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Note on FastTwoSum with Directed Roundings

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Abstract

In [2], Graillat and Jézéquel prove a bound on the maximal error for the FastTwoSum algorithm with directed roundings. We improve that bound by a factor 2.

We recall the FastTwoSum algorithm (note there are several possible variants, we choose the one from [2, Algorithm 1]):

Algorithm 1 (FastTwoSum)

Input: *p*-bit floating-point numbers *a*, *b* with $|a| \ge |b|$ **Output:** *p*-bit floating-point numbers *x*, *y* such that x + y approximates a + b1: $x = \circ(a + b)$ 2: $z = \circ(x - a)$ 3: $y = \circ(b - z)$

It is well-known that FastTwoSum is exact, i.e., x + y equals a + b, when rounding is to nearest. However, the case of other rounding modes, in particular the IEEE 754 directed roundings (towards zero, towards $+\infty$ and towards $-\infty$) has been less studied. A precise error bound in that later case is mandatory to design corrected rounding routines as in the CORE-MATH project [4].

Here, we consider that all floating-point numbers have the same precision p, there is no underflow nor overflow. The roundings for the three additions/subtractions can be any faithful rounding: for example in $x = \circ(a + b)$, x is either the rounding towards $-\infty$ of a + b or the rounding towards $+\infty$ (in particular if a + b is exact there is only one possible result). This implies that the error x - (a + b) is bounded in two ways:

$$|x - (a + b)| < 2u|x|$$
 and $|x - (a + b)| < 2u|a + b|$,

where $u = 2^{-p}$ (and similarly for the two subtractions). Also, as in [2], we can have different roundings for the three operations, for example towards $+\infty$ for $x = \circ(a + b)$, to nearest for $z = \circ(x - a)$ (note that this subtraction is always exact, see for example Lemma 2.5 from [1]), and towards $-\infty$ for $y = \circ(b - z)$.

We recall the result from [2, Proposition 3.2]:

Proposition 1 [2, Proposition 3.2] Let x and y be the floating-point addition of a and b and the correction both computed by Algorithm 1 using directed rounding. Let e be the error on x: a + b = x + e. Then:

$$|e - y| \le 4u^2 |a + b|$$
 and $|e - y| \le 4u^2 |x|$

Assuming $2^{e_a-1} \leq |a| < 2^{e_a}$ and $2^{e_b-1} \leq |b| < 2^{e_b}$, we define the exponent of a to be e_a and that of b to be e_b , thus the exponent difference is $k := e_a - e_b$, which is non-negative since $|b| \leq |a|$. We prove the following improved result, where we denote the FastTwoSum error e - y by ε for simplicity:

Theorem 1 Let x and y be the output of Algorithm 1. Let ε be the corresponding error: $\varepsilon = (x + y) - (a + b)$. Then:

$$|\varepsilon| \le 2u^2 |a+b|$$
 and $|\varepsilon| \le 2u^2 |x|$.

Moreover, if the exponent difference between a and b does not exceed p, Algorithm FastTwoSum is error-free.

The second statement of Theorem 1 extends Lemma 2.6 from [1], which proves that FastTwoSum is exact when the exponent difference does not exceed p-1. The second statement also follows from Remark 1 of [3], which translates for radix $\beta = 2$ and our notations to: if $e_b \leq e_a \leq e_b + p$, and the first rounding $x = \circ(a + b)$ is faithful, then FastTwoSum is exact. Nevertheless, to be self-content, we give an independent proof.

Proof: We first prove the second statement of the theorem. We can assume without lack of generality that a is non-negative, and ulp(a) = 1, i.e., $2^{p-1} \le a < 2^p$, thus a is an integer. Taking the same notations as in [2], we denote by e the error in x = o(a+b): x = a+b-e. By Lemma 2.5 from [1], we know that the second operation z = o(x - a) is exact, thus z = b - e. It follows that y = o(e). By definition of the rounding, we have |e| < ulp(x). Let $k \le p$ be the exponent difference between a and b, then b is an integer multiple of 2^{-k} . Since a is also an integer multiple of 2^{-k} , so is x = o(a + b). Then e = a + b - x is an integer multiple of 2^{-k} : $e = m \cdot 2^{-k}$ with m integer. We now distinguish two cases: $ulp(x) \le 1$ or ulp(x) = 2 (ulp(x) cannot be larger than 2 since $|b| \le |a|$). If $ulp(x) \le 1$, then |e| < ulp(x) yields $|m| < 2^k$, thus e is exactly representable in precision p, thus y = e. If ulp(x) = 2 and k < p, then |e| < ulp(x) yields $|m| < 2^{k+1} \le 2^p$, thus again e is exactly representable in precision p, and y = e. The last case ulp(x) = 2 and k = p can only occur when 0 < b < 1 and $a = 2^p - 1$. In that case $x = 2^p = a + 1$, z = 1, thus y = o(b - 1). Since b is an integer multiple of 2^{-p} , we have $b = m \cdot 2^{-p}$ with $2^{p-1} \le m < 2^p$, thus $b - z = (m - 2^p) \cdot 2^{-p}$ with $-2^{p-1} \le m - 2^p < 0$. Again, b - z is exactly representable, thus y = e.

To prove the first statement of the Theorem, we can thus assume the exponent difference k is at least p + 1, otherwise $\varepsilon = 0$. This means that |b| < ulp(a)/2 = 1/2, thus x is either a, a + 1, a - 1 or a - 1/2 (the latter case can only occur when $a = 2^{p-1}$).

- If x = a, then z = 0 and y = b, thus FastTwoSum is exact.
- If x = a + 1, then z = 1, and $y = \circ(b-1)$; this can only occur when b > 0. Since 0 < b < 1/2, we have -1 < b-1 < -1/2, thus when computing $y = \circ(b-z) = \circ(b-1)$ the rounding error is less than $ulp(1/2) = 2^{-p} = u$. Thus $|\varepsilon| < u$, and since $|x| \ge |a+b| \ge 2^{p-1} = 1/(2u)$, we have $1 \le 2u|a+b| \le 2ux$, thus $|\varepsilon| < 2u^2|a+b| \le 2u^2x$.
- If x = a 1, then z = -1, and $y = \circ(b + 1)$; this can only occur when b < 0. Since -1/2 < b < 0, we have 1/2 < b + 1 < 1, thus when computing $y = \circ(b z) = \circ(b + 1)$ the rounding error is less than $ulp(1/2) = 2^{-p} = u$. Thus $|\varepsilon| < u$. This case can only occur when $a > 2^{p-1}$, otherwise if $a = 2^{p-1}$, the *p*-bit number a 1/2 would be closer to a b. Thus $|x|, |a + b| \ge 2^{p-1} = 1/(2u)$, and we get the desired bound as above.

• The last case $a = 2^{p-1}$ and x = a - 1/2 can only occur when b < 0. Then z = -1/2, and thus $y = \circ(b + 1/2), -1/2 < b < 0$, thus 0 < b + 1/2 < 1/2. When computing $y = \circ(b - z) = \circ(b + 1/2)$ the rounding error is less than $ulp(1/4) = 2^{-p-1} = u/2$. Thus $|\varepsilon| < u/2$, and since $|a + b| \ge |x| \ge 2^{p-2} = 1/(4u)$, we deduce $1 \le 4u|x| \le 4u|a + b|$, and thus $|\varepsilon| < 2u^2|x| \le 2u^2|a + b|$.

The bound of Theorem 1 is tight: if we consider $a = 2^{p-1}$, $b = 2^{p-1-k}$ for k > p, and rounding towards $+\infty$ for all operations, we get x = a + 1, thus z = 1, and $y = \circ(2^{p-1-k} - 1) = -1 + 2^{-p}$, thus $x + y = 2^{p-1} + 2^{-p}$ whereas $a + b = 2^{p-1} + 2^{p-1-k}$. The error is $\varepsilon = 2^{-p} - 2^{p-1-k}$: $\varepsilon/|x|$ and $\varepsilon/|a + b|$ are very close to $2u^2$. This example also shows the tightness of the bound when all roundings are the same.

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