



HAL
open science

Kantorovich-Rubinstein Quasi-Metrics III: Spaces of Sublinear and Superlinear Previsions

Jean Goubault-Larrecq

► **To cite this version:**

Jean Goubault-Larrecq. Kantorovich-Rubinstein Quasi-Metrics III: Spaces of Sublinear and Superlinear Previsions. *Topology and its Applications*, 2022, pp.108259. <10.1016/j.topol.2022.108259>. <hal-03787192>

HAL Id: hal-03787192

<https://inria.hal.science/hal-03787192v1>

Submitted on 24 Sep 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Kantorovich-Rubinstein Quasi-Metrics III: Spaces of Sublinear and Superlinear Previsions

Jean Goubault-Larrecq

*Université Paris-Saclay, CNRS, ENS Paris-Saclay, Laboratoire Méthodes Formelles,
91190, Gif-sur-Yvette, France.*

goubault@lsv.fr

Abstract

Previsions are certain kinds of functionals which model mixtures of probabilistic and non-deterministic choice. For every continuous complete quasi-metric X, d , we show that the spaces of (sub)normalized sublinear previsions and of (sub)normalized superlinear previsions on X , with the Kantorovich-Rubinstein quasi-metrics d_{KR}^a , are continuous complete. Additionally, the d_{KR}^a -Scott topology coincides with the weak topology. If X, d is algebraic complete, then we show that those spaces of previsions are algebraic complete, too, and that finite pointwise suprema (resp., infima) of simple valuations form a strong basis.

Keywords: Quasi-metric, prevision, Kantorovich-Rubinstein quasi-metric, Vietoris topology (lower, upper)

2020 MSC: 54E99, 54H30

1. Introduction

A prevision on a space X is a certain kind of functional F , mapping lower semi-continuous functions $h: X \rightarrow \overline{\mathbb{R}}_+$ to an extended non-negative real number $F(h) \in \overline{\mathbb{R}}_+$ [4]. If X is equipped with a quasi-metric d , then one can equip the space $\mathbf{P}X$ of all previsions on X with a quasi-metric d_{KR}^a , defined by the formula:

$$d_{\text{KR}}^a(G, G') \stackrel{\text{def}}{=} \sup_{h \in \mathcal{L}_1^a X} \mathbf{d}_{\mathbb{R}}(G(h), G'(h)), \quad (1)$$

for any $a > 0$, where $\mathcal{L}_1^a X$ denotes the set of a -bounded 1-Lipschitz continuous maps from X to $\overline{\mathbb{R}}_+$, and $\mathbf{d}_{\mathbb{R}}(x, y) \stackrel{\text{def}}{=} \max(x - y, 0)$. (We use the

convention that $d_{\mathbb{R}}(x, y) = 0$ when $x = y$, even when $x = y = +\infty$.) This notion, as well as a closely related quasi-metric d_{KR} , was studied in Part I [11], where we concentrated on (sub)normalized *linear* previsions, or equivalently (sub)normalized continuous valuations (a close cousin of the notion of (sub)probability measures); and in Part II [12], where we concentrated on *discrete* previsions, or equivalently certain hyperspaces and powerdomains. To be precise, in Part II, we studied the equivalent of the Hoare and Smyth powerdomains, respectively the spaces of non-empty closed subsets of X , and of non-empty compact saturated subsets of X , corresponding to so-called discrete sublinear previsions, and discrete superlinear previsions.

In each case, and up to a few technical points which we will make precise in Section 2, we showed that for every continuous (resp., algebraic) complete quasi-metric space X, d , the corresponding space of previsions is continuous (resp., algebraic) complete with respect to the d_{KR} and the d_{KR}^a quasi-metrics. Moreover, the natural topology induced by the quasi-metric, which we claim is the d_{KR}^a -Scott (resp., d_{KR} -Scott) topology, to be recalled below, coincides with natural topologies: the weak topology in the case of spaces of (sub)normalized linear previsions, the lower and upper Vietoris topologies respectively for hyperspaces.

Our aim is to show similar results for spaces of (sub)normalized sublinear previsions and (sub)normalized superlinear previsions, without the requirement of discreteness¹. The resulting spaces are natural models of mixed non-deterministic and probabilistic choice in denotational semantics [6]. We deal with sublinear previsions in Section 3, and with superlinear previsions in Section 4.

2. Preliminaries

We refer the reader to [3, 5] for general information on topology and domain theory. The second reference contains some required information on quasi-metric space theory, for which the reader is advised to read [13], as well as [11, 12].

¹An unpublished version of the present work is available on arXiv [7]. The present paper covers Sections 14.2 and 14.3.

2.1. Topology

A *directed complete partial order*, or *dcpo*, is a partially ordered set P in which every directed family has a supremum. A subset U of P is *Scott-open* if and only if it is upwards-closed, and every directed family whose supremum is in U intersects U . The Scott-open subsets form the *Scott topology*. A *Scott-continuous* map between two dcpos is a monotonic map that preserves directed suprema; equivalently, it is a continuous map between the two dcpos equipped with their Scott topology.

A compact subset of a topological space X is one such that one can extract a finite subcover from any open cover; we do not assume any form of separation. A subset A of X is *saturated* if and only if it is the intersection of its open neighborhoods, if and only if A is upwards closed in the specialization ordering of X .

A *stably compact* space is a sober, locally compact, compact, and coherent space, where coherence means that the intersection of any two compact saturated subsets is compact (and saturated). The *cocompact topology* on a stably compact space X has the compact saturated subsets of X as closed subsets. We write X^d for X with its cocompact topology. The *patch topology* on a stably compact space X is the join of the original and cocompact topologies; it turns X into a compact Hausdorff space X^{patch} .

A simple example is $\overline{\mathbb{R}}_+$, the set of extended non-negative reals, with the Scott topology of its usual ordering. Apart from $\overline{\mathbb{R}}_+$ and the empty set, the open subsets of $\overline{\mathbb{R}}_+$ are the intervals $]a, +\infty]$, $a \in \mathbb{R}_+$. This is a stably compact space, whose compact saturated subsets are the empty set and the intervals $[a, +\infty]$, $a \in \overline{\mathbb{R}}_+$. Then $(\overline{\mathbb{R}}_+)^{\text{patch}}$ has the usual Hausdorff topology.

A *retraction* of Y onto X in a given category is a pair of morphisms:

$$Y \begin{array}{c} \xrightarrow{r} \\ \xleftarrow{s} \end{array} X$$

such that $r \circ s = \text{id}_X$. The morphism s is the *section*, and r itself will often be called a retraction; then X is a *retract* of Y . A *topological retract* is a retract in the category of topological spaces and continuous maps. An *order-retract* is a retract in the category of partially ordered sets and monotonic maps. A *Scott retract* is a retract in the category of dcpos and Scott-continuous maps. We will also use a notion of 1-Lipschitz retract, which we will introduce later.

2.2. Quasi-Metrics

A *quasi-metric* on a set X is a map $d: X \times X \rightarrow \overline{\mathbb{R}}_+$ satisfying: $d(x, x) = 0$; $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequality); $d(x, y) = d(y, x) = 0$ implies $x = y$. The pair X, d is then called a *quasi-metric space*. Its *specialization ordering* is given by $x \leq^d y$ if and only if $d(x, y) = 0$; the name comes from the fact that \leq^d is the specialization ordering of X in several topologies, notably the open ball topology and the d -Scott topology (see below).

We will always see $\overline{\mathbb{R}}_+$ as a quasi-metric space, with quasi-metric $\mathbf{d}_{\mathbb{R}}$ defined by $\mathbf{d}_{\mathbb{R}}(x, y) \stackrel{\text{def}}{=} 0$ if $x \leq y$, $\mathbf{d}_{\mathbb{R}}(+\infty, y) \stackrel{\text{def}}{=} +\infty$ if $y \neq +\infty$, $\mathbf{d}_{\mathbb{R}}(x, y) \stackrel{\text{def}}{=} x - y$ if $x > y$ and $x \neq +\infty$. Its specialization ordering is the usual ordering.

A *formal ball* on X, d is a pair $(x, r) \in X \times \mathbb{R}_+$; x is the *center* and r is the *radius*. Formal balls are organized into a poset $\mathbf{B}(X, d)$ by letting $(x, r) \leq^{d^+} (y, s)$ if and only if $d(x, y) \leq r - s$ [2, 15]. There is also a quasi-metric d^+ on $\mathbf{B}(X, d)$, defined by $d^+((x, r), (y, s)) \stackrel{\text{def}}{=} \max(d(x, y) - r + s, 0)$, and \leq^{d^+} is its specialization ordering.

A quasi-metric space is *complete* (or *Yoneda-complete*) if and only if $\mathbf{B}(X, d)$ is a dcpo. It is *continuous complete* if and only if $\mathbf{B}(X, d)$ is a continuous dcpo. There is a third, stronger notion of completeness, algebraic completeness, which we will describe below. All three notions coincide on metric spaces.

Given a quasi-metric space X, d , there is an injective map $\eta_X: x \mapsto (x, 0)$ from X to $\mathbf{B}(X, d)$, and the *d -Scott topology* on X is the coarsest that makes it continuous. Unless specified otherwise, this is the topology we will assume for every quasi-metric space. The specialization ordering of the d -Scott topology is \leq^d . When d is a metric, the d -Scott topology coincides with the familiar open ball topology. The $\mathbf{d}_{\mathbb{R}}$ -Scott topology on $\overline{\mathbb{R}}_+$ coincides with the Scott topology of its ordering.

A quasi-metric space X, d is *standard* if and only if, for every directed family of formal balls $(x_i, r_i)_{i \in I}$, for every $s \in \mathbb{R}_+$, $(x_i, r_i)_{i \in I}$ has a supremum in $\mathbf{B}(X, d)$ if and only if $(x_i, r_i + s)_{i \in I}$ has a supremum in $\mathbf{B}(X, d)$. Writing the supremum of the former as (x, r) , we then have that $r = \inf_{i \in I} r_i$, and the supremum of the latter is $(x, r + s)$ —this holds not only for $s \in \mathbb{R}_+$, but for every $s \geq -r$. Every metric space is standard, and all complete quasi-metric spaces are standard.

A quasi-metric space X, d is *algebraic complete* if and only if it is complete and every point is the d -limit of a Cauchy net of d -finite points. We will

not use that definition, and instead we will use the following. We write $B_{x,r}^d \stackrel{\text{def}}{=} \{y \in X \mid d(x,y) < r\}$ for the open ball centered at x with radius r in X . We consider open balls $B_{(x,0),\epsilon}^{d^+}$ in $\mathbf{B}(X,d)$. If those open balls are Scott-open for every $\epsilon > 0$, then we say that x is a *center point*. For example, the center points of $\overline{\mathbb{R}}_+$ are the points of \mathbb{R}_+ ; every point in a metric space is a center point. When X,d is standard, the d -finite points are exactly the center points [13, Lemma 5.7]. Then, a quasi-metric space X,d is *algebraic* if and only if it is standard and for every $x \in X$, $(x,0)$ is the supremum of some directed family of formal balls (x_i,r_i) where each x_i is a center point. For example, $\overline{\mathbb{R}}_+, d_{\mathbb{R}}$ is algebraic complete.

Refining this, a *strong basis* \mathcal{B} of a standard quasi-metric space X,d is a set of center points such that for every $x \in X$, $(x,0)$ is the supremum of some directed family of formal balls (x_i,r_i) where each x_i is in \mathcal{B} . A standard quasi-metric space is algebraic if and only if it has a strong basis. The largest one is then the collection of all center points.

2.3. Lipschitz and Lipschitz continuous functions

For every $\alpha \in \mathbb{R}_+$, and for every map $f: X,d \rightarrow Y,\partial$ between quasi-metric spaces, let $\mathbf{B}^\alpha(f)$ map $(x,r) \in \mathbf{B}(X,d)$ to $(f(x),\alpha r) \in \mathbf{B}(Y,\partial)$. Then f is α -Lipschitz if and only if $\mathbf{B}^\alpha(f)$ is monotonic, and α -Lipschitz continuous if and only if $\mathbf{B}^\alpha(f)$ is Scott-continuous [9, Definition 2.3]. The function f is α -Lipschitz if and only if $\partial(f(x),f(y)) \leq \alpha d(x,y)$ for all $x,y \in X$. When both X,d and Y,∂ are standard, $f: X \rightarrow Y$ is α -Lipschitz continuous if and only if f is both α -Lipschitz, and continuous from X with its d -Scott topology to Y with its ∂ -Scott topology [9, Proposition 2.5].

A useful criterion for α -Lipschitz continuity for $\overline{\mathbb{R}}_+$ -valued maps is given by Lemma 2.7 of [9]: if X,d is standard, then $h: X,d \rightarrow \overline{\mathbb{R}}_+, d_{\mathbb{R}}$ is α -Lipschitz continuous if and only if $h': \mathbf{B}(X,d) \rightarrow \mathbb{R} \cup \{+\infty\}$, $h'(x,r) \stackrel{\text{def}}{=} h(x) - \alpha r$, is Scott-continuous.

Given any standard quasi-metric space X,d , every α -Lipschitz continuous map from X,d to $\overline{\mathbb{R}}_+$ is lower semicontinuous. We write $\mathcal{L}_\alpha(X,d)$, or $\mathcal{L}_\alpha X$ for short, for the subspace of $\mathcal{L}X$ consisting of α -Lipschitz continuous maps, with the topology induced by the Scott topology from $\mathcal{L}X$. Note that this may or may not coincide with the Scott topology on $\mathcal{L}_\alpha(X,d)$. Let $\mathcal{L}_\infty(X,d) \stackrel{\text{def}}{=} \bigcup_{\alpha>0} \mathcal{L}_\alpha(X,d)$, also with the subspace topology induced from $\mathcal{L}X$.

We write $\mathcal{L}_\alpha^a(X,d)$, or $\mathcal{L}_\alpha^a X$, for the subspace of $\mathcal{L}_\alpha X$ of those α -Lipschitz continuous maps from X,d to $\overline{\mathbb{R}}_+$ that take their values in $[0,\alpha a]$. We let

$\mathcal{L}_\infty^b(X, d) \stackrel{\text{def}}{=} \bigcup_{\alpha > 0} \mathcal{L}_\alpha^a(X, d)$ be the subspace of all bounded maps in $\mathcal{L}_\infty(X, d)$.

An important result, which we will use several times, is the following.

Theorem 2.1 (Lemma 7.3, item 4, and Corollary 7.7, item 1, of [10]).

Given any continuous complete quasi-metric space X, d , and any number $a > 0$, $\mathcal{L}_\alpha^a(X, d)$ and $\mathcal{L}_\alpha(X, d)$ are stably compact. For every $n \in \mathbb{N}$, for all $a_1, \dots, a_n \in \mathbb{R}_+$, and for all center points x_1, \dots, x_n of X, d , the map $f \mapsto \sum_{i=1}^n a_i f(x_i)$ is continuous from $\mathcal{L}_\alpha(X, d)^d$ and from $\mathcal{L}_\alpha^a(X, d)^d$ to $(\overline{\mathbb{R}_+})^d$.

We will call *1-Lipschitz continuous retraction* any retraction in the category of standard quasi-metric spaces and 1-Lipschitz continuous maps. The main use of this notion is the following result [13, Theorem 7.9]: the continuous complete quasi-metric spaces are exactly the 1-Lipschitz continuous retracts of algebraic complete quasi-metric spaces.

2.4. Previsions

We write $\mathcal{L}X$ for the space of lower semicontinuous maps from a space X to $\overline{\mathbb{R}_+}$. Elements of $\mathcal{L}X$ are ordered pointwise, and $\mathcal{L}X$ is topologized with the Scott topology of that ordering.

A *prevision* on a topological space X is a Scott-continuous map F from $\mathcal{L}X$ to $\overline{\mathbb{R}_+}$ that is *positively homogeneous*, namely: $F(\alpha h) = \alpha F(h)$ for all $\alpha \in \mathbb{R}_+$, $h \in \mathcal{L}X$. It is:

- *sublinear* if $F(h + h') \leq F(h) + F(h')$ holds for all $h, h' \in \mathcal{L}X$,
- *superlinear* if $F(h + h') \geq F(h) + F(h')$ holds,
- *linear* if it is both sublinear and superlinear,
- *subnormalized* if $F(\alpha \mathbf{1} + h) \leq \alpha + F(h)$ holds, where $\mathbf{1}$ is the constant 1 map,
- *normalized* if $F(\alpha \mathbf{1} + h) = \alpha + F(h)$ holds,
- *discrete* if $F(f \circ h) = f(F(h))$ for every $h \in \mathcal{L}X$ and every strict lower semicontinuous map $f \in \mathcal{L}\overline{\mathbb{R}_+}$ —a map f is strict if and only if $f(0) = 0$.

The previsions of the form $h \mapsto \int_{x \in X} h(x) d\nu$, where ν is a continuous valuation on X , are all linear, and those are the only linear previsions. Continuous valuations are a close cousin to the notion of measure, and coincide with them

on large classes of spaces [1]. We write $\mathbf{V}X$ for the space of linear previsions on X .

Among them, and equating linear previsions with continuous valuations, the *Dirac valuations* δ_x are the maps $h \mapsto h(x)$. The *simple valuations* are the finite non-negative linear combinations of Dirac valuations $\sum_{i=1}^n a_i \delta_{x_i}$, namely the continuous valuations of the form $h \mapsto \sum_{i=1}^n a_i h(x_i)$, where $n \in \mathbb{N}$, $a_i \in \mathbb{R}_+$, and $x_i \in X$. Given a subset \mathcal{B} of X , we will say that a simple valuation is *supported on \mathcal{B}* if and only if for every i such that $a_i \neq 0$, the point x_i is in \mathcal{B} .

The space $\mathbf{P}X$ of all previsions on X is given the *weak topology*, defined as the coarsest topology such that $F \mapsto F(h)$ is lower semicontinuous for every $h \in \mathcal{L}X$. It has subbasic open sets of the form $[h > a] \stackrel{\text{def}}{=} \{F \in \mathbf{P}X \mid F(h) > a\}$, for each $h \in \mathcal{L}X$ and $a \in \mathbb{R}_+$. All the subspaces of previsions that we will consider will have the corresponding subspace topology, which we again call the weak topology, and whose subbasic open sets will again be written as $[h > a]$.

Defining $\mathbf{P}f$ for every continuous map f by $\mathbf{P}f(F)(h) \stackrel{\text{def}}{=} F(h \circ f)$ turns \mathbf{P} into an endofunctor on the category of topological spaces.

The *Hoare hyperspace* \mathcal{H}_0X (resp., $\mathcal{H}X$) is the space of all (resp., non-empty) closed subsets of X , with the *lower Vietoris topology*. That topology is generated by subbasic open sets $\diamond U = \{C \in \mathcal{H}_0X \mid C \cap U \neq \emptyset\}$, where U ranges over the open subsets of X . The map $C \mapsto F^C$, where $F^C(h) \stackrel{\text{def}}{=} \sup_{x \in C} h(x)$ for every $h \in \mathcal{L}X$, is a homeomorphism of \mathcal{H}_0X (resp., $\mathcal{H}X$) onto the space of discrete sublinear (resp., and normalized) previsions on X with the weak topology [12, Lemma 3.4]. This is the object part of a functor defined on morphisms by $\mathcal{H}_0(f)(C) \stackrel{\text{def}}{=} cl(f[C])$, where cl is closure and $f[C]$ is the image of C under f .

Symmetrically, the *Smyth hyperspace* $\mathcal{Q}X$ is the space of all non-empty compact saturated subsets of X , with the *upper Vietoris topology*, generated by basic open sets $\square U \stackrel{\text{def}}{=} \{Q \in \mathcal{Q}X \mid Q \subseteq U\}$, where U ranges over the open subsets of X . When X is sober, the map $Q \mapsto F_Q$, where $F_Q(h) \stackrel{\text{def}}{=} \min_{x \in Q} h(x)$, is a homeomorphism of $\mathcal{Q}X$ onto the space of normalized discrete superlinear previsions on X with the weak topology [12, Lemma 3.6]. This is the object part of a functor defined on morphisms by $\mathcal{Q}(f)(Q) \stackrel{\text{def}}{=} \uparrow f[Q]$, where \uparrow denotes upward closure in the specialization preordering of the target space.

2.5. Quasi-metrics on spaces of previsions

We have already defined the quasi-metric d_{KR}^a in Equation (1). Now that we have defined $\mathcal{L}_1^a X$ formally, this makes sense. The variant d_{KR} is defined by the following formula, where $\mathcal{L}_1^a X$ is replaced by $\mathcal{L}_1 X$.

$$d_{\text{KR}}(G, G') \stackrel{\text{def}}{=} \sup_{h \in \mathcal{L}_1 X} d_{\mathbb{R}}(G(h), G'(h)). \quad (2)$$

The definition of d_{KR} only requires the knowledge of the values of each prevision G, G' on elements h of $\mathcal{L}_{\infty}(X, d)$ (even of just $\mathcal{L}_1(X, d)$). This leads one to define a notion of \mathcal{L} -*prevision* on X , namely any positively homogeneous, continuous map from $\mathcal{L}_{\infty}(X, d)$ to $\overline{\mathbb{R}}_+$ [12, Definition 4.3]. Notions of sublinearity, discreteness, etc., carry over to \mathcal{L} -previsions; for discreteness, one must require f to be strict and in $\mathcal{L}_{\infty}\overline{\mathbb{R}}_+$, not just $\mathcal{L}\overline{\mathbb{R}}_+$. Then, and provided that X, d is standard, there is a bijection $G \mapsto \overline{G}$ from $\mathbf{P}X$ onto $\mathcal{L}\mathbf{P}X$, whose inverse is given by restriction to $\mathcal{L}_{\infty}(X, d)$; that bijection preserves and reflects all the properties of sublinearity, superlinearity, linearity, normalization, subnormalization, and discreteness [12, Proposition 4.5]. Defining d_{KR} on $\mathcal{L}\mathbf{P}X$ by the same formula (2), this bijection turns into an isometry, namely a distance-preserving isomorphism between $\mathbf{P}X$ and $\mathcal{L}\mathbf{P}X$, as well as between their subspaces of sublinear, discrete, etc., previsions.

In order to show that a given space of previsions on a given standard quasi-metric space X, d is complete with respect to d_{KR} , it is therefore equivalent to show that the corresponding isometric subspace Y of $\mathcal{L}\mathbf{P}X$ is complete with respect to d_{KR} , namely that $\mathbf{B}(Y, d_{\text{KR}})$ is a dcpo. In other words, we consider an arbitrary directed family $(G_i, r_i)_{i \in I}$ in $\mathbf{B}(Y, d_{\text{KR}})$, and we must show that it has a supremum. A natural candidate is the *naive supremum* (G, r) of the family, where:

$$G(h) \stackrel{\text{def}}{=} \sup_{i \in I} (G_i(h) + \alpha r - \alpha r_i) \quad (3)$$

for every $\alpha > 0$ and every $h \in \mathcal{L}_{\alpha} X$, and $r \stackrel{\text{def}}{=} \inf_{i \in I} r_i$. This defines G on $\mathcal{L}_{\infty} X = \bigcup_{\alpha > 0} \mathcal{L}_{\alpha} X$, and one can check that the definition does not actually depend on the chosen α such that h is in $\mathcal{L}_{\alpha} X$.

Proposition 4.7 of [12] states that (G, r) is indeed the required supremum, if G defines a continuous map from $\mathcal{L}_{\infty} X$ to $\overline{\mathbb{R}}_+$ with its Scott topology. (We recall that X, d is assumed to be standard.) In order to obtain our

completeness results of Part I [11] and II [12], we systematically chose this route.

A similar approach works with d_{KR}^a . We must consider the space $\mathcal{L}^b\mathbf{P}X$ of \mathcal{L}^b -previsions instead, namely positively homogeneous continuous map from $\mathcal{L}_\infty^b X$ to $\overline{\mathbb{R}}_+$. We define naive suprema of directed families in $\mathbf{B}(Y, d_{\text{KR}}^a)$ by the same formula (3), where h now ranges over $\mathcal{L}_\alpha^a X$, $\alpha > 0$. Assuming X, d standard, Proposition 4.15 of [12] states that if G is continuous from $\mathcal{L}_\infty^b(X, d)$ to $\overline{\mathbb{R}}_+$, then the naive supremum (G, r) is the desired supremum.

The main results of [11] that we will require are the following. This synthesizes Theorems 4.15, 8.5, 9.5, 5.7, 6.1, Corollary 8.3 and Corollary 8.6 there.

Theorem 2.2 (Completeness for linear previsions [11]). *For every continuous complete quasi-metric space X, d , the spaces $\mathbf{V}_{\leq 1}X$ and \mathbf{V}_1X are continuous complete with respect to the d_{KR}^a quasi-metric ($a > 0$). Directed suprema of formal balls are computed as naive suprema. The d_{KR}^a -Scott topology coincides with the weak topology.*

If X, d is algebraic complete, then $\mathbf{V}_{\leq 1}X$ and \mathbf{V}_1X are algebraic complete under d_{KR}^a . The simple (sub)normalized valuations supported on \mathcal{B} form a strong basis, for any given strong basis \mathcal{B} of X, d .

$\mathbf{V}_{\leq 1}, d_{\text{KR}}^a$ and $\mathbf{V}_1, d_{\text{KR}}^a$ define endofunctors on the category of continuous complete quasi-metric spaces and 1-Lipschitz continuous maps, whose action on morphisms is that of \mathbf{P} : for every 1-Lipschitz continuous map $f: X, d \rightarrow Y, \partial$ between continuous complete quasi-metric spaces, $\mathbf{P}f$ is 1-Lipschitz continuous from $\mathbf{V}_{\leq 1}X, d_{\text{KR}}^a$ to $\mathbf{V}_{\leq 1}Y, \partial_{\mathcal{H}}^a$ and from $\mathbf{V}_1X, d_{\text{KR}}^a$ to $\mathbf{V}_1Y, \partial_{\mathcal{H}}^a$.

There are analogous results with the d_{KR} quasi-metric, but which require additional assumptions. We only consider the d_{KR}^a quasi-metrics on spaces of sublinear and superlinear previsions, not d_{KR} , here.

On discrete previsions, the most natural quasi-metric is d_{KR} , not d_{KR}^a . Correspondingly, the main results of [12] are as follows. We separate the cases of the Hoare and the Smyth hyperspaces.

Let X, d be a quasi-metric space. On the Hoare hyperspace \mathcal{H}_0X (where X has the d -Scott topology), the natural definition of quasi-metric is $d_{\mathcal{H}}(C, C') \stackrel{\text{def}}{=} \sup_{x \in C} d(x, C')$, where the supremum of an empty set is taken to be 0 [12, Definition 5.3]. One would expect $d(x, C')$ to be defined as $\inf_{y \in C'} d(x, y)$ (where the infimum of an empty set is $+\infty$), and while this holds in several

important cases, the actual definition is as the supremum of those $r \in \mathbb{R}_+$ such that $(x, r) \in \widehat{U}$, where U is the complement of C and \widehat{U} is the largest Scott-open subset of $\mathbf{B}(X, d)$ such that $\eta_X^{-1}(\widehat{U}) = U$. We will not need to understand this at that level of detail. We will need to know that, given that X, d is standard, the specialization ordering of $d_{\mathcal{H}}$ is inclusion, namely that $d_{\mathcal{H}}(C, C') = 0$ if and only if $C \subseteq C'$ [12, Lemma 5.7, item 2], and the following results, which summarize Proposition 5.8, Remark 5.9, Theorem 5.25, Proposition 5.12, Remark 5.13, Theorem 5.30, Theorem 5.22, and Corollary 5.26 of [12] in this order.

Theorem 2.3 (Completeness for the Hoare hyperspaces [12]). *For every standard quasi-metric space X, d , the map $C \mapsto F^C$ is an isometry between $\mathcal{H}_0X, d_{\mathcal{H}}$ (resp., $d_{\mathcal{H}}^a, a > 0$) and the space of discrete sublinear previsions on X with the d_{KR} (resp., d_{KR}^a) quasi-metric.*

If X, d is continuous complete, then \mathcal{H}_0X and $\mathcal{H}X$ are continuous complete under the $d_{\mathcal{H}}$ and $d_{\mathcal{H}}^a$ quasi-metrics. Through the $C \mapsto F^C$ isometry, suprema of directed families of formal balls are computed as naive suprema. The $d_{\mathcal{H}}$ -Scott, $d_{\mathcal{H}}^a$ -Scott, and lower Vietoris topologies all coincide.

If X, d is algebraic complete, then so are \mathcal{H}_0X and $\mathcal{H}X$ under both $d_{\mathcal{H}}$ and $d_{\mathcal{H}}^a$. The closed sets of the form $\downarrow E$, where E is any finite (and non-empty in the case of $\mathcal{H}X$) set of points of \mathcal{B} form a strong basis, where \mathcal{B} is any given strong basis of X, d .

$\mathcal{H}_0, d_{\mathcal{H}}$ defines an endofunctor on the category of continuous (resp., algebraic) complete quasi-metric spaces and 1-Lipschitz continuous maps, and similarly with \mathcal{H} instead of \mathcal{H}_0 or with $d_{\mathcal{H}}^a$ instead of $d_{\mathcal{H}}$: for every 1-Lipschitz continuous map $f: X, d \rightarrow Y, \partial$ between continuous complete quasi-metric spaces, \mathcal{H}_0f is a 1-Lipschitz continuous map from $\mathcal{H}_0X, d_{\mathcal{H}}$ to $\mathcal{H}_0Y, \partial_{\mathcal{H}}$ and similarly with \mathcal{H} instead of \mathcal{H}_0 or with $d_{\mathcal{H}}^a, \partial_{\mathcal{H}}^a$ instead of $d_{\mathcal{H}}, \partial_{\mathcal{H}}$.

Dually, we define $d_{\mathcal{Q}}$ and $d_{\mathcal{Q}}^a$ on $\mathcal{Q}X$ (where X is given the d -Scott topology) by $d_{\mathcal{Q}}(Q, Q') \stackrel{\text{def}}{=} d_{KR}(F_Q, F_{Q'})$ [12, Definition 6.1] and $d_{\mathcal{Q}}^a(Q, Q') \stackrel{\text{def}}{=} d_{KR}^a(F_Q, F_{Q'})$. If X, d is standard, then we have the Hausdorff-like formula $d_{\mathcal{Q}}(Q, Q') = \sup_{x' \in Q'} \inf_{x \in Q} d(x, x')$ [12, Lemma 6.4], and $d_{\mathcal{Q}}^a(Q, Q') = \min(d_{\mathcal{Q}}(Q, Q'), a)$ [12, Remark 6.5]. We recall that the Hausdorff, or Pompeiu-Hausdorff, distance between two sets E and E' is defined as $\max(\sup_{x \in E} \inf_{x' \in E'} d(x, x'), \sup_{x' \in E'} \inf_{x \in E} d(x, x'))$.

We will also need to know that, when X, d is standard, the specialization ordering of $d_{\mathcal{Q}}$ is reverse inclusion, namely that $d_{\mathcal{Q}}(Q, Q') = 0$ if and only if

$Q \supseteq Q'$ [12, Lemma 6.7]. We will also rely on the following results, which one can find as Lemma 6.4, 6.6, Remark 6.5, Proposition 6.9, Remark 6.10, Theorems 6.23, 6.26, 6.19, Remark 6.20, and Corollary 6.24 of [12] in this order. The fact that $Q \mapsto F_Q$ is an isometry is by definition of d_Q and d_Q^a . Note that every continuous complete quasi-metric space X, d is sober in its d -Scott topology [13, Proposition 4.1].

Theorem 2.4 (Completeness for the Smyth hyperspace [12]). *For every quasi-metric space X, d , the map $Q \mapsto F_Q$ is an isometry between $\mathcal{Q}X, d_Q$ (resp., d_Q^a , $a > 0$) and the space of discrete normalized superlinear previsions on X with the d_{KR} (resp., d_{KR}^a) quasi-metric.*

For every standard quasi-metric space X, d , and for all $Q, Q' \in \mathcal{Q}X$, $d_Q(Q, Q') = \sup_{x' \in Q'} \min_{x \in Q} d(x, x')$, and $d_Q^a(Q, Q') = \min(a, d_Q(Q, Q'))$.

If X is also sober in its d -Scott topology, then $\mathcal{Q}X$ is complete under both the d_Q and d_Q^a quasi-metrics. Through the $Q \mapsto F_Q$ isometry, suprema of directed families of formal balls are computed as naive suprema.

If X, d is continuous complete, then $\mathcal{Q}X$ is continuous complete under both d_Q and d_Q^a . The d_Q -Scott, d_Q^a -Scott, and upper Vietoris topologies all coincide.

If X, d is algebraic complete, then so is $\mathcal{Q}X$ under both d_Q and d_Q^a . The compact saturated sets of the form $\uparrow E$, where E is any finite non-empty set of points of \mathcal{B} form a strong basis, where \mathcal{B} is any given strong basis of X, d .

\mathcal{Q}, d_Q defines an endofunctor on the category of continuous (resp., algebraic) complete quasi-metric spaces and 1-Lipschitz continuous maps, and similarly with d_Q^a instead of d_Q : for every 1-Lipschitz continuous map $f: X, d \rightarrow Y, \partial$ between continuous complete quasi-metric spaces, $\mathcal{Q}f$ is a 1-Lipschitz continuous map from $\mathcal{Q}X, d_Q$ to $\mathcal{Q}Y, \partial_Q$ and from $\mathcal{Q}X, d_Q^a$ to $\mathcal{Q}Y, \partial_Q^a$.

3. Sublinear Previsions

In order to study sublinear previsions, we recall from [6] that sublinear previsions are in bijection with the non-empty closed convex subsets of linear previsions, under some slight assumptions.

More precisely, let r_{AP} be the function that maps every set E of linear previsions on a space X to the prevision $h \in \mathcal{L}X \mapsto \sup_{G \in E} G(h)$. The latter is always a sublinear prevision, which is subnormalized, resp. normalized, as soon as E is a non-empty set of subnormalized, resp. normalized linear previsions.

In the reverse direction, for every sublinear prevision F , let $s_{\text{AP}}(F)$ be the set of all linear previsions G such that $G \leq F$, namely such that $G(h) \leq F(h)$ for every $h \in \mathcal{L}X$. If F is subnormalized, we write $s_{\text{AP}}^{\leq 1}(F)$ for the set of all subnormalized linear previsions $G \leq F$, and if F is normalized, we write $s_{\text{AP}}^1(F)$ for the set of all normalized linear previsions $G \leq F$.

We shall write s_{AP}^\bullet instead of s_{AP} , $s_{\text{AP}}^{\leq 1}$, or s_{AP}^1 , when the superscript is meant to be implied. Again, we equate linear previsions with continuous valuations. Correspondingly, we shall write $\mathbf{V}_\bullet X$ for the space of linear previsions (resp., subnormalized, normalized, depending on the subscript) on X , with the weak topology.

Corollary 3.12 of [6] states that, under the assumption that $\mathcal{L}X$ is locally convex, that is, if every $h \in \mathcal{L}X$ has a base of convex open neighborhoods, then:

- r_{AP} is continuous from the space of sublinear (resp., subnormalized sublinear, resp., normalized sublinear) previsions on X , with the weak topology, to $\mathcal{H}(\mathbf{V}_\bullet X)$ with its lower Vietoris topology;
- s_{AP}^\bullet is continuous in the reverse direction;
- $r_{\text{AP}} \circ s_{\text{AP}}^\bullet = \text{id}$;
- $\text{id} \leq s_{\text{AP}}^\bullet \circ r_{\text{AP}}$.

Hence the space of (possibly subnormalized, or normalized) sublinear previsions, with the weak topology, occurs as a topological retract of $\mathcal{H}(\mathbf{V}_\bullet X)$.

Letting $\mathcal{H}^{cvx}(\mathbf{V}_\bullet X)$ denote the subspace of those closed sets in $\mathcal{H}(\mathbf{V}_\bullet X)$ that are convex, Theorem 4.11 of [6] additionally states that r_{AP} and s_{AP}^\bullet define a homeomorphism between $\mathcal{H}^{cvx}(\mathbf{V}_\bullet X)$ and the corresponding space of (possibly subnormalized, or normalized) sublinear previsions, under the same local convexity assumption. That assumption is satisfied on continuous Yoneda-complete quasi-metric spaces, because of the following.

Proposition 3.1. *Given any continuous complete quasi-metric space X, d seen as a topological space in its d -Scott topology, $\mathcal{L}X$ is locally convex.*

PROOF. Lemma 13.6 of [1] states that $\mathcal{L}X$ is locally convex provided that X is LCS-complete. Every continuous complete quasi-metric space is domain-complete in its d -Scott topology [1, Theorem 4.1], and every domain-complete space is LCS-complete [1, Proposition 3.3]. \square

Hence simply assuming X, d continuous complete will allow us to use the facts stated above on r_{AP} and s_{AP}^\bullet , which we shall do without further mention.

The following is a naturality property for r_{AP} .

Lemma 3.2. *For every continuous map $f: X \rightarrow Y$, $r_{\text{AP}} \circ \mathcal{H}_0(\mathbf{P}f) = \mathbf{P}f \circ r_{\text{AP}}$.*

PROOF. For every closed subset \mathcal{C} of the relevant set $\mathbf{V}_\bullet X$ of linear previsions, and for every $k \in \mathcal{L}Y$, $r_{\text{AP}}(\mathcal{H}_0(\mathbf{P}f)(\mathcal{C}))(k) = \sup_{H \in \mathcal{H}_0(\mathbf{P}f)(\mathcal{C})} H(k) = \sup_{H \in \text{cl}(\mathbf{P}f[\mathcal{C}])} H(k)$. The map $\varphi: H \mapsto H(k)$ is lower semicontinuous from $\mathcal{H}_0(\mathbf{V}_\bullet Y)$ to $\overline{\mathbb{R}}_+$, since $\varphi^{-1}(]b, +\infty]) = [k > b]$ for every $b \in \mathbb{R}_+$. We use the easily proved fact that, for every lower semicontinuous map φ and every set A , $\sup_{H \in \text{cl}(A)} \varphi(H) = \sup_{H \in A} \varphi(H)$. (Hint: the left-hand side is strictly larger than any given $b \in \mathbb{R}_+$ if and only if $\text{cl}(A)$ intersects the open set $\varphi^{-1}(]b, +\infty])$, if and only if A intersects $\varphi^{-1}(]b, +\infty])$.) Hence $r_{\text{AP}}(\mathcal{H}_0(\mathbf{P}f)(\mathcal{C}))(k) = \sup_{H \in \mathbf{P}f[\mathcal{C}]} H(k) = \sup_{G \in \mathcal{C}} \mathbf{P}f(G)(k) = \sup_{G \in \mathcal{C}} G(k \circ f)$. \square

Lemma 3.3. *Let X, d be a continuous complete quasi-metric space, and $\alpha > 0$. Let \bullet be either “ ≤ 1 ” or “1”.*

For each $h \in \mathcal{L}_\alpha(X, d)$, the map $H: G \in \mathbf{V}_\bullet X \mapsto G(h)$ is α -Lipschitz continuous, when $\mathbf{V}_\bullet X$ is equipped with the d_{KR} quasi-metric.

If $h \in \mathcal{L}_\alpha^a(X, d)$, where $a > 0$, then it is also α -Lipschitz continuous with respect to the d_{KR}^a quasi-metric.

PROOF. We first check that it is α -Lipschitz: for all $G, G' \in \mathbf{V}_\bullet X$, $\mathbf{d}_{\mathbb{R}}(H(G), H(G')) = \mathbf{d}_{\mathbb{R}}(G(h), G'(h)) \leq d_{\text{KR}}(G, G')$, by definition of d_{KR} . If additionally h is bounded from above by a , then $\mathbf{d}_{\mathbb{R}}(G(h), G'(h)) \leq d_{\text{KR}}^a(G, G')$, by definition of d_{KR}^a .

We now build $H': (G, r) \mapsto H(G) - \alpha r = G(h) - \alpha r$. In order to show α -Lipschitz continuity, we first recall that $\mathbf{V}_\bullet X$ is complete, both with d_{KR} and with d_{KR}^a (Theorem 2.2); moreover, directed suprema are naive suprema. Since $\mathbf{V}_\bullet X, d_{\text{KR}}$ is complete, it is standard, so we only have to show that H' is Scott-continuous. For every directed family $(G_i, r_i)_{i \in I}$ of formal balls on $\mathbf{V}_\bullet X$, with (naive) supremum (G, r) , we have $r = \inf_{i \in I} r_i$ and $G(h) = \sup_{i \in I} (G_i(h) - \alpha r_i + \alpha r)$. It follows that $H'(G, r) = G(h) - \alpha r = \sup_{i \in I} (G_i(h) - \alpha r_i) = \sup_{i \in I} H'(G_i, r_i)$. \square

The proof of the next lemma uses a minimax theorem. Here and later, we will use the following version, which is the special case of Theorem 3.3 of

[8] described in Remark 3.4 of the same paper. A function $f: A \times B \rightarrow \overline{\mathbb{R}}_+$ is (Ky Fan) *convex in its first argument* if and only if for all $a_1, a_2 \in A$ and for every $\alpha \in]0, 1[$, there is a point $a \in A$ such that for every $b \in B$, $f(a, b) \leq \alpha f(a_1, b) + (1 - \alpha)f(a_2, b)$. It is (Ky Fan) *concave in its second argument* if and only if for all $b_1, b_2 \in B$ and for every $\alpha \in]0, 1[$, there is a point $b \in B$ such that for every $a \in A$, $f(a, b) \geq \alpha f(a, b_1) + (1 - \alpha)f(a, b_2)$.

Theorem 3.4 ([8], **Theorem 3.3**, **Remark 3.4**). *Let A be a non-empty compact topological space and B be a non-empty set. Let $f: A \times B \rightarrow \mathbb{R} \cup \{+\infty\}$ be a map that is convex in its first argument, concave in its second argument, and such that $f(-, b)$ is lower semicontinuous for every $b \in B$. Then:*

$$\sup_{b \in B} \inf_{a \in A} f(a, b) = \inf_{a \in A} \sup_{b \in B} f(a, b),$$

and the infimum on the right-hand side is attained.

The key point in this theorem is that A is not assumed Hausdorff. Although we will use it in a case where A is Hausdorff in Lemma 3.5, we will later use it in cases where A is not.

We recall that a simple valuation $\sum_{i=1} a_i \delta_{x_i}$ on a space X is supported on a set \mathcal{B} if and only if for every i such that $a_i \neq 0$, x_i is in \mathcal{B} . We say that it is *supported on center points* if it is supported on the set of center points; this requires that X be a quasi-metric space.

Lemma 3.5. *Let X, d be a continuous quasi-metric space. Let \bullet be either “ ≤ 1 ” or “ 1 ”, and $a > 0$. For every simple valuation G supported on center points in $\mathbf{V}_\bullet X$, for every $\alpha > 0$, and for every convex subset \mathcal{C}' of $\mathbf{V}_\bullet X$,*

$$\sup_{h \in \mathcal{L}_\alpha^a X} \inf_{G' \in \mathcal{C}'} (G(h) - G'(h)) = \inf_{G' \in \mathcal{C}'} \sup_{h \in \mathcal{L}_\alpha^a X} (G(h) - G'(h)).$$

PROOF. Let us note that, when h is in $\mathcal{L}_\alpha^a X$, $G(h) - G'(h)$ makes sense: since $h \in \mathcal{L}_\alpha^a(X, d)$, and since G and G' are subnormalized, $G(h) - G'(h)$ lies in $[-a\alpha, a\alpha]$. This also shows that the function $f: (h, G') \mapsto -G(h) + G'(h)$ is real-valued.

By Theorem 2.1, $A \stackrel{\text{def}}{=} \mathcal{L}_\alpha^a(X, d)^{\text{patch}}$ is compact (and Hausdorff). We use $B \stackrel{\text{def}}{=} \mathcal{C}'$. Then f is a map from $A \times B$ to \mathbb{R} , which is linear in both arguments, in the sense that it preserves sums and scalar products by non-negative reals. In particular, f is convex in its first argument and concave in its second argument.

We claim that $h \mapsto f(h, G')$ is lower semicontinuous. Recall that G is a simple valuation $\sum_{i=1}^n a_i \delta_{x_i}$. The second part of Theorem 2.1 states that G is continuous from $A = \mathcal{L}_\alpha^a(X, d)^d$ to $(\overline{\mathbb{R}}_+)^d$. Because the non-trivial open subsets of $(\overline{\mathbb{R}}_+)^d$ are of the form $[0, t[$, $t \in \mathbb{R}_+$, $h \mapsto G(h)$ is also continuous from A to \mathbb{R} with the Scott topology of the reverse ordering. Hence $h \mapsto -G(h)$ is continuous from A to \mathbb{R} , the latter with its Scott topology; namely, $h \mapsto -G(h)$ is lower semicontinuous. The map $h \mapsto G'(h)$ is also lower semicontinuous, by definition of a prevision. Since addition is lower semicontinuous from $\mathbb{R} \times \mathbb{R}$ to \mathbb{R} , $h \mapsto -G(h) + G'(h)$ is also lower semicontinuous, and that is the map $h \mapsto f(h, G')$.

Theorem 3.4 then implies that $\sup_{G' \in \mathcal{C}'} \inf_{h \in \mathcal{L}_\alpha^a(X, d)} (-G(h) + G'(h')) = \inf_{h \in \mathcal{L}_\alpha^a(X, d)} \sup_{G' \in \mathcal{C}'} (-G(h) + G'(h'))$. Taking opposites yields the desired result. \square

Lemma 3.6. *Let Y be a topological space. Let \bullet be either “ ≤ 1 ” or “1”. The closure of any convex subset of $\mathbf{V}_\bullet Y$, in the weak topology, is convex.*

PROOF. This follows from the fact that the closure of convex sets in semi-topological cones is convex [14, Lemma 4.10 (a)]. The reader who would like to prove the lemma directly is invited to consider a convex subset \mathcal{C} of $\mathbf{V}_\bullet Y$, such that there are two elements H_1, H_2 in its closure and some $\alpha \in]0, 1[$ such that $\alpha H_1 + (1 - \alpha)H_2$ is not in the closure of \mathcal{C} . Next, consider the continuous map $f: (G_1, G_2) \in \mathbf{V}_\bullet Y \times \mathbf{V}_\bullet Y \mapsto \alpha G_1 + (1 - \alpha)G_2 \in \mathbf{V}_\bullet Y$. Then $f(H_1, H_2)$ has an open neighborhood \mathcal{V} that does not intersect \mathcal{C} , hence we can find open neighborhoods \mathcal{U}_1 and \mathcal{U}_2 of H_1 and H_2 respectively whose product is included in $f^{-1}(\mathcal{V})$. \mathcal{U}_1 intersects the closure of \mathcal{C} , hence it also intersects \mathcal{C} , say at G_1 . Similarly, \mathcal{U}_2 intersects \mathcal{C} at G_2 , so that $f(G_1, G_2)$ is in \mathcal{C} , by convexity; but that contradicts the fact that $(G_1, G_2) \in \mathcal{U}_1 \times \mathcal{U}_2 \subseteq f^{-1}(\mathcal{V})$. \square

Proposition 3.7. *Let X, d be a continuous complete quasi-metric space. Let \bullet be either “ ≤ 1 ” or “1”, and $a > 0$. For all $\mathcal{C}, \mathcal{C}' \in \mathcal{H}(\mathbf{V}_\bullet X)$,*

$$(d_{KR}^a)_{\mathcal{H}}(\mathcal{C}, \mathcal{C}') \geq d_{KR}^a(r_{\text{AP}}(\mathcal{C}), r_{\text{AP}}(\mathcal{C}')), \quad (4)$$

with equality if \mathcal{C}' is convex.

Before we give the proof, we note the following. The topology on $\mathbf{V}_\bullet X$ is implicitly assumed to be the weak topology. Under the assumptions of

Proposition 3.7, this coincides with the d_{KR}^a -Scott topology, by Theorem 2.2. Hence \mathcal{C} and \mathcal{C}' are also closed in the d_{KR}^a -Scott topology, so that $(d_{\text{KR}}^a)_{\mathcal{H}}(\mathcal{C}, \mathcal{C}')$ makes sense.

PROOF. *The inequality.* We develop the left-hand side. Since X, d is continuous complete, we can use Theorem 2.2, so that $\mathbf{V}\bullet X, d_{\text{KR}}^a$ is complete. In particular, it is standard, so we can apply the first part of Theorem 2.3. That justifies the first of the following equalities, the others are by definition:

$$\begin{aligned} (d_{\text{KR}}^a)_{\mathcal{H}}(\mathcal{C}, \mathcal{C}') &= (d_{\text{KR}}^a)_{\text{KR}}(F^{\mathcal{C}}, F^{\mathcal{C}'}) \\ &= \sup_{H \in \mathcal{L}_1(\mathbf{V}\bullet X, d_{\text{KR}}^a)} \mathbf{d}_{\mathbb{R}}(F^{\mathcal{C}}(H), F^{\mathcal{C}'}(H)) \\ &= \sup_{H \in \mathcal{L}_1(\mathbf{V}\bullet X, d_{\text{KR}}^a)} \mathbf{d}_{\mathbb{R}}(\sup_{G \in \mathcal{C}} H(G), \sup_{G' \in \mathcal{C}'} H(G')). \end{aligned}$$

The right-hand side of (4) is equal to:

$$\begin{aligned} d_{\text{KR}}^a(r_{\text{AP}}(\mathcal{C}), r_{\text{AP}}(\mathcal{C}')) &= \sup_{h \in \mathcal{L}_1^a X} \mathbf{d}_{\mathbb{R}}(r_{\text{AP}}(\mathcal{C})(h), r_{\text{AP}}(\mathcal{C}')(h)) \\ &= \sup_{h \in \mathcal{L}_1^a X} \mathbf{d}_{\mathbb{R}}(\sup_{G \in \mathcal{C}} G(h), \sup_{G' \in \mathcal{C}'} G'(h)). \end{aligned}$$

Among the elements H of $\mathcal{L}_1(\mathbf{V}\bullet X, d_{\text{KR}}^a)$, we find those obtained from $h \in \mathcal{L}_1^a(X, d)$ by letting $H(G) \stackrel{\text{def}}{=} G(h)$, according to Lemma 3.3. Therefore the left-hand side of (4) is larger than or equal to its right-hand side.

The equality, assuming X, d algebraic. We now assume that \mathcal{C}' is convex. We will show that $(d_{\text{KR}}^a)_{\mathcal{H}}(\mathcal{C}, \mathcal{C}') = d_{\text{KR}}^a(r_{\text{AP}}(\mathcal{C}), r_{\text{AP}}(\mathcal{C}'))$, and we do that first in the special case where X, d is algebraic complete. If the inequality were strict, then there would be a real number t such that $(d_{\text{KR}}^a)_{\mathcal{H}}(\mathcal{C}, \mathcal{C}') > t > d_{\text{KR}}^a(r_{\text{AP}}(\mathcal{C}), r_{\text{AP}}(\mathcal{C}'))$. Note that the second inequality implies that t is (strictly) positive.

Using the definition of $d_{\mathcal{H}}$, $(d_{\text{KR}}^a)_{\mathcal{H}}(\mathcal{C}, \mathcal{C}') = \sup_{G \in \mathcal{C}} d_{\text{KR}}^a(G, \mathcal{C}')$, so $d_{\text{KR}}^a(G, \mathcal{C}') > t$ for some $G \in \mathcal{C}$.

On a standard quasi-metric space X, d , the map $d(\cdot, C)$ is 1-Lipschitz continuous for every d -Scott closed set C [13, Lemma 6.11]. Moreover, $d(x, C) \leq \inf_{y \in C} d(x, y)$, with equality if x is a center point [13, Proposition 6.12].

Since $\mathbf{V}\bullet X, d_{\text{KR}}^a$ is complete hence standard, the map $d_{\text{KR}}^a(\cdot, \mathcal{C}')$ is 1-Lipschitz continuous. By Theorem 2.2, $(G, 0)$ is the (naive) supremum of

a directed family $(G_i, r_i)_{i \in I}$ where each G_i is a simple (sub)normalized valuation supported on center points. Since $d_{\text{KR}}^a(-, \mathcal{C}')$ is 1-Lipschitz continuous, the map $(F, r) \mapsto d_{\text{KR}}^a(F, \mathcal{C}') - r$ is Scott-continuous, so $d_{\text{KR}}^a(G, \mathcal{C}') = \sup_{i \in I} d_{\text{KR}}^a(G_i, \mathcal{C}') - r_i$. It follows that $d_{\text{KR}}^a(G_i, \mathcal{C}') - r_i > t$ for some $i \in I$. We recall that $d_{\text{KR}}^a(G_i, \mathcal{C}') \leq \inf_{G' \in \mathcal{C}'} d_{\text{KR}}^a(G_i, G')$; in fact, equality holds since G_i is a center point. In any case, $\inf_{G' \in \mathcal{C}'} d_{\text{KR}}^a(G_i, G') > r_i + t$.

Expanding the definition of d_{KR}^a , it follows that $\inf_{G' \in \mathcal{C}'} \sup_{h \in \mathcal{L}_1^a(X)} \mathbf{d}_{\mathbb{R}}(G_i(h), G'(h)) > r_i + t$. Then there is a real number $t' > t + r_i$ such that, for every $G' \in \mathcal{C}'$, there is an $h \in \mathcal{L}_1^a(X, d)$ such that $\mathbf{d}_{\mathbb{R}}(G_i(h), G'(h)) > t'$; in particular, since $t' > t > 0$ and therefore $G'(h)$ cannot be equal to $+\infty$, $G_i(h) - G'(h)$ makes sense and is strictly larger than t' . Working in reverse, we obtain that $\inf_{G' \in \mathcal{C}'} \sup_{h \in \mathcal{L}_1^a(X)} (G_i(h) - G'(h)) - r_i > t$.

We now use Lemma 3.5 and we obtain that $\sup_{h \in \mathcal{L}_1^a(X)} \inf_{G' \in \mathcal{C}'} (G_i(h) - G'(h)) - r_i > t$. Hence there is an $h \in \mathcal{L}_1^a(X, d)$ such that $\inf_{G' \in \mathcal{C}'} (G_i(h) - G'(h)) - r_i > t$. That can be written equivalently as $G_i(h) - \sup_{G' \in \mathcal{C}'} G'(h) - r_i > t$. Since $(G_i, r_i) \leq d_{\text{KR}}^{a+}(G, 0)$, in particular $G_i(h) \leq G(h) + r_i$, so $G(h) - \sup_{G' \in \mathcal{C}'} G'(h) > t$. This certainly implies that $\sup_{G \in \mathcal{C}} G(h) - \sup_{G' \in \mathcal{C}'} G'(h) > t$, and therefore that $\mathbf{d}_{\mathbb{R}}(\sup_{G \in \mathcal{C}} G(h), \sup_{G' \in \mathcal{C}'} G'(h)) > t$. However, we had assumed that $t > d_{\text{KR}}^a(r_{\text{AP}}(\mathcal{C}), r_{\text{AP}}(\mathcal{C}')) = \sup_{h \in \mathcal{L}_1^a(X)} \mathbf{d}_{\mathbb{R}}(\sup_{G \in \mathcal{C}} G(h), \sup_{G' \in \mathcal{C}'} G'(h))$, leading to a contradiction.

This shows $(d_{\text{KR}}^a)_{\mathcal{H}}(\mathcal{C}, \mathcal{C}') = d_{\text{KR}}^a(r_{\text{AP}}(\mathcal{C}), r_{\text{AP}}(\mathcal{C}'))$ when \mathcal{C}' is convex, assuming that X, d is algebraic complete.

The equality, in the continuous case. We now deal with the general case, where \mathcal{C}' is still convex, but X, d is continuous complete. We recall that, since X, d is continuous, it arises as a 1-Lipschitz continuous retract of some algebraic complete quasi-metric space Y, ∂ [13, Theorem 7.9]. Let r be the retraction (not to be confused with r_{AP}), and s be the section. By Lemma 3.2, $r_{\text{AP}} \circ \mathcal{H}_0(\mathbf{P}s) = \mathbf{P}s \circ r_{\text{AP}}$.

Another fact we need to observe is that, when \mathcal{C}' is convex, $\mathcal{H}_0(\mathbf{P}s)(\mathcal{C}') = cl(\mathbf{P}s[\mathcal{C}'])$ is convex. It is easy to see that $\mathbf{P}s[\mathcal{C}']$ is convex: for any two elements $\mathbf{P}s(G_1), \mathbf{P}s(G_2)$ where $G_1, G_2 \in \mathcal{C}'$, for every $\alpha \in [0, 1]$, $\alpha \mathbf{P}s(G_1) + (1 - \alpha) \mathbf{P}s(G_2)$ maps every $k \in \mathcal{L}Y$ to $\alpha G_1(k \circ s) + (1 - \alpha) G_2(k \circ s)$, and is therefore equal to $\mathbf{P}s(\alpha G_1 + (1 - \alpha) G_2)$, which is in $\mathbf{P}s[\mathcal{C}']$. Then we apply Lemma 3.6.

Since $r \circ s = \text{id}_X$, and since \mathcal{H}_0 and \mathbf{P} are functors on the category of

topological spaces, $(\mathcal{H}_0 \circ \mathbf{P})(r) \circ (\mathcal{H}_0 \circ \mathbf{P})(s)$ is the identity map. Hence:

$$\begin{aligned} (d_{\text{KR}}^a)_{\mathcal{H}}(\mathcal{C}, \mathcal{C}') &= (d_{\text{KR}}^a)_{\mathcal{H}}((\mathcal{H}_0 \circ \mathbf{P})(r)((\mathcal{H}_0 \circ \mathbf{P})(s)(\mathcal{C})), (\mathcal{H}_0 \circ \mathbf{P})(r)((\mathcal{H}_0 \circ \mathbf{P})(s)(\mathcal{C}')))) \\ &\leq (d_{\text{KR}}^a)_{\mathcal{H}}(\mathcal{H}_0(\mathbf{P}s)(\mathcal{C}), \mathcal{H}_0(\mathbf{P}s)(\mathcal{C}')), \end{aligned}$$

since $(\mathcal{H}_0 \circ \mathbf{P})(r)$ is 1-Lipschitz. Indeed, by the final parts of Theorem 2.2 and Theorem 2.3, $\mathbf{P}r$ is 1-Lipschitz continuous with respect to the d_{KR}^a quasi-metric, and then $\mathcal{H}_0(\mathbf{P}r)$ is 1-Lipschitz continuous with respect to the $(d_{\text{KR}}^a)_{\mathcal{H}}$ quasi-metric.

We have already shown that (4) is an equality on algebraic complete spaces, provided the second argument is convex. Since Y, ∂ is algebraic complete, and since $\mathcal{H}_0(\mathbf{P}s)(\mathcal{C}') = cl(s[\mathcal{C}'])$ is convex, we obtain that:

$$(d_{\text{KR}}^a)_{\mathcal{H}}(\mathcal{C}, \mathcal{C}') \leq d_{\text{KR}}^a(r_{\text{AP}}(\mathcal{H}_0(\mathbf{P}s)(\mathcal{C})), r_{\text{AP}}(\mathcal{H}_0(\mathbf{P}s)(\mathcal{C}'))).$$

We recall that $r_{\text{AP}} \circ \mathcal{H}_0(\mathbf{P}s) = \mathbf{P}s \circ r_{\text{AP}}$, so $(d_{\text{KR}}^a)_{\mathcal{H}}(\mathcal{C}, \mathcal{C}') \leq d_{\text{KR}}^a(\mathbf{P}s(r_{\text{AP}}(\mathcal{C})), \mathbf{P}s(r_{\text{AP}}(\mathcal{C}')))$, and the latter is less than or equal to $d_{\text{KR}}^a(r_{\text{AP}}(\mathcal{C}), r_{\text{AP}}(\mathcal{C}'))$ since $\mathbf{P}s$ is 1-Lipschitz with respect to the d_{KR}^a quasi-metric, by the last part of Theorem 2.3. Since by (4), the other inequality $(d_{\text{KR}}^a)_{\mathcal{H}}(\mathcal{C}, \mathcal{C}') \geq d_{\text{KR}}^a(r_{\text{AP}}(\mathcal{C}), r_{\text{AP}}(\mathcal{C}'))$ holds, equality follows. \square

Lemma 3.8. *Let X, d be a continuous complete quasi-metric space. Let \bullet be either “ ≤ 1 ” or “1”, and $a > 0$. Then:*

1. r_{AP} is 1-Lipschitz from $\mathcal{H}(\mathbf{V}\bullet X)$, $(d_{\text{KR}}^a)_{\mathcal{H}}$ to the space of (sub)normalized sublinear previsions on X with the d_{KR}^a quasi-metric;
2. s_{AP}^\bullet preserves distances on the nose; in particular, s_{AP}^\bullet is 1-Lipschitz.

PROOF. (1) The first part of Proposition 3.7 implies that r_{AP} is 1-Lipschitz. (2) For any two (sub)normalized sublinear previsions F and F' , $s_{\text{AP}}^\bullet(F') = \{G \in \mathbf{V}\bullet X \mid G \leq F'\}$ is always convex, so the second part of Proposition 3.7 implies that:

$$d_{\text{KR}}^a(F, F') = d_{\text{KR}}^a(r_{\text{AP}}(s_{\text{AP}}^\bullet(F)), r_{\text{AP}}(s_{\text{AP}}^\bullet(F'))) = (d_{\text{KR}}^a)_{\mathcal{H}}(s_{\text{AP}}^\bullet(F), s_{\text{AP}}^\bullet(F')).$$

\square

The next lemma states that r_{AP} is not just 1-Lipschitz, but 1-Lipschitz continuous.

Lemma 3.9. *Let X, d be a continuous complete quasi-metric space. Let \bullet be either “ ≤ 1 ” or “1”, and $a > 0$. Then r_{AP} is 1-Lipschitz continuous from $\mathcal{H}(\mathbf{V}\bullet X), (d_{\text{KR}}^a)_{\mathcal{H}}$ to the space of (sub)normalized sublinear previsions on X with the d_{KR}^a quasi-metric.*

PROOF. By Lemma 3.8 (1), it is 1-Lipschitz. In particular, $\mathbf{B}^1(r_{\text{AP}})$ is monotonic. Similarly $\mathbf{B}^1(s_{\text{AP}}^\bullet)$ is monotonic. Since $r_{\text{AP}} \circ s_{\text{AP}}^\bullet = \text{id}$, $\mathbf{B}^1(r_{\text{AP}}) \circ \mathbf{B}^1(s_{\text{AP}}^\bullet) = \text{id}$. Since $\text{id} \leq s_{\text{AP}}^\bullet \circ r_{\text{AP}}$, i.e., since $\mathcal{C} \subseteq s_{\text{AP}}^\bullet(r_{\text{AP}}(\mathcal{C}))$ for every \mathcal{C} , we also have that $(d_{\text{KR}}^a)_{\mathcal{H}}(\mathcal{C}, s_{\text{AP}}^\bullet(r_{\text{AP}}(\mathcal{C}))) = 0$ since the specialization ordering of $(d_{\text{KR}}^a)_{\mathcal{H}}$ is inclusion, hence $(\mathcal{C}, r) \leq^{(d_{\text{KR}}^a)_{\mathcal{H}}^+} (s_{\text{AP}}^\bullet(r_{\text{AP}}(\mathcal{C})), r) = \mathbf{B}^1(s_{\text{AP}}^\bullet)(\mathbf{B}^1(r_{\text{AP}})(\mathcal{C}, r))$ for every $r \in \mathbb{R}_+$.

This shows that $\mathbf{B}^1(r_{\text{AP}})$ is left adjoint to $\mathbf{B}^1(s_{\text{AP}}^\bullet)$. We conclude because left adjoints preserve all existing suprema, hence in particular directed suprema. \square

We require the following order-theoretic lemma.

Lemma 3.10. *Every order-retract of a dcpo is a dcpo. Precisely, for any two monotonic maps $p: B \rightarrow A$ and $e: A \rightarrow B$ such that $p \circ e = \text{id}_A$, if B is a dcpo, then A is a dcpo. Moreover, the supremum of any directed family $(a_i)_{i \in I}$ in A is $p(\sup_{i \in I} e(a_i))$.*

PROOF. This is an easy exercise, but since one may be surprised by the fact that we do not need p or e to be Scott-continuous for that, we give the complete proof. First $p(\sup_{i \in I} e(a_i))$ is larger than $p(e(a_i)) = a_i$ for every $i \in I$, since p is monotonic. Then, if b is any other upper bound of $(a_i)_{i \in I}$ in A , then $e(a_i) \leq e(b)$ for every $i \in I$, because e is monotonic, so $\sup_{i \in I} e(a_i) \leq e(b)$. It follows that $p(\sup_{i \in I} e(a_i)) \leq p(e(b)) = b$. \square

Proposition 3.11 (Completeness, Sublinear Previsions, Algebraic Case).

Let X, d be an algebraic complete quasi-metric space. Let \bullet be either “ ≤ 1 ” or “1”, and $a > 0$. The space of (sub)normalized sublinear previsions on X with the d_{KR}^a quasi-metric is complete, and directed suprema of formal balls are computed as naive suprema.

PROOF. By Theorem 2.2, $\mathbf{V}\bullet X, d_{\text{KR}}^a$ is algebraic complete (and the d_{KR}^a -Scott topology coincides with the weak topology), so we can use Theorem 2.3 and conclude that $\mathcal{H}(\mathbf{V}\bullet X), (d_{\text{KR}}^a)_{\mathcal{H}}$ is complete, and that directed suprema of formal balls are computed as naive suprema. In particular, $\mathbf{B}(\mathcal{H}(\mathbf{V}\bullet X), (d_{\text{KR}}^a)_{\mathcal{H}})$ is a dcpo. Moreover, $p \stackrel{\text{def}}{=} \mathbf{B}^1(r_{\text{AP}})$ and $e \stackrel{\text{def}}{=} \mathbf{B}^1(s_{\text{AP}}^\bullet)$ are monotonic by

Lemma 3.8, and form an order-retraction. By Lemma 3.10, the poset of formal balls of the space of (sub)normalized sublinear previsions is a dcpo, and that shows that the latter space is complete.

We compute the supremum of a directed family $(F_i, r_i)_{i \in I}$ of formal balls, where each F_i is a (sub)normalized sublinear prevision. This is $p(\sup_{i \in I} e(F_i, r_i))$, where p and e are as above. For each $i \in I$, $e(F_i, r_i) = (s_{\text{AP}}^\bullet(F_i), r_i)$. The closed set $\mathcal{C}_i = s_{\text{AP}}^\bullet(F_i)$ has an associated discrete sublinear prevision $F^{\mathcal{C}_i}$, defined as mapping $H \in \mathcal{L}(\mathbf{V}\bullet X)$ to $\sup_{G \in \mathcal{C}_i} H(G) = \sup_{G \in \mathbf{V}\bullet X, G \leq F_i} H(G)$. Since directed suprema are naive suprema in $\mathbf{B}(\mathcal{H}(\mathbf{V}\bullet X), (d_{\text{KR}}^a)_{\mathcal{H}})$, $\sup_{i \in I} e(F_i, r_i)$ is (\mathcal{C}, r) where $r = \inf_{i \in I} r_i$ and $F^{\mathcal{C}}(H) = \sup_{i \in I} (F^{\mathcal{C}_i}(H) - \alpha r_i + \alpha r) = \sup_{i \in I} (\sup_{G \in \mathbf{V}\bullet X, G \leq F_i} H(G) - \alpha r_i + \alpha r)$, for every $H \in \mathcal{L}_\alpha^a(\mathbf{V}\bullet X, d_{\text{KR}}^a)$. Then $p(\sup_{i \in I} e(F_i, r_i))$ is the formal ball (F, r) , where $F = r_{\text{AP}}(\mathcal{C})$. In particular, for every $h \in \mathcal{L}_\alpha^a(X, d)$, $F(h) = \sup_{G \in \mathcal{C}} G(h)$. The map $H: G \mapsto G(h)$ is in $\mathcal{L}_\alpha^a(\mathbf{V}\bullet X, d_{\text{KR}}^a)$ by Lemma 3.3 (and since it is clearly upper bounded by αr), and:

$$\begin{aligned}
F(h) &= \sup_{G \in \mathcal{C}} H(G) \\
&= F^{\mathcal{C}}(H) \\
&= \sup_{i \in I} \left(\sup_{G \in \mathbf{V}\bullet X, G \leq F_i} H(G) - \alpha r_i + \alpha r \right) \\
&= \sup_{i \in I} \left(\sup_{G \in \mathbf{V}\bullet X, G \leq F_i} G(h) - \alpha r_i + \alpha r \right) \\
&= \sup_{i \in I} (F_i(h) - \alpha r_i + \alpha r).
\end{aligned}$$

□

Lemma 3.12. *Let X, d be an algebraic complete quasi-metric space. Let \bullet be either “ ≤ 1 ” or “1”, and $a > 0$. Then s_{AP}^\bullet is 1-Lipschitz continuous from the space of (sub)normalized sublinear previsions on X with the d_{KR}^a quasi-metric to $\mathcal{H}(\mathbf{V}\bullet X), (d_{\text{KR}}^a)_{\mathcal{H}}$.*

PROOF. By Theorem 2.2, the d_{KR}^a -Scott topology coincides with the weak topology on $\mathbf{V}\bullet X$, and $\mathbf{V}\bullet X, d_{\text{KR}}^a$ is algebraic complete, so we can use Theorem 2.3 and conclude that the $(d_{\text{KR}}^a)_{\mathcal{H}}$ -Scott topology coincides with the lower Vietoris topology on $\mathcal{H}(\mathbf{V}\bullet X)$.

We use Proposition 5.27 of [12], which states (in particular) that if X, d is standard, if the space of (sub)normalized sublinear previsions on X is complete in the d_{KR}^a quasi-metric, and if directed suprema of formal balls

on the latter are computed as naive suprema, then the d_{KR}^a -Scott topology is finer than the weak topology. Proposition 3.11 gives us exactly the assumptions needed in order to apply this result, so the d_{KR}^a -Scott topology on our space of (sub)normalized sublinear previsions is finer than the weak topology. Since s_{AP}^\bullet is continuous from that space of sublinear previsions with the weak topology to $\mathcal{H}(\mathbf{V}_\bullet X)$ with its lower Vietoris topology, it is therefore also continuous with respect to the d_{KR}^a -topology on the former, and the $(d_{\text{KR}}^a)_\mathcal{H}$ -Scott (=weak) topology on the latter.

We now know that s_{AP}^\bullet is both 1-Lipschitz and continuous with respect to the d_{KR}^a -Scott and $(d_{\text{KR}}^a)_\mathcal{H}$ -Scott topologies. Since $\mathcal{H}(\mathbf{V}_\bullet X)$, $(d_{\text{KR}}^a)_\mathcal{H}$ is complete (by Theorems 2.2 and 2.3), and since our space of (sub)normalized sublinear previsions is complete under d_{KR}^a , too (Proposition 3.11), they are standard. It follows that s_{AP}^\bullet is 1-Lipschitz continuous. \square

In order to extend that to the continuous case, we require the following.

Lemma 3.13. *Let X, d be a 1-Lipschitz continuous retract of an algebraic complete quasi-metric space Y, ∂ , with retraction $r: Y \rightarrow X$ and section $s: X \rightarrow Y$. Let \mathcal{B} be a strong basis of Y, ∂ .*

Then X, d is continuous complete, and for every $x \in X$, $(x, 0)$ is the supremum of a directed family of formal balls whose centers are points of the form $r(y)$ with y taken from \mathcal{B} .

If additionally $d(r(y), x) = \partial(y, s(x))$ for all $x \in X$ and $y \in Y$, then X, d is algebraic complete, and a strong basis of X, d is given by the image of \mathcal{B} under r .

PROOF. That X, d is continuous complete is clear, since the continuous complete quasi-metric spaces are exactly the retracts of algebraic complete quasi-metric spaces.

Let \mathcal{A} be the image of \mathcal{B} by r . Lemma 5.6 of [11] states that, given a standard algebraic quasi-metric space Y, ∂ with strong basis \mathcal{B} , the set $\mathring{\mathcal{B}}$ of formal balls whose centers are in \mathcal{B} is a basis of the continuous dcpo $\mathbf{B}(Y, \partial)$. Then the image of $\mathring{\mathcal{B}}$ by $\mathbf{B}^1(r)$ is a basis of $\mathbf{B}(X, d)$, because the image of a basis of a continuous dcpo by a Scott retraction is a basis (see Exercise 5.1.44 of [5], for example). The image of $\mathring{\mathcal{B}}$ is the set $\mathring{\mathcal{A}}$ of formal balls whose centers are in \mathcal{A} . It follows that, for every $x \in X$, $(x, 0)$ is the supremum of a directed family of formal balls whose centers are in \mathcal{A} .

In order to show that \mathcal{A} is a strong basis, it remains to show that the elements $a = r(b)$ of \mathcal{A} are center points, under our additional assumption

that $d(r(y), x) = \partial(y, s(x))$ for all $x \in X$ and $y \in Y$. Let us consider the open ball $B_{(a,0), < \epsilon}^{d^+}$. This is the set of formal balls (x, t) on X such that $d(a, x) < \epsilon - t$, or equivalently $\partial(b, s(x)) < \epsilon - t$, using the additional assumption. Hence $B_{(a,0), < \epsilon}^{d^+} = \mathbf{B}^1(s)^{-1}(B_{(b,0), < \epsilon}^{\partial^+})$. This is open since b is a center point and $\mathbf{B}^1(s)$ is continuous. \square

We also require the following easy fact.

Fact 3.14. *If A is a space with two topologies \mathcal{O}_1 and \mathcal{O}_2 , and both embed into a topological space B by the same topological embedding, then $\mathcal{O}_1 = \mathcal{O}_2$.*

Theorem 3.15 (Continuous Completeness for Sublinear Previsions).

Let X, d be a continuous complete quasi-metric space, and $a > 0$.

The space of all subnormalized (resp., normalized) sublinear previsions on X with the d_{KR}^a quasi-metric is continuous complete.

It occurs as a 1-Lipschitz continuous retract of $\mathcal{H}(\mathbf{V}_{\leq 1}X), (d_{KR}^a)_{\mathcal{H}}$ (resp., $\mathcal{H}(\mathbf{V}_1X), (d_{KR}^a)_{\mathcal{H}}$), and as an isometric copy of $\mathcal{H}^{cvx}(\mathbf{V}_{\leq 1}X), (d_{KR}^a)_{\mathcal{H}}$ (resp., $\mathcal{H}^{cvx}(\mathbf{V}_1X), (d_{KR}^a)_{\mathcal{H}}$) through r_{AP} and $s_{\text{AP}}^{\leq 1}$ (resp., s_{AP}^1).

The d_{KR}^a -Scott topology on the space of subnormalized (resp., normalized) sublinear previsions coincides with the weak topology.

PROOF. We have already seen that r_{AP} and s_{AP}^{\bullet} are continuous, with respect to the weak topologies. By Lemma 3.9 and Lemma 3.12, they are also 1-Lipschitz continuous with respect to $(d_{KR}^a)_{\mathcal{H}}$ on $\mathcal{H}(\mathbf{V}_{\bullet}X)$ and d_{KR}^a on the space of (sub)normalized sublinear previsions.

We also recall that $r_{\text{AP}} \circ s_{\text{AP}}^{\bullet} = \text{id}$. This shows the 1-Lipschitz continuous retract part, and also the isometric copy part, since s_{AP}^{\bullet} and r_{AP} restrict to bijections with the subspace $\mathcal{H}^{cvx}(\mathbf{V}_{\bullet}X)$.

We turn to continuity. By Theorem 2.2, $\mathbf{V}_{\bullet}X, d_{KR}^a$ is continuous complete. We can then use Theorem 2.3 to conclude that $\mathcal{H}(\mathbf{V}_{\bullet}X), (d_{KR}^a)_{\mathcal{H}}$ is continuous complete. Since every 1-Lipschitz continuous retract of a continuous complete quasi-metric space is continuous complete, the space of all subnormalized (resp., normalized) sublinear previsions on X with the d_{KR}^a quasi-metric is continuous complete.

Let us show that the d_{KR}^a -Scott topology on the space of (sub)normalized sublinear previsions is the weak topology. We use Fact 3.14, with the set of (sub)normalized sublinear previsions for A , s_{AP}^{\bullet} for i , and $\mathcal{H}(\mathbf{V}_{\bullet}X)$ for B , which we topologize with the weak topology. We have two topologies on A . With the weak topology, s_{AP}^{\bullet} is a section, hence a topological embedding.

With the d_{KR}^a -Scott topology, s_{AP}^\bullet is also a section, this time into B with its $(d_{KR}^a)_\mathcal{H}$ -Scott topology. But the latter coincides with the weak topology, using Theorem 2.2 and Theorem 2.3.

We did not need to show that directed suprema of formal balls were computed as naive suprema to prove the latter. Still, that is true.

Lemma 3.16. *Let X, d be a continuous complete quasi-metric space, and $a > 0$. The suprema of directed families $(F_i, r_i)_{i \in I}$ of formal balls with respect to the d_{KR}^a quasi-metric, where each F_i is a (sub)normalized sublinear prevision on X , are naive suprema.*

PROOF. Let $\alpha > 0$, and h be in $\mathcal{L}_\alpha^a(X, d)$. By Theorem 3.15, $\mathbf{B}^1(r_{AP})$ is Scott-continuous and $\mathbf{B}^1(r_{AP}) \circ \mathbf{B}^1(s_{AP}^\bullet)$ is the identity. The supremum of $(F_i, r_i)_{i \in I}$ is then $\mathbf{B}^1(r_{AP})(\mathcal{C}, r) = (r_{AP}(\mathcal{C}), r)$, where $(\mathcal{C}, r) = \sup_{i \in I} (s_{AP}^\bullet(F_i), r_i)$. By Theorem 2.3, this is a naive supremum, meaning that $r = \inf_{i \in I} r_i$ and $F^C(H) = \sup_{i \in I} (F^{s_{AP}^\bullet(F_i)}(H) + \alpha r - \alpha r_i)$ for every $H \in \mathcal{L}_\alpha(\mathbf{V}_\bullet X, d_{KR}^a)$. In other words, $F^C(H) = \sup_{i \in I} (\sup_{G \in s_{AP}^\bullet(F_i)} H(G) + \alpha r - \alpha r_i)$. Taking $H(G) \stackrel{\text{def}}{=} G(h)$ (see Lemma 3.3), we obtain:

$$\begin{aligned} F^C(H) &= \sup_{i \in I} (\sup_{G \in s_{AP}^\bullet(F_i)} G(h) + \alpha r - \alpha r_i) \\ &= \sup_{i \in I} (r_{AP}(s_{AP}^\bullet(F_i))(h) + \alpha r - \alpha r_i) \\ &= \sup_{i \in I} (F_i(h) + \alpha r - \alpha r_i), \end{aligned}$$

where the last equalities are by definition of r_{AP} , and because $r_{AP} \circ s_{AP}^\bullet = \text{id}$. By using the definition of F^C , then of H , then of r_{AP} , we obtain $F^C(H) = \sup_{G \in \mathcal{C}} H(G) = \sup_{G \in \mathcal{C}} G(h) = r_{AP}(\mathcal{C})(h)$, showing that $r_{AP}(\mathcal{C})(h) = \sup_{i \in I} (F_i(h) + \alpha r - \alpha r_i)$, as desired. \square

We can now conclude on the situation with algebraic complete spaces.

Theorem 3.17 (Algebraic Completeness for Sublinear Previsions).

Let X, d be an algebraic complete quasi-metric space, with a strong basis \mathcal{B} . Let $a > 0$.

The space of all subnormalized (resp., normalized) sublinear previsions on X with the d_{KR}^a quasi-metric is algebraic complete.

All the sublinear previsions of the form $h \mapsto \max_{i=1}^m \sum_{j=1}^{n_i} a_{ij} h(x_{ij})$, with $m \geq 1$, $\sum_{j=1}^{n_i} a_{ij} \leq 1$ (resp., $= 1$) for every i , and where each x_{ij} is in \mathcal{B} , are center points, and form a strong basis.

PROOF. For every $\mathcal{C} \in \mathcal{H}(\mathbf{V}_\bullet X)$ and every (sub)normalized sublinear prevision F' , we have:

$$\begin{aligned} d_{\text{KR}}^a(r_{\text{AP}}(\mathcal{C}), F') &= d_{\text{KR}}^a(r_{\text{AP}}(\mathcal{C}), r_{\text{AP}}(\mathcal{C}')) \quad \text{where } \mathcal{C}' = s_{\text{AP}}^\bullet(F') \\ &= (d_{\text{KR}}^a)_{\mathcal{H}}(\mathcal{C}, \mathcal{C}') \quad \text{by Proposition 3.7, since } \mathcal{C}' \text{ is convex} \\ &= (d_{\text{KR}}^a)_{\mathcal{H}}(\mathcal{C}, s_{\text{AP}}^\bullet(F')). \end{aligned}$$

This is exactly the additional assumption we require to apply the final part of Lemma 3.13. It follows that the space of (sub)normalized sublinear previsions is algebraic complete, with a strong basis of elements of the form $r_{\text{AP}}(\downarrow\{\sum_{j=1}^{n_i} a_{ij}\delta_{x_{ij}} \mid 1 \leq i \leq m\})$, where $m \geq 1$, each $\sum_{j=1}^{n_i} a_{ij}\delta_{x_{ij}}$ is a (sub)normalized simple valuation and each x_{ij} is in \mathcal{B} . An easy check shows that those elements are exactly the maps $h \mapsto \max_{i=1}^m \sum_{j=1}^{n_i} a_{ij}h(x_{ij})$. \square

Lemma 3.18. *Let X, d and Y, ∂ be two continuous complete quasi-metric spaces, and $f: X, d \mapsto Y, \partial$ be a 1-Lipschitz continuous map. Let also $a > 0$. $\mathbf{P}f$ restricts to a 1-Lipschitz continuous map from the space of normalized, resp. subnormalized sublinear previsions on X to the space of normalized, resp. subnormalized sublinear previsions on Y , with the d_{KR}^a and ∂_{KR}^a quasi-metrics.*

PROOF. Lemma 4.21 of [12] states (in particular) that, given any 1-Lipschitz map f between quasi-metric spaces X, d and Y, ∂ , $\mathbf{P}f$ is 1-Lipschitz between the respective spaces of (sub)normalized sublinear previsions with the d_{KR}^a and ∂_{KR}^a quasi-metrics. Lemma 4.22 of [12] states that, if f is 1-Lipschitz continuous, then $\mathbf{B}^1(\mathbf{P}f)$ preserves naive suprema.

By Lemma 3.16, suprema of directed families of formal balls of (sub)normalized sublinear previsions are naive suprema, so $\mathbf{B}^1(\mathbf{P}f)$ is Scott-continuous. It follows that $\mathbf{P}f$ is 1-Lipschitz continuous. \square

With Theorem 3.17 and Theorem 3.15, we obtain the following.

Corollary 3.19. *There is an endofunctor on the category of continuous (resp., algebraic) complete quasi-metric spaces and 1-Lipschitz continuous maps, which sends every object X, d to the space of (sub)normalized sublinear previsions on X with the d_{KR}^a -Scott topology ($a > 0$), and every 1-Lipschitz continuous map f to $\mathbf{P}f$. \square*

4. Superlinear Previsions

Superlinear previsions are dealt with in a very similar fashion, although, curiously, some of the steps will differ in essential ways. We recall from [6] that superlinear previsions are in bijection with the non-empty compact saturated convex subsets of linear previsions, as follows.

Let r_{DP} be the function that maps every non-empty set E of linear previsions to the functional $h \in \mathcal{L}X \mapsto \inf_{G \in E} G(h)$. The latter is a superlinear prevision as soon as E is compact; it is subnormalized, resp. normalized, as soon as E is a non-empty set of subnormalized, resp. normalized linear previsions.

In the reverse direction, for every superlinear prevision F , let $s_{\text{DP}}(F)$ be the set of all linear previsions G such that $G \geq F$. If F is subnormalized, we write $s_{\text{DP}}^{\leq 1}(F)$ for the set of all subnormalized linear previsions $G \geq F$, and if F is normalized, we write $s_{\text{DP}}^1(F)$ for the set of all normalized linear previsions $G \geq F$. As in Section 3, we use the notation s_{DP}^\bullet .

Proposition 3.22 of [6] states that:

- r_{DP} is continuous from the space of superlinear (resp., subnormalized superlinear, resp., normalized superlinear) previsions on X to $\mathcal{Q}(\mathbf{V}_\bullet X)$ with its lower Vietoris topology, and where $\mathbf{V}_\bullet X$ has the weak topology;
- s_{DP}^\bullet is continuous in the reverse direction;
- $r_{\text{DP}} \circ s_{\text{DP}}^\bullet = \text{id}$;
- $\text{id} \geq s_{\text{DP}}^\bullet \circ r_{\text{DP}}$.

Hence the space of (possibly subnormalized, or normalized) superlinear previsions occurs as a topological retract of $\mathcal{Q}(\mathbf{V}_\bullet X)$. This holds for all topological spaces X , with no restriction.

Let $\mathcal{Q}^{\text{cvx}}(\mathbf{V}_\bullet X)$ denote the subspace of $\mathcal{Q}(\mathbf{V}_\bullet X)$ consisting of convex sets. Theorem 4.15 of [6] additionally states that r_{DP} and s_{DP}^\bullet define a homeomorphism between $\mathcal{Q}^{\text{cvx}}(\mathbf{V}_\bullet X)$ and the corresponding space of (possibly subnormalized, or normalized) superlinear previsions.

The following is where we require a minimax theorem on a non-Hausdorff space. Although this looks very similar to Lemma 3.5, the compact space we use in conjunction with our minimax theorem is not $\mathcal{L}_\alpha^a(X, d)^{\text{patch}}$, but a compact convex space \mathcal{Q} of linear previsions. The latter is almost never Hausdorff.

Lemma 4.1. *Let X, d be a continuous quasi-metric space. Let \bullet be either “ ≤ 1 ” or “1”, and $a > 0$. For every continuous valuation G' in $\mathbf{V}_\bullet X$, for every $\alpha > 0$, for every compact convex subset \mathcal{Q} of $\mathbf{V}_\bullet X$ (with the weak topology),*

$$\sup_{h \in \mathcal{L}_\alpha^a X} \inf_{G \in \mathcal{Q}} (G(h) - G'(h)) = \inf_{G \in \mathcal{Q}} \sup_{h \in \mathcal{L}_\alpha^a X} (G(h) - G'(h)).$$

PROOF. Let $A \stackrel{\text{def}}{=} \mathcal{Q}$. This is compact by assumption. Let $B \stackrel{\text{def}}{=} \mathcal{L}_\alpha^a(X, d)$, and let $f: (G, h) \in A \times B \mapsto (G(h) - G'(h))$. This is a function that takes its values in $[-a, a]$, hence in \mathbb{R} . It is also lower semicontinuous on G , since the set of elements $G \in \mathbf{V}_\bullet X$ such that $f(G, h) > t$ is equal to $[h > G'(h) + t]$, an open set by the definition of the weak topology.

The function f is certainly convex in its first argument and concave in its second argument, being linear in both. Theorem 3.4 then implies that $\sup_{h \in \mathcal{L}_\alpha^a X} \inf_{G \in \mathcal{Q}} (G(h) - G'(h)) = \inf_{G \in \mathcal{Q}} \sup_{h \in \mathcal{L}_\alpha^a X} (G(h) - G'(h))$. \square

Proposition 4.2. *Let X, d be a continuous complete quasi-metric space. Let \bullet be either “ ≤ 1 ” or “1”, and $a > 0$. For all $\mathcal{Q}, \mathcal{Q}' \in \mathcal{Q}(\mathbf{V}_\bullet X)$, where $\mathbf{V}_\bullet X$ has the weak topology,*

$$(d_{KR}^a)_\mathcal{Q}(\mathcal{Q}, \mathcal{Q}') \geq d_{KR}^a(r_{\text{DP}}(\mathcal{Q}), r_{\text{DP}}(\mathcal{Q}')), \quad (5)$$

with equality if \mathcal{Q} is convex.

As with Proposition 3.7, we note that the weak topology coincides with the d_{KR}^a -Scott topology on $\mathbf{V}_\bullet X$, by Theorem 2.2, which allows us to make sense of the left-hand side.

PROOF. The proof is extremely close to Proposition 3.7. We develop the left-hand side by using the definition of $d_\mathcal{Q}$ in terms of d_{KR} :

$$\begin{aligned} (d_{KR}^a)_\mathcal{Q}(\mathcal{Q}, \mathcal{Q}') &= (d_{KR}^a)_{KR}(F_\mathcal{Q}, F_{\mathcal{Q}'}) \\ &= \sup_{H \in \mathcal{L}_1(\mathbf{V}_\bullet X, d_{KR}^a)} \mathbf{d}_\mathbb{R}(F_\mathcal{Q}(H), F_{\mathcal{Q}'}(H)) \\ &= \sup_{H \in \mathcal{L}_1(\mathbf{V}_\bullet X, d_{KR}^a)} \mathbf{d}_\mathbb{R}(\inf_{G \in \mathcal{Q}} H(G), \inf_{G' \in \mathcal{Q}'} H(G')). \end{aligned}$$

The right-hand side is equal to:

$$\begin{aligned} d_{KR}^a(r_{\text{DP}}(\mathcal{Q}), r_{\text{DP}}(\mathcal{Q}')) &= \sup_{h \in \mathcal{L}_1^a X} \mathbf{d}_\mathbb{R}(r_{\text{DP}}(\mathcal{Q})(h), r_{\text{DP}}(\mathcal{Q}')(h)) \\ &= \sup_{h \in \mathcal{L}_1^a X} \mathbf{d}_\mathbb{R}(\inf_{G \in \mathcal{Q}} G(h), \inf_{G' \in \mathcal{Q}'} G'(h)). \end{aligned}$$

Among the elements H of $\mathcal{L}_1(\mathbf{V}_\bullet X, d_{\text{KR}}^a)$, we find those obtained from $h \in \mathcal{L}_1^a(X, d)$ by letting $H(G) \stackrel{\text{def}}{=} G(h)$, according to Lemma 3.3. That shows inequality (5).

We now assume \mathcal{Q} convex. If the inequality were strict, then there would be a real number t such that $(d_{\text{KR}}^a)_{\mathcal{Q}}(\mathcal{Q}, \mathcal{Q}') > t > d_{\text{KR}}^a(r_{\text{DP}}(\mathcal{Q}), r_{\text{DP}}(\mathcal{Q}'))$. Note that the second inequality implies that t is (strictly) positive.

Now $(d_{\text{KR}}^a)_{\mathcal{Q}}(\mathcal{Q}, \mathcal{Q}')$ can be expressed à la Pompeiu-Hausdorff as $\sup_{G' \in \mathcal{Q}'} \inf_{G \in \mathcal{Q}} d_{\text{KR}}^a(G, G')$, so there is a linear prevision G' in \mathcal{Q}' such that $\inf_{G \in \mathcal{Q}} d_{\text{KR}}^a(G, G') > t$, namely such that $\inf_{G \in \mathcal{Q}} \sup_{h \in \mathcal{L}_1^a X} \mathbf{d}_{\mathbb{R}}(G(h), G'(h)) > t$. This means that there is a real number $t' > t$ such that for every $G \in \mathcal{Q}$, there is an $h \in \mathcal{L}_1^a(X, d)$ such that $\mathbf{d}_{\mathbb{R}}(G(h), G'(h)) > t'$, and since $t' > t > 0$, $G(h) - G'(h) > t'$. This implies that $\inf_{G \in \mathcal{Q}} \sup_{h \in \mathcal{L}_1^a X} (G(h) - G'(h)) > t$. We now use Lemma 4.1 and obtain $\sup_{h \in \mathcal{L}_1^a X} \inf_{G \in \mathcal{Q}} (G(h) - G'(h)) > t$. Therefore, there is a map h in $\mathcal{L}_1^a(X, d)$ such that $\inf_{G \in \mathcal{Q}} (G(h) - G'(h)) > t$, namely, $\inf_{G \in \mathcal{Q}} G(h) - G'(h) > t$. This implies $\inf_{G \in \mathcal{Q}} G(h) - \inf_{G' \in \mathcal{Q}'} G'(h) > t$, that is, $r_{\text{DP}}(\mathcal{Q})(h) - r_{\text{DP}}(\mathcal{Q}')(h) > t$. Recalling that $t > 0$, this entails $\mathbf{d}_{\mathbb{R}}(r_{\text{DP}}(\mathcal{Q})(h), r_{\text{DP}}(\mathcal{Q}')(h)) > t$. However, this is impossible since $d_{\text{KR}}^a(r_{\text{DP}}(\mathcal{Q}), r_{\text{DP}}(\mathcal{Q}')) = \sup_{h \in \mathcal{L}_1^a X} \mathbf{d}_{\mathbb{R}}(r_{\text{DP}}(\mathcal{Q})(h), r_{\text{DP}}(\mathcal{Q}')(h))$ is smaller than t . \square

Lemma 4.3. *Let X, d be a continuous complete quasi-metric space. Let \bullet be either “ ≤ 1 ” or “1”, and $a > 0$. Then:*

1. r_{DP} is 1-Lipschitz from $\mathcal{Q}(\mathbf{V}_\bullet X)$, $(d_{\text{KR}}^a)_{\mathcal{Q}}$ to the space of (sub)normalized superlinear previsions on X with the d_{KR}^a quasi-metric;
2. s_{DP}^\bullet preserves distances on the nose; in particular, s_{AP}^\bullet is 1-Lipschitz.

PROOF. (1) is by the first part of Proposition 4.2. By the second part of Proposition 4.2, using the fact that $s_{\text{DP}}^\bullet(F)$ is convex,

$$(d_{\text{KR}}^a)_{\mathcal{Q}}(s_{\text{DP}}^\bullet(F), s_{\text{DP}}^\bullet(F')) = d_{\text{KR}}^a(r_{\text{DP}}(s_{\text{DP}}^\bullet(F)), r_{\text{DP}}(s_{\text{DP}}^\bullet(F'))) = d_{\text{KR}}^a(F, F'),$$

and this shows (2). \square

Next we show that s_{DP}^\bullet is not just 1-Lipschitz, but 1-Lipschitz continuous.

Lemma 4.4. *Let X, d be a continuous complete quasi-metric space. Let \bullet be either “ ≤ 1 ” or “1”, and $a > 0$. Then s_{DP}^\bullet is 1-Lipschitz continuous from the space of (sub)normalized sublinear previsions on X with the d_{KR}^a quasi-metric to $\mathcal{Q}(\mathbf{V}_\bullet X)$, $(d_{\text{KR}}^a)_{\mathcal{Q}}$.*

PROOF. By Lemma 4.3, $\mathbf{B}^1(r_{\text{DP}})$ and $\mathbf{B}^1(s_{\text{DP}}^\bullet)$ are monotonic. Since $r_{\text{DP}} \circ s_{\text{DP}}^\bullet = \text{id}$, $\mathbf{B}^1(r_{\text{DP}}) \circ \mathbf{B}^1(s_{\text{DP}}^\bullet) = \text{id}$. Since $\text{id} \geq s_{\text{DP}}^\bullet \circ r_{\text{DP}}$, i.e., since $\mathcal{Q} \subseteq s_{\text{DP}}^\bullet(r_{\text{DP}}(\mathcal{Q}))$ for every \mathcal{Q} , we have that $(d_{\text{KR}}^a)_{\mathcal{Q}}(s_{\text{DP}}^\bullet(r_{\text{DP}}(\mathcal{Q})), \mathcal{Q}) = 0$, since the specialization ordering of $(d_{\text{KR}}^a)_{\mathcal{H}}$ is reverse inclusion, so $\mathbf{B}^1(s_{\text{DP}}^\bullet)(\mathbf{B}^1(r_{\text{DP}})(\mathcal{Q}, r)) = (s_{\text{DP}}^\bullet(r_{\text{DP}}(\mathcal{Q})), r) \leq (d_{\text{KR}}^a)_{\mathcal{Q}}^+(\mathcal{Q}, r)$ for every $r \in \mathbb{R}_+$.

This shows that $\mathbf{B}^1(s_{\text{DP}}^\bullet)$ is left adjoint to $\mathbf{B}^1(r_{\text{DP}})$. We conclude because left adjoints preserve all existing suprema, hence in particular directed suprema. \square

Lemma 4.5. *Let X, d be a continuous complete quasi-metric space. Let \bullet be either “ ≤ 1 ” or “ 1 ”, and $a > 0$. Then r_{DP} is 1-Lipschitz continuous from $\mathcal{Q}(\mathbf{V}_\bullet X)$, $(d_{\text{KR}}^a)_{\mathcal{Q}}$ to the space of (sub)normalized sublinear previsions on X with the d_{KR}^a quasi-metric.*

PROOF. By Lemma 4.3, $\mathbf{B}^1(r_{\text{DP}})$ is monotonic. Let us consider a directed family of formal balls $(\mathcal{Q}_i, r_i)_{i \in I}$ on $\mathcal{Q}(\mathbf{V}_\bullet X)$, $(d_{\text{KR}}^a)_{\mathcal{Q}}$, with supremum (\mathcal{Q}, r) . By Theorem 2.2, $\mathbf{V}_\bullet X, d_{\text{KR}}^a$ is continuous complete; by Theorem 2.4, $\mathcal{Q}(\mathbf{V}_\bullet X)$, $(d_{\text{KR}}^a)_{\mathcal{Q}}$ is complete, and directed suprema of formal balls are naive suprema. This means that for every $H \in \mathcal{L}_\alpha^a(\mathbf{V}_\bullet X, d_{\text{KR}}^a)$ ($\alpha > 0$), $F_{\mathcal{Q}}(H) = \sup_{i \in I} (F_{\mathcal{Q}_i}(H) - \alpha r_i + \alpha r)$, and $r = \inf_{i \in I} r_i$. In particular, for $H(G) \stackrel{\text{def}}{=} G(h)$, for any $h \in \mathcal{L}_1^a(X, d)$ (Lemma 3.3), so that $F_{\mathcal{Q}}(H) = \min_{G \in \mathcal{Q}} H(G) = \min_{G \in \mathcal{Q}} G(h)$, we obtain $\min_{G \in \mathcal{Q}} G(h) = \sup_{i \in I} (\min_{G \in \mathcal{Q}_i} G(h) - \alpha r_i + \alpha r)$, namely $r_{\text{DP}}(\mathcal{Q})(h) = \sup_{i \in I} (r_{\text{DP}}(\mathcal{Q}_i)(h) - \alpha r_i + \alpha r)$. This characterizes $(r_{\text{DP}}(\mathcal{Q}), r)$ as the naive supremum of $(r_{\text{DP}}(\mathcal{Q}_i), r_i)_{i \in I}$. We recall from Section 2.5 (or Proposition 4.7 of [12]) that every naive supremum that is continuous as a map from $\mathcal{L}_\infty X$ to $\overline{\mathbb{R}}_+$ is a supremum. Since $r_{\text{DP}}(\mathcal{Q})$ is a superlinear prevision, it is continuous, so $r_{\text{DP}}(\mathcal{Q})$ is the supremum of $(r_{\text{DP}}(\mathcal{Q}_i), r_i)_{i \in I}$. We conclude that $\mathbf{B}^1(r_{\text{DP}})$ is Scott-continuous. \square

Theorem 4.6 (Continuous Completeness for Superlinear Previsions).

Let X, d be a continuous complete quasi-metric space, and $a > 0$.

The space of all subnormalized (resp., normalized) superlinear previsions on X with the d_{KR}^a quasi-metric is continuous complete.

It occurs as a 1-Lipschitz continuous retract of $\mathcal{Q}(\mathbf{V}_{\leq 1} X)$, $(d_{\text{KR}}^a)_{\mathcal{Q}}$ (resp., $\mathcal{Q}(\mathbf{V}_1 X)$, $(d_{\text{KR}}^a)_{\mathcal{Q}}$), and as an isometric copy of $\mathcal{Q}^{\text{cvx}}(\mathbf{V}_{\leq 1} X)$, $(d_{\text{KR}}^a)_{\mathcal{Q}}$ (resp., $\mathcal{Q}^{\text{cvx}}(\mathbf{V}_1 X)$, $(d_{\text{KR}}^a)_{\mathcal{Q}}$) through r_{DP} and $s_{\text{DP}}^{\leq 1}$ (resp., s_{DP}^1).

The d_{KR}^a -Scott topology on the space of subnormalized (resp., normalized) superlinear previsions coincides with the weak topology.

Finally, directed suprema of formal balls are computed as naive suprema.

PROOF. We have already seen that r_{AP} and s_{AP}^\bullet are continuous, with respect to the weak topologies. By Lemma 4.5 and Lemma 4.4, they are also 1-Lipschitz continuous with respect to $(d_{\text{KR}}^a)_\mathcal{Q}$ on $\mathcal{Q}(\mathbf{V}\bullet X)$ and d_{KR}^a on the space of (sub)normalized superlinear previsions.

We also recall that $r_{\text{DP}} \circ s_{\text{DP}}^\bullet = \text{id}$. This shows the 1-Lipschitz continuous retract part, and also the isometric copy part, since s_{DP}^\bullet and r_{DP} restrict to bijections with the subspace $\mathcal{Q}^{cvx}(\mathbf{V}\bullet X)$.

Lemma 3.13 tells us that the space of (sub)normalized superlinear previsions on X is continuous complete.

Let us show that the d_{KR}^a -Scott topology on the space of (sub)normalized superlinear previsions is the weak topology. For that, we recall that the $(d_{\text{KR}}^a)_\mathcal{Q}$ -Scott topology coincides with the upper Vietoris topology on $\mathcal{Q}(\mathbf{V}\bullet X)$, and we note that s_{DP}^\bullet is both a topological embedding for the weak and upper Vietoris topologies, and the section part of a 1-Lipschitz continuous retraction, hence a topological embedding for the d_{KR}^a -Scott and $(d_{\text{KR}}^a)_\mathcal{Q}$ -Scott topologies. We conclude by Fact 3.14.

It remains to show that directed suprema of formal balls are computed as naive suprema. The argument is exactly as for Proposition 3.11. We use the fact that $p \stackrel{\text{def}}{=} \mathbf{B}^1(r_{\text{DP}})$ and $e \stackrel{\text{def}}{=} \mathbf{B}^1(s_{\text{DP}}^\bullet)$ form an order-retraction, and that the supremum (F, r) of a directed family $(F_k, r_k)_{k \in K}$ is $p(\sup_{k \in K} e(F_k, r_k))$ by Lemma 3.10. We invoke Theorem 2.2 and Theorem 2.4 to check that the supremum $(\mathcal{Q}, r) = \sup_{k \in K} e(F_k, r_k)$, computed in $\mathbf{B}(\mathcal{Q}(\mathbf{V}\bullet X), (d_{\text{KR}}^a)_\mathcal{Q})$, is a naive supremum. In particular, $r = \inf_{k \in K} r_k$. Let $\mathcal{Q}_k = s_{\text{DP}}^\bullet(F_k)$. Then $F_{\mathcal{Q}_k}(H) = \inf_{G \in \mathcal{Q}_k} H(G) = \inf_{G \in \mathbf{V}\bullet X, G \geq F_k} H(G)$ for every $H \in \mathcal{L}_\alpha^a(\mathbf{V}\bullet X, d_{\text{KR}}^a)$. We have $(F, r) = p(\mathcal{Q}, r)$, so $F = r_{\text{DP}}(\mathcal{Q})$, and that implies that $F(h) = \inf_{G \in \mathcal{Q}} G(h)$ for every $h \in \mathcal{L}_\alpha^a(X, d)$. The map $H: G \mapsto G(h)$ is in $\mathcal{L}_\alpha^a(\mathbf{V}\bullet X, d_{\text{KR}}^a)$ by Lemma 3.3, and:

$$\begin{aligned}
F(h) &= \inf_{G \in \mathcal{Q}} H(G) \\
&= F_{\mathcal{Q}}(H) \\
&= \sup_{i \in I} \left(\inf_{G \in \mathbf{V}\bullet X, G \geq F_i} H(G) - \alpha r_i + \alpha r \right) \\
&= \sup_{i \in I} \left(\inf_{G \in \mathbf{V}\bullet X, G \geq F_i} G(h) - \alpha r_i + \alpha r \right) \\
&= \sup_{i \in I} (F_i(h) - \alpha r_i + \alpha r).
\end{aligned}$$

□

As with sublinear previsions, algebraicity is preserved, too.

Theorem 4.7 (Algebraic Completeness for Superlinear Previsions).

Let X, d be an algebraic complete quasi-metric space, with a strong basis \mathcal{B} . Let $a > 0$.

The space of all subnormalized (resp., normalized) superlinear previsions on X with the d_{KR}^a quasi-metric is algebraic complete.

All the superlinear previsions of the form $h \mapsto \min_{i=1}^m \sum_{j=1}^{n_i} a_{ij} h(x_{ij})$, with $m \geq 1$, $\sum_{j=1}^{n_i} a_{ij} \leq 1$ (resp., $= 1$) for every i , and where each x_{ij} is in \mathcal{B} , are center points, and form a strong basis.

PROOF. We cannot appeal to the final part of Lemma 3.13 in order to show algebraicity. Indeed, we would need to show $d_{KR}^a(r_{DP}(\mathcal{Q}), F') = (d_{KR}^a)_{\mathcal{Q}}(\mathcal{Q}, s_{DP}^{\bullet}(F'))$ to this end, but Proposition 4.2 only allows us to conclude the similarly looking, but different equality $d_{KR}^a(F, r_{DP}(\mathcal{Q}')) = (d_{KR}^a)_{\mathcal{Q}}(s_{DP}^{\bullet}(F), \mathcal{Q}')$.

We can still use the fact that for every (sub)normalized superlinear prevision F , $(F, 0)$ is the supremum of a directed family of formal balls (with respect to the d_{KR}^a quasi-metric) whose centers are of the form $(r_{DP}(\mathcal{Q}))$, where \mathcal{Q} is taken from a strong basis of $\mathcal{Q}(\mathbf{V}_{\bullet}X)$. This is by Theorem 2.2 and Theorem 2.4, which also tell us that we can take \mathcal{Q} of the form $\uparrow\{\sum_{j=1}^{n_1} a_{1j} \delta_{x_{1j}}, \dots, \sum_{j=1}^{n_m} a_{mj} \delta_{x_{mj}}\}$ where $m \geq 1$ and each x_{ij} is in \mathcal{B} . Then $r_{DP}(\mathcal{Q})$ is exactly one of the superlinear previsions mentioned in the theorem, namely it is of the form $h \mapsto \min_{i=1}^m \sum_{j=1}^{n_i} a_{ij} h(x_{ij})$.

In order to show that our spaces of superlinear previsions are algebraic, it remains to show that $\mathring{F}: h \mapsto \min_{i=1}^m \sum_{j=1}^{n_i} a_{ij} h(x_{ij})$, where $m \geq 1$ and each x_{ij} is a center point, is itself a center point. We cannot use the final part of Lemma 3.13. Instead we will show this directly. In other words, we show that $B_{(\mathring{F}, 0), \epsilon}^{d_{KR}^a}$ is Scott-open for every $\epsilon > 0$. To start with, it is upwards closed: every open ball in a space of formal balls is.

Let us consider a directed family $(F_k, r_k)_{k \in K}$ where each F_k is a (sub)normalized superlinear prevision, with supremum (F, r) in $B_{(\mathring{F}, 0), \epsilon}^{d_{KR}^a}$. For every $h \in \mathcal{L}_1^a(X, d)$, $d_{\mathbb{R}}(\mathring{F}(h), F(h)) < \epsilon - r$, that is, $\epsilon > r$ and $\mathring{F}(h) - \epsilon + r < F(h)$. In particular, and using the fact that the supremum (F, r) is a naive supremum (final part of Theorem 4.6), there is a $k \in K$ such that $\mathring{F}(h) - \epsilon + r_k < F_k(h)$. We can also pick k such that $\epsilon - r_k > 0$, since $r = \inf_{k \in K} r_k$. By directedness, we can pick $k \in K$ that satisfies both conditions at the same time.

We note that \mathring{F} is not just Scott-continuous, but that it is also continuous from $\mathcal{L}_{\alpha}^a(X, d)^d$ to $(\overline{\mathbb{R}}_+)^d$, for any $\alpha > 0$. Indeed, by the second part of Theorem 2.1, the maps $f \mapsto \sum_{j=1}^{n_i} a_{ij} h(x_{ij})$ are continuous from $\mathcal{L}_{\alpha}^a(X, d)^d$

to $(\overline{\mathbb{R}_+})^d$ for every i , $1 \leq i \leq m$. Then \mathring{F} is continuous, because \max is continuous from $(\overline{\mathbb{R}_+})^d \times (\overline{\mathbb{R}_+})^d$ to $(\overline{\mathbb{R}_+})^d$; this is easy to see from first principles, noting that the proper open subsets of $(\overline{\mathbb{R}_+})^d$ are the intervals $[0, t[$, $t \in \mathbb{R}_+$.

It follows that the map $h \in \mathcal{L}_1^a(X, d)^{\text{patch}} \mapsto F_k(h) - \mathring{F}(h)$ is lower semi-continuous. Therefore, the set V_k of all $h \in \mathcal{L}_1^a(X, d)$ such that $\mathring{F}(h) - \epsilon + r_k < F_k(h)$ is open in $\mathcal{L}_1^a(X, d)^{\text{patch}}$. We have seen that, for every $h \in \mathcal{L}_1^a(X, d)$, there is a $k \in K$ such that $\epsilon - r_k > 0$ and $\mathring{F}(h) - \epsilon + r_k < F_k(h)$. In other words, $(V_k)_{k \in K, \epsilon - r_k > 0}$ forms an open cover of $\mathcal{L}_1^a(X, d)^{\text{patch}}$. The latter is stably compact by Theorem 2.1, so we can extract a finite subcover. This gives us a finite set J of indices from K such that for every $h \in \mathcal{L}_1^a(X, d)$, there is a $j \in J$ such that $\mathring{F}(h) - \epsilon + r_j < F_j(h)$, and $\epsilon - r_j > 0$. Using the fact that $(F_k, r_k)_{k \in K}$ is directed, there is even a single $k \in K$ such that $\epsilon - r_k > 0$ and $(F_j, r_j) \leq^{d_{\text{KR}}^a} (F_k, r_k)$ for every $j \in J$, and that implies that for every $h \in \mathcal{L}_1^a(X, d)$, $\mathring{F}(h) - \epsilon + r_k < F_k(h)$. That shows that (F_k, r_k) is in $B_{(\mathring{F}, 0), \epsilon}^{d_{\text{KR}}^a}$, finishing our proof that $B_{(\mathring{F}, 0), \epsilon}^{d_{\text{KR}}^a}$ is Scott-open. Therefore $\mathring{F}: h \mapsto \min_{i=1}^m \sum_{j=1}^{n_i} a_{ij} h(x_{ij})$ is a center point. \square

Lemma 4.8. *Let X, d and Y, ∂ be two continuous complete quasi-metric spaces, and $f: X, d \mapsto Y, \partial$ be a 1-Lipschitz continuous map. Let also $a > 0$. $\mathbf{P}f$ restricts to a 1-Lipschitz continuous map from the space of normalized, resp. subnormalized superlinear previsions on X to the space of normalized, resp. subnormalized superlinear previsions on Y , with the d_{KR}^a and ∂_{KR}^a quasi-metrics.*

PROOF. As in the proof of Lemma 3.18, we use Lemmata 4.21 and 4.22 of [12] in order to show that $\mathbf{P}f$ is a 1-Lipschitz map between spaces of (sub)normalized superlinear previsions, with the d_{KR}^a and ∂_{KR}^a quasi-metrics, and that $\mathbf{B}^1(\mathbf{P}f)$ preserves naive suprema. By Theorem 4.6, suprema of directed families of formal balls of (sub)normalized sublinear previsions are naive suprema, so $\mathbf{B}^1(\mathbf{P}f)$ is Scott-continuous. It follows that $\mathbf{P}f$ is 1-Lipschitz continuous. \square

With Theorem 4.7 and Theorem 4.6, we obtain the following.

Corollary 4.9. *There is an endofunctor on the category of continuous (resp., algebraic) complete quasi-metric spaces and 1-Lipschitz continuous maps, which sends every object X, d to the space of (sub)normalized superlinear previsions on X with the d_{KR}^a -Scott topology ($a > 0$), and every 1-Lipschitz continuous map f to $\mathbf{P}f$. \square*

Acknowledgments

I thank Vasile Berinde for drawing my attention that credit should be given to D. Pompeiu (1905) for proposing a definition of a metric between planar curves, which F. Hausdorff then rigorously defined and extended in 1914. I also thank the anonymous referee.

References

- [1] de Brecht, M., Goubault-Larrecq, J., Jia, X., Lyu, Z., 2019. Domain-complete and LCS-complete spaces. *Electronic Notes in Theoretical Computer Science* 345, 3–35, proc. 8th International Symposium on Domain Theory (ISDT'19), also available as arXiv report 1902.11142 [math.GN].
- [2] Edalat, A., Heckmann, R., 1998. A computational model for metric spaces. *Theoretical Computer Science* 193, 53–73.
- [3] Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M., Scott, D. S., 2003. *Continuous Lattices and Domains*. Vol. 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press.
- [4] Goubault-Larrecq, J., Sep. 2007. Continuous previsions. In: Duparc, J., Henzinger, T. A. (Eds.), *Proceedings of the 16th Annual EACSL Conference on Computer Science Logic (CSL'07)*. Springer-Verlag LNCS 4646, Lausanne, Switzerland, pp. 542–557.
- [5] Goubault-Larrecq, J., 2013. *Non-Hausdorff Topology and Domain Theory—Selected Topics in Point-Set Topology*. Vol. 22 of *New Mathematical Monographs*. Cambridge University Press.
- [6] Goubault-Larrecq, J., 6 2016. Isomorphism theorems between models of mixed choice. *Mathematical Structures in Computer Science FirstView*, 1–36, to appear.
URL http://journals.cambridge.org/article_S0960129515000547
- [7] Goubault-Larrecq, J., 2017. Complete quasi-metrics for hyperspaces, continuous valuations, and previsions, arXiv 1707.03784, v2, <https://arxiv.org/abs/1707.03784>; v6, July 2020.

- [8] Goubault-Larrecq, J., 2017. A non-Hausdorff minimax theorem. *Minimax Theory and its Applications* 3 (1), 73–80.
- [9] Goubault-Larrecq, J., 2019. Formal ball monads. *Topology and its Applications* 263, 372–391.
URL <http://www.sciencedirect.com/science/article/pii/S0166864119302160>
- [10] Goubault-Larrecq, J., 2020. Some topological properties of spaces of Lipschitz continuous maps on quasi-metric spaces. *Topology and its Applications* 282, 107281.
URL <https://www.sciencedirect.com/science/article/pii/S0166864120302248>
- [11] Goubault-Larrecq, J., 2021. Kantorovich-Rubinshtein quasi-metrics I: Spaces of measures and of continuous valuations. *Topology and its Applications* 295, 107673.
URL <https://www.sciencedirect.com/science/article/pii/S0166864121000870>
- [12] Goubault-Larrecq, J., 2022. Kantorovich-Rubinshtein quasi-metrics II: Hyperspaces and powerdomains. *Topology and its Applications* 305, 107885.
URL <https://www.sciencedirect.com/science/article/abs/pii/S016686412100300X>
- [13] Goubault-Larrecq, J., Ng, K. M., Oct. 2017. A few notes on formal balls. *Logical Methods in Computer Science* 13 (4:18), 1–34, special issue of Domain Theory and Applications (the Domains XII Workshop), Cork, Ireland, August 2015.
- [14] Keimel, K., 2008. Topological cones: Functional analysis in a T_0 -setting. *Semigroup Forum* 77 (1), 109–142.
- [15] Weihrauch, K., Schreiber, U., 1981. Embedding metric spaces into cpo's. *Theoretical Computer Science* 16 (1), 5–24.