\mathsf{GGL}_{\mathsf{C}'})$ where C' is the Boolean completion of $\mathcal{O}(\mathsf{C})$. Moreover the acyclic formulas of Δ are effectively computable from the analytic structural rules of the calculus.*

Proof: It is a direct consequence of the previous theorem. □

D Proofs of Lemmata 13.5, 13.6, 13.7, 13.8 and Theorem 13.9

Lemma D.1. *The following rules are derivable in PL^0 for all $i,j \in [\![1;3]\!]$:*

$$\frac{X\,[\otimes_j]\,Y \vdash U}{X\,[\otimes_i]\,Y \vdash U} \qquad \frac{Z \vdash X\,[\oplus_j]\,Y}{Z \vdash X\,[\oplus_i]\,Y} \qquad \mathsf{TI}_{i[\otimes_i]j}$$

Proof: We only prove it for $i = 2$ and $j = 3$, the proof for the other indices being similar.

$$\begin{array}{cl}
(X\,[\otimes_3]\,Y) \vdash Z & \\ \hline\hline
X \vdash (Y\,[\supset_1]\,Z) & (\mathsf{dr}_1) \\ \hline\hline
(X\,[\otimes_2]\,Y) \vdash (Y\,[\supset_1]\,Z) & \mathsf{K_i} \\ \hline\hline
((X\,[\otimes_2]\,Y)\,[\otimes_3]\,Y) \vdash Z & (\mathsf{dr}_1) \\ \hline\hline
(Y\,[\otimes_3]\,(X\,[\otimes_2]\,Y)) \vdash Z & \mathsf{Cl_i} \\ \hline\hline
Y \vdash ((X\,[\otimes_2]\,Y)\,[\supset_1]\,Z) & (\mathsf{dr}_1) \\ \hline\hline
(Y\,[\otimes_2]\,X) \vdash ((X\,[\otimes_2]\,Y)\,[\supset_1]\,Z) & \mathsf{K_i} \\ \hline\hline
(X\,[\otimes_2]\,Y) \vdash ((X\,[\otimes_2]\,Y)\,[\supset_1]\,Z) & \mathsf{Cl_i} \\ \hline\hline
((X\,[\otimes_2]\,Y)\,[\otimes_3]\,(X\,[\otimes_2]\,Y)) \vdash Z & (\mathsf{dr}_1) \\ \hline\hline
(X\,[\otimes_2]\,Y) \vdash Z & \mathsf{WI_i}
\end{array}$$

□

Lemma D.2. *The following rules are derivable in PL^0: for all $i \in \{1,2,3\}$,*

$$\begin{array}{c}
(X\,[\prec_i]\,Y) \vdash Z \\ \hline\hline
(X\,[\otimes_i] * Y) \vdash Z
\end{array}
\qquad
\begin{array}{c}
(X\,[\succ_i]\,Y) \vdash Z \\ \hline\hline
(*X\,[\otimes_i]\,Y) \vdash Z
\end{array}
\qquad
\begin{array}{c}
X \vdash (Y\,[\subset_i]\,Z) \\ \hline\hline
X \vdash (Y\,[\oplus_i] * Z)
\end{array}
\qquad
\begin{array}{c}
X \vdash (Y\,[\supset_i]\,Z) \\ \hline\hline
X \vdash (*Y\,[\oplus_i]\,Z)
\end{array}$$

$$\begin{array}{c}
(X\,[\prec_i']\,Y) \vdash Z \\ \hline\hline
(X\,[\otimes_i] * Y) \vdash Z
\end{array}
\qquad
\begin{array}{c}
(X\,[\succ_i']\,Y) \vdash Z \\ \hline\hline
(*X\,[\otimes_i]\,Y) \vdash Z
\end{array}
\qquad
\begin{array}{c}
X \vdash (Y\,[\subset_i']\,Z) \\ \hline\hline
X \vdash (Y\,[\oplus_i] * Z)
\end{array}
\qquad
\begin{array}{c}
X \vdash (Y\,[\supset_i']\,Z) \\ \hline\hline
X \vdash (*Y\,[\oplus_i]\,Z)
\end{array}$$

$$\begin{array}{c}
(X\,[\uparrow_i]\,Y) \vdash Z \\ \hline\hline
(*X\,[\otimes_i] * Y) \vdash Z
\end{array}
\qquad
\begin{array}{c}
X \vdash (Y\,[\downarrow_i]\,Z) \\ \hline\hline
X \vdash (*Y\,[\oplus_i] * Z)
\end{array}$$

Proof: We only provide some of the proofs, the others can be obtained with slight modifications.

$$\begin{array}{cl}
X \vdash (Y\,[\supset_1]\,Z) & \\ \hline\hline
(X\,[\otimes_3]\,Y) \vdash Z & (\mathsf{dr}_1) \\ \hline\hline
Y \vdash (X\,[\supset_2']\,Z) & (\mathsf{dr}_1) \\ \hline\hline
(X\,[\prec_2']\,Z) \vdash *Y & (\mathsf{dr}_2') \\ \hline\hline
X \vdash (*Y\,[\otimes_1]\,Z) & (\mathsf{dr}_1)
\end{array}
\qquad
\begin{array}{cl}
(X\,[\succ_1]\,Y) \vdash Z & \\ \hline\hline
Y \vdash (Z\,[\otimes_3]\,X) & (\mathsf{dr}_1) \\ \hline\hline
(Z\,[\prec_2']\,Y) \vdash X & (\mathsf{dr}_1) \\ \hline\hline
*X \vdash (Z\,[\supset_2']\,Y) & (\mathsf{dr}_2') \\ \hline
(*X\,[\otimes_1]\,Y) \vdash Z & (\mathsf{dr}_1)
\end{array}$$

$$\begin{array}{cl}
(X\,[\uparrow_3]\,Y) \vdash Z & \\ \hline\hline
*Z \vdash (X\,[\otimes_3]\,Y) & (\mathsf{dr}_2') \\ \hline\hline
(X\,[\succ_2'] * Z) \vdash Y & (\mathsf{dr}_1) \\ \hline\hline
*Y \vdash (X\,[\subset_2'] * Z) & (\mathsf{dr}_2') \\ \hline\hline
(*Y\,[\otimes_1] * Z) \vdash X & (\mathsf{dr}_1) \\ \hline\hline
*X \vdash (*Y \downarrow_1 *Z) & (\mathsf{dr}_2') \\ \hline\hline
*Z \vdash (*X \downarrow_3 *Y) & (\mathsf{dr}_1) \\ \hline\hline
(*X\,[\otimes_3] * Y) \vdash Z & (\mathsf{dr}_2')
\end{array}
\qquad
\begin{array}{cl}
Z \vdash (X\,[\downarrow_3]\,Y) & \\ \hline\hline
(X\,[\otimes_3]\,Y) \vdash *Z & (\mathsf{dr}_2') \\ \hline\hline
Y \vdash (X\,[\supset_2'] * Z) & (\mathsf{dr}_1) \\ \hline\hline
(X\,[\prec_1'] * Z) \vdash *Y & (\mathsf{dr}_2') \\ \hline\hline
X \vdash (*Y\,[\otimes_1] * Z) & (\mathsf{dr}_1) \\ \hline\hline
(*Y\,[\uparrow_1] * Z) \vdash *X & (\mathsf{dr}_2') \\ \hline\hline
(*X\,[\uparrow_3] * Y) \vdash *Z & (\mathsf{dr}_1) \\ \hline\hline
Z \vdash (*X\,[\otimes_3] * Y) & (\mathsf{dr}_2')
\end{array}$$

Note that we do not use the structural rules in the proofs. □

Lemma D.3. *The following rules are derivable in PL^0:*

$$
\begin{array}{c} [\sim_1]\,X \vdash Y \\ \hline\hline [\sim_2]\,X \vdash Y \\ \hline\hline {*X} \vdash Y \end{array}
\qquad
\begin{array}{c} X \vdash [\bot_1]\,Y \\ \hline\hline X \vdash [\bot_2]\,Y \\ \hline\hline X \vdash {*Y} \end{array}
\qquad
\begin{array}{c} [\Diamond^-]\,X \vdash Y \\ \hline\hline [\Diamond]\,X \vdash Y \\ \hline\hline X \vdash Y \end{array}
\qquad
\begin{array}{c} X \vdash [\Box^-]\,Y \\ \hline\hline X \vdash [\Box]\,Y \\ \hline\hline X \vdash Y \end{array}
$$

Proof: We only prove it for $[\sim_2]\,,[\Box]\,,[\bot_2]$ and $[\Diamond^-]$ (the last two derivations use the first two).

$$
\begin{array}{cl} [\sim_2]\,X \vdash Y & \\ \hline\hline [\sim_1]\,Y \vdash X & (\mathsf{dr}_1) \\ \hline\hline {*Y} \vdash X & \mathsf{N}^3_{\mathrm{i}},\mathsf{N}^4_{\mathrm{i}} \\ \hline\hline {*X} \vdash Y & (\mathsf{dr}_2') \end{array}
\qquad
\begin{array}{cl} X \vdash [\Box]\,Y & \\ \hline\hline [\sim_1]\,Y \vdash {*X} & (\mathsf{dr}_2') \\ \hline\hline {*Y} \vdash {*X} & \mathsf{N}^3_{\mathrm{i}},\mathsf{N}^4_{\mathrm{i}} \\ \hline\hline X \vdash Y & (\mathsf{dr}_2') \end{array}
\qquad
\begin{array}{cl} X \vdash [\bot_2]\,Y & \\ \hline\hline [\Diamond^-]\,Y \vdash {*X} & (\mathsf{dr}_2') \\ \hline\hline Y \vdash [\Box]\,{*X} & (\mathsf{dr}_1) \\ \hline\hline Y \vdash {*X} & \\ \hline\hline X \vdash {*Y} & (\mathsf{dr}_2') \end{array}
\qquad
\begin{array}{cl} [\Diamond^-]\,X \vdash Y & \\ \hline\hline {*Y} \vdash [\bot_2]\,X & (\mathsf{dr}_2') \\ \hline\hline {*Y} \vdash {*X} & \\ \hline\hline X \vdash Y & (\mathsf{dr}_2') \end{array}
$$

The other inferences are obtained using the display rules. □

Lemma D.4. *In PL^0, the introduction rules for $\neg = \sim_1$ are equivalent to the following:*

$$
\frac{{*\varphi} \vdash U}{\neg\varphi \vdash U}
\qquad\qquad
\frac{U \vdash {*\varphi}}{U \vdash \neg\varphi}
$$

Proof: It follows from the previous lemma and the following theorems and rules:

$$
\frac{U \vdash {*\varphi}}{U \vdash -_1\,\varphi}
\qquad
\frac{{*\varphi} \vdash U}{\sim_1\,\varphi \vdash U}
\qquad
\sim_1\,\varphi \vdash -_1\,\varphi
\qquad
-_1\,\varphi \vdash \sim_1\,\varphi
$$

□

Theorem D.5. *Any proof in PL^0 can be translated into a proof in PL, and vice versa. Hence, PL^0 proves the same theorems of $\mathcal{S}_{\mathsf{C}}$ as the display calculus PL of Figure 9.*

Proof: It follows from our previous explanations, lemmas and theorems. □