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Correspondence Theory of Atomic Logics

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Abstract

The correspondence theory for the framework of atomic and molecular logics is developed on the basis of the work of Goranko & Vakarelov. First, we show that modal polyadic logics can be embedded into atomic logics. Using this embedding, we reformulate the notion of inductive formulas introduced by Goranko & Vakarelov into our framework. This allows us to prove correspondence theorems for atomic logics by adapting their results. We apply our general results to FDE, the family of relevance logics, intuitionistic logic and intermediate logics. We obtain novel axiomatizations of these logics that include Boolean connectives.

Keywords : Universal logic; Hilbert calculus; Sahlqvist correspondence theory; inductive formula; relevance logics; FDE; intermediate logics; intuitionistic logic

1 Introduction

The display, sequent and Hilbert calculi that we introduced in [6] are sound and complete for *basic* atomic and molecular logics. That is, we are able to axiomatize the validities of logics whose class of models or frames do not satisfy any specific conditions (such as reflexivity, transitivity, *etc.*). It constitutes an advancement because, like for their bisimulation notions, we can automatically obtain the display and Hilbert calculi of any *basic* atomic logic once its set of connectives is given. However, this has limitations. Indeed, we would also like to obtain automatically a sound and complete axiomatization of any atomic or molecular logic defined on *any* class of models or frames, when it is possible. Solving this problem amounts to developing a correspondence theory for atomic and molecular logics.

Correspondence theory started to be developed with modal logic as initial object of study [54, 58, 40, 42, 34]. It investigates to what extent specific properties of the semantics can be reformulated in terms of the validity of specific formulas or inference rules. More precisely, it addresses the following kinds of questions. When does the truth of a given formula in a frame correspond to a first-order property in that frame? When does the validity of a formula on a class of frames correspond to the fact that this class of frames satisfies a specific first-order property? Which class of frames defined by a first-order property can be defined by a formula of the logic considered? It has been, since its inception with modal logic, extended and applied to other non-classical logics, such as tense logic [40, 41, 42]. A general and unified theory of correspondence

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has also been propounded by Palmigiano & Al. [16, 35, 22]. Like Kracht, they introduced a calculus for correspondence by extending the language with nominals. This enables to compute the minimal valuation and the first-order correspondent of a formula of a given non-classical logic automatically by means of an algorithm called ALBA. This algorithm is in fact a generalisation of the algorithm SQEMA and stems from ideas coming originally from Conradie, Goranko, and Vakarelov’s series of earlier papers on algorithmic correspondence in modal logic [18, 19, 20].¹

In that article, we do not intend to propose another unified correspondence theory but simply to apply well-known results for modal polyadic logics [34] to our atomic logics. Hence, our overall methodology is somehow different. In order to apply the unified correspondence techniques of Palmigiano & Al. to a given logical framework, one has to recognize the logic as a normal DLE-logic introduced in [35] or a fragment thereof (*e.g.* in [21, Example 1.2] or in [17] where PEARL, an adaptation of the algorithms SQEMA and ALBA for relevance logic, is introduced). However, it is not always possible, for example for temporal logic.² Here, instead of tackling the full range of non-classical logics and developing a general correspondence theory which can be instantiated, adapted and applied to each of these non-classical logics, we have first found a way to represent uniformly, faithfully and systematically non-classical logics by means of our atomic and molecular logics. Now we develop a correspondence theory for these specific ‘paradigmatic’ logics.

Indirectly, our correspondence theory for atomic logics should and does lead to an induced correspondence theory for many non-classical logic. We illustrate this claim with some well-known non-classical (propositional) logics: FDE, the family of relevance logics, intuitionistic logic and intermediate (superintuitionistic) logics. We show how we can automatically define sound and complete Hilbert calculi for these logics, and infinitely many more, by simply running our correspondence algorithms.

Structure of the article. In Section 2, we recall atomic logics, in Section 3 molecular logics, in Section 4 their Boolean versions and in Section 5 their natural extension/generalization with information orders. In Section 6, we show that modal polyadic logics and atomic logics whose connectives are all of dimension $(1, 1, \dots, 1)$ are equally expressive. In Section 7, we introduce group actions on the set of atomic connectives, they play an important role in the definitions of calculi for atomic logics. Then, in Section 8, we introduce sound and complete Hilbert calculi for basic atomic logics based on the results of [6]. In Section 9, we define universal and existential molecular connectives. This allows us in Section 10 to adapt the definitions of Goranko & Vakarelov [34] to our setting and reformulate their main correspondence results in terms of atomic logics. In that section, we also transfer our results to atomic logics with information orders (Section 10.3). In Section 11, we apply our general results to First Degree Entailment, the family of relevance logics, intuitionistic and intermediate logics and find out novel axiomatizations of these logics that include Boolean connectives. We end in Section 12 by discussing related works and conclude.

Note. The article is self-contained. It is the second in a series of articles on the proof and correspondence theory of atomic and molecular logics, starting with [6] and continuing with [5]. Omitted proofs are in the appendix or in the companion articles.

¹See also https://store.fmi.uni-sofia.bg/fmi/logic/sqema/sqema_gwt_20180317_2/index.html. This work extends the algorithm SCAN of Gabbay and Ohlbach [31] and the algorithm DLS of Szalas [56].

²The scope of the unified correspondence theory can be widened, so that the modal connectives do not need to be normal, but can also be regular [22, 46] and monotone or without any order-theoretic properties [15].

2 Atomic Logics

Molecular logics are logics whose primitive connectives are compositions of connectives of atomic logics. Atomic logics are logics in which the truth conditions of connectives are defined by first-order (FO for short) formulas of the form $\forall x_1 \dots x_n (\pm_1 Q_1 x_1 \vee \dots \vee \pm_n Q_n x_n \vee \pm R x_1 \dots x_n x)$ or $\exists x_1 \dots x_n (\pm_1 Q_1 x_1 \wedge \dots \wedge \pm_n Q_n x_n \wedge \pm R x_1 \dots x_n x)$ where the \pm_i s and \pm are either empty or \neg . Likewise, propositional letters are defined by first-order formulas of the form $\pm R x$. They are viewed as 0-ary connectives (which is why we note them R and not Q) and the \pm in front of them stands for the fact that they can stand for literals. We will represent the structure of these formulas by means of so-called *skeletons* whose various arguments capture the different features and patterns from which they can be redefined completely. But first, we introduce some notations.

Notations. We use the symbol \triangleq for equality by definition instead of the assignment symbol $:=$ often encountered in the literature (the Δ above the equality symbol $=$ is the Greek ‘D’ of Definition). \mathbb{N}^* denotes the set of natural numbers without 0 and, for all $m, n \in \mathbb{N}$ such that $m \leq n$, $\llbracket m; n \rrbracket$ denotes the set of natural numbers $\{m, m+1, \dots, n\}$. \mathfrak{S}_n denotes the group of permutations over the set $\{1, \dots, n\}$. Permutations are generally denoted σ, τ , the identity permutation is denoted Id and σ^- stands for the inverse permutation of the permutation σ . For example, the permutation $\sigma = (3, 1, 2)$ is the permutation that maps 1 to 3, 2 to 1 and 3 to 2. We recall that $\mathbb{Z}/2\mathbb{Z}$ denotes the field of the integers modulo 2 (also known as the dihedral group of order 2).³ When viewed as a multiplicative group, its elements will be denoted in the sequel $+$ and $-$ and its operation \cdot is such that $+\cdot- = -\cdot+ = -$ and $-\cdot- = +\cdot+ = +$. For brevity, when we use the notation $+, -$, we often omit the \cdot and write for example $+- = -$ for $+\cdot- = -$. See [50] for relevant details on group theory.

For example, the truth condition of the connectives \square of modal logic is the first-order formula $\forall y (P y \vee \neg R x y)$. Its ‘skeleton’ is $((1, 1), +, +, -, (2, 1))$, which we also write for better readability $((1, 1), \forall, +, -, (2, 1))$. The so-called ‘dimension signature’ $(1, 1)$ corresponds to the fact that this connective takes as input a formula of dimension 1, represented by the predicate P of arity 1, and yields another formula of dimension 1, because the first-order formula has a single free variable representing the state where the resulting formula is evaluated. The ‘quantification signature’ \forall corresponds to the universal quantification \forall in front of the formula and is represented by $+$ ($-$ represents the existential quantification \exists). The ‘tonicity signature’ is $+$ because there is no negation in front of P. The $-$ in the skeleton corresponds to the negation \neg in front of $\neg R x y$. Finally, by convention, the natural order for elements appearing in a relation is $R y z x$ or $R y x$, and more generally $R x_1 \dots x_n x$, where the free variable x denotes the state where the formula is evaluated. So in this example, $R y x$ is transformed into $R x y$, which explains the introduction of the permutation $(2, 1) \in \mathfrak{S}_2$ which swaps x and y .

Definition 1 (Atomic connectives and skeletons). The set SKL_0 of *propositional letter skeletons* and the sets SKL_n of *skeletons of arity* $n \in \mathbb{N}^*$ are defined as follows:

$$\begin{aligned} \text{SKL}_0 &\triangleq \mathbb{N}^* \times \mathbb{Z}/2\mathbb{Z}^2 \\ \text{SKL}_n &\triangleq \mathbb{N}^{*n+1} \times \mathbb{Z}/2\mathbb{Z}^{n+2} \times \mathfrak{S}_{n+1}. \end{aligned}$$

³The group $\mathbb{Z}/2\mathbb{Z}$ was used to define atomic connectives by Espejo-Boix [30, Definition 3]. He used both the additive and multiplicative operations of the field $\mathbb{Z}/2\mathbb{Z}$ to reformulate the central group action for atomic logics [2, Definition 18] in terms of a matrix product over $\mathbb{Z}/2\mathbb{Z}^{n+1}$.

The set of *atomic skeletons* is the set $\text{SKL} \triangleq \bigcup_{n \in \mathbb{N}} \text{SKL}_n$. They can be represented by tuples $(k_1, \dots, k_{n+1}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$, or $(\bar{k}, \mathbb{A}, \pm)$ if it is a propositional letter skeleton, where $(k_1, \dots, k_{n+1}) \in \mathbb{N}^{*n+1}$, often denoted \bar{k} , is called the *dimension signature*, $\mathbb{A} \in \{+, -\}$ is called the *quantification signature*, $(\pm_1, \dots, \pm_n) \in \{+, -\}^n$ is called the *tonicity signature*, $\pm \in \{+, -\}$ is called the *relation signature* and σ is called the *permutation signature*. The tuple $(\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_n)$ is called the *signature* or *trace* of the skeleton; the tonicity and quantification signatures are in correspondence with Dunn's notion of trace [28] (see [2, Definition 15] for details). The quantification signature \mathbb{A} will often be denoted \forall if it is $+$ and \exists if it is $-$. The *arity* of a propositional letter skeleton is 0 and its *dimension* is k . The *input dimensions* and (*output*) *dimension* of a connective skeleton $\bullet \in \text{SKL}_n$ of arity $n \in \mathbb{N}^*$ are k_1, \dots, k_n and k_{n+1} respectively.

Let \mathbb{I} be an arbitrary but fixed set; in this article we assume that $\mathbb{N} \subseteq \mathbb{I}$. The set ATM_n of *atomic connectives of arity* $n \in \mathbb{N}$ is defined as follows:

$$\text{ATM}_n \triangleq \{(\bullet, i) \mid \bullet \in \text{SKL}_n, i \in \mathbb{I}\}.$$

The set of *atomic connectives* is the set $\text{ATM} \triangleq \bigcup_{n \in \mathbb{N}} \text{ATM}_n$. Those of arity 0, ATM_0 , are also called *propositional letters*. The arity, signature, quantification signature, dimension signature, tonicity signature, relation signature, permutation signature and input and output dimensions of an atomic connective (\bullet, i) are the same as its skeleton \bullet .

If \mathbb{C} is a set of atomic connectives, its set of propositional letters is denoted \mathbb{C}_0 . Propositional letters are denoted p, p_1, p_2, \dots, p_i , etc, skeletons are denoted $\bullet, \bullet_1, \bullet_2, \dots, \bullet_i$, etc. and connectives $\odot, \odot_1, \odot_2, \dots, \odot_i$, etc. The quantification signature of a connective \odot or a skeleton \bullet is denoted $\mathbb{A}(\odot) = \mathbb{A}(\bullet)$ and the j^{th} element of the tonicity signature of \odot and \bullet is denoted $\pm_j(\odot) = \pm_j(\bullet)$. \square

Definition 2 (Residuated skeletons and connectives). Two atomic skeletons $(\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$ and $(\bar{k}', \mathbb{A}', \pm'_1, \dots, \pm'_n, \pm', \sigma')$ are *residuated* when they are of equal arity $n \in \mathbb{N}^*$ and there is $\tau \in \mathfrak{S}_{n+1}$ such that $\sigma' = \tau \circ \sigma$ and $(k'_{\tau(1)}, \dots, k'_{\tau(n+1)}) = (k_1, \dots, k_{n+1})$. Two atomic connectives $(\bullet, i), (\bullet', i') \in \text{ATM}_n$ are residuated when $i = i'$ and their skeletons \bullet and \bullet' are residuated.

Example 1. The skeleton of the implication \supset of the Lambek calculus, represented by the FO formula $\forall yz(\neg Py \vee Qz \vee \neg Rxyz)$, is $((1, 1, 1), \forall, -, +, -, (2, 3, 1))$ and the skeleton of the fusion \otimes of the Lambek calculus, represented by the FO formula $\exists yz(Py \wedge Qz \wedge Ryzx)$, is $((1, 1, 1), \exists, +, +, +, \text{Id})$. Note the permutations associated with the relations: $(2, 3, 1)$ for $Rxyz$ and $\text{Id} = (1, 2, 3)$ for $Ryzx$ because the latter order (y, z, x) is the natural order. As one can also easily notice, \supset and \otimes are residuated.

Definition 3 (Atomic language). Let $\mathbb{C} \subseteq \text{ATM}$ be a set of atomic connectives. The *atomic language* $\mathcal{L}_{\mathbb{C}}$ associated to \mathbb{C} is the smallest set that contains the propositional letters of \mathbb{C} and that is closed under the atomic connectives of \mathbb{C} while respecting the dimensions constraint. That is,

- $\mathbb{C}_0 \subseteq \mathcal{L}_{\mathbb{C}}$;
- for all $\odot \in \mathbb{C}$ of arity $n > 0$ and of dimension signature (k_1, \dots, k_{n+1}) and for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathbb{C}}$ of dimensions k_1, \dots, k_n respectively, we have that $\odot(\varphi_1, \dots, \varphi_n) \in \mathcal{L}_{\mathbb{C}}$ and $\odot(\varphi_1, \dots, \varphi_n)$ is of *dimension* k_{n+1} .

Elements of $\mathcal{L}_{\mathbb{C}}$ are called *atomic formulas* and are denoted φ, ψ, \dots . The *dimension of a formula* $\varphi \in \mathcal{L}_{\mathbb{C}}$ is denoted $k(\varphi)$. A set of atomic connectives \mathbb{C} is *plain* if for all $\odot \in \mathbb{C}$ of

dimension signature (k_1, \dots, k_{n+1}) with $n > 0$ there are propositional letters $p_1, \dots, p_n \in \mathcal{C}_0$ of dimensions k_1, \dots, k_n respectively. In the sequel, we assume that all sets of connectives \mathcal{C} are plain.

Definition 4 (Atomic \mathcal{C} -models and \mathcal{C} -frames). Let $\mathcal{C} \subseteq \text{ATM}$ be a set of atomic connectives. An (atomic) \mathcal{C} -model is a tuple $\mathfrak{M} = (W, \mathfrak{R})$ where W is a non-empty set and \mathfrak{R} is a set of relations over W such that each n -ary connective $\odot \in \mathcal{C}$ of dimension signature (k_1, \dots, k_{n+1}) is associated to a $k_1 + \dots + k_{n+1}$ -ary relation $R_\odot \in \mathfrak{R}$ and such that for all connectives $\odot, \odot' \in \mathcal{C}$ we have that $R_\odot = R_{\odot'}$ iff \odot and \odot' are residuated. An assignment is a tuple $(w_1, \dots, w_k) \in W^k$ for some $k \in \mathbb{N}^*$, generally denoted \bar{w} . The set of assignments of a \mathcal{C} -model \mathfrak{M} is denoted $\omega(\mathfrak{M}, \mathcal{C})$. A pointed \mathcal{C} -model (\mathfrak{M}, \bar{w}) is a \mathcal{C} -model \mathfrak{M} together with an assignment \bar{w} and, in that case, we say that (\mathfrak{M}, \bar{w}) is of dimension k . The class of all pointed \mathcal{C} -models is denoted $\mathcal{C}_\mathcal{C}$.

A (pointed) atomic \mathcal{C} -frame is a (pointed) atomic $(\mathcal{C} - \text{ATM}_0)$ -model. The class of all pointed \mathcal{C} -frames is denoted $\mathcal{F}_\mathcal{C}$.

Definition 5 (Atomic logics). Let $\mathcal{C} \subseteq \text{ATM}$ be a set of atomic connectives and let $\mathfrak{M} = (W, \mathfrak{R})$ be a \mathcal{C} -model. We define the interpretation function of $\mathcal{L}_\mathcal{C}$ in \mathfrak{M} , denoted $\llbracket \cdot \rrbracket^{\mathfrak{M}} : \mathcal{L}_\mathcal{C} \rightarrow \bigcup_{k \in \mathbb{N}^*} W^k$, inductively as follows: for all propositional letters $p \in \mathcal{C}$ of dimension k , all connectives $\odot \in \mathcal{C}$ of skeleton $(k_1, \dots, k_{n+1}, \mathbb{E}, \pm_1, \dots, \pm_n, \pm, \sigma)$ of arity $n > 0$, for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_\mathcal{C}$,

$$\begin{aligned} \llbracket p \rrbracket^{\mathfrak{M}} &\triangleq \begin{cases} R_p & \text{if } \pm = + \\ W^k - R_p & \text{if } \pm = - \end{cases} \\ \llbracket \odot(\varphi_1, \dots, \varphi_n) \rrbracket^{\mathfrak{M}} &\triangleq f_\odot(\llbracket \varphi_1 \rrbracket^{\mathfrak{M}}, \dots, \llbracket \varphi_n \rrbracket^{\mathfrak{M}}) \end{aligned}$$

where the function f_\odot is defined as follows. For all $W_1 \in \mathcal{P}(W^{k_1}), \dots, W_n \in \mathcal{P}(W^{k_n})$, $f_\odot(W_1, \dots, W_n) \triangleq \{\bar{w}_{n+1} \in W^{k_{n+1}} \mid \mathcal{C}^\odot(W_1, \dots, W_n, \bar{w}_{n+1})\}$ where $\mathcal{C}^\odot(W_1, \dots, W_n, \bar{w}_{n+1})$ is called the truth condition of \odot and is defined as follows:

- if $\mathbb{E} = \forall$: “ $\forall \bar{w}_1 \in W^{k_1} \dots \bar{w}_n \in W^{k_n}$
 $(\bar{w}_1 \uparrow_1 W_1 \vee \dots \vee \bar{w}_n \uparrow_n W_n \vee R_\odot^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}_{n+1})$ ”;
- if $\mathbb{E} = \exists$: “ $\exists \bar{w}_1 \in W^{k_1} \dots \bar{w}_n \in W^{k_n}$
 $(\bar{w}_1 \uparrow_1 W_1 \wedge \dots \wedge \bar{w}_n \uparrow_n W_n \wedge R_\odot^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}_{n+1})$ ”;

where, for all $j \in \llbracket 1; n \rrbracket$, $\bar{w}_j \uparrow_j W_j \triangleq \begin{cases} \bar{w}_j \in W_j & \text{if } \pm_j = + \\ \bar{w}_j \notin W_j & \text{if } \pm_j = - \end{cases}$ and $R_\odot^{\pm\sigma} \bar{w}_1 \dots \bar{w}_{n+1}$ holds iff $\pm R_\odot \bar{w}_{\sigma(1)} \dots \bar{w}_{\sigma(n+1)}$ holds, with the notations $+R_\odot \triangleq R_\odot$ and $-R_\odot \triangleq W^{k_1 + \dots + k_{n+1}} - R_\odot$. We extend the definition of the interpretation function $\llbracket \cdot \rrbracket^{\mathfrak{M}}$ to \mathcal{C} -frames as follows: for all $\varphi \in \mathcal{L}_\mathcal{C}$ and all \mathcal{C} -frames \mathfrak{F} ,

$$\llbracket \varphi \rrbracket^{\mathfrak{F}} \triangleq \bigcap \left\{ \llbracket \varphi \rrbracket^{(\mathfrak{F}, V)} \mid V \text{ a set of } n\text{-ary relations over } W \text{ such that } (\mathfrak{F}, V) \text{ is a } \mathcal{C}\text{-model} \right\}$$

Finally, if $\mathcal{E}_\mathcal{C} \subseteq \mathcal{C}_\mathcal{C}$ is a class of pointed \mathcal{C} -models, or pointed \mathcal{C} -frames, the satisfaction relation $\Vdash \subseteq \mathcal{E}_\mathcal{C} \times \mathcal{L}_\mathcal{C}$ is defined as follows: for all $\varphi \in \mathcal{L}_\mathcal{C}$ and all $(\mathfrak{M}, \bar{w}) \in \mathcal{E}_\mathcal{C}$, $((\mathfrak{M}, \bar{w}), \varphi) \in \Vdash$ iff $\bar{w} \in \llbracket \varphi \rrbracket^{\mathfrak{M}}$. We usually write $(\mathfrak{M}, \bar{w}) \Vdash \varphi$ instead of $((\mathfrak{M}, \bar{w}), \varphi) \in \Vdash$ and we say that φ is true in (\mathfrak{M}, \bar{w}) . The logic $(\mathcal{L}_\mathcal{C}, \mathcal{E}_\mathcal{C}, \Vdash)$ is the atomic logic associated to $\mathcal{E}_\mathcal{C}$ and \mathcal{C} . Logics of the form $(\mathcal{L}_\mathcal{C}, \mathcal{C}_\mathcal{C}, \Vdash)$ are called basic atomic logics.

Example 2. A simple example of an atomic logic is modal logic, where $\mathcal{C} = \{p, \top, \perp, \neg, \wedge, \vee, \rightarrow, \diamond_i, \square_i \mid i \in \mathbb{I}\}$. We spell this example out in some detail:

- p is a proposition letter of dimension 1 and \top, \perp are the proposition letters $((1, \exists, +), 0)$ and $((1, \forall, -), 0)$ respectively;
- \neg is the connective $((1, 1), \exists, -, +, \text{Id}), 0$;
- $\wedge, \vee, \rightarrow$ are the connectives $((1, 1, 1), \exists, +, +, +, \text{Id}), 0$, $((1, 1, 1), \forall, +, +, -, \text{Id}), 0$ and $((1, 1, 1), \forall, -, +, -, (3, 1, 2)), 0$ respectively;
- \diamond_i, \square_i are the connectives $((1, 1), \exists, +, +, (2, 1)), i$ and $((1, 1), \forall, +, -, (2, 1)), i$ respectively;
- the C-models $\mathfrak{M} = (W, \mathfrak{R}) \in \mathcal{E}_C$ are such that $R_{\neg} \triangleq \{(w, w) \mid w \in W\}$, $R_{\wedge} = R_{\vee} = R_{\rightarrow} \triangleq \{(w, w, w) \mid w \in W\}$, $R_{\diamond_i} = R_{\square_i}$ for all $i \in \mathbb{I}$ and $R_{\top} = R_{\perp} = W$.

With these conditions on the C-models of \mathcal{E}_C , for all $(\mathfrak{M}, w) \in \mathcal{E}_C$, for all $i \in \mathbb{I}$,

$w \in \llbracket \diamond_i \varphi \rrbracket^{\mathfrak{M}}$	iff	$\exists v (v \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \wedge R_{\diamond_i} wv)$
$w \in \llbracket \square_i \varphi \rrbracket^{\mathfrak{M}}$	iff	$\forall v (v \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \vee \neg R_{\square_i} wv)$
$w \in \llbracket \wedge(\varphi, \psi) \rrbracket^{\mathfrak{M}}$	iff	$\exists vu (v \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \wedge u \in \llbracket \psi \rrbracket^{\mathfrak{M}} \wedge R_{\wedge} vuw)$
	iff	$w \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \wedge w \in \llbracket \psi \rrbracket^{\mathfrak{M}}$
$w \in \llbracket \vee(\varphi, \psi) \rrbracket^{\mathfrak{M}}$	iff	$\forall vu (v \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \vee u \in \llbracket \psi \rrbracket^{\mathfrak{M}} \vee \neg R_{\vee} vuw)$
	iff	$w \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \vee w \in \llbracket \psi \rrbracket^{\mathfrak{M}}$
$w \in \llbracket \rightarrow(\varphi, \psi) \rrbracket^{\mathfrak{M}}$	iff	$\forall vu (v \notin \llbracket \varphi \rrbracket^{\mathfrak{M}} \vee u \in \llbracket \psi \rrbracket^{\mathfrak{M}} \vee \neg R_{\rightarrow} wvu)$
	iff	$w \notin \llbracket \varphi \rrbracket^{\mathfrak{M}} \vee w \in \llbracket \psi \rrbracket^{\mathfrak{M}}$
$w \in \llbracket \neg \varphi \rrbracket^{\mathfrak{M}}$	iff	$\exists v (v \notin \llbracket \varphi \rrbracket^{\mathfrak{M}} \wedge R_{\neg} vw)$
	iff	$w \notin \llbracket \varphi \rrbracket^{\mathfrak{M}}$
$w \in \llbracket \top \rrbracket^{\mathfrak{M}}$	iff	$w \in R_{\top}$
	iff	always
$w \in \llbracket \perp \rrbracket^{\mathfrak{M}}$	iff	$w \in W - R_{\perp}$
	iff	never

Example 3 (Temporal arrow logic). Arrow logic and temporal arrow logics are atomic logics which extend polyadic modal logics. They are based on the set of atomic connectives $C = \{p, \iota\delta, \lambda, \neg_2, \vee_2, \otimes_2, \circ_2\}$ where

- p is a propositional letter and \neg_2, \vee_2 are the Boolean negation and disjunction respectively, all of dimension 2;
- $\iota\delta$ and λ are propositional letters, both of dimension 2;
- \otimes_2 and \circ_2 are atomic connectives of skeletons $((2, 2), \exists, +, +, (2, 1))$ and $((2, 2, 2), \exists, +, +, +, \text{Id})$ respectively;
- the C-models $\mathfrak{M} = (W, \mathfrak{R}) \in \mathcal{C}_C$ are such that $R_{\circ_2} \triangleq \{(u, w), (w, v), (u, v) \mid u, v, w \in W\}$, $R_{\otimes_2} \triangleq \{(u, v), (v, u) \mid u, v \in W\}$, $R_{\iota\delta} \triangleq \{(u, u) \mid u \in W\}$ and R_{λ} is a 2-ary relation over W .

With these conditions on the C-models of \mathcal{E}_C , for all $(\mathfrak{M}, w) \in \mathcal{E}_C$,

$(u, v) \in \llbracket \iota\delta \rrbracket^{\mathfrak{M}}$	iff	$u = v$
$(u, v) \in \llbracket \lambda \rrbracket^{\mathfrak{M}}$	iff	$(u, v) \in R_{\lambda}$
$(u, v) \in \llbracket \otimes_2 \varphi \rrbracket^{\mathfrak{M}}$	iff	$\exists (u', v') ((u', v') \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \wedge ((u', v'), (u, v)) \in R_{\otimes_2})$
	iff	$(v, u) \in \llbracket \varphi \rrbracket^{\mathfrak{M}}$
$(u, v) \in \llbracket \varphi \circ_2 \psi \rrbracket^{\mathfrak{M}}$	iff	$\exists (u', v')(u'', v'') ((u', v') \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \wedge (u'', v'') \in \llbracket \psi \rrbracket^{\mathfrak{M}} \wedge ((u', v''), (u', v''), (u, v)) \in R_{\circ_2})$
	iff	$\exists w ((u, w) \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \wedge (w, v) \in \llbracket \psi \rrbracket^{\mathfrak{M}})$. \square

Example 4 (Dummy connectives). We can define ‘dummy’ connectives, in the sense that these connectives do not affect the truth value of the formulas that they take as argument. They will be used in the definition of molecular connectives and will serve as ‘markers’ so that we can ‘plug’ the same formula at different places of a (molecular) connective. They are defined as follows:

$$\text{ATM}_x \triangleq \{(((k, k), \exists, +, +, \text{Id}), i), (((k, k), \forall, +, -, \text{Id}), i) \mid k, i \in \mathbb{N}^*\}$$

These ‘dummy’ connectives will be denoted x_i^k in general and the set of their skeletons, denoted x^k , is denoted SKL_x . Any C-model $\mathfrak{M} = (W, \mathfrak{R})$ containing some of these connectives is such that $R_{x_i^k} = \{(\bar{w}, \bar{w}) \mid \bar{w} \in W^k\}$. Hence, for all $\varphi \in \mathcal{L}_C$ and all $\bar{w} \in W^k$, we have that $\bar{w} \in \llbracket x_i^k(\varphi) \rrbracket^{\mathfrak{M}}$ iff $\bar{w} \in \llbracket \varphi \rrbracket^{\mathfrak{M}}$.

The following theorem can be easily proven by polynomially reducing the satisfiability problem of basic atomic logic to the satisfiability problem of modal logic.

Theorem 1 ([6]). *Every basic atomic logic is decidable and in PSPACE.*

3 Molecular Logics

Molecular logics are basically logics whose primitive connectives are compositions of atomic connectives and in which it is possible to repeat the same atomic connective at different places in the connective. That is why we call them ‘molecular’, just as molecules are compositions of atoms in chemistry.

Definition 6 (Molecular connectives and skeletons). The set MOL of *molecular connectives* is the smallest set such that:

- $\text{ATM}_x \cup \text{ATM}_0 \subseteq \text{MOL}$;
- for all atomic connectives $\odot \in \text{ATM} - \text{ATM}_x$ of arity $n > 0$ and dimension signature (k_1^0, \dots, k_n^0, k) and all $\otimes_1, \dots, \otimes_n \in \text{MOL}$ of output dimensions or dimensions (if they are propositional letters) k_1^0, \dots, k_n^0 respectively, the string of symbols $\odot(\otimes_1, \dots, \otimes_n)$ is a molecular connective of MOL of *output dimension* k .

If $\otimes \in \text{MOL}$, we define its *decomposition tree* as follows. If $\otimes = p \in \text{ATM}_0$ or $\otimes = x_i^k \in \text{ATM}_x$, then its decomposition tree T_{\otimes} is the tree consisting of a single node labeled with p or x_i^k respectively. If $\otimes = \odot(\otimes_1, \dots, \otimes_n) \in \text{MOL}$ then its decomposition tree T_{\otimes} is the tree defined inductively as follows: the root of T_{\otimes} is \otimes and it is labeled with \odot and one sets edges between that root and the roots $\otimes_1, \dots, \otimes_n$ of the decomposition trees $T_{\otimes_1}, \dots, T_{\otimes_n}$ respectively. A *subconnective* of \otimes is any vertex of the decomposition tree T_{\otimes} . The set of *atomic connectives associated to a set C of molecular connectives* is the set of labels different from x_i^k of the decomposition trees of the molecular connectives of C .

If \otimes is a molecular connective then the tuple $(x_{i_1}^{k_1}, \dots, x_{i_l}^{k_l})$ of dummy connectives which appear (possibly with some repetitions) in the leaves of the decomposition tree T_{\otimes} in the in-order traversal of T_{\otimes} is called the *dummy signature* of \otimes . The same tuple in the same order but without repetition is called the *input signature* of \otimes .⁴ The number l is called the *width* of \otimes . If $(x_{i_1}^{k_1}, \dots, x_{i_m}^{k_m})$ is the input signature of a molecular connective \otimes of output dimension k then the *dimension signature* of \otimes is (k_1, \dots, k_m, k) and its *arity* is m . We also define the *quantification signature* $\mathbb{A}(\otimes)$ of $\otimes = \odot(\otimes_1, \dots, \otimes_n)$ by $\mathbb{A}(\otimes) \triangleq \mathbb{A}(\odot)$.

⁴For example, if the dummy signature of some molecular connective \otimes is $(x_1^1, x_2^1, x_1^1, x_2^2, x_1^3, x_1^1)$ then its associated input signature is $(x_1^1, x_1^2, x_2^2, x_1^3)$.

A *molecular skeleton* is defined exactly like a molecular connective except that in the above definition the term “connective” has to be replaced uniformly by the term “skeleton” and ATM by SKL. Likewise, the arity, dimension signature, output dimension, dummy signature, input signature, quantification signature and decomposition tree of a molecular skeleton are defined like those for molecular connectives. The molecular skeleton $*$ associated to a molecular connective $\otimes \in \text{MOL}$ is defined inductively as follows:

$$* \triangleq \begin{cases} \cdot & \text{if } \otimes = \odot \in \text{ATM} \\ \cdot(*_1, \dots, *_n) & \text{if } \otimes = \odot(\otimes_1, \dots, \otimes_n). \end{cases}$$

Molecular connectives will be generally denoted $\otimes, \otimes_1, \otimes_2, \dots, \otimes_i$, etc. and molecular skeletons $*, *_1, *_2, \dots, *_i$, etc. \square

Note that the vertices of the decomposition tree of a molecular connective are molecular connectives. Moreover, two different molecular connectives may have the same molecular skeleton, such as $\diamond_1(\square_1(x_1^1) \vee \square_2(x_2^1))$ and $\diamond_2(\square_1(x_1^1) \vee \square_3(x_2^1))$. One needs to introduce the connective x_i^k either in order to deal with molecular connectives whose skeletons are for example of the form $\odot(p, x_i^k)$ where $p \in \text{ATM}_0$ is a propositional letter, or to deal with molecular connectives in which the same argument(s) appear at different places, like for example in $\odot(x_1^k, \dots, x_1^k)$ which is in fact of arity 1.

Definition 7 (Molecular language). Let $\mathbf{C} \subseteq \text{MOL}$ be a set of molecular connectives. The *molecular language* $\mathcal{L}_{\mathbf{C}}$ associated to \mathbf{C} is the smallest set such that

- the propositional letters of \mathbf{C} , denoted \mathbf{C}_0 , belong to $\mathcal{L}_{\mathbf{C}}$;
- for all $\otimes \in \mathbf{C}$ of output dimension k and input signature $(x_{i_1}^{k_1}, \dots, x_{i_m}^{k_m})$ and for all $\varphi_1, \dots, \varphi_m \in \mathcal{L}_{\mathbf{C}}$ of dimensions k_1, \dots, k_m respectively, the string of symbols denoted $\otimes(\varphi_1, \dots, \varphi_m)$, which is the string of symbols \otimes where the dummy atomic connectives $x_{i_1}^{k_1}, \dots, x_{i_m}^{k_m}$ appearing in \otimes are uniformly replaced by $\varphi_1, \dots, \varphi_m$ respectively, belongs to $\mathcal{L}_{\mathbf{C}}$ and $\otimes(\varphi_1, \dots, \varphi_m)$ is of *dimension* k .

Elements of $\mathcal{L}_{\mathbf{C}}$ are called *molecular formulas* and are denoted $\varphi, \psi, \alpha, \dots$. The *dimension* of a formula $\varphi \in \mathcal{L}_{\mathbf{C}}$ is denoted $k(\varphi)$. We use the same abbreviations as for the atomic language. We also assume in the sequel that all our molecular connectives are plain (as in Definition 3). \square

Definition 8 (Molecular logic). Let $\mathbf{C} \subseteq \text{MOL}$ be a set of molecular connectives. A (*molecular*) \mathbf{C} -*model* \mathfrak{M} is a \mathbf{C}' -model \mathfrak{M} where \mathbf{C}' is the set of atomic connectives associated to \mathbf{C} . The class of all pointed molecular \mathbf{C} -models is also denoted $\mathcal{C}_{\mathbf{C}}$, like for atomic logics.

The truth conditions for molecular connectives are defined naturally from the truth conditions of atomic connectives. We define the *interpretation function* of $\mathcal{L}_{\mathbf{C}}$ in \mathfrak{M} , denoted $\llbracket \cdot \rrbracket^{\mathfrak{M}} : \mathcal{L}_{\mathbf{C}} \rightarrow \bigcup_{k \in \mathbb{N}^*} W^k$, inductively as follows: for all propositional letters $p \in \mathbf{C}$ of skeleton $(k, \mathbb{A}, \pm, \sigma)$, all molecular connectives $\odot(\otimes_1, \dots, \otimes_n) \in \mathbf{C}$ of arity $m > 0$ and input signature $(x_{i_1}^{k_1}, \dots, x_{i_m}^{k_m})$ and all $k, i \in \mathbb{N}^*$, for all $\varphi, \varphi_1, \dots, \varphi_m \in \mathcal{L}_{\mathbf{C}}$,

$$\begin{aligned} \llbracket p \rrbracket^{\mathfrak{M}} &\triangleq \pm R_p \\ \llbracket x_i^k(\varphi) \rrbracket^{\mathfrak{M}} &\triangleq \llbracket \varphi \rrbracket^{\mathfrak{M}} \\ \llbracket \odot(\otimes_1, \dots, \otimes_n)(\varphi_1, \dots, \varphi_m) \rrbracket^{\mathfrak{M}} &\triangleq f_{\odot}(\llbracket \otimes_1(\varphi_1^1, \dots, \varphi_{m_1}^1) \rrbracket^{\mathfrak{M}}, \dots, \llbracket \otimes_n(\varphi_1^n, \dots, \varphi_{m_n}^n) \rrbracket^{\mathfrak{M}}) \end{aligned}$$

where for all $j \in \llbracket 1; n \rrbracket$, if $(x_{i_{j1}}^{k_{j1}}, \dots, x_{i_{jm_j}}^{k_{jm_j}})$ is the input signature of \otimes_j then $(\varphi_1^j, \dots, \varphi_{m_j}^j) = (\varphi_{j_1}, \dots, \varphi_{j_{m_j}})$.

If \mathcal{E}_C is a class of pointed C-models, the triple $(\mathcal{L}_C, \mathcal{E}_C, \models)$ is a logic called the *molecular logic associated to \mathcal{E}_C and C*. \square

As one can easily notice, every atomic logic can be canonically mapped to an equi-expressive molecular logic: each atomic connective \odot of dimension signature (k_1, \dots, k_n, k) of the given atomic logic has to be transformed into the molecular connective of skeleton $\odot(x_1^{k_1}, \dots, x_n^{k_n})$. We recall that the x_i^k are in fact specific atomic connectives whose associated relations are the identity relations (see Example 4).

Example 5 (Temporal logic). The *temporal language* \mathcal{L}_{TL} is defined inductively by the following grammar in BNF:

$$\mathcal{L}_{\text{TL}} : \varphi ::= \top \mid \perp \mid p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid U(\varphi, \psi) \mid S(\varphi, \psi)$$

where $p \in \text{ATM}_0$. A *temporal model* is a tuple $\mathfrak{M} = (W, \{<, P_1, \dots, P_n, \dots\})$ where: W is a non-empty set; $< \subseteq W \times W$ is a binary relation over W ; $P_1, \dots, P_n, \dots \subseteq W$ are unary relations over W .

We write $w \in \mathfrak{M}$ for $w \in W$ by abuse and the pair (\mathfrak{M}, w) is called a *pointed temporal model*. The class of all pointed temporal models is denoted \mathcal{C}_{TL} . We define the *satisfaction relation* $\models_{\text{TL}} \subseteq \mathcal{C}_{\text{TL}} \times \mathcal{L}_{\text{TL}}$ by the following *truth conditions*. Below, we write $(\mathfrak{M}, w) \models \varphi$ for $((\mathfrak{M}, w), \varphi) \in \models_{\text{TL}}$. For all temporal models $\mathfrak{M} = (W, \{<, P_1, \dots, P_n, \dots\})$, all $w \in \mathfrak{M}$, all $\varphi, \psi \in \mathcal{L}_{\text{TL}}$ and all $p_i \in \text{SKL}_0$,

$$\begin{aligned} (\mathfrak{M}, w) \models \top & \quad \text{always;} \\ (\mathfrak{M}, w) \models \perp & \quad \text{never;} \\ (\mathfrak{M}, w) \models p_i & \quad \text{iff } P_i(w) \text{ holds;} \\ (\mathfrak{M}, w) \models \neg\varphi & \quad \text{iff it is not the case that } (\mathfrak{M}, w) \models \varphi; \\ (\mathfrak{M}, w) \models (\varphi \wedge \psi) & \quad \text{iff } (\mathfrak{M}, w) \models \varphi \text{ and } (\mathfrak{M}, w) \models \psi; \\ (\mathfrak{M}, w) \models U(\varphi, \psi) & \quad \text{iff there is } v \in W \text{ such that } w < v \text{ and } (\mathfrak{M}, v) \models \varphi \text{ and} \\ & \quad \text{for all } u \in W \text{ such that } w < u < v, (\mathfrak{M}, u) \models \psi; \\ (\mathfrak{M}, w) \models S(\varphi, \psi) & \quad \text{iff there is } v \in W \text{ such that } v < w \text{ and } (\mathfrak{M}, v) \models \varphi \text{ and} \\ & \quad \text{for all } u \in W \text{ such that } v < u < w, (\mathfrak{M}, u) \models \psi. \end{aligned}$$

The triple $(\mathcal{L}_{\text{TL}}, \mathcal{C}_{\text{TL}}, \models_{\text{TL}})$ forms a logic, that we call *temporal logic*.

Encoding of temporal logic as a molecular logic. Let us consider the skeletons defined by the following first-order formulas:

$$\begin{aligned} \bullet_1(x) & \triangleq \exists y z z' (P y \wedge Q z z' \wedge R y z z' x) \\ \bullet'_1(x) & \triangleq \exists y z z' (P y \wedge Q z z' \wedge R x z z' y) \\ \bullet_2(x, x') & \triangleq \forall y (P y \vee \neg S y x x') \end{aligned}$$

\odot_1, \odot'_1 and \odot_2 are atomic skeletons (this is independent from the definitions of R and S). The connectives of skeletons $\otimes = \odot_1(x_1^1, \odot_2(x_2^1))$ and $\otimes' = \odot'_1(x_1^1, \odot_2(x_2^1))$ are normal molecular connectives standing for the ‘Until’ and the ‘Since’ temporal operators U and S . Let $\mathfrak{M}_1 = (W_1, \{<_1, P\})$ and $\mathfrak{M}_2 = (W_2, \{<_2, P\})$ be two temporal models. We represent these temporal models by the C-models $\mathfrak{M}_1^{U,S} = (W_1, \{R_1, S_1, P\})$ and $\mathfrak{M}_2^{U,S} = (W_2, \{R_2, S_2, P\})$ respectively such that for all $v_1, u_1, u'_1, w_1, w'_1 \in W_1$,

$$R_1 v_1 u_1 u'_1 w_1 \text{ iff } w_1 <_1 v_1, w_1 = u_1 \text{ and } v_1 = u'_1 \quad (1)$$

$$S_1 v_1 w_1 w'_1 \text{ iff } w_1 <_1 v_1 <_1 w'_1 \quad (2)$$

and likewise for R_2 and S_2 of \mathfrak{M}_2 . One can show that for all $\varphi \in \mathcal{L}_C$ and all $w_1 \in W_1$, $(\mathfrak{M}_1, w_1) \models \varphi$ iff $(\mathfrak{M}_1^{U,S}, w_1) \Vdash \varphi$ (and likewise for \mathfrak{M}_2 and $\mathfrak{M}_2^{U,S}$). \square

4 Boolean atomic and molecular logics.

Atomic and molecular logics do not include Boolean connectives as primitive connectives. In fact, they can be defined in terms of specific atomic connectives, as follows.

Definition 9 (Boolean connectives). The *Boolean connectives* called *conjunctions*, *disjunctions*, *negations* and *Boolean constants* (of dimension k) are the atomic connectives denoted, respectively $\text{BLN} \triangleq \{\wedge_k, \vee_k, \rightarrow_k, \top_k, \perp_k, \neg_k \mid k \in \mathbb{N}^*\}$ where

$$\begin{aligned} \wedge_k &\triangleq ((k, k, k, \exists, +, +, +, \text{Id}), 0) & \top_k &\triangleq ((k, \exists, +), 0) \\ \vee_k &\triangleq ((k, k, k, \forall, +, +, -, \text{Id}), 0) & \perp_k &\triangleq ((k, \forall, -), 0) \\ \neg_k &\triangleq ((k, k, \exists, -, +, \text{Id}), 0) & \rightarrow_k &\triangleq ((k, k, k, \forall, -, +, (3, 1, 2)), 0) \end{aligned}$$

In any C-model $\mathfrak{M} = (W, \mathfrak{R})$ containing Boolean connectives, the associated relation of any \vee_k , \wedge_k or \rightarrow_k is $R_{\wedge_k} = R_{\vee_k} = R_{\rightarrow_k} \triangleq \{(\bar{w}, \bar{w}, \bar{w}) \mid \bar{w} \in W^k\}$, the associated relation of \neg_k is $R_{\neg_k} \triangleq \{(\bar{w}, \bar{w}) \mid \bar{w} \in W^k\}$ and the associated relation of any \top_k or \perp_k is $R_{\perp_k} = R_{\top_k} \triangleq W^k$. We will often omit the subscript k in $\wedge_k, \vee_k, \rightarrow_k, \top_k, \perp_k, \neg_k$ when it is clear from the context and simply write $\wedge, \vee, \rightarrow, \top, \perp, \neg$. We define BLN^* to be the set of all atomic connectives which are residuated with some connective of BLN , $\text{BLN}^- \triangleq \{\odot \in \text{BLN}^* \mid \pm_j(\odot) = - \text{ for some } j \in \{1, 2\}\}$ those that have a negative tonicity in their argument(s) and BLN_k those of dimension k .

We say that a set of atomic connectives C is *complete for truth constants, conjunction and disjunction* (resp. *negation*) when it contains all truth constants, conjunctions and disjunctions $\top_k, \perp_k, \wedge_k, \vee_k$ (resp. Boolean negation \neg_k), for k ranging over all input types and output types of the atomic connectives of C . We say that a set C of atomic connectives is *Boolean* when it contains all conjunctions, disjunctions, material implications, constants as well as negations $\wedge_k, \vee_k, \rightarrow_k, \top_k, \perp_k, \neg_k$, for k ranging over all input dimensions and output dimensions of the connectives of C . The *Boolean completion* of a set of atomic connectives C is the smallest set of connectives including C which is Boolean. A *Boolean atomic logic* is an atomic logic such that its set of connectives is Boolean. \square

Proposition 1. *Let C be a set of atomic connectives containing Boolean connectives. and let $\mathfrak{M} = (W, \mathfrak{R})$ be a C-model. Then, for all $k \in \mathbb{N}^*$, all $\varphi, \psi \in \mathcal{L}_C$, if $k(\varphi) = k(\psi) = k$, then*

$$\begin{aligned} \llbracket \top_k \rrbracket^{\mathfrak{M}} &= W^k & \llbracket \varphi \wedge_k \psi \rrbracket^{\mathfrak{M}} &= \llbracket \varphi \rrbracket^{\mathfrak{M}} \cap \llbracket \psi \rrbracket^{\mathfrak{M}} \\ \llbracket \perp_k \rrbracket^{\mathfrak{M}} &= \emptyset & \llbracket \varphi \vee_k \psi \rrbracket^{\mathfrak{M}} &= \llbracket \varphi \rrbracket^{\mathfrak{M}} \cup \llbracket \psi \rrbracket^{\mathfrak{M}} \\ \llbracket \neg_k \varphi \rrbracket^{\mathfrak{M}} &= W^k - \llbracket \varphi \rrbracket^{\mathfrak{M}} & \llbracket \varphi \rightarrow_k \psi \rrbracket^{\mathfrak{M}} &= (W^k - \llbracket \varphi \rrbracket^{\mathfrak{M}}) \cup \llbracket \psi \rrbracket^{\mathfrak{M}} \end{aligned}$$

5 Atomic and molecular logics with information order

Many logics include in their semantics a preorder \sqsubseteq over states of the models, sometimes called an *information order* or an *order relation*, such as the family of relevance logics but also the Kripke semantics of intuitionistic logic. Intuitively, $w \sqsubseteq v$ means that everything that is true at w is true at v . Hence, because of this intuitive interpretation, the valuation of propositional letters p by predicates P should consist of so-called ‘‘cones of situations’’ and should satisfy the following condition, sometimes called the ‘‘persistence’’ condition in intuitionistic logic:

$$\text{if } Pw \text{ and } w \sqsubseteq v \text{ then } Pv. \quad (\text{Heredit}y \text{ condition})$$

Some other conditions must be imposed on the accessibility relations associated to the connectives so that the Heredity condition transfers to all formulas of the language, which is a common *desideratum*, or only to a sub-language of the whole language.

As it turns out, because of its grounding in gaggle theory, these conditions can be automatically defined from the signature of the atomic connectives considered. This information order was part of Dunn's original gaggle theory [28, p. 98-99] and can be very easily and naturally integrated within the framework of atomic and molecular logics.

Definition 10 (Atomic and molecular model with information order). Let C be a set of atomic connectives. A *persistent set of connectives associated to C* is a set of atomic connectives $D \subseteq C$ such that $D \cap C_0 \neq \emptyset$ and $D \cap \text{BLN}^- = \emptyset$, that is, D contains some propositional letter(s) of C and no Boolean connective of negative tonicity.

Let $\mathfrak{M} = (W, \mathfrak{R})$ be an atomic C -model. A *D-information order* over \mathfrak{M} is a preorder $\sqsubseteq \subseteq \bigcup_{k \in \mathbb{N}^*} (W^k \times W^k)$ such that for all $\odot \in D - \text{BLN}^*$ of skeleton $(\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$, all $j \in \llbracket 1; n \rrbracket$ and all $\bar{w}_1, \dots, \bar{w}_j, \bar{w}'_j, \dots, \bar{w}_n, \bar{w}, \bar{w}' \in \omega(\mathfrak{M}, C)$,

- if $R_{\odot}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_j \dots \bar{w}_n \bar{w}$ and $\bar{w}_j \sqsubseteq \bar{w}'_j$ then $R_{\odot}^{\pm\sigma} \bar{w}_1 \dots \bar{w}'_j \dots \bar{w}_n \bar{w}$
- where $\bar{w}_j \sqsubseteq \bar{w}'_j \triangleq \begin{cases} \bar{w}_j \sqsubseteq \bar{w}'_j & \text{if } \pm_j \mathbb{A} = + \\ \bar{w}'_j \sqsubseteq \bar{w}_j & \text{if } \pm_j \mathbb{A} = - \end{cases}$ (Tonicity postulates)
- if $R_{\odot}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}$ and $\bar{w} \sqsubseteq \bar{w}'$ then $R_{\odot}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}'$.

An atomic C -model (C -frame) together with a D -information order \sqsubseteq is called an *atomic C -model (resp. C -frame) with D -information order*. A *molecular model with D -information order* is a molecular model such that its associated C' -atomic model is with D -information order, where $D \subseteq C'$ is such that $D \cap C'_0 \neq \emptyset$ and $D \cap \text{BLN}^- = \emptyset$.

Note that any atomic or molecular model with D -information order satisfies the Heredity condition for the propositional letters p of D since this condition is an instance of the Tonicity postulates for the connectives p .

Proposition 2. *Let C be a set of atomic (molecular) connectives and let \mathfrak{M} be an atomic (resp. molecular) C -model with D -information order \sqsubseteq . Then, for all $\bar{w}, \bar{v} \in \omega(\mathfrak{M}, C)$ and all $\varphi \in \mathcal{L}_D$, if $(\mathfrak{M}, \bar{w}) \Vdash \varphi$ and $\bar{w} \sqsubseteq \bar{v}$ then $(\mathfrak{M}, \bar{v}) \Vdash \varphi$.*

Proof. By induction on φ . □

Information orders allow us to have a more fined-grained account over the class of (non-classical) logics. As one can easily see, if we decide to take \sqsubseteq as the equality $=$, then we just recover all our previous definitions of atomic and molecular models. So, in a sense, atomic and molecular models *with information order* are a generalization of our previous atomic and molecular models. By using information order, we have more freedom and accuracy over the definition of our semantics and therefore we can devise more refined and sometimes more complex (non-classical) logics.

Definition 11 (Atomic and molecular logics with information order). An *atomic (molecular) logic with information order* is an atomic (resp. molecular) logic such that all its atomic (resp. molecular) models have an information order.

Example 6. On the one hand, intuitionistic and superintuitionistic/intermediate logics are canonical examples of atomic logics with information order (see Section 11.4). On the other hand, modal intuitionistic logic is an example of molecular logic with information order.

6 Modal polyadic logics versus atomic logics

Another example of Boolean atomic logic is polyadic modal logics, as we will now show.

6.1 Modal polyadic logic

These definitions are taken verbatim from Goranko & Vakarelov [34], except for the last clause of Definition 15 (we also removed the assumption that the propositional variables are countably infinite).

Definition 12 (Modal similarity type). A purely modal polyadic language \mathcal{L}_τ contains a set of propositional variables VAR , negation \neg , and a *modal similarity type* τ consisting of a set of *basic modal terms* (modalities) with pre-assigned finite arities, including a 0-ary modality ι_0 , a unary one ι_1 and a binary one ι_2 . \square

The intuition behind the three distinguished modalities above is simple: ι_1 will be interpreted as the constant \top and its negation \perp ; ι_1 will be interpreted as our dummy connectives xs [6, Example 3] and ι_2 will be the Boolean disjunction \vee and its dual the Boolean conjunction \wedge .

Definition 13 (Modal polyadic language). By simultaneous mutual induction we define the set of *modal terms* $MT(\tau)$ and their *arity function* ρ , and the set of (*purely*) *modal formulas* $MF(\tau)$ as follows:

- (MT i) Every basic modal term of $MT_0(\tau)$ is a modal term of the predefined arity;
- (MT ii) Every formula containing no propositional variables (hereafter called a *constant formula*) is a 0-ary modal term;
- (MT iii) If $n > 0$, $\alpha, \beta_1, \dots, \beta_n \in MT(\tau)$ and $\rho(\alpha) = n$, then $\alpha(\beta_1, \dots, \beta_n) \in MT(\tau)$ and $\rho(\alpha(\beta_1, \dots, \beta_n)) = \rho(\beta_1) + \dots + \rho(\beta_n)$.

Modal terms of arity 0 will be called *modal constants*.

- (MF i) Every propositional variable is a modal formula;
- (MF ii) Every modal constant is a modal formula;
- (MF iii) If φ is a formula then $\neg\varphi$ is a formula;
- (MF iv) If $\varphi_1, \dots, \varphi_n$ are formulas, α is a modal term and $\rho(\alpha) = n > 0$, then $[\alpha](\varphi_1, \dots, \varphi_n)$ is a modal formula.

We also use the abbreviation $\langle \alpha \rangle(\varphi_1, \dots, \varphi_n) \triangleq \neg[\alpha](\neg\varphi_1, \dots, \neg\varphi_n)$. \square

Definition 14 (Kripke τ -model). A (*Kripke*) τ -frame is a structure $F = (W, \{R_\alpha \mid \alpha \in MT_0(\tau)\})$ where the relations R_α are such that:

- $R_{\iota_0} = W$, $R_{\iota_1} = \{(w, w) \mid w \in W\}$, $R_{\iota_2} = \{(w, w, w) \mid w \in W\}$;
- for every basic modal term $\alpha \in MT_0(\tau)$, $R_\alpha \subseteq W^{\rho(\alpha)+1}$.

A (*Kripke*) τ -model over F is a pair $\mathcal{M} = (F, V)$ where $V : VAR \rightarrow \mathcal{P}(W)$ is a *valuation* of the propositional variables in F . \square

Note that any τ -frame is a C-frame, and vice versa.

Definition 15 (Modal polyadic logic). The *truth condition* of a formula at a state w of a Kripke model \mathcal{M} is defined through the following clauses: for all basic modal term α ,

- $\mathcal{M}, w \models p$ iff $w \in V(p)$;
- $\mathcal{M}, w \models \neg\varphi$ iff not $\mathcal{M}, w \models \varphi$;
- $\mathcal{M}, w \models [\alpha](\varphi_1, \dots, \varphi_n)$ iff for all $u_1, \dots, u_n \in W$ such that $R_\alpha u_1 \dots u_n w$, $\mathcal{M}, u_i \models \varphi_i$ holds for some $i \in \{1, \dots, n\}$;
- $\mathcal{M}, w \models [\alpha(\beta_1, \dots, \beta_n)](\varphi_1^1, \dots, \varphi_1^{k_1}, \dots, \varphi_n^1, \dots, \varphi_n^{k_n})$ iff
 $\mathcal{M}, w \models [\alpha]([\beta_1](\varphi_1^1, \dots, \varphi_1^{k_1}), \dots, [\beta_n](\varphi_n^1, \dots, \varphi_n^{k_n}))$. □

Canonical formulas are (modal) formulas which are true on the canonical frame for a (modal) logic which includes them as theorem, and therefore their first-order frame correspondent holds in the canonical frame as well. Hence, they enable to prove the completeness of various calculi w.r.t. a Kripke semantics.

Theorem 2 (Goranko & Vakarelov [34]). *Every inductive formula is canonical.*⁵

6.2 Equi-expressivity of Boolean atomic logics of dimension 1 and modal polyadic logics

6.2.1 Translation from modal polyadic logics to Boolean atomic logics of dimension 1

Definition 16 (Atomic connectives associated to a modal similarity type). Let τ be a modal similarity type. The set of *atomic connectives associated to τ* , denoted \mathcal{C}_τ , is the set of basic modal terms together with the set of propositional variables VAR as well as the truth constants \top_1 and the Boolean disjunction \vee_1 and negation \neg_1 of Definition 9. □

Definition 17 (\mathcal{C}_τ -model associated to a τ -model). Let $\mathcal{M} = (W, \{R_\alpha \mid \alpha \in MT(\tau)\}, V)$ be a τ -model. The \mathcal{C}_τ -model associated to \mathfrak{M} is the \mathcal{C}_τ -model $T(\mathcal{M}) \triangleq (W, \{R_\alpha \mid \alpha \in MT_0(\tau)\} \cup \{R_p \mid p \in VAR\})$ where $MT_0(\tau)$ is the set of basic modal terms and for all $p \in VAR$, $R_p \triangleq V(p)$.

Definition 18 (Formula associated to a modal polyadic formula). Let τ be a modal similarity type and let φ be a modal formula of $MF(\tau)$. The *formula of $\mathcal{L}_{\mathcal{C}_\tau}$ associated to φ* , denoted $T(\varphi)$, is defined inductively as follows: for all basic modal term $\alpha \in \tau$ (distinct from $\iota_0, \iota_1, \iota_2$) and all $p \in VAR$,

$$\begin{aligned} T(p) &\triangleq p & T(\iota_0) &\triangleq \top \\ T([\iota_1]\varphi) &\triangleq T(\varphi) & T([\iota_2](\varphi, \psi)) &\triangleq \vee(T(\varphi), T(\psi)) \\ T([\alpha(\beta_1, \dots, \beta_n)](\varphi_1, \dots, \varphi_1^{k_1}, \dots, \varphi_n, \dots, \varphi_n^{k_n})) &\triangleq \alpha(T(\varphi_1), \dots, T(\varphi_n)) \\ &\triangleq \alpha(T([\beta_1](\varphi_1^1, \dots, \varphi_1^{k_1})), \dots, T([\beta_n](\varphi_n^1, \dots, \varphi_n^{k_n}))). \end{aligned}$$

Proposition 3. *Let τ be a modal similarity type, let φ be a formula of $MF(\tau)$ and let \mathcal{M} be a τ -model. Then, for all $w \in \mathcal{M}$, it holds that $\mathcal{M}, w \models \varphi$ iff $T(\mathcal{M}), w \models T(\varphi)$.*

Proof. The proof is without specific difficulty. It suffices essentially to check the preservation of truth for every clause of Definition 18. □

⁵Inductive formulas (for polyadic modal logics) are defined by Goranko & Vakarelov [34].

6.2.2 Translation from atomic logics of dimension 1 to modal polyadic logics

Definition 19 (Modal similarity type associated to a set of atomic connectives). Let C be a set of atomic connectives. The *modal similarity type associated to C* , denoted τ_C , consists of the orbits of C as well as ι_0 , ι_1 and ι_2 ; the propositional variables VAR of \mathcal{L}_{τ_C} consist of the propositional letters of C . The basic modal term associated to an orbit containing the connective \odot is denoted α_{\odot} . \square

Definition 20 (τ_C -model associated to a C -model). Let C be a set of atomic connectives and let $\mathfrak{M} = (W, \mathfrak{R})$ be a C -model. The τ_C -model associated to \mathfrak{M} is the τ_C -model $T^-(\mathfrak{M}) \triangleq (W, \{R_\alpha \mid \alpha \in MT_0(\tau)\}, V)$ such that

- for every basic modal term $\alpha \in MT_0(\tau_C)$ corresponding to the orbit of a connective $\odot \in C$ of skeleton $(\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_j, \dots, \pm_n, \pm, \sigma)$, we have that

$$R_\alpha = \begin{cases} -R_{\odot}^{\pm\sigma} & \text{if } \mathbb{A} = \forall; \\ R_{\odot}^{\pm\sigma} & \text{if } \mathbb{A} = \exists; \end{cases}$$

- for all $p \in VAR$, we have that $V(p) = R_p$. \square

Hence, every τ_C -frame based on $T^-(\mathfrak{M})$ is equal to the C -frame based on \mathfrak{M} .

Definition 21 (Modal polyadic formula associated to an atomic formula). Let C be a set of (Boolean) atomic connectives and let φ be a formula of \mathcal{L}_C . The *modal polyadic formula of $MF(\tau_C)$ associated to φ* , denoted $T^-(\varphi)$, is defined inductively as follows:

$$\begin{aligned} T^-(p) &\triangleq p \\ T^-(\top) &\triangleq \iota_0 \\ T^-(\perp) &\triangleq \neg\iota_0 \\ T^-(\neg\varphi) &\triangleq \neg T^-(\varphi) \\ T^-(\varphi \vee \psi) &\triangleq [\iota_2](T^-(\varphi), T^-(\psi)) \\ T^-(\varphi \wedge \psi) &\triangleq \neg[\iota_2](\neg T^-(\varphi), \neg T^-(\psi)) \\ T^-(\odot(\varphi_1, \dots, \varphi_n)) &\triangleq \begin{cases} [\alpha_{\odot}](\pm_1 T^-(\varphi_1), \dots, \pm_n T^-(\varphi_n)) & \text{if } \mathbb{A}(\odot) = \forall \\ \langle \alpha_{\odot} \rangle(\pm_1 T^-(\varphi_1), \dots, \pm_n T^-(\varphi_n)) & \text{if } \mathbb{A}(\odot) = \exists. \end{cases} \end{aligned}$$

where for all $j \in \llbracket 1; n \rrbracket$, we have that $\pm_j \triangleq \begin{cases} \neg & \text{if } \pm_j = - \\ \text{empty} & \text{if } \pm_j = + \end{cases}$. \square

Proposition 4. *Let C be a set of atomic connectives, let φ be a formula of \mathcal{L}_C and let \mathfrak{M} be a C -model. Then, for all $w \in \mathfrak{M}$, $\mathfrak{M}, w \models \varphi$ iff $T^-(\mathfrak{M}), w \Vdash T^-(\varphi)$.*

Proof. The proof is without specific difficulty. It suffices essentially to check the preservation of truth for every clause of Definition 21. \square

In the following theorem, we use the notion of equi-expressivity introduced in [2].

Theorem 3. *The class of modal polyadic logics is as expressive as the class of Boolean atomic logics whose connectives are all of dimension $(1, 1, \dots, 1)$.*

Proof. Because of the two previous propositions, it suffices to prove that for all τ -models \mathcal{M} , $T^-(T(\mathcal{M})) \equiv \mathcal{M}$ and for all C-models \mathfrak{M} , $T(T^-(\mathfrak{M})) \equiv \mathfrak{M}$ (\equiv is the equivalence w.r.t. the appropriate languages). This can be easily checked. \square

Remark 1. One may argue at this stage that there is no point in considering and introducing atomic logics: one should instead simply introduce polyadic modal logics with dimensions since they seem to be equally expressive overall. The reply is that the definition of the *connectives* of atomic logics are much more precise and allow for more diversity and flexibility than with typed polyadic modal logics, because of the parameters that are taken into account to define the connectives, in particular the tonicity signature. The formulas can somehow always be turned to ‘equivalent’ ones in both direction but the connectives and logics defined within the atomic logics framework are more refined and varied, allowing to capture many more non-classical logics. For example, the relevant implication \rightarrow of the Routley-Meyer semantics is an atomic connective but it cannot be reduced to a modality [52]. It also turns out that we can obtain sound and complete axiomatizations (Theorem 4) as well as model-theoretical results in a generic fashion for a much larger class of logics than mere (typed) polyadic modal logics [4]. Finally, our generalization with information orders, based on the tonicity signature, allows us to capture logics satisfying the Heredity condition, such as intuitionistic logic and intermediate logics. This is not possible with polyadic modal logics.

7 Group actions and residuations

In [2], a group action of the symmetric group over the set of gaggle connectives was introduced. In this section, we are going to redefine it in a simpler and more concise way. It will turn out to play a crucial role in the proof theory of atomic logics.

7.1 Group action

If X is a set and G a group, a (*left*) *action of G on X* is a function $\alpha : G \times X \rightarrow X$ given by $(g, x) \mapsto gx$ such that: $x\text{Id} = x$ for all $x \in X$; $(g_1g_2)x = g_1(g_2x)$ for all $x \in X$ and all $g_1, g_2 \in G$. If $x \in X$ and α an action of a group G on X , then the *orbit* of x under α is $\mathcal{O}_\alpha(x) \triangleq \{\alpha(g, x) \mid g \in G\}$. The orbits form a partition of X .

Let G and H be two groups. If α and β are actions of G and H on a set X , then the *free action* $\alpha*\beta$ is the mapping $\alpha*\beta : G*H \times X \rightarrow X$ defined by $\alpha*\beta(g, x) \triangleq \alpha(g_1, \dots, \alpha(\beta(h_{n-1}, \alpha(g_n, x))))$, where $g = g_1h_1 \dots g_nh_n$ is the factorization of g in the free group $G*H$. This definition can be generalized canonically to the case of a finite number of actions $\alpha_1, \dots, \alpha_n$, yielding the mapping $\alpha_1 * \dots * \alpha_n$.

One can easily show that if $\alpha_1, \dots, \alpha_n$ are actions of G_1, \dots, G_n on a set X respectively, then the mapping $\alpha_1 * \dots * \alpha_n$ is an action of the (free) group $G_1 * \dots * G_n$ on X .

7.2 Group actions over atomic connectives

Below we define the group actions $\alpha_n, \beta_n, \gamma_n$ of the symmetric group and the dihedral group over the set of atomic connectives.⁶

⁶The definition of α_n is inspired but slightly different from the group action introduced by Espejo-Boix [30, Definition 2]. We resort here to the extra components \pm_{n+1} and δ_{n+1}^τ , always equal to $+$.

Definition 22. Let $n \in \mathbb{N}^*$, $\bullet = (k_1, \dots, k_{n+1}, \mathbb{E}, \pm_1, \dots, \pm_n, \pm, \sigma) \in \text{SKL}_n$ and $\tau \in \mathfrak{S}_{n+1}$. For all $j \in \llbracket 0; n \rrbracket$, we first define $\Delta_j \triangleq \delta_j^\tau \pm_{\tau^{-1}(j)}$ where

$$\delta_j^\tau \triangleq \begin{cases} + & \text{if } j = \tau(n+1) \\ \delta_{n+1, \tau(n+1)} & \text{otherwise} \end{cases} \quad \delta_{n+1, \tau(n+1)} \triangleq \begin{cases} + & \text{if } n+1 = \tau(n+1) \\ - & \text{otherwise} \end{cases}$$

and we also set $\pm_{n+1} \triangleq +$ and $\delta_{n+1}^\tau \triangleq +$.⁷ Then, we define the function $a_n : \mathfrak{S}_{n+1} \times \text{SKL}_n \rightarrow \text{SKL}_n$ as follows:

$$a_n(\tau, \bullet) \triangleq (k_{\tau^{-1}(1)}, \dots, k_{\tau^{-1}(n+1)}, \Delta_0 \mathbb{E}, \Delta_1 \pm_{\tau^{-1}(1)}, \dots, \Delta_n \pm_{\tau^{-1}(n)}, \Delta_0 \pm, \tau \circ \sigma).$$

The function a_n induces a function $\alpha_n : \text{ATM} \times \mathfrak{S}_{n+1} \rightarrow \text{ATM}$ on the set ATM of connectives of arity n defined by $\alpha_n((\bullet, i), \tau) \triangleq (a_n(\tau, \bullet), i)$. Likewise, we define the functions $\beta_n : \mathbb{Z}/2\mathbb{Z} \times \text{ATM} \rightarrow \text{ATM}$ and $\gamma_n : \mathbb{Z}/2\mathbb{Z} \times \text{ATM} \rightarrow \text{ATM}$ by

$$\beta_n(\pm, (\bullet, i)) \triangleq \begin{cases} (-\bullet, i) & \text{if } \pm = - \\ (\bullet, i) & \text{if } \pm = + \end{cases} \quad \gamma_n(\pm, (\bullet, i)) \triangleq \begin{cases} (\sim \bullet, i) & \text{if } \pm = - \\ (\bullet, i) & \text{if } \pm = + \end{cases}$$

$$\begin{aligned} \text{where } -\bullet &\triangleq (k_1, \dots, k_{n+1}, -\mathbb{E}, -\pm_1, \dots, -\pm_n, -\pm, \sigma) \\ \text{and } \sim \bullet &\triangleq (k_1, \dots, k_{n+1}, \mathbb{E}, \pm_1, \dots, \pm_n, -\pm, \sigma). \end{aligned}$$

$a_n(\tau, \bullet)$, $\alpha_n(\tau, \odot)$ and $\beta_n(\pm, \odot)$ are often denoted $\tau\bullet$, $\tau\odot$ and $\pm\odot$ respectively.

Remark 2. Note that if $n+1 = \tau(n+1)$ then

$$\tau\bullet = (k_{\tau^{-1}(1)}, \dots, k_{\tau^{-1}(n+1)}, \mathbb{E}, \pm_{\tau^{-1}(1)}, \dots, \pm_{\tau^{-1}(n)}, \pm, \tau \circ \sigma). \quad (\text{Res})$$

Definition 23 (Common set of connectives). A set \mathcal{C} of atomic connectives is *common* when for all pairs of residuated connectives $\odot, \odot' \in \mathcal{C}$, we have that $\mathcal{O}_{\alpha_n * \beta_n}(\odot) = \mathcal{O}_{\alpha_n * \beta_n}(\odot')$, that is, $\odot' = \tau_0 - \tau_1 \dots - \tau_m \odot$ for some $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$.

Common sets of atomic connectives correspond to a very large class of logics. As far as we know, all non-classical logics which are atomic logics are based on some common set of connectives.⁸

8 Hilbert calculi for atomic logics

In this section on Hilbert calculi, we define the notion of provability (deducibility) from a set of formulas, *i.e.* $\Sigma \vdash_{\mathcal{P}} \varphi$ as follows. Let $\mathbb{L} = (\mathcal{L}, \mathcal{C}, \Vdash)$ be an atomic logic and let $\Sigma \subseteq \mathcal{L}$ (possibly empty and not necessarily closed under uniform substitution) and $\varphi \in \mathcal{L}$ be of dimension k . Then, we say that φ is *provable from* Σ in a proof system \mathcal{P} for \mathbb{L} , written $\Sigma \vdash_{\mathcal{P}} \varphi$, when there is a proof of φ in the proof system $\mathcal{P} + \Sigma$.

We assume in what follows that \mathcal{C} is a class of pointed atomic models (\mathfrak{M}, \bar{w}) but it could also be a class of pointed atomic frames (\mathfrak{F}, \bar{w}) . We write $(\mathfrak{M}, \bar{w}) \Vdash \Sigma$ when for all $\psi \in \Sigma$, we

⁷Setting δ_{n+1}^τ to $+$ does not play any role in this definition but it might appear in the expressions and play a role if we apply the group action successively to several permutations τ .

⁸An exception might be Levesque's logic of only knowing [43], because we might need to refer to the complement of the epistemic accessibility relation so as to be able to capture it as an atomic logic.

have $(\mathfrak{M}, \bar{w}) \Vdash \psi$. We say that φ is a *logical consequence* of Σ , written $\Sigma \Vdash_{\mathcal{L}} \varphi$, when for all $(\mathfrak{M}, \bar{w}) \in \mathcal{C}$, if $(\mathfrak{M}, \bar{w}) \Vdash \Sigma$ then $(\mathfrak{M}, \bar{w}) \Vdash \varphi$; φ is *valid*, written $\Vdash_{\mathcal{L}} \varphi$, when for all $(\mathfrak{M}, \bar{w}) \in \mathcal{C}$, we have $(\mathfrak{M}, \bar{w}) \Vdash \varphi$; φ is *satisfiable* when there is a model $(\mathfrak{M}, \bar{w}) \in \mathcal{C}$ such that $(\mathfrak{M}, \bar{w}) \Vdash \varphi$. An inference rule of the form “from $\varphi_1, \dots, \varphi_n$ infer φ ” is *valid* when $\varphi_1, \dots, \varphi_n$ being all valid implies that φ is valid. We say that φ is *globally valid* (*globally unsatisfiable*) when for all $(\mathfrak{M}, \bar{w}) \in \mathcal{C}$ and for all $\bar{w}' \in \omega(\mathfrak{M}, \mathcal{C})$ of the same dimension as φ (resp. for no $\bar{w}' \in \omega(\mathfrak{M}, \mathcal{C})$), we have that $(\mathfrak{M}, \bar{w}') \Vdash \varphi$.⁹

The notion of *strong completeness* of a (Hilbert) calculus \mathcal{P} for a logic \mathbf{L} is defined as usual by $\Sigma \Vdash_{\mathcal{L}} \varphi$ implies $\Sigma \vdash_{\mathcal{P}} \varphi$ and, conversely, a calculus \mathcal{P} is *sound* for a logic \mathbf{L} when $\vdash_{\mathcal{P}} \varphi$ implies $\Vdash_{\mathcal{L}} \varphi$.

Definition 24. Let \mathcal{C} be a common set of atomic connectives (complete for Boolean connectives or not). We denote by $\mathcal{P}_{\mathcal{C}}$ the calculus of Figure 1. \square

Theorem 4 (Soundness and strong completeness, [6]). *Let \mathcal{C} be a common set of atomic connectives. The calculus $\mathcal{P}_{\mathcal{C}}$ is sound and strongly complete for the Boolean basic atomic logic $(\mathcal{L}_{\mathcal{C}}, \mathcal{C}_{\mathcal{C}}, \Vdash)$.*

Our calculus $\mathcal{P}_{\mathcal{C}}$ is well-defined and, to be more precise, the axiom A5 does not depend on the particular residuation equation $\odot' = \tau_0 - \tau_1 \dots - \tau_m \odot$ that we pick, as proven by Proposition 4 of [6].

Remark 3. The set \mathcal{C} of atomic connectives is not necessarily Boolean. Yet, the calculus $\mathcal{P}_{\mathcal{C}}$ includes axioms and inference rules with Boolean connectives. This is because for any basic atomic logic, its extension with Boolean connectives is a conservative extension, as proven in [6, Theorem 7] (recalled in Proposition 8 below). So, even if the proof of a formula may contain extra Boolean connectives, if the conclusion of the proof does not contain any, the proven formula is nevertheless valid in the initial logic without Boolean connectives.

Example 7 (Modal logic). If we take $\mathcal{C} = \text{ATM}_0 \cup \text{BLN}_1 \cup \{\Box, \Diamond\}$ to be the set of atomic connectives of modal logic with \Diamond, \Box atomic connectives of skeletons $((2, 1), +, t_1)$ and $((2, 1), -, t_2)$ respectively then we have that $(2, 1) - (2, 1) - \Box = \Diamond$ (\Box and \Diamond are dual of each other, see [2, Proposition 30]) and the \mathcal{C} -models $\mathfrak{M} = (W, \mathfrak{R}) \in \mathcal{E}_{\mathcal{C}}$ are such that $R_{\Diamond} = R_{\Box}$. The Hilbert calculus $\mathcal{P}_{\mathcal{C}}$ that we obtain and which is sound and complete for (this) modal logic is spelled out in Figure 2. Note that this proof system is novel. It can be simplified and is in fact equivalent to the classical proof system of modal logic [10, Definition 1.39] (see [6, Appendix F] for details). \square

9 Universal and existential molecular connectives

Universal and existential molecular connectives are essentially molecular connectives such that the quantification patterns of the quantification signatures of their successive atomic connectives are of the form $\forall \dots \forall$ or $\exists \dots \exists$ respectively. So, they essentially behave as ‘macroscopic’ atomic connectives of quantification signatures \forall or \exists .

First, just as we have tonicity signatures for atomic connectives, we can also define an adaptation of this notion for molecular connectives in general.

Definition 25 (Tonicity signature of a molecular connective). Let \otimes be a molecular connective and let \otimes' be a molecular sub-connective of \otimes . We define the *tonicity of \otimes' w.r.t. \otimes* ,

⁹The distinction between validity/unsatisfiability and *global* validity/unsatisfiability plays a role in some logics which distinguish two kinds of states, such as relevance logic [52] where validity is defined w.r.t. *normal* worlds (see Section 11.2).

• <i>Axiom schemas:</i>	
Any sound and complete axiomatization of propositional logic	CPC
For all $\odot \in \mathbf{C}$ such that $\mathcal{A}(\odot) = \exists$,	
if $\pm_j(\odot) = +$ then	
$\odot(\varphi_1, \dots, \varphi_j \vee \varphi'_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vee \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n)$	A1
if $\pm_j(\odot) = -$ then	
$\odot(\varphi_1, \dots, \varphi_j \wedge \varphi'_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vee \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n)$	A2
For all $\odot \in \mathbf{C}$ such that $\mathcal{A}(\odot) = \forall$,	
if $\pm_j(\odot) = +$ then	
$\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \wedge \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j \wedge \varphi'_j, \dots, \varphi_n)$	A3
if $\pm_j(\odot) = -$ then	
$\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \wedge \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j \vee \varphi'_j, \dots, \varphi_n)$	A4
For all $\odot, \odot' \in \mathbf{C}$ such that $\odot' = \tau_0 - \tau_1 \dots - \tau_m \odot$ for some $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$:	
denoting $\rho = (\tau_0 \circ \tau_1 \circ \dots \circ \tau_m)^-$,	
$S\left(\odot', \pm^{\rho(1)}\varphi_{\rho(1)}, \dots, \pm^{n+1}\odot(\varphi_1, \dots, \varphi_n), \dots, \pm^{\rho(n+1)}\varphi_{\rho(n+1)}\right)$	A5
where $S(\odot, \psi_1, \dots, \psi_n, \psi) \triangleq \begin{cases} \odot(\psi_1, \dots, \psi_n) \rightarrow \psi & \text{if } \mathcal{A}(\odot) = \exists \\ \psi \rightarrow \odot(\psi_1, \dots, \psi_n) & \text{if } \mathcal{A}(\odot) = \forall \end{cases}$	
$\pm\varphi = \begin{cases} \neg\varphi & \text{if } \pm = - \\ \varphi & \text{if } \pm = + \end{cases}$ and for all $j \in \llbracket 1; n+1 \rrbracket$, $\pm^j = \begin{cases} \pm_1^j \pm_2^j \dots \pm_m^j & \text{if } m \neq 0 \\ + & \text{if } m = 0 \end{cases}$	
where for all $i \in \llbracket 1; m \rrbracket$, $\pm_i^j = \begin{cases} - & \text{if } \tau_i \circ \tau_{i+1} \circ \dots \circ \tau_m(j) = n+1 \\ + & \text{otherwise} \end{cases}$	
• <i>Inference rules:</i>	
From φ and $(\varphi \rightarrow \psi)$, infer ψ	MP
For all $\odot \in \mathbf{C}$ such that $\mathcal{A}(\odot) = \exists$,	
if $\pm_j(\odot) = +$ then from $\neg\varphi_j$, infer $\neg\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$	R1
if $\pm_j(\odot) = -$ then from φ_j , infer $\neg\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$	R2
For all $\odot \in \mathbf{C}$ such that $\mathcal{A}(\odot) = \forall$,	
if $\pm_j(\odot) = +$ then from φ_j , infer $\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$	R3
if $\pm_j(\odot) = -$ then from $\neg\varphi_j$, infer $\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$	R4
For all $\odot \in \mathbf{C}$,	
if $\pm_j(\odot) = +$ then	
From $\varphi_j \rightarrow \varphi'_j$, infer $\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n)$	R5
if $\pm_j(\odot) = -$ then	
From $\varphi_j \rightarrow \varphi'_j$, infer $\odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$	R6
$\varphi, \psi, \varphi_1, \dots, \varphi_j, \varphi'_j, \dots, \varphi_n$ range over $\mathcal{L}_{\mathbf{C}^{\text{BLN}}}$, \mathbf{C}^{BLN} is the Boolean completion of \mathbf{C} .	

Figure 1: Hilbert calculus $\mathcal{P}_{\mathbf{C}}$

Any sound and complete axiomatization of propositional logic	(CPC)
$\Diamond(\varphi \vee \psi) \rightarrow \Diamond\varphi \vee \Diamond\psi$	(A1)
$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$	(A3)
$\Diamond\neg\varphi \rightarrow \neg\Box\varphi$	(A5)
$\neg\Diamond\varphi \rightarrow \Box\neg\varphi$	(A5)
From $\neg\varphi$, infer $\neg\Diamond\varphi$	(R1)
From φ , infer $\Box\varphi$	(R3)
From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$	(R5)
From $\varphi \rightarrow \psi$, infer $\Diamond\varphi \rightarrow \Diamond\psi$	(R5)
From φ and $\varphi \rightarrow \psi$, infer ψ	(MP)
The formulas φ, ψ range over $\mathcal{L}_{\mathbf{C}}$, \mathbf{C} are the connectives of Example 7.	

Figure 2: Hilbert calculus $\mathcal{P}_{\mathbf{C}}$ instantiated with the atomic connectives \mathbf{C} of modal logic.

denoted $tn(\otimes', \otimes)$, inductively as follows. If $\otimes = \otimes'$ then $tn(\otimes', \otimes) = +$. Otherwise, if $\otimes = \odot(\otimes_1, \dots, \otimes_n)$ with $\odot = (\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$ and \otimes' appears in the decomposition tree of \otimes_j then $tn(\otimes', \otimes) = \pm_j tn(\otimes', \otimes_j)$. The *tonicity signature* of a molecular connective is the tuple (\pm_1, \dots, \pm_l) of the tonicities $tn(x_i^k, \otimes)$ of the connectives labeling the leafs of the decomposition tree of \otimes of the form x_i^k (possibly with repetition). We say that the tonicity signature (\pm_1, \dots, \pm_l) of a molecular connective \otimes is *coherent* if all tonicities \pm_i s corresponding to the same x_j^k in the decomposition tree of \otimes are equal. In that case, its tonicity signature is also denoted (\pm_1, \dots, \pm_m) by abuse, where \pm_j is the unique tonicity of the corresponding x_j^k appearing in \otimes and is also denoted $\pm_j(\otimes)$.

Example 8. In this section, the connective \supset is the implication of relevance logic and \rightarrow is the (Boolean) material implication (see Definition 9). They both have the same skeleton. If $\otimes = x_1^1 \supset \Box x_2^1$ then $tn(\otimes, x_1^1) = -$ and $tn(\otimes, x_2^1) = tn(\otimes, \Box x_2^1) = +$. Therefore, if $\otimes' = x_1^1 \supset \Box x_1^1$ then the tonicity signature of \otimes' is not coherent.

Definition 26 (Basic universal and existential molecular connective). Let \mathbf{C} be a set of atomic connectives. A molecular connective \otimes based on \mathbf{C} is *basic universal* (resp. *existential*) when the tonicity signature of \otimes is coherent and the following hold:

- $\mathbb{A}(\otimes) = \forall$ (resp. $\mathbb{A}(\otimes) = \exists$);
- for each node of its decomposition tree labeled with $\odot = (\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$ and each of its j th children labeled with some $\odot_j \in \text{MOL}$ such that the subtree generated by this j th children contains at least one x_j^k , we have that $\mathbb{A}(\odot_j) = \pm_j \mathbb{A}$.

Example 9. On the one hand, the molecular connective $\odot(p, x_1^k)$ is a basic universal (resp. existential) molecular connective if $\mathbb{A}(\odot) = \forall$ (resp. $\mathbb{A}(\odot) = \exists$). Likewise, $(x_1^1 \supset \Box x_2^1)$ and $\otimes(\Diamond x_1^1, p)$ are universal and existential molecular connectives respectively. On the other hand, the molecular connectives $\Box\Diamond^{-}x_1^1$ and $(\Box x_1^1 \supset \Box x_2^1)$ are neither universal nor existential molecular connectives.

Definition 27 (Relation associated to a molecular connective). Let \mathbf{C} be a set of molecular connectives and let $\otimes = \odot(\otimes_1, \dots, \otimes_n) \in \mathbf{C}$ be a molecular connective with \odot of skeleton $((k_1^\odot, \dots, k_{n+1}^\odot), \exists, \pm_1^\odot, \dots, \pm_n^\odot, \pm, \sigma)$. Let $(x_{i_1}^{k_{i_1}}, \dots, x_{i_l}^{k_{i_l}})$ be the dummy signature of \otimes . We also denote $k = k_{n+1}^\odot$ the output dimension of \otimes .

Let $\mathfrak{M} = (W, \mathfrak{R})$ be a C-model. We define the relation R_{\circledast} inductively as follows. First, we set $R_{\ominus} = R_{\circ}$ if \ominus is an atomic connective and if $\pm \in \{+, -\}$,

$$\pm p \triangleq \begin{cases} p & \text{if } \pm = + \\ -p & \text{if } \pm = - \end{cases} \text{ and } \pm \circledast (\circledast_1, \dots, \circledast_n) = \begin{cases} \circledast(\circledast_1, \dots, \circledast_n) & \text{if } \pm = + \\ -\circledast(\circledast_1, \dots, \circledast_n) & \text{if } \pm = - \end{cases}.$$

- If the arity m of \circledast is 0 then $R_{\circledast} = \llbracket \circledast \rrbracket^{\mathfrak{M}}$;
- If the arity m of \circledast is greater than 0 and $\mathfrak{E} = \exists$ then for all $\bar{w} \in W^k, \bar{w}_1 \in W^{k_1}, \dots, \bar{w}_l \in W^{k_l}$, we set

$$R_{\circledast} \bar{w}_1 \dots \bar{w}_l \bar{w} \text{ iff } \exists \bar{v}_1 \in W^{k_1^{\circ}}, \dots, \bar{v}_n \in W^{k_n^{\circ}} \\ \left(R_{\circ}^{\pm \sigma} \bar{v}_1 \dots \bar{v}_n \bar{w} \wedge R_{\pm \circledast_1} \bar{w}_1^1 \dots \bar{w}_{l_1}^1 \bar{v}_1 \wedge \dots \wedge R_{\pm \circledast_n} \bar{w}_1^n \dots \bar{w}_{l_n}^n \bar{v}_n \right)$$

where, by definition, for all $j \in \llbracket 1; n \rrbracket$, the dummy signature $(x_{i_{j_1}}^{kj_1}, \dots, x_{i_{j_{l_j}}^{kj_{l_j}}})$ of \circledast_j is a precise sub-sequence of the dummy signature $(x_{i_1}^{k_1}, \dots, x_{i_l}^{k_l})$ of \circledast , so $(\bar{w}_1^j, \dots, \bar{w}_{l_j}^j) = (\bar{w}_{j_1}, \dots, \bar{w}_{j_{l_j}})$ is the corresponding sequence of tuples of states.

- If the arity m of \circledast is greater than 0 and $\mathfrak{E} = \forall$ then for all $\bar{w} \in W^k, \bar{w}_1 \in W^{k_1}, \dots, \bar{w}_l \in W^{k_l}$, we set

$$R_{\circledast} \bar{w}_1 \dots \bar{w}_l \bar{w} \text{ iff } \forall \bar{v}_1 \in W^{k_1^{\circ}}, \dots, \bar{v}_n \in W^{k_n^{\circ}} \\ \left(R_{\circ}^{\pm \sigma} \bar{v}_1 \dots \bar{v}_n \bar{w} \vee R_{\pm \circledast_1} \bar{w}_1^1 \dots \bar{w}_{l_1}^1 \bar{v}_1 \vee \dots \vee R_{\pm \circledast_n} \bar{w}_1^n \dots \bar{w}_{l_n}^n \bar{v}_n \right)$$

where, for all $j \in \llbracket 1; n \rrbracket$, the tuples $\bar{w}_1^j, \dots, \bar{w}_{l_j}^j$ are defined as above.

Unsurprisingly, the semantics of basic universal and existential molecular connectives is similar to the semantics of atomic connectives of quantification signature \forall and \exists respectively. That is what the following proposition shows. In this proposition and the definition above, the molecular connective $\circledast = \circledast(\circledast_1, \dots, \circledast_n)$ yields three different values: n, m and l . They can be all different in general. The value m is the arity of \circledast , n is the arity of \circledast and l is the width of \circledast , the number of leaves of the decomposition tree T_{\circledast} of \circledast . The value m is also the number of different labels x_i^k of the leafs of the decomposition tree T_{\circledast} . Hence, the size l of the tuple of the tonicity signature of a molecular connective \circledast is larger than the arity m of \circledast because in the former case we take into account repetitions of the same x_i^k s.

Proposition 5. *Let C be a set of molecular connectives and let $\circledast \in C$ be a molecular connective of dimension signature (k_1, \dots, k_m, k) , tonicity signature (\pm_1, \dots, \pm_l) and dummy signature $(x_{i_1}^{k_1}, \dots, x_{i_l}^{k_l})$. Let $\mathfrak{M} = (W, \mathfrak{R})$ be a C-model and let $\bar{w} \in W^k$.*

- *If \circledast is a basic existential molecular connective then*

$$\bar{w} \in \llbracket \circledast(\varphi_1, \dots, \varphi_m) \rrbracket^{\mathfrak{M}} \\ \text{iff } \exists \bar{w}_1 \dots \bar{w}_l (\bar{w}_1 \uparrow_1 \llbracket \varphi_{i_1} \rrbracket^{\mathfrak{M}} \wedge \dots \wedge \bar{w}_l \uparrow_l \llbracket \varphi_{i_l} \rrbracket^{\mathfrak{M}} \wedge R_{\circledast} \bar{w}_1 \dots \bar{w}_l \bar{w})$$

- *If \circledast is a basic universal molecular connective then*

$$\bar{w} \in \llbracket \circledast(\varphi_1, \dots, \varphi_m) \rrbracket^{\mathfrak{M}} \\ \text{iff } \forall \bar{w}_1 \dots \bar{w}_l (\bar{w}_1 \uparrow_1 \llbracket \varphi_{i_1} \rrbracket^{\mathfrak{M}} \vee \dots \vee \bar{w}_l \uparrow_l \llbracket \varphi_{i_l} \rrbracket^{\mathfrak{M}} \vee R_{\circledast} \bar{w}_1 \dots \bar{w}_l \bar{w})$$

where the R_{\otimes} s are defined in Definition 27 and for all $j \in [1; l]$,

$$\bar{w}_j \uparrow_j \llbracket \varphi_{i_j} \rrbracket^{\mathfrak{M}} \triangleq \begin{cases} \bar{w}_j \in \llbracket \varphi_{i_j} \rrbracket^{\mathfrak{M}} & \text{if } \pm_j = + \\ \bar{w}_j \notin \llbracket \varphi_{i_j} \rrbracket^{\mathfrak{M}} & \text{if } \pm_j = - \end{cases}.$$

Proof. The proof proceeds by induction on the inductive definition of basic universal and existential molecular connectives. We only prove the existential case, the proof for the universal case is similar.

Assume that $\otimes = \odot(x_1^{k_1}, \dots, x_n^{k_n})$ is an existential molecular connective with \odot of skeleton $(\bar{k}, \exists, \pm_1^{\odot}, \dots, \pm_n^{\odot}, \pm, \sigma)$. Then, $l = n$ and we have that $R_{\otimes} \bar{w}_1 \dots \bar{w}_n \bar{w}$ holds iff there are $\bar{v}_1 \dots \bar{v}_n$ such that $R_{\odot}^{\pm\sigma} \bar{v}_1 \dots \bar{v}_n \bar{w}$ and $\bar{w}_1 = \bar{v}_1$ and \dots and $\bar{w}_n = \bar{v}_n$. Then, we have that $R_{\otimes} \bar{w}_1 \dots \bar{w}_n \bar{w}$ iff $R_{\odot}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}$. However, by definition, we have that for all $\varphi_1, \dots, \varphi_m \in \mathcal{L}_{C^+}$, $\bar{w} \in \llbracket \otimes(\varphi_1, \dots, \varphi_m) \rrbracket^{\mathfrak{M}}$ iff $\exists \bar{w}_1 \dots \bar{w}_n (\bar{w}_1 \uparrow_1 \llbracket \varphi_{i_1} \rrbracket^{\mathfrak{M}} \wedge \dots \wedge \bar{w}_n \uparrow_n \llbracket \varphi_{i_n} \rrbracket^{\mathfrak{M}} \wedge R_{\otimes} \bar{w}_1 \dots \bar{w}_n \bar{w})$. We obtain the result by replacing $R_{\odot}^{\pm\sigma}$ by R_{\otimes} .

Now, we prove the induction step. If $\otimes = \odot(\otimes_1, \dots, \otimes_n)$ is an existential molecular connective with \odot of skeleton $(\bar{k}, \exists, \pm_1^{\odot}, \dots, \pm_n^{\odot}, \pm, \sigma)$ then for all $\varphi_1, \dots, \varphi_m \in \mathcal{L}_{C^+}$, we have that $\bar{w} \in \llbracket \otimes(\varphi_1, \dots, \varphi_m) \rrbracket^{\mathfrak{M}}$

iff $\exists \bar{v}_1 \dots \bar{v}_n (R_{\odot}^{\pm\sigma} \bar{v}_1 \dots \bar{v}_n \bar{w} \wedge \bar{v}_1 \uparrow_1 \llbracket \otimes_1(\varphi_1^1, \dots, \varphi_{i_1}^1) \rrbracket^{\mathfrak{M}} \wedge \dots \wedge \bar{v}_n \uparrow_n \llbracket \otimes_n(\varphi_1^n, \dots, \varphi_{i_n}^n) \rrbracket^{\mathfrak{M}})$ by the truth condition for \odot

iff $\exists \bar{v}_1 \dots \bar{v}_n (R_{\odot}^{\pm\sigma} \bar{v}_1 \dots \bar{v}_n \bar{w} \wedge \bar{v}_1 \in \llbracket \pm_1^{\odot} \otimes_1(\varphi_1^1, \dots, \varphi_{i_1}^1) \rrbracket^{\mathfrak{M}} \wedge \dots \wedge \bar{v}_n \in \llbracket \pm_n^{\odot} \otimes_n(\varphi_1^n, \dots, \varphi_{i_n}^n) \rrbracket^{\mathfrak{M}})$
iff $\exists \bar{v}_1 \dots \bar{v}_n (R_{\odot}^{\pm\sigma} \bar{v}_1 \dots \bar{v}_n \bar{w} \wedge \exists \bar{w}_1^1 \dots \bar{w}_{i_1}^1 (\bar{w}_1^1 \uparrow_1^1 \llbracket \varphi_1^1 \rrbracket^{\mathfrak{M}} \wedge \dots \wedge \bar{w}_{i_1}^1 \uparrow_{i_1}^1 \llbracket \varphi_{i_1}^1 \rrbracket^{\mathfrak{M}} \wedge R_{\pm_1^{\odot} \otimes_1} \bar{w}_1^1 \dots \bar{w}_{i_1}^1 \bar{v}_1) \wedge \dots \wedge \exists \bar{w}_1^n \dots \bar{w}_{i_n}^n (\bar{w}_1^n \uparrow_1^n \llbracket \varphi_1^n \rrbracket^{\mathfrak{M}} \wedge \dots \wedge \bar{w}_{i_n}^n \uparrow_{i_n}^n \llbracket \varphi_{i_n}^n \rrbracket^{\mathfrak{M}} \wedge R_{\pm_n^{\odot} \otimes_n} \bar{w}_1^n \dots \bar{w}_{i_n}^n \bar{v}_n))$

iff $\exists \bar{w}_1^1 \dots \bar{w}_{i_1}^1 \dots \bar{w}_1^n \dots \bar{w}_{i_n}^n (\bar{w}_1^1 \uparrow_1^1 \llbracket \varphi_1^1 \rrbracket^{\mathfrak{M}} \wedge \dots \wedge \bar{w}_{i_1}^1 \uparrow_{i_1}^1 \llbracket \varphi_{i_1}^1 \rrbracket^{\mathfrak{M}} \wedge \dots \wedge \bar{w}_1^n \uparrow_1^n \llbracket \varphi_1^n \rrbracket^{\mathfrak{M}} \wedge \dots \wedge \bar{w}_{i_n}^n \uparrow_{i_n}^n \llbracket \varphi_{i_n}^n \rrbracket^{\mathfrak{M}} \wedge \exists \bar{v}_1 \dots \bar{v}_n (R_{\odot}^{\pm\sigma} \bar{v}_1 \dots \bar{v}_n \bar{w} \wedge R_{\pm_1^{\odot} \otimes_1} \bar{w}_1^1 \dots \bar{w}_{i_1}^1 \bar{v}_1 \wedge \dots \wedge R_{\pm_n^{\odot} \otimes_n} \bar{w}_1^n \dots \bar{w}_{i_n}^n \bar{v}_n))$

iff $\exists \bar{w}_1 \dots \bar{w}_l (\bar{w}_1 \uparrow_1 \llbracket \varphi_{i_1} \rrbracket^{\mathfrak{M}} \wedge \dots \wedge \bar{w}_l \uparrow_l \llbracket \varphi_{i_l} \rrbracket^{\mathfrak{M}} \wedge R_{\otimes} \bar{w}_1 \dots \bar{w}_l \bar{w})$ because the union of the sets of leaves of T_{\otimes_j} for each \otimes_j , which corresponds to the tuples $\bar{w}_1^1 \dots \bar{w}_{i_1}^1 \dots \bar{w}_1^n \dots \bar{w}_{i_n}^n$, is the set of leaves of T_{\otimes} , which is denoted here $\bar{w}_1 \dots \bar{w}_l$. This proves the induction step. \square

The tonicities $+$ or $-$ of a molecular connective encode the isotonicity or antitonicity of its arguments, as the following proposition shows.

Proposition 6. *The rules R5-R6 of Fig. 1 are globally valid for all coherent molecular connective \otimes and the axioms A3-A4 (resp. A1-A2) and the rules R3-R4 (resp. R1-R2) are globally valid for all basic universal (resp. existential) molecular connectives \otimes .*

Proof. The proof of the first part is by induction on the structure of molecular connectives. The base case where \otimes is an atomic connective \odot corresponds to the validity of rules R5-R6. The proof of the second part follows from the previous proposition since the semantics of basic universal and existential molecular connectives is equivalent to the semantics of atomic connectives of quantification signatures \forall and \exists respectively. \square

The definition of basic universal and existential molecular connectives implies that the Boolean negation of a basic universal (resp. existential) molecular connective is a basic existential (resp. universal) molecular connective. However, $\neg_1^1(x_1^1 \supset \Box x_2^1)$ and $\otimes(\Diamond x_1^1, p) \rightarrow \perp$ are *not* basic universal or existential molecular connectives, although they should be: $\neg_1^1(x_1^1 \supset \Box x_2^1)$ should be existential and $\otimes(\Diamond x_1^1, p) \rightarrow \perp$ should be universal. Moreover, intuitively, there should be (molecular) connectives which are both universal and existential: for example the dummy connective x_i^k or the Boolean negation \neg_k^1 (which is equivalent to \neg_k^3). This leads us to the following more general and semantic-based definition. Basically, we consider the axioms and inference rules validated by universal or existential *atomic* connectives to define their molecular versions.

Definition 28 (Universal and existential molecular connective). Let $\mathbf{L} \triangleq (\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ be a Boolean atomic logic based on a set of molecular connectives \mathbf{C} . A molecular connective \otimes based on \mathbf{C} is *universal* (resp. *existential*) in \mathbf{L} when the tonicity signature of \otimes is coherent, the rules R5-R6 of Fig. 1 are globally valid in \mathbf{L} for \otimes and the axioms A3-A4 (resp. A1-A2) and the rules R3-R4 (resp. R1-R2) are globally valid in \mathbf{L} for \otimes .

Note that every basic universal or existential molecular connective is universal or existential respectively because of Proposition 6. The following proposition shows roughly the converse, more precisely that a universal or existential molecular connective can always be turned into an equivalent basic one. Hence, our more general and semantic-based Definition 28 of universal and existential molecular connectives is somehow equivalent to its ‘basic’ syntactic-based Definition 26.

Proposition 7. *For every molecular logic, there is an equally expressive molecular logic where all universal and existential molecular connectives are equivalent to atomic connectives of universal and existential quantification signature respectively.*¹⁰

Proof. Let $\mathbf{L} = (\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ be a molecular logic based on a set of molecular connectives \mathbf{C} . We are going to define an equally expressive logic $\mathbf{L}' = (\mathcal{L}_{C'}, \mathcal{E}_{C'}, \models)$ where C' is the set of molecular connectives \mathbf{C} where all universal and existential molecular connectives \otimes of \mathbf{C} have been replaced by atomic connectives \odot' of the same tonicity signature, dimension signature and quantification signature (and a positive relation signature). This defines a canonical translation t from \mathcal{L}_C to $\mathcal{L}_{C'}$ and its converse t^- . Now, we are going to define the class of C' -models $\mathcal{E}_{C'}$ in two steps. First, we consider the class $\{(\prod_U \mathfrak{M}, \prod_U \bar{w}) \mid (\mathfrak{M}, \bar{w}) \in \mathcal{E}_C\}$ of pointed \mathbf{C} -models consisting of all ultrapowers of \mathbf{C} -models of \mathcal{E}_C , where U is a countably incomplete ultrafilter over a set I (of the same cardinality as the cardinality of the domains of the \mathbf{C} -models of \mathcal{E}_C). Hence, by [?, Theorem 6.1.1], all these pointed models $(\prod_U \mathfrak{M}, \prod_U \bar{w})$ are ω -saturated.¹¹ Second, for each universal or existential molecular connective $\otimes \in \mathbf{C}$, we define in $\prod_U \mathfrak{M}$ the relation R'_\odot as follows: for all $\prod_U \bar{w}_1, \dots, \prod_U \bar{w}_n, \prod_U \bar{w}$ belonging to $\prod_U \mathfrak{M}$,

- if $\mathbb{A}(\otimes) = \exists$: we have that $R'_\odot \prod_U \bar{w}_1 \dots \prod_U \bar{w}_n \prod_U \bar{w}$ iff for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_C$, if $\prod_U \bar{w}_1 \Vdash_1 \llbracket \varphi_1 \rrbracket \prod_U \mathfrak{M}$ and \dots and $\prod_U \bar{w}_n \Vdash_n \llbracket \varphi_n \rrbracket \prod_U \mathfrak{M}$ then $\prod_U \bar{w} \in \llbracket \otimes(\varphi_1, \dots, \varphi_n) \rrbracket \prod_U \mathfrak{M}$;
- if $\mathbb{A}(\otimes) = \forall$: we have that *not* $R'_\odot \prod_U \bar{w}_1 \dots \prod_U \bar{w}_n \prod_U \bar{w}$ iff for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_C$, if $\prod_U \bar{w} \in \llbracket \otimes(\varphi_1, \dots, \varphi_n) \rrbracket \prod_U \mathfrak{M}$ then $\prod_U \bar{w}_1 \Vdash_1 \llbracket \varphi_1 \rrbracket \prod_U \mathfrak{M}$ or \dots or $\prod_U \bar{w}_n \Vdash_n \llbracket \varphi_n \rrbracket \prod_U \mathfrak{M}$

where, for $j \in \llbracket 1; n \rrbracket$, $\prod_U \bar{w}_j \Vdash_j \llbracket \varphi_j \rrbracket \prod_U \mathfrak{M} \triangleq \begin{cases} (\prod_U \mathfrak{M}, \prod_U \bar{w}_j) \Vdash \varphi_j & \text{if } \pm_j = + \\ \text{not } (\prod_U \mathfrak{M}, \prod_U \bar{w}_j) \Vdash \varphi_j & \text{if } \pm_j = - \end{cases}$ and

(\pm_1, \dots, \pm_n) is the tonicity signature of \otimes . These definitions of relations are the same as in the definition of the canonical model \mathfrak{M}^c of [6]. These two steps define a new class of pointed C' -models $\mathcal{E}_{C'}$ where all formulas of $\mathcal{L}_{C'}$ can now be interpreted by means of a satisfaction relation denoted \models . We denote also t the translation mapping from \mathcal{E}_C to $\mathcal{E}_{C'}$. Its converse t^- can be easily defined. We must prove that for all $(\mathfrak{M}, \bar{w}) \in \mathcal{E}_C$ and all $\varphi \in \mathcal{L}_C$, we have that

$$(\mathfrak{M}, \bar{w}) \Vdash \varphi \text{ iff } t(\mathfrak{M}, \bar{w}) \models t(\varphi). \quad (t)$$

The proof is by induction on φ . We only prove it for the case where the outermost molecular connective of φ is a universal molecular connective \otimes (and so its corresponding universal atomic

¹⁰We use [3, Definition 1] for the definition of *equally expressive* logics.

¹¹See [?] for a definition of ultrapowers of \mathbf{C} -models and ω -saturation.

connectives is $\odot' = t(\otimes)$, the proof for the existential case is similar and the proofs for the other molecular connectives are straightforward. That is, we are going to prove that for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_C$,

$$(\mathfrak{M}, \bar{w}) \Vdash \otimes(\varphi_1, \dots, \varphi_n) \text{ iff } t(\mathfrak{M}, \bar{w}) \models \odot'(t(\varphi_1), \dots, t(\varphi_n)). \quad (t_V)$$

The left to right direction of (t_V) follows from the definition of R'_\odot above. For the right to left direction, we are going to reason by contraposition. Assume that we do not have that $(\mathfrak{M}, \bar{w}) \Vdash \otimes(\varphi_1, \dots, \varphi_n)$. We must prove that it is not the case that $(\mathfrak{M}, \bar{w}) \models \odot'(t(\varphi_1), \dots, t(\varphi_n))$, that is, there are $\prod_U \bar{w}_1, \dots, \prod_U \bar{w}_n \in \prod_U \mathfrak{M}$ such that not $R'_\odot \prod_U \bar{w}_1 \dots \prod_U \bar{w}_n \prod_U \bar{w}$ and not $\prod_U \bar{w}_1 \Vdash_1 \llbracket t(\varphi_1) \rrbracket^{t(\mathfrak{M})}$ and \dots and not $\prod_U \bar{w}_n \Vdash_n \llbracket t(\varphi_n) \rrbracket^{t(\mathfrak{M})}$. That is, there are $\prod_U \bar{w}_1, \dots, \prod_U \bar{w}_n \in \prod_U \mathfrak{M}$ such that:

1. for all $\psi_1, \dots, \psi_n \in \mathcal{L}_C$, if $\prod_U \bar{w} \in \llbracket \otimes(\psi_1, \dots, \psi_n) \rrbracket \prod_U \mathfrak{M}$ then $\prod_U \bar{w}_1 \Vdash_1 \llbracket \psi_1 \rrbracket \prod_U \mathfrak{M}$ or \dots or $\prod_U \bar{w}_n \Vdash_n \llbracket \psi_n \rrbracket \prod_U \mathfrak{M}$;
2. not $\prod_U \bar{w}_1 \Vdash_1 \llbracket t(\varphi_1) \rrbracket^{t(\mathfrak{M})}$ and \dots and not $\prod_U \bar{w}_n \Vdash_n \llbracket t(\varphi_n) \rrbracket^{t(\mathfrak{M})}$.

To obtain these states $\prod_U \bar{w}_1, \dots, \prod_U \bar{w}_n$, we proceed in two steps. First, we define maximal consistent sets $\Gamma_1, \dots, \Gamma_n$ of \mathcal{L}_C such that:

- (1') for all $\psi_1, \dots, \psi_n \in \mathcal{L}_C$, if $\bar{w} \in \llbracket \otimes(\psi_1, \dots, \psi_n) \rrbracket^{\mathfrak{M}}$ then $\psi_1 \Vdash_1 \Gamma_1$ or \dots or $\psi_n \Vdash_n \Gamma_n$;
- (2') not $\varphi_1 \Vdash_1 \Gamma_1$ and \dots and not $\varphi_n \Vdash_n \Gamma_n$.

Second, we show that these maximal consistent sets $\Gamma_1, \dots, \Gamma_n$ are true in some states $\prod_U \bar{w}_1, \dots, \prod_U \bar{w}_n$ of $\prod_U \mathfrak{M}$. This will prove our two conditions because in that case (1') entails (1) and, by Induction Hypothesis, (2') entails (2) too.

The first step is carried out thanks to Algorithm 2 of [6] because \otimes is a universal molecular connective and, as such, validates the axioms and inference rules needed to ensure that Algorithm 2 outputs the expected maximal consistent sets $\Gamma_1, \dots, \Gamma_n$ of \mathcal{L}_C (see the proof of [6, Lemma 6]). Conditions (1') and (2') are thus fulfilled by the soundness of Algorithm 2. As for the second step, we need to show that these sets of formulas of \mathcal{L}_C are true in some states $\prod_U \bar{w}_1, \dots, \prod_U \bar{w}_n$ of $\prod_U \mathfrak{M}$. By construction of these sets in Algorithm 2, for all $j \in \llbracket 1; n \rrbracket$ and all finite subsets $S_j \subseteq \Gamma_j$ there is a state $\bar{v}_j \in \mathfrak{M}$ such that $(\mathfrak{M}, \bar{v}_j) \Vdash S_j$, because \otimes fully validates rules R3 and R4 (we use their contrapositions). Then, by definition of the ultrapower, we also have that $(\prod_U \mathfrak{M}, \prod_U \bar{v}_j) \Vdash S_j$, and this holds for all finite subsets S_j of Γ_j . Hence, by ω -saturation of $\prod_U \mathfrak{M}$, there is $\prod_U \bar{w}_j \in \prod_U \mathfrak{M}$ such that $(\prod_U \mathfrak{M}, \prod_U \bar{w}) \Vdash \Gamma_j$. \square

10 Correspondence theory for atomic logics

The proofs and results of Goranko & Vakarelov [34] transfer to the setting of atomic logics. One basically has to replace in the proofs of [34] unary predicates by k -ary predicates, and valuations that assign to each propositional letter a subset of the domain by valuations that assign to each propositional letter a subset of k -tuples of the domain (depending on the dimension k of the propositional letter). One of the main reasons of this more or less direct transfer is that we were able to prove the soundness and completeness of our Hilbert calculi for basic atomic logic w.r.t. a Kripke-style relational semantics using a canonical model construction [6]. It is this canonical model construction which is used to prove that our inductive atomic formulas are canonical. The main difficulty for developing the correspondence theory of atomic logics thus does not reside in

the adaptation of Goranko & Vakarelov's theory to our framework but in proving the soundness and completeness result of atomic logics: it is from this completeness proof that we can adapt Goranko & Vakarelov's correspondence theory, because it also uses basic properties of canonical models and in particular their ω -saturation (see the proof of Theorem 6 in the appendix for details).

This said, we slightly generalize Goranko & Vakarelov's definition of inductive formulas by distinguishing regular formulas from *plain* regular formulas (the latter corresponding to their definition of regular formulas). Our definition of regular formulas now includes an additional and optional clause: a closure under Boolean conjunctions or disjunctions (condition (3) of Definition 30).

10.1 Inductive atomic formulas

In this section, we introduce inductive atomic formulas, which are adaptations to atomic logics of the inductive formulas for polyadic modal logics of Goranko & Vakarelov [34].

Definition 29 (Positive and negative formula). Let $\mathbf{L} \triangleq (\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ be an atomic logic based on a set of atomic connectives C and let $\varphi \in \mathcal{L}_C$. The formula φ can be written as a formula of the form $\otimes(p_1, \dots, p_n)$ where p_1, \dots, p_n are all the propositional letters that occur in φ (possibly with repetitions) and \otimes is a molecular connective (not necessarily belonging to C). We say that φ is *positive* (resp. *negative*) *in* p_j if $tn(\otimes, p_j) = +$ (resp. $tn(\otimes, p_j) = -$) and simply *positive* (resp. *negative*) if for all $j \in \llbracket 1; n \rrbracket$, $tn(\otimes, p_j) = +$ (resp. $tn(\otimes, j, p_j) = -$). \square

Example 10. The formulas $p, \Box p$ and $p \otimes p$ are positive and $\neg p, \Box \neg p$ and $\Diamond \neg p$ are negative. The formulas $p \rightarrow p$ and $\Box p \wedge \Box \neg p$ are neither positive nor negative. \square

Definition 30 (Essentially universal and existential formulas). Let $\mathbf{L} \triangleq (\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ be an atomic logic based on a set of atomic connectives C . An *essentially universal formula* (resp. *essentially existential formula*) of \mathbf{L} is

1. either a negative (resp. positive) formula
2. or a formula of \mathcal{L}_C equivalent to a formula of the form $\otimes(\varphi_1, \dots, \varphi_{i-1}, p, \varphi_{i+1}, \dots, \varphi_n)$ where \otimes is a universal molecular connective (resp. existential molecular connective) such that for all $j \in \llbracket 1; n \rrbracket \setminus \{i\}$,
 - if $tn(\otimes, j) = -$ (resp. $tn(\otimes, j) = +$) then φ_j is a positive formula or a globally valid formula of \mathbf{L}
 - if $tn(\otimes, j) = +$ (resp. $tn(\otimes, j) = -$) then φ_j is a negative formula or a globally unsatisfiable formula of \mathbf{L}
 - $p \in C_0$ is a propositional letter in an arbitrary position $i \in \llbracket 1; n \rrbracket$ but such that $tn(\otimes, i) = +$ (resp. $tn(\otimes, i) = -$).

In that case, p is called the *head* of the essentially universal (resp. existential) formula and the formula is said to be *headed*.

3. or a Boolean disjunction (resp. Boolean conjunction) of formulas of the form (1) or (2) (if these Boolean connectives belong to C).

Formulas of the form (1) or (2) are called *plain essentially universal* (resp. *existential*) *formulas*. \square

Example 11. Propositional letters p are not only essentially existential formulas because they are positive, but also essentially universal formulas because they are equivalent to formulas of the form $x_i^k p$ (since x_i^k can be taken to be a universal molecular connective). Their head is p itself. The formula $p \rightarrow q$ is a plain essentially universal formula with head q , whereas $p \vee r \rightarrow q$ is also an essentially universal formula but it is not plain, because of the antecedent $p \vee r$. Moreover, on the one hand, $((\Box p \rightarrow p) \rightarrow q)$ is a formula of the modal logic KT which is essentially universal because $(\Box p \rightarrow p)$ is valid in KT, its head is q . On the other hand, $((p \rightarrow \Box p) \rightarrow q)$ is neither an essentially universal formula nor an essentially existential formula of the modal logic KT because $p \rightarrow \Box p$ is neither valid nor positive. \square

Definition 31 (Regular formula). Let $\mathbf{L} \triangleq (\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ be an atomic logic based on a set of atomic connectives \mathcal{C} . A *regular formula* φ of \mathbf{L} is a formula $\varphi \in \mathcal{L}_C$ of the form $\varphi = \otimes(\varphi_1, \dots, \varphi_n)$ where \otimes is a universal molecular connective of \mathcal{L}_C and such that for all $j \in \llbracket 1; n \rrbracket$,

- if $tn(\otimes, j) = -$ then φ_j is an essentially universal formula
- if $tn(\otimes, j) = +$ then φ_j is an essentially existential formula.

The headed formulas $\varphi_1, \dots, \varphi_n$ are called the *main components* of φ and the *heads* of φ are the heads of $\varphi_1, \dots, \varphi_n$ (if they exist). A *plain regular formula* is a regular formula where all formulas φ_j are plain. \square

Example 12. We recall that the tonicity signature of \rightarrow is $(-, +)$. The formula $((\Box p \rightarrow p) \rightarrow q) \rightarrow \Box q$ is a regular formula of the modal logic KT because the subformula $((\Box p \rightarrow p) \rightarrow q)$ is essentially universal (see Example 11) and $\Box q$ is essentially existential (it is positive). The formula $((p \rightarrow \Box p) \rightarrow q) \rightarrow \Box q$ is not a regular formula of the modal logic KT because the subformula $((p \rightarrow \Box p) \rightarrow q)$ is not an essentially universal formula (see Example 11). \square

There might indeed be no head of φ if the φ_i s are positive or negative formulas.

Fact 1. Every regular formula is equivalent to the Boolean conjunction of plain regular formulas.

Proof. The proof of this fact follows straightforwardly from the validity, proven in Proposition 6, of Rules R5 and R6 (applied with $\varphi \wedge \varphi' \rightarrow \varphi$, $\varphi \wedge \varphi' \rightarrow \varphi'$, $\varphi \rightarrow \varphi \vee \varphi'$ and $\varphi' \rightarrow \varphi \vee \varphi'$) and axioms A3 and A4. \square

Definition 32 (Essential and inessential atom). An occurrence of a propositional letter in a regular formula φ is *essential* in φ if it is the head of a main component of the formula, otherwise it is *inessential* in φ . A propositional letter in a regular formula φ is *essential* in φ if it has at least one essential occurrence in it, otherwise it is *inessential* in φ . \square

Example 13. The propositional letter p in $p \rightarrow q$ is inessential whereas p is essential in it. The propositional letters p and r are essential in the modal polyadic formula $[\alpha](\neg p, \neg[\alpha](r, \neg\langle\beta\rangle q))$ while q is inessential there. The propositional letters p and q are essential in the modal polyadic formula $[\iota_3](\neg p, \neg[\alpha(\iota_2(\alpha, \alpha))](\neg p, q), \langle\alpha\rangle[\alpha][\alpha]q)$. \square

Definition 33 (Dependency digraph). Given a plain regular formula $\varphi = \otimes(\varphi_1, \dots, \varphi_n)$ with main components $\{\varphi_1, \dots, \varphi_k\}$, the *dependency digraph* of φ is a digraph $G = (V_\varphi, E_\varphi)$ where $V_\varphi = \{p_1, \dots, p_n\}$ is the set of heads of φ and we set $p_i E_\varphi p_j$ iff p_i occurs as an inessential propositional letter in a formula from $\{\varphi_1, \dots, \varphi_k\}$ with a head p_j . A digraph is *acyclic* if it does not contain oriented cycles. \square

Definition 34 (Inductive atomic formula). Let $L \triangleq (\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ be an atomic logic based on a set of atomic connectives C . An *inductive atomic formula* of L is a regular formula φ of L , therefore equivalent to a conjunction $\varphi_1 \wedge \dots \wedge \varphi_n$ of plain regular formulas¹², such that each plain regular formula φ_i has an acyclic dependency digraph. If L is a basic atomic logic then φ is simply called an inductive atomic formula. \square

Example 14. Like in [34], the formula of modal logic $p \wedge \Box(\Diamond p \rightarrow \Box q) \rightarrow \Diamond\Box\Box q$ which, written in the purely modal polyadic language, is $[\iota_3](\neg p, \neg[\alpha(\iota_2(\alpha, \alpha))](-p, q), \langle \alpha \rangle[\alpha][\alpha]q)$ is an inductive atomic formula. The formula $((\Box p \rightarrow p) \rightarrow q) \rightarrow \Box q$ is an inductive atomic formula of the modal logic KT but not an inductive (atomic) formula, also in the sense of [34], because $\Box p \rightarrow p$ is neither a positive nor a valid formula. The Löb formula $(\Box(\Box p \rightarrow p) \rightarrow \Box p)$ and the McKinsey formula $(\Box\Diamond p \rightarrow \Diamond\Box p)$ as well as the axiom K are not Sahlqvist formulas, in fact not even inductive atomic formulas. The Dirk Gently's Principle $(p \supset q) \vee (q \supset p)$ (DGP) of the intermediate logic LC and the formula $p \wedge q \supset p$ are not inductive atomic formulas. Finally, the formula $\zeta \triangleq ((p \supset p) \supset q) \supset q$ from the relevance logic E has a local first-order correspondent but is not an inductive (atomic) formula as proven in [17].

Note that in the Definition 30 of essentially universal and existential formulas, the notions of *global* validity and *global* unsatisfiability (defined at the beginning of Section 8) play a role.

Remark 4. Given an arbitrary atomic logic L , the class of inductive atomic formulas of L might be undecidable if L is undecidable for example. Indeed, to determine whether a given formula is inductive atomic, one needs to decide whether some of its subformulas are valid in L . This is coherent with Chagrova's result [10, Theorem 3.56] stating that it is undecidable whether an arbitrary basic modal formula has a first-order correspondent.

Our definition of inductive atomic formulas is a more or less direct reformulation of the definitions of inductive formulas by Goranko & Vakarelov [34] using the translations of Section 6.2. The main differences and generalizations concerns the way we define universal and existential (molecular) connectives, the fact that we take into account semantic considerations in the definition of inductive atomic formulas (Definitions 30 and 26) and the fact that in clause (3) of Definition 30 we also consider Boolean combinations of essentially universal and existential formulas. Hence, without surprise, we obtain the following result.

Proposition 8. *For all inductive formulas ψ of a modal polyadic logic, we have that $T(\psi)$ is equivalent to an inductive atomic formula.*

10.2 First-order correspondent and canonicity of inductive atomic formulas

Definition 35 (First-order language associated to a C -frame). Let C be a set of atomic connectives and let $F = (W, \mathfrak{R})$ be a C -frame. The associated first-order language with equality, truth constants \top and \perp and a family of predicates $\{R_\odot \mid \odot \in C\}$, with arities matching those of the respective relations in C -frames, will be denoted by \mathcal{L}_{FO}^C . Hereafter we will use the same symbol, R_\odot , for the predicate R_\odot in \mathcal{L}_{FO}^C and for the relation which interprets it in a given C -frame. A formula with free variables \bar{x} will be denoted $\varphi(\bar{x})$. \square

Definition 36 (Local first-order correspondent). Let C be a set of atomic connectives and let $\varphi \in \mathcal{L}_C$. A first-order formula $\chi(\bar{x}) \in \mathcal{L}_{FO}^C$ is a *local first-order correspondent* of φ if for every C -frame $\mathfrak{F} = (W, \mathfrak{R})$ and all $\bar{w} \in \omega(\mathfrak{M}, C)$, $\mathfrak{F}, \bar{w} \Vdash \varphi$ iff $\mathfrak{F} \models \chi[\bar{w}/\bar{x}]$, where $\mathfrak{F} \models \chi[\bar{w}/\bar{x}]$ denotes the first-order truth of $\chi(\bar{x})$ in \mathfrak{F} under the assignment of \bar{w} to the variable \bar{x} . A formula φ is *locally first-order definable* if it has a local first-order correspondent. \square

¹²by Fact 1

Definition 37 (Translation from atomic and molecular logics to first-order logic). Let C be a set of atomic connectives possibly containing Boolean connectives.

Syntax. For all $k \in \mathbb{N}^*$ and all tuples of variables or constants \bar{x} , we define the mappings $ST_{\bar{x}} : \mathcal{L}_{\mathsf{C}}^k \rightarrow \mathcal{L}_{\text{FO}}^{\mathsf{C}}$, where $\mathcal{L}_{\mathsf{C}}^k$ is the set of formulas of \mathcal{L}_{C} of dimension k , as follows: for all $p \in \mathsf{C}$, all $\odot \in \mathsf{C}$ of skeleton $(\sigma, \pm, \mathbb{A}, (k, k_1, \dots, k_n), (\pm_1, \dots, \pm_n))$ and all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$,

$$\begin{aligned} ST_{\bar{x}}(p) &\triangleq \text{P}(\bar{x}) & ST_{\bar{x}}(\varphi \rightarrow_k \psi) &\triangleq ST_{\bar{x}}(\varphi) \rightarrow ST_{\bar{x}}(\psi) & ST_{\bar{x}}(\perp_k) &\triangleq \perp \\ & & ST_{\bar{x}}(\varphi \wedge_k \psi) &\triangleq ST_{\bar{x}}(\varphi) \wedge ST_{\bar{x}}(\psi) & ST_{\bar{x}}(\top_k) &\triangleq \top \\ & & ST_{\bar{x}}(\varphi \vee_k \psi) &\triangleq ST_{\bar{x}}(\varphi) \vee ST_{\bar{x}}(\psi) & ST_{\bar{x}}(\neg_k \varphi) &\triangleq \neg ST_{\bar{x}}(\varphi) \\ ST_{\bar{x}}(\odot(\varphi_1, \dots, \varphi_n)) &\triangleq \mathbb{A}\bar{x}_1 \dots \bar{x}_n (*_1 ST_{\bar{x}_1}(\varphi_1) \times \dots \times *_n ST_{\bar{x}_n}(\varphi_n) \times \mathbf{R}_{\odot}^{\pm\sigma} \bar{x}_1 \dots \bar{x}_n \bar{x}) \end{aligned}$$

where $\bar{x}_1, \dots, \bar{x}_n$ are tuples of free variables of size k_1, \dots, k_n ,

$$\times = \begin{cases} \wedge & \text{if } \mathbb{A} = \exists \\ \vee & \text{if } \mathbb{A} = \forall \end{cases} \text{ and for all } j \in \llbracket 1; n \rrbracket, *_j = \begin{cases} \neg & \text{if } \pm_j = - \\ \text{empty} & \text{if } \pm_j = + \end{cases}$$

We also use the following notations: $\bar{x} \in ST(\varphi)$ for $ST_{\bar{x}}(\varphi)$ and $\bar{x} \notin ST(\varphi)$ for $\neg ST_{\bar{x}}(\varphi)$.

Semantics. Let (\mathfrak{M}, \bar{w}) be a pointed C -model of dimension k with $\bar{w} = (w_1, \dots, w_k)$. Let $\bar{x} = (x_1, \dots, x_k)$ be a tuple of variables and constants of size k . A *pointed structure associated to (\mathfrak{M}, \bar{w}) and \bar{x}* , denoted $ST_{\bar{x}}(\mathfrak{M}, \bar{w})$, is a pointed structure $(\mathfrak{M}, s_{\bar{x}}^{\bar{w}})$ where the assignment $s_{\bar{x}}^{\bar{w}}$ is such that $s_{\bar{x}}^{\bar{w}}(x_1) = w_1, \dots, s_{\bar{x}}^{\bar{w}}(x_k) = w_k$. So, $\bar{w} \in ST(\varphi)$ stands for $(\mathfrak{M}, s_{\bar{x}}^{\bar{w}}) \models ST_{\bar{x}}(\varphi)$.

The above translations canonically extend to molecular logics. Indeed, if C is a set of molecular connectives, every molecular formula of \mathcal{L}_{C} can be viewed as a formula of $\mathcal{L}_{\mathsf{C}^+}$, where C^+ is the set of atomic connectives associated to C . Likewise, any pointed C -model can also be viewed as a pointed C^+ -model. Then, we apply the above translations to obtain the translation of molecular formulas or C -models into FO. \square

Theorem 5. Let $\mathsf{L} \triangleq (\mathcal{L}_{\mathsf{C}}, \mathcal{E}_{\mathsf{C}}, \Vdash)$ be a molecular logic based on a set of molecular connectives C and a class of C -frames \mathcal{E}_{C} . Every inductive atomic formula of L is locally first-order definable. Its local first-order correspondent can be computed effectively.

Proof. Let $\mathsf{L} \triangleq (\mathcal{L}_{\mathsf{C}}, \mathcal{E}_{\mathsf{C}}, \Vdash)$ be a molecular logic based on a set of molecular connectives C and a class of C -frames \mathcal{E}_{C} . Let $\varphi = \otimes(\varphi_1, \dots, \varphi_n)$ be an inductive atomic formula of L and let \mathfrak{F} be a C -frame of \mathcal{E}_{C} . By Fact 1, every inductive atomic formula φ is equivalent to the conjunction of inductive atomic formulas $\varphi_1 \wedge \dots \wedge \varphi_n$ which are also *plain* regular formulas. So, w.l.o.g. we can also assume that φ is a plain regular formula (that is, we have not resorted to clause (3) of Definition 30 to build it). Assume that the tonicity signature of the universal molecular connective \otimes is (\pm_1, \dots, \pm_l) . Then, for all tuples of states \bar{w} of size the dimension of φ , we have that:

$\bar{w} \in ST(\varphi)$ holds

iff $\bar{w} \in ST(\otimes(\varphi_1, \dots, \varphi_n))$

iff $\forall \bar{P} \forall \bar{w}_1 \dots \bar{w}_l (\bar{w}_1 \Vdash_1 ST(\varphi_{i_1}) \vee \dots \vee \bar{w}_l \Vdash_l ST(\varphi_{i_l}) \vee R_{\otimes} \bar{w}_1 \dots \bar{w}_l \bar{w})$

where \bar{P} are the predicate symbols corresponding to the propositional letters appearing in φ .

W.l.o.g. we assume that $i_1 = 1, \dots, i_l = l$ and that the tonicity signature (\pm_1, \dots, \pm_l) of \otimes is such that

$$\begin{aligned} (\pm_1, \dots, \pm_l) &= \underbrace{(-, \dots, -, +, \dots, +, -, \dots, -, +, \dots, +)}_{k'} \\ &\quad \underbrace{\hspace{10em}}_k \\ &\quad \underbrace{\hspace{15em}}_{l'} \\ &\quad \underbrace{\hspace{20em}}_l \end{aligned}$$

with $\varphi_1, \dots, \varphi'_k$ headed essentially universal formulas, $\varphi_{k'+1}, \dots, \varphi_k$ headed essentially existential formulas, $\varphi_{k+1}, \dots, \varphi'_l$ negative formulas, $\varphi_{l'+1}, \dots, \varphi_l$ positive formulas. Then, $\bar{w} \in ST(\otimes(\varphi_1, \dots, \varphi_n))$ holds

iff $\forall \bar{P} \forall \bar{w}_1 \dots \bar{w}_l \left[R_{\otimes}^- \bar{w}_1 \dots \bar{w}_l \bar{w} \vee \bar{w}_1 \notin ST(\varphi_1) \vee \dots \vee \bar{w}'_k \notin ST(\varphi'_k) \vee \bar{w}_{k'+1} \in ST(\varphi_{k'+1}) \vee \dots \vee \bar{w}_k \in ST(\varphi_k) \vee \bar{w}_{k+1} \notin ST(\varphi_{k+1}) \vee \dots \vee \bar{w}'_l \notin ST(\varphi'_l) \vee \bar{w}_{l'+1} \in ST(\varphi_{l'+1}) \vee \dots \vee \bar{w}_l \in ST(\varphi_l) \right]$
by Proposition 5

iff $\forall \bar{P} \forall \bar{w}_1 \dots \bar{w}_l \left[R_{\otimes}^- \bar{w}_1 \dots \bar{w}_l \bar{w} \wedge \bar{w}_1 \in ST(\varphi_1) \wedge \dots \wedge \bar{w}'_k \in ST(\varphi'_k) \wedge \bar{w}_{k'+1} \in ST(-\varphi_{k'+1}) \wedge \dots \wedge \bar{w}_k \in ST(-\varphi_k) \right] \rightarrow (\bar{w}_{k+1} \in ST(-\varphi_{k+1}) \vee \dots \vee \bar{w}'_l \in ST(-\varphi'_l) \vee \bar{w}_{l'+1} \in ST(\varphi_{l'+1}) \vee \dots \vee \bar{w}_l \in ST(\varphi_l))$ but since $-\varphi_{k'+1}, \dots, -\varphi_k$ are headed essentially universal formulas and $\varphi_{k+1}, \dots, \varphi'_l$ are positive formulas, we have that it holds

iff $\forall \bar{P} \forall \bar{w}_1 \dots \bar{w}_l \left[R_{\otimes}^- \bar{w}_1 \dots \bar{w}_l \bar{w} \wedge \bar{w}_1 \in ST(\psi_1) \wedge \dots \wedge \bar{w}_k \in ST(\psi_k) \rightarrow \bar{w}_{k+1} \in ST(\psi_{k+1}) \vee \dots \vee \bar{w}_l \in ST(\psi_l) \right]$

where $\psi_1 = \varphi_1, \dots, \psi'_k = \varphi'_k, \psi_{k'+1} = -\varphi_{k'+1}, \dots, \psi_k = -\varphi_k$ and $\psi_{k+1} = -\varphi_{k+1}, \dots, \psi'_l = -\varphi'_l, \psi_{l'+1} = \varphi_{l'+1}, \dots, \psi_l = \varphi_l$. So, ψ_1, \dots, ψ_k are headed essentially universal formulas and $\psi_{k+1}, \dots, \psi_l$ are positive formulas.

(Note that the transformations that we have made until this point correspond to the so-called “pre-processing” of [34].)

iff $\forall \bar{P} \forall \bar{w}_1 \dots \bar{w}_l \left[R_{\otimes}^- \bar{w}_1 \dots \bar{w}_l \bar{w} \wedge \bar{w}_1 \in ST(\otimes^1(\psi_1^1, \dots, \psi_1^{n_1})) \wedge \dots \wedge \bar{w}_k \in ST(\otimes^k(\psi_k^1, \dots, \psi_k^{n_k})) \rightarrow (\bar{w}_{k+1}, \dots, \bar{w}_l) \in POS(\psi_{k+1}, \dots, \psi_l) \right]$

where $(\bar{w}_{k+1}, \dots, \bar{w}_l) \in POS(\psi_{k+1}, \dots, \psi_l)$ stands for $\bar{w}_{k+1} \in ST(\psi_{k+1}) \vee \dots \vee \bar{w}_l \in ST(\psi_l)$

iff $\forall \bar{P} \forall \bar{w}_1 \dots \bar{w}_l \left[R_{\otimes}^- \bar{w}_1 \dots \bar{w}_l \bar{w} \wedge \forall \bar{v}_1^1 \dots \bar{v}_{n_1}^1 (R_{\otimes^1}^- \bar{v}_1^1 \dots \bar{v}_{n_1}^1 \bar{w}_1 \vee \bar{v}_1^1 \not\vdash_1^1 ST(\psi_1^1) \vee \dots \vee P_1 \bar{v}_{i_1}^1 \vee \dots \vee \bar{v}_{n_1}^1 \not\vdash_{n_1}^1 ST(\psi_{n_1}^1)) \wedge \dots \wedge \forall \bar{v}_1^k \dots \bar{v}_{n_k}^k (R_{\otimes^k}^- \bar{v}_1^k \dots \bar{v}_{n_k}^k \bar{w}_1 \vee \bar{v}_1^k \not\vdash_1^k ST(\psi_1^k) \vee \dots \vee P_k \bar{v}_{i_k}^k \vee \dots \vee \bar{v}_{n_k}^k \not\vdash_{n_k}^k ST(\psi_{n_k}^k)) \rightarrow (\bar{w}_{k+1}, \dots, \bar{w}_l) \in POS(\psi_{k+1}, \dots, \psi_l) \right]$ (3)

We assume w.l.o.g. that for all $j \in \llbracket 1; k \rrbracket$, if the tonicity of \otimes^j is $(\pm_1^j, \dots, \pm_{n_j}^j)$ then $(\pm_1^j, \dots, \pm_{i_j-1}^j) = (-, \dots, -)$ and $(\pm_{i_j}^j, \dots, \pm_{n_j}^j) = (+, \dots, +)$. We also assume that for all $j \in \llbracket 1; k \rrbracket$ and all $i < i_j$, ψ_i is a positive formula which is not valid in \mathbf{L} and for all $i \geq i_j$, ψ_i is a negative formula which is satisfiable in \mathbf{L} . Otherwise, if it was not the case, we could remove

from the expressions that follow these formulas. So, (3) holds

$$\begin{aligned}
& \text{iff } \forall \bar{P} \forall \bar{w}_1 \dots \bar{w}_l \left[R_{\otimes}^- \bar{w}_1 \dots \bar{w}_l \bar{w} \wedge \right. \\
& \quad \forall \bar{v}_1^1 \dots \bar{v}_{n_1}^1 (R_{\otimes_1}^- \bar{v}_1^1 \dots \bar{v}_{n_1}^1 \bar{w}_1 \vee \bar{v}_1^1 \notin ST(\psi_1^1) \vee \dots \vee P_1 \bar{v}_{i_1}^1 \vee \dots \vee \bar{v}_{n_1}^1 \in ST(\psi_{n_1}^1)) \\
& \quad \wedge \dots \wedge \\
& \quad \forall \bar{v}_1^k \dots \bar{v}_{n_k}^k (R_{\otimes_k}^- \bar{v}_1^k \dots \bar{v}_{n_k}^k \bar{w}_1 \vee \bar{v}_1^k \notin ST(\psi_1^k) \vee \dots \vee P_k \bar{v}_{i_1}^k \vee \dots \vee \bar{v}_{n_k}^k \in ST(\psi_{n_k}^k)) \\
& \quad \left. \rightarrow (\bar{w}_{k+1}, \dots, \bar{w}_l) \in POS(\psi_{k+1}, \dots, \psi_l) \right] \\
& \text{iff } \forall \bar{P} \forall \bar{w}_1 \dots \bar{w}_l \left[R_{\otimes}^- \bar{w}_1 \dots \bar{w}_l \bar{w} \right. \\
& \quad \wedge \forall \bar{v}_1^1 \dots \bar{v}_{n_1}^1 \left(R_{\otimes_1}^- \bar{v}_1^1 \dots \bar{v}_{n_1}^1 \bar{w}_1 \wedge \bar{v}_1^1 \in ST(\psi_1^1) \wedge \dots \wedge \bar{v}_{i_1-1}^1 \in ST(\psi_{i_1-1}^1) \right. \\
& \quad \left. \wedge \bar{v}_{i_1+1}^1 \in ST(\neg \psi_{i_1+1}^1) \wedge \dots \wedge \bar{v}_{n_1}^1 \in ST(\neg \psi_{n_1}^1) \rightarrow P_1 \bar{v}_{i_1}^1 \right) \wedge \dots \\
& \quad \wedge \forall \bar{v}_1^k \dots \bar{v}_{n_k}^k \left(R_{\otimes_k}^- \bar{v}_1^k \dots \bar{v}_{n_k}^k \bar{w}_k \wedge \bar{v}_1^k \in ST(\psi_1^k) \wedge \dots \wedge \bar{v}_{i_k-1}^k \in ST(\psi_{i_k-1}^k) \right. \\
& \quad \left. \wedge \bar{v}_{i_k+1}^k \in ST(\neg \psi_{i_k+1}^k) \wedge \dots \wedge \bar{v}_{n_k}^k \in ST(\neg \psi_{n_k}^k) \rightarrow P_k \bar{v}_{i_k}^k \right) \\
& \quad \left. \rightarrow (\bar{w}_{k+1}, \dots, \bar{w}_l) \in POS(\psi_{k+1}, \dots, \psi_l) \right]
\end{aligned} \tag{4}$$

For all $j \in \llbracket 1; k \rrbracket$, $\bar{w}_j \in ST(\psi_j)$ stands for the j^{th} conjunct of the antecedent formula above and is of one of the following two forms:

$$\bar{w}_j \in ST(\psi_j) \triangleq \begin{cases} \forall \bar{v}_1^j \dots \bar{v}_{i_j}^j \left(R_{\otimes_j}^- \bar{v}_1^j \dots \bar{v}_{n_j}^j \bar{w}_j \wedge \bar{v}_1^j \in ST(\psi_1^j) \wedge \dots \right. \\ \quad \left. \wedge \bar{v}_{i_j-1}^j \in ST(\psi_{i_j-1}^j) \wedge \bar{v}_{i_j+1}^j \in ST(\neg \psi_{i_j+1}^j) \wedge \dots \right) & (5) \\ \quad \left. \wedge \bar{v}_{n_j}^j \in ST(\neg \psi_{n_j}^j) \rightarrow P_j \bar{v}_{i_j}^j \right) \\ P_j \bar{w}_j^1 \wedge \dots \wedge P_j \bar{w}_j^k & (6) \end{cases}$$

It is denoted, in the case of (5), $\forall \bar{v}_{i_j}^j \left[ANT_j(P_{m_1}, \dots, P_{m_j}; P_j)(\bar{v}_{i_j}^j) \rightarrow P_j \bar{v}_{i_j}^j \right]$ where $(P_{m_1}, \dots, P_{m_j})$ are the predicate symbols which appear in the antecedent of (5). Because the dependency graph of φ is acyclic, by a topological sort [23, p. 612], the formulas ψ_1, \dots, ψ_k can be linearly ordered $\psi_{j_1} < \dots < \psi_{j_k}$ in such a way that the predicates $(P_{m_1}, \dots, P_{m_j})$ which occur in $ANT_j(P_{m_1}, \dots, P_{m_j}; P_j)$ were among the heads of $\psi_{i_1}, \dots, \psi_{i_{j-1}}$ or were inessential propositional letters. This also implies that $(P_{m_1}, \dots, P_{m_{j_1}})$ is in fact empty.

Thus, we are going to assign to each predicate symbol an interpretation/valuation defined by a λ -term (see [10, p. 155] for details on this notation) in an order which follows the topological sort. But first, for the inessential propositional letters Q , we set:

$$V_m(Q) \triangleq \lambda t. \neg(t = t)$$

because inessential propositional letters are always positive in φ , so their minimal valuation is the empty set. For the other propositional letters, we follow the order given by the topological

sort of the dependency graph of φ :

$$V_m(P_{j_1}) \triangleq \begin{cases} \lambda t. (\bar{w}_j^1 = t \vee \dots \vee \bar{w}_j^k = t) & \text{if } \bar{w}_j \in ST(\psi_j) \text{ is } P_j \bar{w}_j^1 \wedge \dots \wedge P_j \bar{w}_j^k \\ \lambda \bar{v}_{i_{j_1}}^{j_1}. ANT_{j_1}(-; P_{j_1})(\bar{v}_{i_{j_1}}^{j_1}) & \text{otherwise} \end{cases} \quad (7)$$

$$\begin{aligned} V_m(P_{j_2}) &\triangleq \lambda \bar{v}_{i_{j_2}}^{j_2}. ANT_{j_2}(P_{j_1}; P_{j_2})(\bar{v}_{i_{j_2}}^{j_2})[P_{j_1} := V_m(P_{j_1})] \\ &\vdots \\ V_m(P_{j_k}) &\triangleq \lambda \bar{v}_{i_{j_k}}^{j_k}. ANT_{j_k}(P_{j_1}, \dots, P_{j_{k-1}}; P_{j_k})(\bar{v}_{i_{j_k}}^{j_k})[P_{j_1} := V_m(P_{j_1}), \dots, P_{j_{k-1}} := V_m(P_{j_{k-1}})] \end{aligned} \quad (8)$$

The valuation V_m for $(P_{j_1}, \dots, P_{j_k})$ that we obtain is called the *minimal valuation*. After this process, the formula that we obtain is a first-order formula. It is the local first-order correspondent of φ . Moreover, by positivity of $POS(\psi_{k+1}, \dots, \psi_l)$, the minimal valuation is such that for all sets of tuples W_1, \dots, W_k of appropriate sizes, if we denote V the valuation which assigns P_1 to W_1, \dots, P_k to W_k , we have that $\mathfrak{F}, V \models ST(\varphi)$ iff $\mathfrak{F}, V_m \models ST(\varphi)$, the latter being a first-order formula. \square

Theorem 6. *Let $L \triangleq (\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ be a Boolean atomic logic based on a Boolean common set \mathcal{C} of atomic connectives and a set \mathcal{E}_C of pointed C -frames. Let $\Sigma \subseteq \mathcal{L}_C$ be a set of inductive atomic formulas of L such that \mathcal{E}_C is the class of pointed C -frames defined by the first-order correspondents of the inductive atomic formulas of Σ . Then, the calculus $\mathcal{P}_C + \Sigma + (\text{Uniform substitution})$ is sound and strongly complete for L , where $(\text{Uniform substitution})$ is the following rule:*

$$\begin{aligned} &\text{For all } \chi \in \Sigma \text{ and all prop. letters } p_1, \dots, p_k \in \mathcal{C} \text{ and } \alpha_1, \dots, \alpha_k \in \mathcal{L}_C \\ &\text{From } \chi, \text{ infer } \chi[p_1/\alpha_1, \dots, p_k/\alpha_k] \quad (\text{Uniform substitution}) \end{aligned}$$

where $\chi[p_1/\alpha_1, \dots, p_k/\alpha_k]$ is the formula resulting from the uniform substitution in χ of p_1 by α_1 and \dots and p_k by α_k .

Remark 5. Propositional letters p in a proof system can be viewed either as propositional *variables* or propositional *constants*, like the connectives $\iota\delta$ and λ of temporal arrow logic (see Example 3). In the latter case, they are in fact nullary atomic connectives. As such, they should be part of any C -frame even if $p \in \mathcal{C}$. All the results that we have proven hold as well in such a case. The only difference appears when we want to obtain the first-order correspondent of a formula containing a propositional constant p . In that case, the translation of the formula into second-order logic should *not* quantify over the predicates P corresponding to p . Otherwise, everything else remains the same.

10.3 Exploiting the full power of atomic logics

Can we adapt the correspondence theory that we have developed so far for atomic logics without information order to these atomic logics with D -information order? The answer is yes, more or less easily. All our algorithms and theorems transfer to this setting. The only modifications that we have to make in our correspondence algorithms is to replace Expression (7) on p. 30 by the following condition below and to replace the rule of uniform substitution $(\text{Uniform substitution})$ by the rule $(\text{Uniform D-substitution})$ below.

$$V_m(P_{j_1}) \triangleq \begin{cases} \lambda t. (\bar{w}_j^1 = t \vee \dots \vee \bar{w}_j^k = t) & \text{if } p_{j_1} \notin D, \bar{w}_j \in ST(\psi_j) \text{ is } P_j \bar{w}_j^1 \wedge \dots \wedge P_j \bar{w}_j^k \\ \lambda t. (\bar{w}_j^1 \sqsubseteq t \vee \dots \vee \bar{w}_j^k \sqsubseteq t) & \text{if } p_{j_1} \in D, \bar{w}_j \in ST(\psi_j) \text{ is } P_j \bar{w}_j^1 \wedge \dots \wedge P_j \bar{w}_j^k \\ \lambda t. ANT_{j_1}(-; P_{j_1})(t) & \text{otherwise.} \end{cases} \quad (9)$$

Then, we have the following correspondence result, which is the same as Theorem 6 except that we now deal with atomic models *with information orders*, that the correspondence algorithms in the proof of Theorem 5 and the uniform substitution rule (Uniform substitution) have been accordingly adapted as we have just explained.

Theorem 7. *Let $L \triangleq (\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ be a Boolean atomic logic based on a Boolean common set C of atomic connectives and a set \mathcal{E}_C of pointed C -frames with D -information order. Let $\Sigma \subseteq \mathcal{L}_C$ be a set of inductive atomic formulas of L such that \mathcal{E}_C is the class of pointed C -frames defined by the first-order correspondents of the inductive atomic formulas of Σ . Then, the calculus $\mathcal{P}_C + \Sigma + (\text{Uniform } D\text{-substitution})$ is sound and strongly complete for L , where (Uniform D -substitution) is the following rule:*

*For all $\chi \in \Sigma$ and all prop. letters $p_1, \dots, p_k \in D$ and $\alpha_1, \dots, \alpha_k \in \mathcal{L}_D$
and all prop. letters $q_1, \dots, q_l \in C - D$ and $\beta_1, \dots, \beta_l \in \mathcal{L}_C$: (Uniform D -substitution)
From χ , infer $\chi[p_1/\alpha_1, \dots, p_k/\alpha_k, q_1/\beta_1, \dots, q_l/\beta_l]$*

where $\chi[p_1/\alpha_1, \dots, p_k/\alpha_k, q_1/\beta_1, \dots, q_l/\beta_l]$ is the formula resulting from the uniform substitution in χ of p_1 by α_1 and ... and p_k by α_k and q_1 by β_1 and ... and q_l by β_l .

Remark 6. In the formulation of rule (Uniform D -substitution), we could replace in χ the propositional letters p_1, \dots, p_k by formulas schemas ψ_1, \dots, ψ_k ranging over \mathcal{L}_C and reformulate this rule as a uniform substitution rule over an axiom schema instead of a single formula χ . This would be particularly appropriate if $C_0 \cap (C - D) = \emptyset$.

Remark 7. Because of Theorem 8 (that is [5, Theorem 7]), both Theorems 6 and 7 hold even when the atomic logic L is not Boolean.

To sum up, because of its grounding in Gaggle theory [28, 27], the framework of atomic logics can easily and naturally accommodate and integrate information orders. They are used in all the logics dealt with in [9] and the logics studied in [9] can all be represented in the framework of atomic logics. This includes the family of relevance logics, linear logics, intuitionistic logic, affine logics, punctual logics, *etc.* We are going to study some of them in the next section. In particular, we will solve in Section 11.3 the problem 8.4.18 of Bimbó & Dunn [9].

11 Case studies

We are going to study non-classical logics which are all based on the propositional language and the connectives $\neg, \wedge, \vee, \rightarrow$. Because the Boolean negation \neg and the material implication \rightarrow will be given various semantics different from the classical ones, we will employ the notations \sim and \supset to distinguish them, but both classical and non-classical versions of these connectives will often occur in the same language.

Our presentation of First-Degree Entailment is based on Priest [47, §8.5.3] and the one of relevance logics on Badia [8] (itself stemming from Brady [11]).

11.1 Case study: FDE and the Routley star semantics

The language \mathcal{L}_{FDE} of First Degree Entailment (FDE) is based on the connectives \sim, \vee, \wedge defined by the following grammar in BNF:

$$\mathcal{L}_{\text{FDE}} : \varphi ::= p \mid \sim \varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi)$$

where $p \in \text{ATM}_0$. A *Routley interpretation* is a structure $\mathfrak{M} = (W, \star, v)$ where W is a non-empty set of worlds, $\star : W \rightarrow W$ is an *involutive* function, that is a function such that $\star(\star(w)) = w$,¹³ and $v : \text{ATM}_0 \times W \rightarrow \{0, 1\}$ is a function called an *assignment* which is extended to an assignment of truth values for all formulas by the truth conditions:

$$\begin{aligned} v(\varphi \wedge \psi, w) &= 1 & \text{iff} & & v(\varphi, w) = 1 \text{ and } v(\psi, w) = 1 \\ v(\varphi \vee \psi, w) &= 1 & \text{iff} & & v(\varphi, w) = 1 \text{ or } v(\psi, w) = 1 \\ v(\sim \varphi, w) &= 1 & \text{iff} & & v(\varphi, \star w) = 0. \end{aligned}$$

A *pointed Routley interpretation* is a pair (\mathfrak{M}, w) where $w \in W$. We denote the set of all pointed Routley interpretations by \mathcal{E}_{FDE} and the satisfaction relation \models is defined by $((\mathfrak{M}, w), \varphi) \in \models$ iff $v(\varphi, w) = 1$. The triple $\text{FDE} \triangleq (\mathcal{L}_{\text{FDE}}, \mathcal{E}_{\text{FDE}}, \models)$ is a logic called First Degree Entailment.

Encoding FDE as an atomic logic. The involutive function \star of a Routley interpretation can be seen as a binary relation and we can naturally associate atomic connectives to it. Then, we will need to impose some conditions on this relation by means of specific axioms so that this relation is in fact an involutive function.

We introduce the atomic connectives $\text{C}_{\text{FDE}} = \text{ATM}_0 \cup \text{BLN}_1 \cup \{\sim^1, \sim^2, \sim^3\}$ where we recall that $\text{BLN}_1 \triangleq \{\top, \perp, \neg, \wedge, \vee, \rightarrow\}$ and we set

$$\sim^1 \triangleq ((1, 1, \exists, -, +, \text{Id}), 1) \quad \sim^2 \triangleq ((1, 1, \exists, -, +, (2\ 1)), 1) \quad \sim^3 \triangleq ((1, 1, \forall, -, -, \text{Id}), 1).$$

The atomic connectives \sim^1, \sim^2, \sim^3 do not have any constraint (for the moment) on their associated relation. Instead of introducing only one atomic connective \sim , we introduce three atomic connectives which are all residuated. This is because we will need all of them to be able to impose conditions on their associated relation $R_{\sim^1} = R_{\sim^2} = R_{\sim^3}$ so that this relation is the graph of an involutive function. These connectives are residuated as we said. For example, $\sim^2 = (2\ 1) \sim^1$. We spell out the Hilbert calculus $\mathcal{P}_{\text{C}_{\text{FDE}}}$ in Fig. 3 and we obtain 6 instances of the Axiom A5. Note that the instances of Axioms A2 and A4 are weaker than the classical De Morgan laws.

If we apply the correspondence algorithm in the proof of Theorem 5 to the formulas of Fig. 4, we obtain the following computations:

Example 15 (Axiom $\text{N}_{1,2}$). $\sim^1 p \rightarrow \sim^2 p$ is an inductive atomic formula.

- $\forall P \exists y (S y x \wedge \neg P y) \rightarrow \exists z (S x z \wedge \neg P z)$
 $\forall P \forall y [(S y x \wedge \neg P y) \rightarrow \exists z (S x z \wedge \neg P z)]$
 $\forall P \forall y [\forall z (S x z \rightarrow P z) \rightarrow (S y x \rightarrow P z)]$
 $\forall P \forall y [S y x \wedge \forall z (S x z \rightarrow P z) \rightarrow P y]$
- $\sigma(P) \triangleq \lambda t. S x t$
 $\forall y (S y x \rightarrow S x y).$

Example 16 (Axiom $\text{N}_{2,1}$). $\sim^2 p \rightarrow \sim^1 p$ is an inductive atomic formula.

- $\forall P [\exists y (S x y \wedge \neg P y) \rightarrow \exists z (S z x \wedge \neg P z)]$
 $\forall P [\forall z (S z x \rightarrow P z) \rightarrow \forall y (S x y \rightarrow P y)]$
 $\forall P \forall y [S x y \wedge \forall z (S z x \rightarrow P z) \rightarrow P y]$

¹³ $\star(w)$ will be often denoted $\star w$ or w^\star .

Any sound and complete axiomatization of propositional logic (CPC)	
$\sim^1 (\varphi \wedge \psi) \rightarrow \sim^1 \varphi \vee \sim^1 \psi$	(A2)
$\sim^2 (\varphi \wedge \psi) \rightarrow \sim^2 \varphi \vee \sim^2 \psi$	(A2)
$\sim^3 \varphi \wedge \sim^3 \psi \rightarrow \sim^3 (\varphi \vee \psi)$	(A4)
$\sim^2 \sim^1 \varphi \rightarrow \varphi$	$\sim^1 \sim^2 \varphi \rightarrow \varphi$ (A5)
$\sim^1 \neg \varphi \rightarrow \neg \sim^3 \varphi$	$\neg \sim^1 \varphi \rightarrow \sim^3 \neg \varphi$ (A5)
$\neg \varphi \rightarrow \sim^3 \neg \sim^2 \varphi$	$\sim^2 \neg \sim^3 \varphi \rightarrow \neg \varphi$ (A5)
From φ , infer $\neg \sim^1 \varphi$	(R2)
From φ , infer $\neg \sim^2 \varphi$	(R2)
From $\neg \varphi$, infer $\sim^3 \varphi$	(R4)
From $\varphi \rightarrow \psi$, infer $\sim^i \psi \rightarrow \sim^i \varphi$, for all $i \in \llbracket 1; 3 \rrbracket$	(R6)
From φ and $\varphi \rightarrow \psi$, infer ψ	(MP)
The formulas φ, ψ range over $\mathcal{L}_{\mathcal{C}_{\text{FDE}}}$.	

Figure 3: Hilbert calculus $\mathcal{P}_{\mathcal{C}_{\text{FDE}}}$ instantiated with the atomic connectives \mathcal{C}_{FDE} .

- $\sigma(P) \triangleq \lambda t. Stx$
 $\forall y(Sxy \rightarrow Syx)$.

Example 17 (Axiom $\mathbf{N}_{1,3}$). $\sim^1 p \rightarrow \sim^3 p$ is an inductive atomic formula.

- $\forall P [\exists y(Syx \wedge \neg Py) \rightarrow \forall z(Szx \rightarrow \neg Pz)]$
 $\forall P [\exists z(Szx \wedge Pz) \rightarrow \forall y(Syx \rightarrow Py)]$
 $\forall P \forall zy [(Szx \wedge Pz) \rightarrow (Syx \rightarrow Py)]$
 $\forall P \forall zy [Szx \wedge Syx \wedge Pz \rightarrow Py]$
- $\sigma(P) \triangleq \lambda t. z = t$
 $\forall zy [Szx \wedge Syx \rightarrow z = y]$.

Example 18 (Axiom $\mathbf{N}_{3,1}$). $\sim^3 p \rightarrow \sim^1 p$ is an inductive atomic formula.

- $\forall P [\forall y(Syx \rightarrow \neg Py) \rightarrow \exists z(Szx \wedge \neg Pz)]$
 $\forall P [\forall z(Szx \rightarrow Pz) \rightarrow \exists y(Syx \wedge Py)]$
- $\sigma(P) \triangleq \lambda t. Stx$
 $\exists y(Syx \wedge Syx)$
 $\exists y Syx$.

The Hilbert calculus \mathcal{P}^{FDE} of Fig. 5 is the calculus $\mathcal{P}_{\mathcal{C}_{\text{FDE}}} + \{\mathbf{N}_{1,2}, \mathbf{N}_{2,1}, \mathbf{N}_{1,3}, \mathbf{N}_{3,1}\}$ where \sim^1, \sim^2, \sim^3 have been uniformly replaced by $\sim = \sim^1$.

Proposition 9. *The calculus \mathcal{P}^{FDE} of Fig. 5 is sound and strongly complete for FDE.*

Proof. It is a direct application of Theorem 6. It follows from the fact that the first-order conditions of $\mathbf{N}_{1,2}, \mathbf{N}_{2,1}, \mathbf{N}_{1,3}, \mathbf{N}_{3,1}$ are such that they impose on S to be the graph of an involutive function \star . Moreover, these conditions entail on the syntactic side that for all formulas φ , $\sim^1 \varphi \leftrightarrow \sim^2 \varphi$ and $\sim^1 \varphi \leftrightarrow \sim^3 \varphi$ are provable in $\mathcal{P}_{\mathcal{C}_{\text{FDE}}} + \{\mathbf{N}_{1,2}, \mathbf{N}_{2,1}, \mathbf{N}_{1,3}, \mathbf{N}_{3,1}\}$, thus allowing us to identify these four connectives into a single one \sim . This replacement is possible because of the Replacement lemma [6, Lemma 9]. \square

Inductive atomic axiom	Local first-order correspondent $\chi(x)$	
$\sim^1 p \rightarrow \sim^2 p$	$\forall y (S y x \rightarrow S x y)$	$N_{1,2}$
$\sim^2 p \rightarrow \sim^1 p$	$\forall y (S x y \rightarrow S y x)$	$N_{2,1}$
$\sim^1 p \rightarrow \sim^3 p$	$\forall y z (S y x \wedge S z x \rightarrow y = z)$	$N_{1,3}$
$\sim^3 p \rightarrow \sim^1 p$	$\exists y S y x$	$N_{3,1}$

Figure 4: Axioms for FDE.

Any sound and complete axiomatization of propositional logic (CPC)		
$\sim (\varphi \wedge \psi) \rightarrow (\sim \varphi \vee \sim \psi)$		(A2)
$(\sim \varphi \wedge \sim \psi) \rightarrow \sim (\varphi \vee \psi)$		(A4)
$\sim \sim \varphi \rightarrow \varphi$		(A5)
$\sim \neg \varphi \rightarrow \neg \sim \varphi$	$\neg \sim \varphi \rightarrow \sim \neg \varphi$	(A5)
$\neg \varphi \rightarrow \sim \neg \sim \varphi$	$\sim \neg \sim \varphi \rightarrow \neg \varphi$	(A5)
From φ , infer $\neg \sim \varphi$		(R2)
From $\neg \varphi$, infer $\sim \varphi$		(R4)
From $\varphi \rightarrow \psi$, infer $\sim \psi \rightarrow \sim \varphi$		(R6)
From φ and $\varphi \rightarrow \psi$, infer ψ		(MP)
The formulas φ, ψ range over $\mathcal{L}_{\text{C}_{\text{FDE}}}$.		

Figure 5: Hilbert calculus \mathcal{P}^{FDE} , sound and strongly complete for FDE.

Our Hilbert calculus \mathcal{P}^{FDE} does not contain any instance of the Rule (Uniform substitution) although it should apply, according to Theorem 6, to the formulas $\text{N}_{1,2}$, $\text{N}_{2,1}$, $\text{N}_{1,3}$, $\text{N}_{3,1}$. It has been omitted and been replaced by axiom schemas instead. So, here, we apply Remark 6. Moreover, because of our replacement of all connectives \sim^1, \sim^2, \sim^3 by a single one \sim , these axioms schemas are then confused with axioms from the base logic $\mathcal{P}_{\text{C}_{\text{FDE}}}$.

According to the classification of Dunn [29], \sim is a De Morgan negation because it validates Axioms A2 and A4. In this setting with Boolean conjunction (and disjunction), our analysis shows that the De Morgan negation reappears as the most primitive negation if we assume these axioms. This fact was already noticed by Dunn [29, Section 9].

Remark 8. Our Hilbert calculus \mathcal{P}^{FDE} includes as well the Boolean connectives \top , \perp , \neg and \rightarrow which are part of $\mathcal{L}_{\text{C}_{\text{FDE}}}$ but not of \mathcal{L}_{FDE} . One might then think that we will therefore derive in \mathcal{P}^{FDE} more or fewer of the validities of FDE than we should. However, this is not the case, because the logic $(\mathcal{L}_{\text{C}_{\text{FDE}}}, \mathcal{E}_{\text{FDE}}, \models)$ is a conservative extension of FDE according to the following theorem. Indeed, the axioms $\text{N}_{1,2}, \text{N}_{2,1}, \text{N}_{1,3}, \text{N}_{3,1}$ are $\mathcal{I}_2(\mathcal{P}_{\text{C}_{\text{FDE}}})$ inductive atomic formulas. This remark applies to all calculi of this section.

Theorem 8 (Theorem 7, [5]). *Let $\mathcal{C}_1, \mathcal{C}_2$ be two common sets of atomic connectives such that $\mathcal{C}_1 \subseteq \mathcal{C}_2$. If the class of pointed \mathcal{C}_2 -frames $\mathcal{E}_{\mathcal{C}_2}$ of the logic $(\mathcal{L}_{\mathcal{C}_2}, \mathcal{E}_{\mathcal{C}_2}, \models)$ is definable by a set of acyclic formulas of $\mathcal{I}_2(\mathcal{P}_{\mathcal{C}_1^{\text{BLN}}})$ (where $\mathcal{C}_1^{\text{BLN}}$ is the Boolean completion of \mathcal{C}_1) then $(\mathcal{L}_{\mathcal{C}_2}, \mathcal{E}_{\mathcal{C}_2}, \models)$ is a conservative extension of $(\mathcal{L}_{\mathcal{C}_1}, \mathcal{E}_{\mathcal{C}_2}, \models)$.*

11.2 Case study: Basic relevance logic B

The *relevance language* \mathcal{L}_{R} is built up using the connectives **t** (the Ackermann constant), \sim , \wedge , \vee , \supset (implication), \otimes (fusion) and $p \in \text{ATM}_0$:

$$\mathcal{L}_{\text{R}} : \varphi ::= p \mid \mathbf{t} \mid \sim \varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \supset \varphi) \mid (\varphi \otimes \varphi)$$

We present the Routley-Meyer semantics for basic relevance logic with an (induced) information order. It is taken from Badia [8]) which stems from Brady [11, p. 80]. The Routley-Meyer semantics for the basic relevance logic **B** can also be used as a basis for all relevance logics (see next section).

A *Routley-Meyer frame* for \mathcal{L}_{R} is a structure $\mathfrak{F} = (W, R, \star, T)$ where W is a non-empty set, $\emptyset \neq T \subseteq W$ is the set of *normal states*, $\star : W \rightarrow W$ is an involutive function, and $R \subseteq W \times W \times W$ satisfies **p1-p6** below. In the standard way, we will abbreviate $\exists w(Tw \wedge Rvw)$ by $u \leq v$. Then, for all $u, u', v, v', w, w' \in W$,

- p1.** $w \leq w$
- p2.** if $u \leq u'$ and $Ru'vw$ then $Ruvw$
- p3.** if $v \leq v'$ and $Ruw'w$ then $Ruvw$
- p4.** if $w \leq w'$ and $Ruvw$ then $Ruw'w'$
- p5.** if $u \leq v$ then $v^* \leq u^*$
- p6.** T is upward closed under \leq .¹⁴

¹⁴In the first publications [52, 51] and many others, T is assumed to be (generated by) a single element 0.

The relation \leq is reflexive and transitive.¹⁵ A *Routley-Meyer model* for \mathcal{L}_R based on a Routley-Meyer frame \mathfrak{F} is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $V : \text{ATM}_0 \rightarrow \mathcal{P}(W)$ is a function such that for any $p \in \text{ATM}_0$, $V(p)$ is upward closed under the relation \leq , that is, $u \in V(p)$ and $u \leq v$ implies that $v \in V(p)$. By abuse, we write $w \in \mathfrak{M}$ and $w \in \mathfrak{F}$ when $w \in W$. We define the satisfaction relation \models between pointed Routley-Meyer models and formulas of \mathcal{L}_R as follows:

$$\begin{array}{lll}
(\mathfrak{M}, w) \models \mathbf{t} & \text{iff} & \text{there is } v \in T \text{ such that } v \leq w \\
(\mathfrak{M}, w) \models p & \text{iff} & w \in V(p) \\
(\mathfrak{M}, w) \models \sim \varphi & \text{iff} & \text{not } (\mathfrak{M}, \star w) \models \varphi \\
(\mathfrak{M}, w) \models (\varphi \wedge \psi) & \text{iff} & (\mathfrak{M}, w) \models \varphi \text{ and } (\mathfrak{M}, w) \models \psi \\
(\mathfrak{M}, w) \models (\varphi \vee \psi) & \text{iff} & (\mathfrak{M}, w) \models \varphi \text{ or } (\mathfrak{M}, w) \models \psi \\
(\mathfrak{M}, w) \models (\varphi \supset \psi) & \text{iff} & \text{for every } u, v \in W \text{ such that } Ruv, \text{ if } (\mathfrak{M}, u) \models \varphi \text{ then } \\
& & (\mathfrak{M}, v) \models \psi \\
(\mathfrak{M}, w) \models (\varphi \otimes \psi) & \text{iff} & \text{there are } u, v \in W \text{ such that } Ruv, (\mathfrak{M}, u) \models \varphi \text{ and } \\
& & (\mathfrak{M}, v) \models \psi.
\end{array}$$

If \mathfrak{F} is a Routley-Meyer frame, the satisfaction relation between pointed Routley-Meyer frames (\mathfrak{F}, w) and formulas φ of \mathcal{L}_R is defined by $(\mathfrak{F}, w) \models \varphi$ iff $(\mathfrak{M}, w) \models \varphi$ for all Routley-Meyer model \mathfrak{M} based on \mathfrak{F} . A formula φ is said to be *valid* if $(\mathfrak{F}, w) \models \varphi$ for all Routley-Meyer frames \mathfrak{F} and all $w \in T$, and *globally valid* if $(\mathfrak{F}, w) \models \varphi$ for all Routley-Meyer frames \mathfrak{F} and all $w \in \mathfrak{F}$. We denote by \mathcal{F}_B the class of all pointed Routley-Meyer frames (\mathfrak{F}, w) where w ranges over the normal states T of \mathfrak{F} . The triple $\mathbf{B} \triangleq (\mathcal{L}_R, \mathcal{F}_B, \models)$ is a logic called *basic relevance logic*.

Encoding the basic relevance logic \mathbf{B} as an atomic logic. We consider atomic models with information order. We introduce the atomic connectives $\mathbf{C}_B = \text{ATM}_0 \cup \text{BLN}_1 \cup \{\mathbf{t}, \sim, \supset, \otimes\}$ where \sim is the negation of FDE and

$$\begin{array}{ll}
\otimes \triangleq (((1, 1, 1), \exists, +, +, +, (1, 2, 3)), 1) & \mathbf{t} \triangleq ((1, \exists, +), 1) \\
\supset \triangleq (((1, 1, 1), \forall, -, +, -, (3, 1, 2)), 1).
\end{array}$$

The persistent set of connectives associated to \mathbf{C}_B are $\mathbf{D} \triangleq \text{ATM}_0 \cup \{\wedge, \vee\} \cup \{\mathbf{t}, \supset, \otimes\}$. The connectives \otimes and \supset are residuated. If we apply the correspondence algorithm in the proof of Theorem 5 (where the definition of (7) is replaced by (9)) to the formulas $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$ of Fig. 6, we obtain the following computations.

Example 19 (Axiom \mathbf{T}_1). $\mathbf{t} \rightarrow (q \supset q)$ is an inductive atomic formula.

- $\forall Q \forall yz (Tx \wedge Rxyz \wedge Qy \rightarrow Qz)$
- $\sigma(Q) \triangleq \lambda t. y \sqsubseteq t$
 $\forall yz (Tx \wedge Rxyz \rightarrow y \sqsubseteq z).$

Example 20 (Axiom \mathbf{T}_2). $p \rightarrow (\mathbf{t} \otimes p)$ is an inductive atomic formula.

- $\forall P (Px \rightarrow \exists yz (Ryzx \wedge Ty \wedge Pz))$
- $\sigma(P) \triangleq \lambda t. x \sqsubseteq t$
 $\exists yz (Ryzx \wedge Ty \wedge x \sqsubseteq z)$
 $\exists y (Ryxx \wedge Ty)$ because of the Tonicity postulates.

¹⁵We can see this as follows. By **p1**, we have reflexivity. Now if $x \leq y$ (i.e. $\exists u(Tu \wedge Rux)$) and $y \leq z$ (i.e. $\exists v(Tv \wedge Rvy)$), by **p3**, we have that $\exists w(Tw \wedge Rwx)$, i.e. $x \leq z$.

Inductive atomic axiom	Local first-order correspondent $\chi(x)$	
$\mathbf{t} \rightarrow (q \supset q)$	$\forall yz (Tx \wedge Rxyz \rightarrow y \sqsubseteq z)$	T_1
$p \rightarrow (\mathbf{t} \otimes p)$	$\exists y (Ryxx \wedge Ty)$	T_2
$\mathbf{t} \rightarrow ((p_1 \supset \sim p_2) \rightarrow (p_2 \supset \sim p_1))$	$\forall y_1y_2 (Tx \wedge Rxy_1y_2 \rightarrow Rxy_2^*y_1^*)$	T_3

Figure 6: Axioms for basic relevance logic B.

Example 21 (Axiom T_3). $\varphi = \mathbf{t} \rightarrow ((p_1 \supset \sim p_2) \rightarrow (p_2 \supset \sim p_1))$ is an inductive atomic formula.

- $\forall P_1P_2(Tx \rightarrow (\forall x_1x_2 (Rxx_1x_2 \wedge P_1x_1 \rightarrow \neg P_2x_2^*) \rightarrow \forall y_1y_2 (Rxy_1y_2 \wedge P_2y_1 \rightarrow \neg P_1y_2^*)))$
 $\forall P_1P_2(Tx \rightarrow \forall y_1y_2 (\forall x_1x_2 (Rxx_1x_2 \wedge P_1x_1 \rightarrow \neg P_2x_2^*) \wedge Rxy_1y_2 \wedge P_1y_2^* \rightarrow \neg P_2y_1))$
 $\forall P_1P_2(Tx \rightarrow \forall y_1y_2 (\neg \forall x_1x_2 (Rxx_1x_2 \wedge P_1x_1 \rightarrow \neg P_2x_2^*) \vee \neg (Rxy_1y_2 \wedge P_1y_2^*) \vee \neg P_2y_1))$
 $\forall P_1P_2(Tx \rightarrow \forall y_1y_2 (P_2y_1 \wedge Rxy_1y_2 \wedge P_1y_2^* \rightarrow \exists x_1x_2 (Rxx_1x_2 \wedge P_1x_1 \wedge P_2x_2^*)))$
 $\forall P_1P_2 \forall y_1y_2 (Tx \wedge P_2y_1 \wedge Rxy_1y_2 \wedge P_1y_2^* \rightarrow \exists x_1x_2 (Rxx_1x_2 \wedge P_1x_1 \wedge P_2x_2^*))$
- $\sigma(P_1) \triangleq \lambda t. y_2^* \sqsubseteq t$ and $\sigma(P_2) \triangleq \lambda t. y_1 \sqsubseteq t$
 $\forall y_1y_2 (Tx \wedge Rxy_1y_2 \rightarrow \exists (Rxx_1x_2 \wedge y_2^* \sqsubseteq x_1 \wedge y_1 \sqsubseteq x_2^*))$
 $\forall y_1y_2 (Tx \wedge Rxy_1y_2 \rightarrow \exists (Rxx_1x_2 \wedge y_2^* \sqsubseteq x_1 \wedge x_2 \sqsubseteq y_1^*))$ because \star is involutive
 $\forall y_1y_2 (Tx \wedge Rxy_1y_2 \rightarrow Rxy_2^*y_1^*)$ because of the Tonicity postulates.

The calculus \mathcal{P}^B of Fig. 7 is in fact $\mathcal{P}^{\text{FDE}} + \mathcal{P}_{\{\mathbf{t}, \otimes, \supset\}} + \{\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3\} + (\text{RT}) + (\text{Uniform D-substitution})$ where RT is the following. It somehow connects the global validity of formulas with their validity in B, as shown in the proof of the next theorem.

From $\mathbf{t} \rightarrow \varphi$, infer φ RT

Theorem 9. *The calculus \mathcal{P}^B of Fig. 7 is sound and complete for the basic relevance logic B.*

Proof. The soundness part is routine. As for the completeness part, let us define the Hilbert calculus $\mathcal{P}_0^B \triangleq \mathcal{P}^B - (\text{RT})$. It is in fact the calculus $\mathcal{P}_0^B = \mathcal{P}^{\text{FDE}} + \mathcal{P}_{\{\mathbf{t}, \otimes, \supset\}} + \{\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3\} + (\text{Uniform D-substitution})$, that is $\mathcal{P}_{\mathbf{C}_B} + \{\mathsf{N}_{1,2}, \mathsf{N}_{2,1}, \mathsf{N}_{1,3}, \mathsf{N}_{3,1}, \mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3\} + (\text{Uniform D-substitution})$ where \sim^1, \sim^2, \sim^3 have been uniformly replaced by $\sim = \sim^1$. According to Theorem 7, \mathcal{P}_0^B is sound and strongly complete w.r.t. the class \mathcal{E}_B of pointed \mathbf{C}_B -frames (with information order) defined by the first-order correspondents of $\{\mathsf{N}_{1,2}, \mathsf{N}_{2,1}, \mathsf{N}_{1,3}, \mathsf{N}_{3,1}, \mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3\}$. We are going to show that this is the class of all pointed Routley-Meyer frames where the designated/pointed state w ranges over *all* states of the Routley-Meyer frames: $\mathcal{E}_B \triangleq \{(\mathfrak{F}, v) \mid (\mathfrak{F}, w) \in \mathcal{F}_B, v \in \mathfrak{F}\}$.

The first-order conditions (T_1) and (T_2) are equivalent to the following condition: $x \sqsubseteq y$ iff $\exists z (Tz \wedge Rzxy)$ (1). (To see that, if we assume that $x \sqsubseteq y$, then by the Tonicity postulates and (T_2) we have that $\exists z (Rzxy \wedge Tz)$ holds. Conversely, if this holds, then by (T_1) we have that $x \sqsubseteq y$.) This shows that the information order \sqsubseteq in a \mathbf{C}_B -frame $\mathfrak{F} = (W, \{R, S\}, \sqsubseteq)$ can indeed be seen as the same abbreviation as in a Routley-Meyer frame. Now, conditions $\{\mathsf{N}_{1,2}, \mathsf{N}_{2,1}, \mathsf{N}_{1,3}, \mathsf{N}_{3,1}\}$ entail that \star is the graph of an involutive function. Conditions **p1-p4** and **p6** are fulfilled: they

All axioms and inference rules of \mathcal{P}_{FDE}	$(\mathcal{P}^{\text{FDE}})$
$\psi \otimes (\varphi \vee \varphi') \rightarrow (\psi \otimes \varphi) \vee (\psi \otimes \varphi')$	(A1)
$(\varphi \vee \varphi') \otimes \psi \rightarrow (\varphi \otimes \psi) \vee (\varphi' \otimes \psi)$	(A1)
$(\psi \supset \varphi) \wedge (\psi \supset \varphi') \rightarrow (\psi \supset (\varphi \wedge \varphi'))$	(A3)
$(\varphi \supset \psi) \wedge (\varphi' \supset \psi) \rightarrow ((\varphi \vee \varphi') \supset \psi)$	(A4)
$((\varphi \supset \psi) \otimes \varphi) \rightarrow \psi$	(A5)
$\psi \rightarrow (\varphi \supset (\psi \otimes \varphi))$	(A5)
From $\neg\varphi$, infer $\neg(\varphi \otimes \psi)$	(R1)
From $\neg\varphi$, infer $\neg(\psi \otimes \varphi)$	(R1)
From φ , infer $\psi \supset \varphi$	(R3)
From $\neg\varphi$, infer $\varphi \supset \psi$	(R4)
From $\varphi \rightarrow \varphi'$, infer $(\varphi \otimes \psi) \rightarrow (\varphi' \otimes \psi)$	(R5)
From $\varphi \rightarrow \varphi'$, infer $(\psi \otimes \varphi) \rightarrow (\psi \otimes \varphi')$	(R5)
From $\varphi \rightarrow \varphi'$, infer $(\psi \supset \varphi) \rightarrow (\psi \supset \varphi')$	(R5)
From $\varphi \rightarrow \varphi'$, infer $(\varphi' \supset \psi) \rightarrow (\varphi \supset \psi)$	(R6)
From φ and $\varphi \rightarrow \psi$, infer ψ	(MP)
$\mathbf{t} \rightarrow (\chi \supset \chi)$	(T ₁)
$\varphi \rightarrow (\mathbf{t} \otimes \varphi)$	(T ₂)
$\mathbf{t} \rightarrow ((\varphi \supset \sim \varphi') \rightarrow (\varphi' \supset \sim \varphi))$	(T ₃)
From $\mathbf{t} \rightarrow \varphi$, infer φ	(RT)
The formulas χ, χ' range over \mathcal{L}_{D} and $\varphi, \varphi', \psi, \psi'$ range over \mathcal{L}_{C} .	

Figure 7: Hilbert calculus \mathcal{P}^{B} , sound and complete for \mathbf{B} .

are just the Tonicity postulates for the D-information order \sqsubseteq . Finally, condition **p5** is also fulfilled because of the first-order correspondent of (T₃).

So, for all $\varphi \in \mathcal{L}_{\text{R}}$, if φ is globally valid in \mathbf{B} then φ is provable in \mathcal{P}_0^{B} by completeness. Now, if we assume that φ is valid in \mathbf{B} then $\mathbf{t} \rightarrow \varphi$ is globally valid in \mathbf{B} . Therefore, $\mathbf{t} \rightarrow \varphi$ is provable in \mathcal{P}_0^{B} . Then, by Rule RT, φ is provable in \mathcal{P}^{B} . \square

Proposition 10. *The following theorems and inference rule are derivable in \mathcal{P}^{B} : for all $\varphi, \psi \in \mathcal{L}_{\text{D}}$,*

$$\mathbf{t} \qquad \varphi \supset \varphi \qquad \varphi \rightarrow \psi \text{ iff } \varphi \supset \psi.$$

Proof. The theorem \mathbf{t} follows from RT since $\mathbf{t} \rightarrow \mathbf{t}$ is a theorem of Propositional Logic (CPC).

The theorem $\varphi \supset \varphi$ follows from it and T_1 . Now, we prove the two-way rule:

1. $\varphi \supset \varphi$	Hyp.	1. $\varphi \rightarrow \psi$	Hyp.
2. $\psi \rightarrow (\varphi \rightarrow \psi)$	CPC	2. $\mathbf{t} \rightarrow (\varphi \supset \varphi)$	$T_1, \varphi \in \mathcal{L}_D$
3. $\varphi \supset (\varphi \rightarrow \psi)$	R5, 1, 2	3. $\mathbf{t} \rightarrow \mathbf{t}$	CPC
4. $\varphi \rightarrow (\mathbf{t} \otimes \varphi)$	T_2	4. \mathbf{t}	RT, 3
5. $\mathbf{t} \rightarrow (\varphi \supset (\varphi \rightarrow \psi))$	CPC, 3	5. $\varphi \supset \varphi$	MP, 4, 2
6. $\mathbf{t} \otimes \varphi \rightarrow (\varphi \supset (\varphi \rightarrow \psi)) \otimes \varphi$	R5, 5	6. $(\varphi \supset \varphi) \rightarrow (\varphi \supset \psi)$	R5
7. $\varphi \rightarrow (\varphi \supset (\varphi \rightarrow \psi)) \otimes \varphi$	Gen. MP, 4, 6	7. $\varphi \supset \psi$	MP, 5, 6
8. $(\varphi \supset (\varphi \rightarrow \psi)) \otimes \varphi \rightarrow (\varphi \rightarrow \psi)$	A5		
9. $\varphi \rightarrow (\varphi \rightarrow \psi)$	Gen. MP, 7, 8		
10. $(\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$	CPC		
11. $\varphi \rightarrow \psi$	MP, 9, 10.		\square

Note that the two-way rule of Proposition 10 is not valid for the whole language \mathcal{L}_{CB} , only for the (positive) relevance language \mathcal{L}_D . For example, $\neg p \rightarrow \neg p$ is valid but $\neg p \supset \neg p$ is not. Likewise, $\varphi \rightarrow (\psi \supset \varphi)$ is not valid, although $\varphi \rightarrow (\psi \rightarrow \varphi)$ is, because $\varphi \supset (\psi \supset \varphi)$ is not valid.

11.3 Case study: Relevance logics **R**, **T**, **TW**, *etc.*

The local first-order correspondents of Fig. 8 have been obtained by applying the correspondence algorithm in the proof of Theorem 5 to the respective inductive atomic formulas (where the definition of (7) is replaced by (9)). Unsurprisingly, the first-order formulas that we obtained are equal to those that can be found in the literature [47]. We only provide as example the computations of this algorithm for the inductive atomic formulas A9 and A10.

Example 22 (Axiom A9). $\varphi = (p \supset q) \rightarrow ((q \supset r) \supset (p \supset r))$ is an inductive atomic formula.

- $\forall PQR[\forall x_1x_2(Rxx_1x_2 \wedge Px_1 \rightarrow Qx_2) \rightarrow (\forall y_1y_2(Rxy_1y_2 \wedge \forall z_1z_2(Ry_1z_1z_2 \wedge Qz_1 \rightarrow Rz_2) \rightarrow \forall w_1w_2(Ry_2w_1w_2 \wedge Pw_1 \rightarrow R w_2)))]$

It is an inductive atomic formula, its graph is $pE_\varphi qE_\varphi r$ (the second occurrence of p is essential because equivalent to $x_1^1 p$, x_1^1 being an universal connective):

$$\forall PQR[\forall y_1y_2w_1w_2(\forall x_1x_2(Rxx_1x_2 \wedge Px_1 \rightarrow Qx_2) \wedge \forall z_1z_2(Ry_1z_1z_2 \wedge Qz_1 \rightarrow Rz_2) \wedge Rxy_1y_2 \wedge Ry_2w_1w_2 \wedge Pw_1 \rightarrow R w_2)]$$

- $\sigma(P) \triangleq \lambda t. w_1 \sqsubseteq t$
 $\forall QR[\forall y_1y_2w_1w_2(\forall x_1x_2(Rxx_1x_2 \wedge w_1 \sqsubseteq x_1 \rightarrow Qx_2) \wedge \forall z_1z_2(Ry_1z_1z_2 \wedge Qz_1 \rightarrow Rz_2) \wedge Rxy_1y_2 \wedge Ry_2w_1w_2 \rightarrow R w_2)]$
- $\sigma(Q) \triangleq \lambda t. \exists x_1(Rxx_1t \wedge w_1 \sqsubseteq x_1)$
 $\forall R\forall y_1y_2w_1w_2(\forall z_1z_2(Ry_1z_1z_2 \wedge Rxx_1z_1 \wedge w_1 \sqsubseteq x_1 \rightarrow Rz_2) \wedge Rxy_1y_2 \wedge Ry_2w_1w_2 \rightarrow R w_2)$
- $\sigma(R) \triangleq \lambda t. \exists u(Ry_1ut \wedge Rxx_1u \wedge w_1 \sqsubseteq x_1)$
 $\forall y_1y_2w_1w_2(Rxy_1y_2 \wedge Ry_2w_1w_2 \rightarrow \exists(Rxx_1u \wedge Ry_1uw_2 \wedge w_1 \sqsubseteq x_1))$
 $\forall y_1w_1w_2(\exists u(Rxy_1u \wedge R w_1w_2) \rightarrow \exists u(Rxw_1u \wedge Ry_1uw_2))$ by the Tonicity postulates
 $\forall yzw(R(xy)zw \rightarrow Ry(xz)w)$.

Example 23 (Axiom A10). $\varphi = (p \supset q) \rightarrow ((r \supset p) \supset (r \supset q))$ is an inductive atomic formula.

- $\forall PQR[\forall x_1x_2(Rxx_1x_2 \wedge Px_1 \rightarrow Qx_2) \rightarrow \forall y_1y_2(Rxy_1y_2 \wedge \forall z_1z_2(Ry_1z_1z_2 \wedge Rz_1 \rightarrow Pz_2) \rightarrow \forall w_1w_2(Ry_2w_1w_2 \wedge Rw_1 \rightarrow Qw_2))]$
 $\forall PQR\forall y_1y_2w_1w_2[\forall x_1x_2(Rxx_1x_2 \wedge Px_1 \rightarrow Qx_2) \wedge Rxy_1y_2 \wedge \forall z_1z_2(Ry_1z_1z_2 \wedge Rz_1 \rightarrow Pz_2) \wedge Ry_2w_1w_2 \wedge Rw_1 \rightarrow Qw_2]$
 It is an inductive atomic formula, its graph is $rE_\varphi pE_\varphi q$.
- $\sigma(R) \triangleq \lambda t.w_1 \sqsubseteq t$
 $\forall PQ\forall y_1y_2w_1w_2[\forall x_1x_2(Rxx_1x_2 \wedge Px_1 \rightarrow Qx_2) \wedge Rxy_1y_2 \wedge \forall z_1z_2(Ry_1z_1z_2 \wedge w_1 \sqsubseteq z_1 \rightarrow Pz_2) \wedge Ry_2w_1w_2 \rightarrow Qw_2]$
- $\sigma(P) \triangleq \lambda t.\exists z_1(Ry_1z_1t \wedge w_1 \sqsubseteq z_1)$
 $\forall Q\forall y_1y_2w_1w_2[\forall x_1x_2(Rxx_1x_2 \wedge \exists z_1(Ry_1z_1x_1 \wedge w_1 \sqsubseteq z_1) \rightarrow Qx_2) \wedge Rxy_1y_2 \wedge Ry_2w_1w_2 \rightarrow Qw_2]$
 $\forall Q\forall y_1y_2w_1w_2[\forall x_1x_2z_1(Rxx_1x_2 \wedge Ry_1z_1x_1 \wedge w_1 \sqsubseteq z_1 \rightarrow Qx_2) \wedge Rxy_1y_2 \wedge Ry_2w_1w_2 \rightarrow Qw_2]$
- $\sigma(Q) \triangleq \lambda t.\exists x_1z_1(Rxx_1t \wedge Ry_1z_1x_1 \wedge w_1 \sqsubseteq z_1)$
 $\forall y_1y_2w_1w_2[Rxy_1y_2 \wedge Ry_2w_1w_2 \rightarrow \exists x_1z_1(Rxx_1w_2 \wedge Ry_1z_1x_1 \wedge w_1 \sqsubseteq z_1)]$
 $\forall y_1w_1w_2[\exists y_2(Rxy_1y_2 \wedge Ry_2w_1w_2) \rightarrow \exists x_1(Rxx_1w_2 \wedge Ry_1w_1x_1)]$ by the Tonicity postulates
 $\forall y_1w_1w_2(R(xy_1)w_1w_2 \rightarrow Rx(y_1w_1)w_2)$.

Theorem 10. *Let $\Sigma \subseteq \{A8, A9, \dots, A16\}$. The Hilbert calculus $\mathcal{P}^B + \Sigma + (\text{Uniform D-substitution})$ is sound and complete for the relevance logic $L \triangleq (\mathcal{L}_{\mathcal{C}_B}, \mathcal{F}_R^\Sigma, \models)$, where \mathcal{F}_R^Σ is the class of pointed Routley-Meyer frames (\mathfrak{F}, w) (with w a normal state) validating the first-order conditions corresponding to Σ in Fig. 8.*

Proof. It is basically the same as the proof of Theorem 9. The first-order conditions corresponding to Σ are global and hold for all states x of the domain of the Routley-Meyer frames, for the same reason as with B. They are just added to those first-order conditions $N_{1,2}, N_{2,1}, N_{1,3}, N_{3,1}, T_1, T_2, T_3$ characterizing B. \square

As we did before, the axioms A8-A16 could be replaced with axioms schemas by substituting the propositional letters p, q, r by formulas ranging over \mathcal{L}_D , because of the Rule (Uniform D-substitution). Likewise, the material implication \rightarrow in these axioms (which is the outermost connective) could be replaced by the relevant implication \supset because of Proposition 10 (third item). In doing so, we would obtain the classical axioms A8-A16 of relevance logic, recalled in [47] for example.

11.4 Case study: Intuitionistic and intermediate logics

In this section, we assume some familiarity with intuitionistic logic (see [5, Section 15.1] for the necessary background). We consider atomic logics *with* information order. The encoding of intuitionistic and intermediate logics into atomic logics is similar to the encoding of intuitionistic logic into update logic of [1, Section 8].¹⁶ The underlying idea is to impose conditions on the ternary relation associated to the atomic connective \supset by means of specific axioms so that for all u, v, w ,

$$R_\supset uvw \text{ iff } u \sqsubseteq w \text{ and } v \sqsubseteq w \quad (10)$$

¹⁶A similar encoding had also been proposed by Bimbó & Dunn [9, p. 291-295].

Inductive atomic axiom	Local first-order correspondent $\chi(x)$	
$(p \supset \sim q) \rightarrow (q \supset \sim p)$	$\forall yz (Rxyz \rightarrow Rxz^*y^*)$	A8
$(p \supset q) \supset ((q \supset r) \rightarrow (p \supset r))$	$\forall yzw (R(xy)zw \rightarrow Ry(xz)w)$	A9
$(p \supset q) \rightarrow ((r \supset p) \supset (r \supset q))$	$\forall yzw (R(xy)zw \rightarrow Rx(yz)w)$	A10
$(p \supset (p \supset q)) \rightarrow (p \supset q)$	$\forall xyz (Ryz \rightarrow R(xy)yz)$	A11
$p \rightarrow ((p \supset q) \supset q)$	$\forall yz (Rxyz \rightarrow Ryxz)$	A12
$\mathbf{t} \rightarrow (p \vee \sim p)$	$Tx \rightarrow x^* \sqsubseteq x$	A13
$(p \supset \sim p) \rightarrow \sim p$	$(Tx \rightarrow x^* \sqsubseteq x) \wedge (\neg Tx \rightarrow Rxx^*x)$	A14
$p \rightarrow (q \supset p)$	$\forall yz (Rxyz \rightarrow x \sqsubseteq z)$	A15
$p \rightarrow (p \supset p)$	$\forall yz (Rxyz \rightarrow (x \sqsubseteq z \vee y \sqsubseteq z))$	A16
$(p \wedge (p \supset q)) \rightarrow p$	$Rxxx$	PMP
$p \supset p$	$\forall yz (Rxyz \rightarrow y \sqsubseteq z)$	ID
$p \vee \neg_h p$	$\forall yz (Rxyz \rightarrow y \sqsubseteq x)$	PEM
$\neg_h p \vee \neg_h \neg_h p$	$\forall x_1 x_2 y_1 y_2 (Rxx_1 x_2 \wedge Rxy_1 y_2$ $\rightarrow \exists w Ry_1 x_1 w)$	wPEM
$(p \supset p) \supset (\neg_h p \supset \neg_h q)$	$\forall x_1 x_2 y_1 y_2 z_1 z_2 (Rxx_1 x_2 \wedge Rxy_1 y_2 \wedge Ry_2 z_1 z_2$ $\rightarrow z_1 \sqsubseteq x_2 \vee \exists w Ry_1 x_1 w)$	wDGP
$\neg_h (p \wedge q) \supset (\neg_h q \vee \neg_h p)$	$\forall x_1 x_2 y_1 y_2 z_1 z_2 (Rxx_1 x_2 \wedge Rxy_1 y_2 \wedge Rx_2 z_1 z_2$ $\rightarrow \exists w (Rx_2 z_1 w \wedge Rx_2 y_1 w))$	DM4
<p>where $R(xy)zw \triangleq \exists u (Rxyu \wedge Ruzw)$ and $\neg_h \varphi \triangleq (\varphi \supset \perp)$ $Rx(yz)w \triangleq \exists u (Ryzu \wedge Rxuw)$.</p>		

Figure 8: Axioms and their local first-order correspondents.

DW	=	B+A8	T	=	TW + A11+ A14
DWX	=	DW+A13		=	TWX+A11+A14]
TW	=	DW +A9+A10	RW	=	TW + A12
TWX	=	TW + A13	R	=	RW+A11 [=T+A12]
	[=	DWX +A9+A10]	RWK	=	RW+A15
			RM	=	R+A16

Figure 9: Classical relevance logics.

This condition will ensure that the semantics for \supset with this kind of ternary relation will coincide with the semantics of the intuitionistic implication. Indeed, if (10) holds then we have that

$$\begin{aligned}
(\mathfrak{M}, w) \Vdash \varphi \supset \psi \text{ iff for all } u, v \in \mathfrak{M} \text{ such that } R_{\supset} wuv, \text{ if } (\mathfrak{M}, u) \Vdash \varphi \text{ then } (\mathfrak{M}, v) \Vdash \psi \\
\text{iff for all } v \in \mathfrak{M} \text{ such that } w \sqsubseteq v, \text{ if } (\mathfrak{M}, v) \Vdash \varphi \text{ then } (\mathfrak{M}, v) \Vdash \psi.
\end{aligned}$$

11.4.1 Intuitionistic logic

For the encoding of intuitionistic logic into an atomic logic with information order, we consider the set of atomic connectives $C_{IPC} \triangleq ATM_0 \cup BLN_1 \cup \{\supset\}$ where $\supset = (((1, 1, 1), \forall, -, +, -, (3, 1, 2)), 1)$. The persistent set of connectives associated to C_{IPC} are $D_{IPC} \triangleq ATM_0 \cup \{\wedge, \vee\} \cup \{\supset\}$. Condition (10) is then enforced on C_{IPC} -frames by choosing adequately the axioms and more precisely by including in the calculus the Axioms PMP, ID and A15. The proof of that is then very similar to the proof of [1, Lemma 20]. Applying Theorem 7, we obtain the following result (we recall that $\mathcal{P}_{C_{IPC}}$ is defined by replacing C with C_{IPC} in Fig. 1).

Proposition 11. *The calculus $\mathcal{P}^{IPC} \triangleq \mathcal{P}_{C_{IPC}} + \{PMP, ID, A15\}$ is sound and strongly complete for intuitionistic logic.*

Note that, again, our calculus for intuitionistic logic includes axioms and inference rules with Boolean connectives, and in particular Boolean negation. But Remark 3 (and Theorem 8) applies here too (our axioms are inductive atomic formulas). In fact, our calculus \mathcal{P}^{IPC} is a genuine combination of classical and intuitionistic logic which preserves exactly all theorems of classical and intuitionistic logics when restricted to their respective languages. On the one hand, it derives all the theorems of the logic $C + J$ of Fariñas del Cerro & Herzig [26].¹⁷ On the other hand, it is more comprehensive than the more recent system S of Niki & Omori [44] because the latter only considers a classical extension of “positive” intuitionistic logic (without the absurdity constant \perp and therefore without the intuitionistic negation).

11.4.2 Intermediate and superintuitionistic logics

A superintuitionistic logic is a propositional logic extending intuitionistic logic. Intermediate logics are consistent superintuitionistic logics. So, they are logics between intuitionistic logic and classical propositional logic. There exists a continuum of different intermediate logics [38] and just as many such logics exhibit the disjunction property.¹⁸ Intermediate logics form a complete lattice with intuitionistic logic as the least element and classical logic as the greatest element.¹⁹

As is well known, classical logic is obtained from intuitionistic logic by adding the principle of excluded middle (PEM). It turns out that if we add the corresponding first-order condition to the condition (10) characterizing the relation associated to \supset in intuitionistic logic, this amounts to have a ternary relation R satisfying the following condition: for all u, v, w ,

$$R_{\supset}uvw \text{ iff } u \equiv v \text{ and } v \sqsubseteq w \quad (11)$$

where $u \equiv v$ is a shorthand for $u \sqsubseteq v$ and $v \sqsubseteq u$. Assuming condition (11), the intuitionistic implication \supset then collapses with the material implication \rightarrow of classical propositional logic. Indeed, one can prove that for all $\varphi, \psi \in \mathcal{L}_{C_{IPC}}$, the formula $(\varphi \supset \psi) \leftrightarrow (\varphi \rightarrow \psi)$ is valid in any frame satisfying Condition (11) (note that PMP is the left to right direction of this implication).

We can weaken (PEM) in many ways and thus obtain a plethora of intermediate logics. For example, the logic KC, also called Jankov’s logic [37] is the logic obtained from intuitionistic logic by adding Axiom wPEM (weak PEM) or equivalently wDGP (weak Dirk Gently’s Principle) or

¹⁷Fariñas del Cerro & Herzig [26] used a different definition of the standard notion of conservative extension: what should be preserved in a conservative extension in their definition is the validity of the axiom schemas of intuitionistic logic, not so much its theorems.

¹⁸The disjunction property for a logic L is the following: if $\varphi \vee \psi$ is provable/valid in L then either φ is provable/valid in L or ψ is provable/valid in L .

¹⁹A *complete lattice* is a partially ordered set (L, \leq) such that every subset A of L has both a greatest lower bound (the *infimum*, also called the *meet*) and a least upper bound (the *supremum*, also called the *join*) in (L, \leq) . The *meet* is denoted $\bigwedge A$, and the *join* is denoted $\bigvee A$.

DM4 (generalized 4th De Morgan law). As one can easily notice, the first-order frame correspondents in Fig. 8 are all different for these axioms. However, if we assume condition (10), then they are all equivalent to the following condition:

$$\text{if } u \sqsubseteq v \text{ and } u \sqsubseteq w \text{ then there is } x \text{ such that } v \sqsubseteq x \text{ and } w \sqsubseteq x \quad (\text{Confluence})$$

This is a condition that characterizes the frames of the logic KC [39, Table 1].²⁰

12 Related works and conclusion

Related works. Greco et al. [35] also adapted and generalized the framework of Goranko & Vakarelov [34] dealing with the correspondence theory of polyadic modal logics to their DLE-logics, just as we did it with our atomic and molecular logics. In particular, they introduced the notion of inductive inequality which is the counterpart in their setting to the notion of Goranko & Vakarelov’s inductive formula. It is therefore not surprising that we come up with notions and results which are very similar to their notions.²¹ Unlike theirs, our notions are genuine instances of the notions introduced by Goranko & Vakarelov [34] and operate and apply directly at the level of the Kripke-style relational semantics, like [34], because we set a formal connection between our atomic logics and modal polyadic logics (Proposition 8). It is in fact the soundness and completeness of our calculi w.r.t. a Kripke-style relational semantics which allows to import directly the results and notions of Goranko & Vakarelov [34] in our framework. This is different from what has been done by Greco et al. [35] where such a formal connection is absent. They do not connect their work to the work of Goranko & Vakarelov as tightly as we do because they simply do not provide a Kripke semantics to their DLE-logics. Finally, since they do not use a Kripke-style relational semantics, they do not consider models with information order. This makes the study of intermediate logics rather difficult within their framework and, in fact, they do not deal with such logics.

As for the case study of relevance logics, Restall [48, 49] briefly proved some correspondence theory results, connecting display logic structural rules with Hilbert axioms and corresponding frame conditions. But Badia [8] was one of the first to develop systematically the correspondence theory of relevance logics (another forerunner is Sekhi [55] but for modal extensions of relevance logics). He introduced a Sahlqvist fragment for relevance logic and showed how to compute the first-order correspondents of the formulas of this fragment. This work was extended by Conradie & Goranko [17] who introduced a larger fragment of so-called “inductive formulas” by applying general result of the “unified correspondence theory” [21] discussed above. This inductive fragment turns out to be a proper fragment of our inductive atomic formulas for relevance logics (because we also consider in our definitions of inductive atomic formulas semantic features dealing with the validity of formulas). However, none of them proved the canonicity of the formulas of their Sahlqvist or inductive class and they thus cannot state a theorem similar to

²⁰This condition is also sometimes replaced by other formulations, such as: “KC is complete w.r.t. finite rooted frames with unique top points” [25, p. 64] or KC is the logic “characterized by the class of directed [intuitionistic Kripke] frames” [24, Proposition 3.2]. These formulations are equivalent because of the properties of S4.2 [14].

²¹To be more precise, the tonicity of molecular connectives corresponds to their *sign* inherited by the leaves in the *signed generation tree* [35, Definition 14]; essentially universal molecular formulas are concatenations of normal operators that behave ‘like boxes’ and correspond to their positive *PIA formulas* or negative *skeleton* [35, Definition 15]; essentially existential molecular formulas are compositions of normal operators that behave ‘like diamonds’ and correspond to their negative *PIA formulas* or positive *skeleton* [35, Definition 15]; the universal molecular connective part is the *skeleton* part of [35, Definition 15], and the several essentially existential and universal formulas attached to it are the maximal *PIA formulas* containing the *critical* occurrences (these occurrences are called essential here and in [34]). Every branch of a regular formula is *good* [35, Definition 15] since the lower part of the formula is all PIA, and the upper is all skeleton.

our Theorems 6 or 7. Moreover, the PEARL algorithm of [17] is non-deterministic, unlike ours. Finally, they do not assume that the Routley star \star is involutive and thus they do not faithfully capture the family of relevance logics as we do. In the same vein, the “Universal logic” of Brady [12], and to be more precise his logic DJ^dQ , could also be reformulated as an atomic logic because of its grounding in the Routley-Meyer semantics of relevance logics [53]. An encoding similar to those of Sections 11.2 and 11.3 could be introduced.

Concluding remarks. Atomic and molecular logics capture logics such that the truth conditions of their connectives are expressible by formulas of first-order logic. As such, this represents a very large class of non-classical logics. However, among these logics, there might be atomic logics to which our correspondence theory does not apply. This is the case in particular for logics which are not axiomatizable or finitely axiomatizable. Venema’s CDT logic [59] is an example of atomic logic which is not finitely axiomatizable [36]. Conversely, there are also normal modal logics defined proof-theoretically which are not complete for any class of frames [57]. There are in fact as many incomplete modal logics as there are complete ones [10] [13, Chapter 6]. This said, it might be possible to give a relational semantics to these logics by means of specific molecular connectives. Our framework of atomic and molecular logics may provide a relational semantics for modal logics which are also incomplete w.r.t. the standard Kripke semantics. In fact, we already know that there are consistent normal modal logics that are incomplete w.r.t. any class of Kripke frames but complete w.r.t. a class of neighborhood frames [33, 32][45, Section 2.3.3]. Yet, in general, the modal connective \Box of the neighborhood semantics can be seen as a molecular connective: the alternation of quantifiers \exists and \forall in the truth conditions of the modal operator \Box corresponds to the composition of two different atomic connectives. Molecular connectives generalize the so-called ‘bundled modalities’ [60] and therefore could be used to provide a relational semantics for logics which are incomplete w.r.t. the standard Kripke semantics.

Finally, the ‘correspondence’ Theorem 5 applies not only to atomic logics but also to any molecular logic. However, the ‘canonicity’ Theorems 6 and 7 apply only to atomic logics and not to molecular logics. This is because in the proof of this theorem, we use the completeness of the Hilbert calculi for basic atomic logics and in particular the canonical model built for that proof. Such completeness results and methods have not been developed yet for molecular logics. They are a preliminary step for the development of a full correspondence theory of molecular logics.

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Appendix: Proofs of Theorems 6 and 7

Lemma 1. *Let C be a set of atomic connectives and let \otimes be a basic universal molecular connective based on C of tonicity signature (\pm_1, \dots, \pm_l) and arity m . Then, for all $\Gamma_1, \dots, \Gamma_{l+1}$ of the canonical model \mathfrak{M}^c defined in [6], we have that $(\Gamma_1, \dots, \Gamma_{l+1}) \in R_{\otimes}^-$ iff for all $\varphi_1, \dots, \varphi_m \in \mathcal{L}_C$, if $\otimes(\varphi_1, \dots, \varphi_m) \in \Gamma_{l+1}$ then $\varphi_{i_1} \uparrow \Gamma_1$ or ... or $\varphi_{i_l} \uparrow \Gamma_l$ where for all $j \in \llbracket 1; l \rrbracket$, $\varphi_{i_j} \uparrow \Gamma_j \triangleq$*

$$\begin{cases} \varphi_{i_j} \in \Gamma_j & \text{if } \pm_j = + \\ \varphi_{i_j} \notin \Gamma_j & \text{if } \pm_j = - \end{cases}$$

Proof. By induction on the number i of atomic connectives composing \otimes . If $i = 1$ then \otimes is an atomic connective \odot . So, the expected result is just the definition of the accessibility relation R_{\odot} of the canonical model (see the appendix of [6]).

Now, we deal with the induction step $i + 1$. Assume that \otimes is a basic universal molecular connective of the form $\otimes = \odot(\otimes_1, \dots, \otimes_n)$ of dimension signature (k, k_1, \dots, k_m) , width l and tonicity signature (\pm_1, \dots, \pm_l) with \odot an atomic connective of skeleton $(\sigma, \pm, \forall, \bar{k}, (\pm_1^{\odot}, \dots, \pm_n^{\odot}))$. Then for all $\Gamma, \Gamma_1, \dots, \Gamma_l \in \mathfrak{M}^c$, we have by Definition 27 that $R_{\otimes}\Gamma_1 \dots \Gamma_l \Gamma$ iff for all $\Gamma'_1, \dots, \Gamma'_n \in \mathfrak{M}^c$ ($R_{\odot}^{\pm\sigma}\Gamma'_1 \dots \Gamma'_n \Gamma$ or $R_{\pm_1^{\odot}\otimes_1}\Gamma_1^1 \dots \Gamma_l^1 \Gamma'_1$ or ... or $R_{\pm_n^{\odot}\otimes_n}\Gamma_1^n \dots \Gamma_l^n \Gamma'_n$) where $\Gamma_1^1 \dots \Gamma_l^1, \dots, \Gamma_1^n \dots \Gamma_l^n \in$

$\{\Gamma_1, \dots, \Gamma_l\}$. Thus, $R_{\otimes}^- \Gamma_1 \dots \Gamma_l \Gamma$ iff there are $\Gamma'_1, \dots, \Gamma'_n \in \mathfrak{M}^c$ such that $(R_{\odot}^{-\pm\sigma} \Gamma'_1 \dots \Gamma'_n \Gamma$ and $R_{\pm_1^{\odot} \otimes_1}^- \Gamma'_1 \dots \Gamma'_{l_1} \Gamma'_1$ and \dots and $R_{\pm_n^{\odot} \otimes_n}^- \Gamma'_1 \dots \Gamma'_{l_n} \Gamma'_n$).

Assume that $(\Gamma_1, \dots, \Gamma_l, \Gamma) \in R_{\otimes}^-$. We are going to prove that for all $\varphi_1, \dots, \varphi_m \in \mathcal{L}_C$, if $\otimes(\varphi_1, \dots, \varphi_m) \in \Gamma$ then $\varphi_{i_1} \pitchfork_1 \Gamma_1$ or \dots or $\varphi_{i_l} \pitchfork_l \Gamma_l$. Note that $\otimes(\varphi_1, \dots, \varphi_m)$ is of the form $\odot(\otimes_1(\varphi_1^1, \dots, \varphi_{l_1}^1), \dots, \otimes_n(\varphi_1^n, \dots, \varphi_{l_n}^n))$. Then, we have that $\otimes(\varphi_1, \dots, \varphi_m) \in \Gamma$ iff $\odot(\otimes_1(\varphi_1^1, \dots, \varphi_{l_1}^1), \dots, \otimes_n(\varphi_1^n, \dots, \varphi_{l_n}^n)) \in \Gamma$. So, by definition of $R_{\odot}^{-\pm\sigma}$, we have $\pm_1^{\odot} \otimes_1(\varphi_1^1, \dots, \varphi_{l_1}^1) \in \Gamma'_1$ and \dots and $\pm_n^{\odot} \otimes_n(\varphi_1^n, \dots, \varphi_{l_n}^n) \in \Gamma'_n$. Now, because we also have $R_{\pm_1^{\odot} \otimes_1}^- \Gamma'_1 \dots \Gamma'_{l_1} \Gamma'_1$ and \dots and $R_{\pm_n^{\odot} \otimes_n}^- \Gamma'_1 \dots \Gamma'_{l_n} \Gamma'_n$, by Induction Hypothesis, if $(\pm_1^1, \dots, \pm_{l_1}^1), \dots, (\pm_1^n, \dots, \pm_{l_n}^n)$ are the tonicity signatures of $\otimes_1, \dots, \otimes_n$ respectively then we have $\varphi_{i_1} \pitchfork'_1 \Gamma_1$ and \dots and $\varphi_{i_l} \pitchfork'_l \Gamma_l$

where for all $j \in \llbracket 1; l \rrbracket$, if $\varphi_{i_j} = \varphi_i^k$ then $\varphi_{i_j} \pitchfork'_j \Gamma_j \triangleq \begin{cases} \varphi_{i_j} \in \Gamma_j & \text{if } \pm_k^{\odot} \pm_i^k = + \\ \varphi_{i_j} \notin \Gamma_j & \text{if } \pm_k^{\odot} \pm_i^k = - \end{cases}$. (This is well-defined by the coherence condition of the universal molecular connective.) That is, for all

$j \in \llbracket 1; l \rrbracket$, $\varphi_{i_j} \pitchfork'_j \Gamma_j$ iff $\varphi_{i_j} \pitchfork_j \Gamma_j \triangleq \begin{cases} \varphi_{i_j} \in \Gamma_j & \text{if } \pm_j = + \\ \varphi_{i_j} \notin \Gamma_j & \text{if } \pm_j = - \end{cases}$ because $\pm_j = \pm_k^{\odot} \pm_i^k$ by definition of the tonicity signature. This proves the left to right direction of the inductive step.

Reciprocally, assume that for all $\varphi_1, \dots, \varphi_m \in \mathcal{L}_C$, if $\otimes(\varphi_1, \dots, \varphi_m) \in \Gamma_{l+1}$ then $\varphi_{i_1} \pitchfork_1 \Gamma_1$ or \dots or $\varphi_{i_l} \pitchfork_l \Gamma_l$, i.e.

$$\text{if } \neg\varphi_{i_1} \pitchfork_1 \Gamma_1 \text{ and } \dots \text{ and } \neg\varphi_{i_l} \pitchfork_l \Gamma_l \text{ then } \neg \otimes(\varphi_1, \dots, \varphi_m) \in \Gamma_{l+1}. \quad (*)$$

Then, we define for all $j \in \llbracket 1; n \rrbracket$ the set $\Gamma'_{j,0} \triangleq \left\{ \neg \pm_j^{\odot} \otimes_j(\psi_1, \dots, \psi_{l_j}) \mid \neg\psi_1 \pitchfork_1 \Gamma_1^j, \dots, \neg\psi_{l_j} \pitchfork_{l_j} \Gamma_{l_j}^j \right\}$.

Using the Induction Hypothesis on $\pm_j^{\odot} \otimes_j$, one can prove that $\Gamma'_{j,0}$ is \mathcal{P}_C -consistent for all $j \in \llbracket 1; n \rrbracket$. Moreover, for all $\neg\chi_1 \in \Gamma'_{1,0}, \dots, \neg\chi_n \in \Gamma'_{n,0}$, we have that $\neg \odot(\chi_1, \dots, \chi_n)$ belongs to Γ by the assumption (*) (for formulas χ_1, \dots, χ_n such that $\odot(\chi_1, \dots, \chi_n)$ does belong to the language \mathcal{L}_C , since not all combinations work out). Hence, partially, we already have the required condition so that ' $R_{\odot}^{-\pm\sigma} \Gamma'_{1,0} \dots \Gamma'_{n,0} \Gamma$ ' holds (see [6] for the definition of R_{\odot}). Then, given an enumeration of all tuples of formulas (ψ_1, \dots, ψ_n) of \mathcal{L}_C , we can add incrementally a tuple (ψ_1, \dots, ψ_n) to $\Gamma'_{1,0} \times \dots \times \Gamma'_{n,0}$ if its formulas retain the consistency of $\Gamma'_{1,0}, \dots, \Gamma'_{n,0}$ and if $\odot(\psi_1, \dots, \psi_n)$ belongs to Γ . Therefore, these \mathcal{P}_C -consistent sets $\Gamma'_{1,0}, \dots, \Gamma'_{n,0}$ can be extended into maximal \mathcal{P}_C -consistent sets $\Gamma'_1, \dots, \Gamma'_n$, using the definition of R_{\odot} (because $R_{\odot} = R_{\odot} = R_{\odot}$, note that $-\odot$ has an existential quantification signature), in such a way that we have $R_{\odot}^{-\pm\sigma} \Gamma'_1 \dots \Gamma'_n \Gamma (= R_{\odot}^{-\pm\sigma} \Gamma'_1 \dots \Gamma'_n \Gamma)$. Moreover, we also have that $R_{\pm_1^{\odot} \otimes_1}^- \Gamma'_1 \dots \Gamma'_{l_1} \Gamma'_1$ and \dots and $R_{\pm_n^{\odot} \otimes_n}^- \Gamma'_1 \dots \Gamma'_{l_n} \Gamma'_n$ because by construction $\Gamma'_{1,0} \subseteq \Gamma'_1, \dots, \Gamma'_{n,0} \subseteq \Gamma'_n$ so that for all $j \in \llbracket 1; n \rrbracket$, for all $\psi_1, \dots, \psi_{l_j} \in \mathcal{L}_C$, if $\pm_j^{\odot} \otimes_j(\psi_1, \dots, \psi_{l_j}) \in \Gamma'_j$ then $\psi_1 \pitchfork_1 \Gamma_1^j, \dots, \psi_{l_j} \pitchfork_{l_j} \Gamma_{l_j}^j$ (which is the condition required to set $R_{\pm_j^{\odot} \otimes_j}^- \Gamma_1^j \dots \Gamma_{l_j}^j \Gamma'_j$). This proves the other direction of the induction step. \square

Theorem 6. Let $L \triangleq (\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ be an atomic logic based on a set C of atomic connectives and a set \mathcal{E}_C of pointed C -frames. Let $\Sigma \subseteq \mathcal{L}_C$ be a set of inductive atomic formulas of L such that \mathcal{E}_C is the class of pointed C -frames defined by the first-order correspondents of the inductive atomic formulas of Σ . Then, the calculus $\mathcal{P}_C + \Sigma$ is sound and strongly complete for L .

Proof. The proof is similar to the proof of [34]. The main difference is that we do not resort to general frames and we do not use a compactness argument, we stay within the canonical model and use its ω -saturation (the notions of compactness and ω -saturation are close enough notions to obtain the desired proof). The other difference is that we deal with dimensions and tuples of states instead of single states like in [34] but this does not change the layout of the proof.

Soundness is proven without difficulty. As for completeness, we must prove that any $\mathcal{P}_C + \Sigma$ -consistent set $\Gamma \subseteq \mathcal{L}_C$ is satisfiable in a pointed C -frame of \mathcal{E}_C . We consider the simpler case where Σ consists of a single formula χ in order to highlight the main ideas; the general case where Σ is an arbitrary set follows easily. We are going to show that the canonical frame underlying the canonical model \mathfrak{M}^c of [6, Definition 29] (where k - \mathcal{P}_C -consistency is replaced by $\mathcal{P}_C + \{\chi\}$ -consistency) fulfills the expected requirements. W.l.o.g., we assume that all universal connectives that appear in χ are basic (if it is not the case, we can always replace them by basic ones and change the initial set of atomic connectives C accordingly by taking another set of atomic connectives C'). Let us take the canonical model $\mathfrak{M}^c = (W^c, \mathfrak{R}^c)$ associated to the set of atomic connectives C defined in [6] and let \mathfrak{F}^c be the C -frame underlying the canonical C -model \mathfrak{M}^c . We know by assumption that $\mathfrak{M}^c \models \chi$ and we want to show that $\mathfrak{F}^c \models \chi$. As we showed in the proof of Theorem 5, if $(\mathfrak{F}^c, V_m) \models \chi$ for the minimal valuation V_m (defined in the proof of Theorem 5, (3)) then $(\mathfrak{F}^c, V) \models \chi$ for any valuation V , and thus $\mathfrak{F}^c \models \chi$. The result then follows from Theorem 5. So, we need to show that $(\mathfrak{F}^c, V_m) \models \chi$ (4). The problem is that V_m is not necessarily the valuation of the canonical model \mathfrak{M}^c . We are going to show a weaker lemma which will nevertheless imply that (4) holds. First, note that because of Fact 1 we can assume w.l.o.g. that χ is a *plain* regular formula. This weaker lemma is the following:

Lemma 2. *Let V_m be the ‘minimal’ valuation of Expressions (7) and (8) defined for the canonical model \mathfrak{M}^c of [6, Definition 29].*

- a. *for all $i \in \llbracket 1; k \rrbracket$, $V_m(P_{j_i})$ is an intersection of sets $\llbracket \varphi \rrbracket^{\mathfrak{M}^c}$ for φ ranging over \mathcal{L}_C ;*
- b. *for every positive formula $\pi \in \mathcal{L}_C$, $\llbracket \pi \rrbracket^{(\mathfrak{F}^c, V_m)} = \bigcap_{V_m \subseteq V} \llbracket \pi \rrbracket^{(\mathfrak{F}^c, V)}$ where the intersection ranges over all valuations V extending V_m and such that for all $i \in \llbracket 1; k \rrbracket$, $V(P_{j_i})$ is $\llbracket \varphi \rrbracket^{\mathfrak{M}^c}$ for some $\varphi \in \mathcal{L}_C$.*

Indeed, if we prove a. and b., then, because χ is of a specific shape (see Expression (4) in the proof of Theorem 5) and because of the rule (Uniform substitution), χ will hold in (\mathfrak{F}^c, V_m) . Now, we prove both a. and b. by an intertwined induction on $i \in \{1, \dots, k\}$.

- $i = 1$.

- a. $V_m(P_{j_1})$ is of the form $V_m(P_{j_1}) = \{\bar{w}_{j_1}^1, \dots, \bar{w}_{j_1}^k\}$ or $V_m(P_{j_1}) = \{\bar{v} \mid R_{\otimes^{j_1}}^- \bar{v} \bar{w}_{j_1}\}$. In the former case, we prove that

$$\{\bar{w}_{j_1}^1, \dots, \bar{w}_{j_1}^k\} = \bigcap_{\varphi \in \mathcal{L}_C} \left\{ \llbracket \varphi \rrbracket^{\mathfrak{M}^c} \mid \bar{w}_{j_1}^1 \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c} \text{ and } \dots \text{ and } \bar{w}_{j_1}^k \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c} \right\}$$

and this will prove a. The left to right inclusion is immediate. As for the right to left implication, assume that $\bar{v} \in \bigcap_{\varphi \in \mathcal{L}_C} \left\{ \llbracket \varphi \rrbracket^{\mathfrak{M}^c} \mid \bar{w}_{j_1}^1 \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c} \text{ and } \dots \text{ and } \bar{w}_{j_1}^k \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c} \right\}$ and assume towards a contradiction that $\bar{v} \notin \{\bar{w}_{j_1}^1, \dots, \bar{w}_{j_1}^k\}$. Then, there are $\varphi^1, \dots, \varphi^k \in \mathcal{L}_C$ such that $\bar{w}_{j_1}^1 \in \llbracket \varphi^1 \rrbracket^{\mathfrak{M}^c}$ but $\bar{v} \notin \llbracket \varphi^1 \rrbracket^{\mathfrak{M}^c}$ and \dots and $\bar{w}_{j_1}^k \in \llbracket \varphi^k \rrbracket^{\mathfrak{M}^c}$ but $\bar{v} \notin \llbracket \varphi^k \rrbracket^{\mathfrak{M}^c}$ by definition of the canonical model \mathfrak{M}^c . Now, take $\varphi \triangleq \varphi^1 \vee \dots \vee \varphi^k \in \mathcal{L}_C$. Then, we should have that $\bar{v} \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c}$, because $\bar{w}_{j_1}^1 \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c}$ and \dots and $\bar{w}_{j_1}^k \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c}$. This is impossible by assumption.

In the latter case, \otimes^{j_1} is a basic universal molecular connective by assumption. So, by Fact 1, we have that $R_{\otimes^{j_1}}^- \bar{v} \bar{w}_{j_1}$ iff for all $\varphi \in \mathcal{L}_C$, if $\bar{w}_{j_1} \in \llbracket \otimes^{j_1} \varphi \rrbracket^{\mathfrak{M}^c}$ then $\bar{v} \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c}$. So, $V_m(P_{j_1}) = \bigcap_{\varphi \in \mathcal{L}_C} \left\{ \llbracket \varphi \rrbracket^{\mathfrak{M}^c} \mid \bar{w}_{j_1} \in \llbracket \otimes^{j_1} \varphi \rrbracket^{\mathfrak{M}^c} \right\}$ and this proves a.

b. same as the proof of Fact 2 for the induction step $i + 1$.

• induction step: $i + 1$.

b. We first prove b. and then a.

Fact 2. Let \mathcal{C} be a set of atomic connectives. For any positive formula $\pi \in \mathcal{L}_{\mathcal{C}}$, for all \bar{v} , we have that $\bar{v} \in ST(\pi)[SUB]$ iff it holds that $\bar{v} \in ST(\pi[sub])$ for all $\varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_{\mathcal{C}}$ such that

$$V_m(P_{j_1}) \subseteq \llbracket \varphi_{j_1} \rrbracket^{\mathfrak{M}^c} \text{ and } \dots \text{ and } V_m(P_{j_{i-1}}) \subseteq \llbracket \varphi_{j_{i-1}} \rrbracket^{\mathfrak{M}^c} \quad (5)$$

$$\begin{aligned} \text{where } SUB &\triangleq [P_{j_1} := V_m(P_{j_1}), \dots, P_{j_{i-1}} := V_m(P_{j_{i-1}})] \\ \text{sub} &\triangleq [p_{j_1}/\varphi_{j_1}, \dots, p_{j_{i-1}}/\varphi_{j_{i-1}}] \end{aligned}$$

Proof. (Fact 2) The left to right direction follows from the positivity of π . We prove the right to left direction by induction on π . The proof is the same as [34, Lemma 55] (relying on Esakia's lemma 53).

• if $\pi = p$ is a propositional letter of $\{p_{j_1}, \dots, p_{j_{i-1}}\}$ then it follows straightforwardly from the Induction Hypothesis of a., namely $V_m(p) = \bigcap_{i \in I} \llbracket \varphi_i \rrbracket$ for some family of formulas of $\mathcal{L}_{\mathcal{C}}$ indexed by some set I .

• If $\pi = \otimes(\rho_1, \dots, \rho_m)$ with \otimes existential molecular connective of tonicity signature (\pm_1, \dots, \pm_l) where $\pm_1 = \dots = \pm_{l'} = +$ and $\pm_{l'+1} = \dots = \pm_m = -$ for some $1 \leq l' \leq l$, then we have that $\mathfrak{M}^c, \bar{v} \models \otimes(\rho_1, \dots, \rho_m)[sub]$ for all $\varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_{\mathcal{C}}$ such that (5)

iff for all $\varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_{\mathcal{C}}$ such that (5), $\exists \bar{v}_1, \dots, \bar{v}_l (R_{\otimes} \bar{v}_1 \dots \bar{v}_l \bar{v} \wedge \bar{v}_1 \in ST(\rho_{i_1}[sub]) \wedge \dots \wedge \bar{v}_{l'} \in ST(\rho_{i_{l'}}[sub]) \wedge \bar{v}_{l'+1} \notin ST(\rho_{i_{l'+1}}[sub]) \wedge \dots \wedge \bar{v}_l \notin ST(\rho_{i_l}[sub]))$ (where $\rho_{i_1}, \dots, \rho_{i_{l'}}$ are positive and $\rho_{i_{l'+1}}, \dots, \rho_{i_l}$ are negative)

iff $\exists \bar{v}_1 \dots \bar{v}_l (R_{\otimes} \bar{v}_1 \dots \bar{v}_l \bar{v} \wedge \bar{v}_1 \in ST(\rho_{i_1}[sub]) \wedge \dots \wedge \bar{v}_{l'} \in ST(\rho_{i_{l'}}[sub]) \wedge \bar{v}_{l'+1} \in ST(\neg \rho_{i_{l'+1}}[sub]) \wedge \dots \wedge \bar{v}_l \in ST(\neg \rho_{i_l}[sub]))$.

Now, by ω -saturation of \mathfrak{M}^c , there are $\bar{v}_1, \dots, \bar{v}_l$ such that $R_{\otimes} \bar{v}_1 \dots \bar{v}_l \bar{v}$ and for all $\varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_{\mathcal{C}}$ such that (5), it holds that $\bar{v}_1 \in ST(\rho_{i_1}[sub])$ and \dots and $\bar{v}_{l'} \in ST(\rho_{i_{l'}}[sub])$ and $\bar{v}_{l'+1} \in ST(\neg \rho_{i_{l'+1}}[sub])$ and \dots and $\bar{v}_l \in ST(\neg \rho_{i_l}[sub])$.

So, by Induction Hypothesis, because $\rho_1, \dots, \rho_{l'}$ and $\neg \rho_{l'+1}, \dots, \neg \rho_m$ are positive, there are $\bar{v}_1, \dots, \bar{v}_l$ such that $R_{\otimes} \bar{v}_1 \dots \bar{v}_l \bar{v}$ and $\bar{v}_1 \in ST(\rho_{i_1}[SUB])$ and \dots and $\bar{v}_{l'} \in ST(\rho_{i_{l'}}[SUB])$ and $\bar{v}_{l'+1} \in ST(\rho_{i_{l'+1}}[SUB])$ and \dots and $\bar{v}_l \in ST(\rho_{i_l}[SUB])$.

That is, $\bar{v} \in ST(\otimes(\rho_1, \dots, \rho_m))[SUB]$.

• If $\pi = \otimes(\rho_1, \dots, \rho_m)$ with \otimes universal molecular connective of tonicity signature (\pm_1, \dots, \pm_l) with $\pm_1 = \dots = \pm_{l'} = +$ and $\pm_{l'+1} = \dots = \pm_l = -$ for some $1 \leq l' \leq l$, then $\mathfrak{M}^c, \bar{v} \models \otimes(\rho_1, \dots, \rho_m)[sub]$ for all $\varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_{\mathcal{C}}$ such that (5)

iff for all $\varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_{\mathcal{C}}$ such that (5), $\forall \bar{v}_1, \dots, \bar{v}_l (R_{\otimes} \bar{v}_1 \dots \bar{v}_l \bar{v} \vee \bar{v}_1 \in ST(\rho_{i_1}[sub]) \vee \dots \vee \bar{v}_{l'} \in ST(\rho_{i_{l'}}[sub]) \vee \bar{v}_{l'+1} \notin ST(\rho_{i_{l'+1}}[sub]) \vee \dots \vee \bar{v}_l \notin ST(\rho_{i_l}[sub]))$ where $\rho_{i_1}, \dots, \rho_{i_{l'}}$ are positive and $\rho_{i_{l'+1}}, \dots, \rho_{i_l}$ are negative.

Then, for all $\bar{v}_1, \dots, \bar{v}_l \left(R_{\otimes} \bar{v}_1 \dots \bar{v}_l \bar{v} \text{ or } \bar{v}_1 \in ST(\rho_{i_1}[\text{sub}]) \text{ for all } \varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_C \text{ such that (5) or } \dots \text{ or } \bar{v}_{l'} \in ST(\rho_{i_{l'}}[\text{sub}]) \text{ for all } \varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_C \text{ such that (5) or } \bar{v}_{l'+1} \in ST(\neg\rho_{i_{l'+1}}[\text{sub}]) \text{ for all } \varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_C \text{ such that (5) or } \dots \text{ or } \bar{v}_l \in ST(\neg\rho_{i_l}[\text{sub}]) \text{ for all } \varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_C \text{ such that (5)} \right)$.

So, for all $\bar{v}_1, \dots, \bar{v}_l \left(R_{\otimes} \bar{v}_1 \dots \bar{v}_l \bar{v} \text{ or } \bar{v}_1 \in ST(\rho_{i_1})[\text{SUB}] \text{ or } \dots \text{ or } \bar{v}_{l'} \in ST(\rho_{i_{l'}})[\text{SUB}] \text{ or } \bar{v}_{l'+1} \in ST(\neg\rho_{i_{l'+1}})[\text{SUB}] \text{ or } \dots \text{ or } \bar{v}_l \in ST(\neg\rho_{i_l})[\text{SUB}] \right)$ by Induction Hypothesis.

Thus, $((\mathfrak{F}^c, V_m), \bar{v}) \Vdash ST(\otimes(\rho_1, \dots, \rho_m))$ i.e. $\bar{v} \in ST(\otimes(\rho_1, \dots, \rho_m))[\text{SUB}]$. \square

a. For better readability, let us denote j_i by j . We define $V_m(P_j) \triangleq \left\{ \bar{v}_{i_j} \mid \exists \bar{v}_1 \dots \bar{v}_{n_j} (R_{\otimes^j} \bar{v}_1 \dots \bar{v}_{n_j} \bar{w}_j \wedge \bar{v}_1 \in ST(\pi_1) \wedge \dots \wedge \bar{v}_{i_j-1} \in ST(\pi_{i_j-1}) \wedge \bar{v}_{i_j+1} \notin ST(\nu_{i_j+1}) \wedge \dots \wedge \bar{v}_{n_j} \notin ST(\nu_{n_j}))[\text{SUB}] \right\}$ where $\pi_1, \dots, \pi_{i_j-1}$ are positive and $\nu_{i_j+1}, \dots, \nu_{n_j}$ are negative formulas of \mathcal{L}_C . We are going to prove the following lemma.

Fact 3. *Let $j = j_i$. Then,*

$$V_m(P_j) = \bigcap \left\{ \llbracket \varphi \rrbracket^{\mathfrak{M}^c} \mid \bar{w}_j \in \llbracket \otimes^j(\neg\pi_1, \dots, \varphi, \dots, \neg\nu_{n_j})[p_{j_1}/\varphi_{j_1}, \dots, p_{j_{i-1}}/\varphi_{j_{i-1}}] \rrbracket^{\mathfrak{M}^c} \right. \\ \left. \text{for some } \varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_C \text{ such that} \right. \\ \left. V_m(P_{j_1}) \subseteq \llbracket \varphi_{j_1} \rrbracket^{\mathfrak{M}^c} \text{ and } \dots \text{ and } V_m(P_{j_{i-1}}) \subseteq \llbracket \varphi_{j_{i-1}} \rrbracket^{\mathfrak{M}^c} \right\}. \quad (6)$$

That is, $\bar{v} \in V_m(p_{j_i})$ iff for all $\varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_C$, all $\varphi \in \mathcal{L}_C$, if $V_m(P_{j_1}) \subseteq \llbracket \varphi_{j_1} \rrbracket^{\mathfrak{M}^c}$ and \dots and $V_m(P_{j_{i-1}}) \subseteq \llbracket \varphi_{j_{i-1}} \rrbracket^{\mathfrak{M}^c}$ (5) and $\otimes^j(\neg\pi_1, \dots, \varphi, \dots, \neg\nu_{n_j})[\text{sub}] \in \bar{w}_j$ then $\varphi \in \bar{v}$.

Proof. (Fact 3) Left to right inclusion \subseteq . $\bar{v} \in V_m(P_j)$ iff there are $\bar{v}_1, \dots, \bar{v}_{n_j}$ such that $R_{\otimes^j} \bar{v}_1 \dots \bar{v}_{n_j} \bar{w}_j$ and $\bar{v}_1 \in ST(\pi_1)$ and \dots and $\bar{v}_{n_j} \notin ST(\nu_{n_j})$ where $P_{j_1}, \dots, P_{j_{i-1}}$ are interpreted in the canonical model by the sets $V_m(P_{j_1}), \dots, V_m(P_{j_{i-1}})$ respectively. So, there are $\bar{v}_1, \dots, \bar{v}_{n_j}$ such that $R_{\otimes^j} \bar{v}_1 \dots \bar{v}_{n_j} \bar{w}_j$ and for all $\varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_C$ such that (5), $\bar{v}_1 \in ST(\pi_1[\text{sub}])$ and \dots and $\bar{v}_{n_j} \notin ST(\nu_{n_j}[\text{sub}])$ by positivity of $\pi_1, \dots, \neg\nu_{n_j}$. Indeed, for all $l \in \llbracket 1; n_j \rrbracket$, we have that $\bar{v}_l \in ST(\pi_l)$ implies $\bar{v}_l \in ST(\pi_l[\text{sub}])$ for all $\varphi_{j_1}, \dots, \varphi_{j_{i-1}}$ such that (5) by positivity of π_l .

Thus, for all $\varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_C$ such that (5), there are $\bar{v}_1, \dots, \bar{v}_{n_j}$ such that $R_{\otimes^j} \bar{v}_1 \dots \bar{v}_{n_j} \bar{w}_j$ and $\bar{v}_1 \in ST(\pi_1[\text{sub}])$ and \dots and $\bar{v}_{n_j} \notin ST(\nu_{n_j}[\text{sub}])$. (7)

Hence, let φ and $\varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_C$ be such that (5) and assume that $\otimes^j(\neg\pi_1, \dots, \varphi, \dots, \neg\nu_{n_j})[\text{sub}]$. Then, by Fact 2, because we have $R_{\otimes^j} \bar{v}_1 \dots \bar{v}_{n_j} \bar{w}_j$, we also have $\neg\pi_1[\text{sub}] \in \bar{v}_1$ or \dots or $\varphi \in \bar{v}_{i_j}$ or \dots or $\neg\nu[\text{sub}] \notin \bar{v}_{n_j}$ i.e. $\bar{v}_1 \notin ST(\pi_1[\text{sub}])$ or \dots or $\bar{v}_{n_j} \in ST(\nu_{n_j}[\text{sub}])$ or $\varphi \in \bar{v}_{i_j}$. So, by (7), we have that $\varphi \in \bar{v}_{i_j}$.

Right to left inclusion \supseteq . Let \bar{v} be such that for all $\varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_C$ such that (5), for all $\varphi \in \mathcal{L}_C$, $\otimes^j(\neg\pi_1, \dots, \varphi, \dots, \neg\nu_{n_j})[\text{sub}] \in \bar{w}_j$ implies $\varphi \in \bar{v}$.

Let $\varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_C$ be such that (5). Assume that for all $\bar{v}_1, \dots, \bar{v}_{n_j}$, we have that $R_{\otimes^j} \bar{v}_1 \dots \bar{v}_{n_j} \bar{w}_j$ or $\bar{v}_1 \in ST(\neg\pi_1[\text{sub}])$ or \dots or $\bar{v}_{i_j-1} \in ST(\neg\pi_{i_j-1}[\text{sub}])$ or $\bar{v}_{i_j+1} \notin ST(\nu_{i_j+1}[\text{sub}])$ or \dots or $\bar{v}_{n_j} \notin ST(\nu_{n_j}[\text{sub}])$. Then, $(\mathfrak{M}^c, \bar{w}_j) \Vdash \otimes^j(\neg\pi_1, \dots, \neg\pi_{i_j-1}, \perp, \neg\nu_{i_j+1}, \dots, \neg\nu_{n_j})[\text{sub}]$. Therefore, by assumption, $\varphi \in \llbracket \perp \rrbracket^{\mathfrak{M}^c}$, which is impossible. So, for all $\varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_C$ such that (5), there are $\bar{v}_1, \dots, \bar{v}_{n_j}$ such that $R_{\otimes^j} \bar{v}_1 \dots \bar{v}_{n_j} \bar{w}_j$ and $\bar{v}_1 \in ST(\pi_1[\text{sub}])$ and \dots and $\bar{v}_{i_j-1} \in ST(\pi_{i_j-1}[\text{sub}])$ and $\bar{v}_{i_j+1} \notin ST(\nu_{i_j+1}[\text{sub}])$ and \dots and $\bar{v}_{n_j} \notin ST(\nu_{n_j}[\text{sub}])$.

Now, for all $\varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_C$ such that (5) and all $\varphi'_{j_1}, \dots, \varphi'_{j_{i-1}} \in \mathcal{L}_C$ such that (5), we have that $\varphi_{j_1} \wedge \varphi'_{j_1}, \dots, \varphi_{j_{i-1}} \wedge \varphi'_{j_{i-1}} \in \mathcal{L}_C$ fulfills (5) as well. So, for all $(\varphi_{j_1}^1, \dots, \varphi_{j_{i-1}}^1), \dots, (\varphi_{j_1}^n, \dots, \varphi_{j_{i-1}}^n)$

of \mathcal{L}_C such that for all $l \in \llbracket 1; n \rrbracket$, $\bar{v}_1 \in ST(\pi_1)[\text{sub}]$ and \dots and $\bar{v}_{i_j-1} \in ST(\pi_{i_j-1})[\text{sub}]$ and $\bar{v}_{i_j+1} \in ST(\neg\nu_{i_j+1})[\text{sub}]$ and \dots and $\bar{v}_{n_j} \in ST(\neg\nu_{n_j})[\text{sub}]$. So, by ω -saturation of the canonical model \mathfrak{M}^c , there exist $\bar{v}_1, \dots, \bar{v}_{n_j}$ such that $R_{\otimes^j} \bar{v}_1 \dots \bar{v}_{n_j} \bar{w}_j$ and for all $\varphi_{j_1}, \dots, \varphi_{j_{i-1}} \in \mathcal{L}_C$ such that (5) holds, we have that $\bar{v}_1 \in ST(\pi_1)[\text{sub}]$ and \dots and $\bar{v}_{i_j-1} \in ST(\pi_{i_j-1})[\text{sub}]$ and $\bar{v}_{i_j+1} \in ST(\neg\nu_{i_j+1})[\text{sub}]$ and \dots and $\bar{v}_{n_j} \in ST(\neg\nu_{n_j})[\text{sub}]$. Then, by Fact 2, there exist $\bar{v}_1, \dots, \bar{v}_{n_j}$ such that $\bar{v}_1 \in ST(\pi_1)[\text{SUB}]$ and \dots and $\bar{v}_{i_j-1} \in ST(\pi_{i_j-1})[\text{SUB}]$ and $\bar{v}_{i_j+1} \in ST(\neg\nu_{i_j+1})[\text{SUB}]$ and \dots and $\bar{v}_{n_j} \in ST(\neg\nu_{n_j})[\text{SUB}]$. That is, $\bar{v} \in V_m(P_{j_i})$. \square

So, we have proven the induction step and therefore a. and b. as well. \square

Theorem 7. *Let $L \triangleq (\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ be a Boolean atomic logic based on a Boolean common set \mathcal{C} of atomic connectives and a set \mathcal{E}_C of pointed \mathcal{C} -frames with D-information order. Let $\Sigma \subseteq \mathcal{L}_C$ be a set of inductive atomic formulas of L such that \mathcal{E}_C is the class of pointed \mathcal{C} -frames defined by the first-order correspondents of the inductive atomic formulas of Σ . Then, the calculus $\mathcal{P}_C + \Sigma + (\text{Uniform D-substitution})$ is sound and strongly complete for L , where (Uniform D-substitution) is the following rule:*

*For all $\chi \in \Sigma$ and all prop. letters $p_1, \dots, p_k \in D$ and $\alpha_1, \dots, \alpha_k \in \mathcal{L}_D$
and all prop. letters $q_1, \dots, q_l \in C - D$ and $\beta_1, \dots, \beta_l \in \mathcal{L}_C$: (Uniform D-substitution)
From χ , infer $\chi[p_1/\alpha_1, \dots, p_k/\alpha_k, q_1/\beta_1, \dots, q_l/\beta_l]$*

where $\chi[p_1/\alpha_1, \dots, p_k/\alpha_k, q_1/\beta_1, \dots, q_l/\beta_l]$ is the formula resulting from the uniform substitution in χ of p_1 by α_1 and \dots and p_k by α_k and q_1 by β_1 and \dots and q_l by β_l .

Proof. We take up the proof of Theorem 6 and follow the same method. We must (1) add to the canonical model \mathfrak{M}^c a D-information order \sqsubseteq^c , (2) prove that the minimal valuation V_m defined by Expressions (9) and (8) fulfills the Heredity condition with that D-information order \sqsubseteq^c and (3) prove that it still satisfies Conditions (a) and (b) in the proof of Lemma 2 on p. 50 but with formulas of \mathcal{L}_D instead of \mathcal{L}_C if $P_{j_i} \in D$ and with formulas of \mathcal{L}_C if $P_{j_i} \in C - D$ (because of our specific rule (Uniform D-substitution)).

1. We take up the definition of the canonical model \mathfrak{M}^c in [6] and define the preorder \sqsubseteq^c on W^c as follows: for all $\Gamma, \Gamma' \in W^c$, we set

$$\Gamma \sqsubseteq^c \Gamma' \text{ iff for all } \varphi \in \mathcal{L}_D \text{ we have that } \varphi \in \Gamma \text{ implies } \varphi \in \Gamma'$$

One can easily prove using the definition of the (canonical) relations \mathfrak{R}^c that this preorder \sqsubseteq^c is an information order, that is, it satisfies the Tonicity postulates.

2. We prove it by distinguishing two cases. The first case is when $V_m(P_{j_1})$ is defined by $\lambda t. (\bar{w}_{j_1}^1 \sqsubseteq^c t \vee \dots \vee \bar{w}_{j_1}^k \sqsubseteq^c t)$. It holds trivially: this set is closed under \sqsubseteq^c because it is transitive. The second case is when $V_m(P_{j_i})$ is defined by $\lambda t. ANT_{j_1}(P_{j_1}, \dots, P_{j_{i-1}}; P_{j_i})(t)$, it also holds because the i^{th} argument of the tonicity $\pm_{i_{j_i}}(\odot^{j_i})$ of \odot^{j_i} is $+$. So, in Expression (5), we can derive from $R_{\odot^{j_i}}^- \bar{v}_1^{j_i} \dots \bar{v}_{i_{j_i}}^{j_i} \dots \bar{v}_{n_{j_i}}^{j_i}$ and $\bar{v}_{i_{j_i}}^{j_i} \sqsubseteq^c \bar{w}_{i_{j_i}}^{j_i}$ and the Tonicity postulates (generalized to universal molecular connectives) that $R_{\odot^{j_i}}^- \bar{v}_1^{j_i} \dots \bar{w}_{i_{j_i}}^{j_i} \dots \bar{v}_{n_{j_i}}^{j_i}$. Hence, from $ANT_{j_1}(-; P_{j_1})(w)$ and $w \sqsubseteq^c v$, one can derive $ANT_{j_1}(P_{j_1}, \dots, P_{j_{i-1}}; P_{j_i})(v)$. That is from $w \in V_m(P_{j_i})$ and $w \sqsubseteq^c v$, we derive that $v \in V_m(P_{j_i})$, that is the Heredity condition for P_{j_i} .
3. To prove Conditions (a) and (b) of Lemma 2, we follow the same inductive reasoning as in the original proof. The only difference in the proof resides in the base case for Condition (a). In that case, $V_m(P_{j_1})$ is of one of the following three forms:

- (a) $V_m(P_{j_1}) = \{\bar{w} \mid \bar{w}_{j_1}^1 = \bar{w} \text{ or } \dots \text{ or } \bar{w}_{j_1}^k = \bar{w}\}$ or
- (b) $V_m(P_{j_1}) = \{\bar{w} \mid \bar{w}_{j_1}^1 \sqsubseteq^c \bar{w} \text{ or } \dots \text{ or } \bar{w}_{j_1}^k \sqsubseteq^c \bar{w}\}$ or
- (c) $V_m(P_{j_1}) = \{\bar{v} \mid R_{\otimes_{j_1}}^- \bar{v} \bar{w}_{j_1}\}$.

The first and third subcases are proven like in the proof of Lemma 2 on p. 50. In the second subcase, we are going to prove that $V_m(P_{j_1}) \triangleq \{\bar{w} \mid \bar{w}_{j_1}^1 \sqsubseteq^c \bar{w} \text{ or } \dots \text{ or } \bar{w}_{j_1}^k \sqsubseteq^c \bar{w}\} = \bigcap_{\varphi \in \mathcal{L}_D} \{\llbracket \varphi \rrbracket^{\mathfrak{M}^c} \mid \bar{w}_{j_1}^1 \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c} \text{ and } \dots \text{ and } \bar{w}_{j_1}^k \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c}\}$. The left to right inclusion follows from the fact that $\{\bar{w} \mid \bar{w}_{j_1}^1 \sqsubseteq^c \bar{w}\} \subseteq \bigcap_{\varphi \in \mathcal{L}_D} \{\llbracket \varphi \rrbracket^{\mathfrak{M}^c} \mid \bar{w}_{j_1}^1 \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c}\}$ by Proposition 2. Conversely, let $\bar{v} \in \bigcap_{\varphi \in \mathcal{L}_D} \{\llbracket \varphi \rrbracket^{\mathfrak{M}^c} \mid \bar{w}_{j_1}^1 \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c} \text{ and } \dots \text{ and } \bar{w}_{j_1}^k \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c}\}$ and assume towards a contradiction that we have neither $\bar{w}_{j_1}^1 \sqsubseteq^c \bar{v}$ nor \dots nor $\bar{w}_{j_1}^k \sqsubseteq^c \bar{v}$. Then, by our definition of \sqsubseteq^c , there are $\varphi^1, \dots, \varphi^k \in \mathcal{L}_D$ such that $\bar{w}_{j_1}^i \in \llbracket \varphi^i \rrbracket^{\mathfrak{M}^c}$ but $\bar{v} \notin \llbracket \varphi^i \rrbracket^{\mathfrak{M}^c}$ for all $i \in \{1, \dots, k\}$. We take $\varphi \triangleq \varphi^1 \vee \dots \vee \varphi^k \in \mathcal{L}_D$ (because $\vee \in C^-$). Then, $\bar{w}_{j_1}^1 \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c}$ and \dots and $\bar{w}_{j_1}^k \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c}$ but $\bar{v} \notin \llbracket \varphi \rrbracket^{\mathfrak{M}^c}$. This is impossible by our assumption. So, $V_m(P_{j_1}) \triangleq \{\bar{w} \mid \bar{w}_{j_1}^1 \sqsubseteq^c \bar{w} \text{ or } \dots \text{ or } \bar{w}_{j_1}^k \sqsubseteq^c \bar{w}\} = \bigcap_{\varphi \in \mathcal{L}_D} \{\llbracket \varphi \rrbracket^{\mathfrak{M}^c} \mid \bar{w}_{j_1}^1 \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c} \text{ and } \dots \text{ and } \bar{w}_{j_1}^k \in \llbracket \varphi \rrbracket^{\mathfrak{M}^c}\}$. \square