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► **To cite this version:**

Frederic Mazenc, Michael Malisoff. Almost Finite-Time Observers for Parameters and State Variables of Nonlinear Systems. Joint IFAC Conference: SSSC – TDS – LPVS 2022, Sep 2022, Montreal, Canada. hal-03781119v1

**HAL Id: hal-03781119**

**<https://inria.hal.science/hal-03781119v1>**

Submitted on 20 Sep 2022 (v1), last revised 26 Oct 2022 (v2)

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# Almost Finite-Time Observers for Parameters and State Variables of Nonlinear Systems

Frédéric Mazenc and Michael Malisoff

**Abstract**—We provide new classes of finite-time and fast-converging observers for continuous-time nonlinear systems that contain unknown constant parameters and measurement delays. When the dynamics are affine in the unmeasured state, we obtain fixed time observers for states and for the model parameters, i.e., the finite convergence time is independent of the initial state of the dynamics. When the dynamics contain a nonlinearity that depends on the unmeasured state, we instead obtain asymptotic convergence of our observers, whose rate of convergence converges to infinity as the growth rate of this nonlinearity converges to zero, so in this case we call the observers almost finite-time observers. Our examples illustrate the ease with which our assumptions are verified in practice.

**Index Terms**—Observers, nonlinear, estimation

## I. INTRODUCTION

This work continues our search (which we started in [1], [2], [3], [4], [5], [6], and [7]) for finite-time observers for nonlinear systems under suitable structural conditions. While our prior works provided arbitrarily fast convergence of the observers for states of perturbed systems (which gave fixed finite-time convergence in the special case where the uncertainties are zero and where the dynamics were affine in the unmeasured state) here we consider a more difficult problem where in addition to requiring finite-time state observers, one must identify unknown model parameters. This note is motivated by the ubiquity of unmeasured states in engineering applications, and the value of using observers to substitute for unmeasured states in controllers; see, e.g., [8], [9], [10], [11], [12], [13], and [14].

An important setting where unknown constant model parameters occur is in artificial neural network expansions, where an unknown time-varying function is expressed as a linear combination of known basis functions with unknown constant weights, where the goal is to compute values of the unknown weights and thereby identify the unknown uncertainty; see, e.g., Section IV-A below. One standard approach to estimating unknown model parameters is adaptive control, which often calls for persistency of excitation (or PE) conditions (but see [15], which used an adaptive controller and barrier Lyapunov functions to estimate an unknown curvature in a marine robotic planar curve tracking problem, without PE conditions). On the other hand, adaptive control usually does not give fixed finite-time parameter identification, and is not amenable to the estimation of unmeasured states.

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Supported by US National Science Foundation Grant 2009659.

This motivates this note, which addresses the preceding challenges in cases where there is also an affine output measurement with measurement delays. In Section II, we express our assumptions using integrals and invertibility conditions. This makes it possible to prove fixed finite time convergence when the dynamics are affine in the unmeasured state, and almost finite convergence in the context of [6] when there are nonlinearities depending on the unmeasured state. As in the global convergence results in [6], we assume that the nonlinearity that depends on the unmeasured state satisfies a global Lipschitzness condition, and then we place bounds on a corresponding global Lipschitz constant. However, since [6] did not identify unknown model parameters, we believe that this note adds significant value as compared to [6].

Then in Section III, we show how to express our assumptions and observers without integrals. We illustrate our results in Section IV, and we close in Section V with our conjectures and research plans. This work improves on our preliminary conference version [16], which was confined to cases where the nonlinearity only depended on the measured state and time, and which only allowed one unknown parameter in the dynamics when measurement delays were present.

We use standard notation, which we simplify when no confusion would arise. The dimensions of our Euclidean spaces are arbitrary, unless we indicate otherwise, and  $|\cdot|$  is the standard Euclidean norm and the corresponding matrix norm. Also, we set  $g_t(s) = g(t+s)$  for functions  $g$  and all  $s$  and  $t$  such that  $t+s$  is in the domain of  $g$ ,  $|f|_J$  denotes the usual supremum over any interval  $J$  in the domain of functions  $f$ , and  $|f|_\infty$  is the supremum of a function  $f$ .

## II. MAIN OBSERVER DESIGN

### A. Studied system

We consider the system

$$\begin{cases} \dot{\xi}(t) &= A\xi(t) + \varphi(C\xi(t), u(t)) + \Delta(\xi(t), t) \\ &+ \sum_{j=1}^q \epsilon_j \gamma_j(t) \\ y_\xi(t) &= C\xi(t-h) \end{cases} \quad (1)$$

where  $\xi$  is valued in  $\mathbb{R}^n$ ,  $y_\xi$  is valued in  $\mathbb{R}$ ,  $\varphi$  and  $\Delta$  are locally Lipschitz continuous nonlinear functions (which play different roles in our analysis, so we do not combine them with the drift term  $A\xi$  to obtain one function of  $(\xi, t)$ ), the real constants  $\epsilon_j$  are unknown,  $u$  is a known input,  $h \geq 0$  is a known constant delay,  $A \in \mathbb{R}^{n \times n}$  and  $C \in \mathbb{R}^{1 \times n}$  are constant real matrices, and the functions  $\gamma_j : \mathbb{R} \rightarrow \mathbb{R}^n$  are locally bounded and piecewise continuous.

Our first assumption is this standard observability condition, we precludes the possibility of extending the dynamics using the additional equations  $\dot{\epsilon}_j = 0$  for  $j = 1, 2, \dots, q$  to reduce the problem to one of identifying unknown states:

*Assumption 1:* The pair  $(A, C)$  is observable.

Let us introduce a constant  $\tau > 0$ , the matrix

$$E = \int_{-\tau}^0 e^{A^\top s} C^\top C e^{A(s-h)} ds \quad (2)$$

which is invertible because of Assumption 1 (by [17, Section 3.5]), and the functional

$$\kappa(\phi) = \int_{-\tau}^0 e^{A^\top r} C^\top C \int_{r-h}^0 e^{A(r-h-\ell)} \phi(\ell) d\ell dr. \quad (3)$$

We also use the function  $W : \mathbb{R} \rightarrow \mathbb{R}^q$  that is defined by

$$\begin{aligned} W(t) &= [W_1(t), \dots, W_q(t)]^\top, \text{ where} \\ W_i(t) &= \int_{t-T-h}^{t-h} C[AE^{-1}\kappa(\gamma_{i,m}) + \gamma_i(m)] dm \\ &\text{for } i = 1, \dots, q, \end{aligned} \quad (4)$$

where  $T > 0$  is a constant, and  $\gamma_{i,m}$  is defined by  $\gamma_{i,m}(s) = \gamma_i(m+s)$  for real values  $m$  and  $s$  and  $i = 1, \dots, q$ .

Our second assumption is as follows (but see Section III for ways to express the  $W_i$ 's in terms of states of dynamical extensions instead of integrals):

*Assumption 2:* There exist  $q$  nonnegative constants  $\tau_1, \dots, \tau_q$  such that  $\tau_i < \tau_j$  for pairs  $(i, j)$  with  $j > i$  such that the matrix valued function  $M : \mathbb{R} \rightarrow \mathbb{R}^{q \times q}$  defined by

$$M(t) = [W(t - \tau_1) \quad \dots \quad W(t - \tau_q)] \quad (5)$$

is invertible for all  $t \geq \tau_q + T + \tau$ .

In terms of the above notation and the constant

$$\bar{\beta} = \int_{-\tau}^0 \int_{\ell-h}^0 |E^{-1} e^{A^\top \ell} C^\top C e^{A(\ell-h-m)}| dmd\ell \quad (6)$$

and the  $\mathbb{R}^{n \times q}$ -valued function

$$\kappa^\sharp(t) = [E^{-1}\kappa(\gamma_{1,t}) \quad \dots \quad E^{-1}\kappa(\gamma_{q,t})] \quad (7)$$

and the inverse values  $M(t)^{-1}$  for the matrix  $M(t)$  that are defined for all  $t \geq \tau_q + T + \tau$ , our last assumption is:

*Assumption 3:* The function  $\Delta$  in (1) admits a constant  $L_\Delta \geq 0$  such that  $|\Delta(\xi_a, t) - \Delta(\xi_b, t)| \leq L_\Delta |\xi_a - \xi_b|$  for all  $\xi_a \in \mathbb{R}^n$ ,  $\xi_b \in \mathbb{R}^n$ , and  $t \in \mathbb{R}$ . Also, the function  $\kappa_*(t) = \kappa^\sharp(t)(M(t)^{-1})^\top$  is bounded, and the bound

$$L_\Delta (\bar{\beta} \{T\sqrt{q}|\kappa_*|_\infty |CA| + 1\} + T|\kappa_*|_\infty |C|\sqrt{q}) < 1 \quad (8)$$

is satisfied.

Our smallness condition (8) on the uniform global Lipschitz constant  $L_\Delta$  for  $\Delta$  will allow us to use a contractivity approach from [18] to obtain our result. It holds with  $L_\Delta = 0$  when (1) is affine in the unmeasured state and where we can therefore use  $\Delta = 0$ . When  $\Delta = 0$ , the proof of our theorem will show that the theorem remains true if we remove the assumption that the function  $\kappa_*$  from Assumption 3 is bounded. This affineness property in the unmeasured state occurs in notable applications, e.g., manipulator and nonholonomic systems in chained form after changes in variables (as in [3] and [4]), vibrating membranes (as in [1] and [2]), and more. Also, the nonzero  $\Delta$  case covers bilinear systems with bounded controls, i.e., systems having products

of state and bounded control components, which commonly arise in power electronics [19]. See also Section IV for systems with nonzero  $\Delta$ 's that satisfy our assumptions, and Remark 2 for other ways to relax our assumptions.

### B. Simplification

We use the dynamic extension

$$\dot{\hat{\xi}}(t) = A\hat{\xi}(t) + \varphi(y_\xi(t), u(t)) + \Delta(\hat{\xi}(t), t) \quad (9)$$

and the new variable

$$x(t) = \xi(t) - \hat{\xi}(t). \quad (10)$$

By combining (1) and (9), we obtain the system

$$\begin{cases} \dot{x}(t) = Ax(t) + \Delta_d(t) + \epsilon_1 \gamma_1(t) + \dots + \epsilon_q \gamma_q(t) \\ y(t) = Cx(t-h) \end{cases} \quad (11)$$

$$\text{where } \Delta_d(t) = \Delta(x(t) + \hat{\xi}(t), t) - \Delta(\hat{\xi}(t), t), \quad (12)$$

and where  $y$  is known since  $y_\xi(t)$  and  $\hat{\xi}$  are being measured.

Hence, to find  $\xi(t)$  and the  $\epsilon_j$ 's, we focus on (11), to compute  $x$  and the  $\epsilon_j$  in fixed time when  $\Delta$  is the zero function, and to estimate  $x(t)$  and the  $\epsilon_j$ 's in the almost fixed time way defined above when  $\Delta$  is not the zero function.

### C. Almost Fixed Time Observer

In terms of our assumptions and notation from the preceding subsection, we next introduce the functionals

$$\Psi(y_t) = \int_{t-\tau}^t e^{A^\top(s-t)} C^\top y(s) ds, \quad (13)$$

$$\mu(t) = y(t) - y(t-T) - CAE^{-1} \int_{t-h-T}^{t-h} \Psi(y_m) dm, \quad (14)$$

$$\text{and } \mathcal{V}(y_t) = [\mu(t - \tau_1) \quad \dots \quad \mu(t - \tau_q)] \quad (15)$$

for values  $\tau_1, \dots, \tau_q$  that satisfy our requirements from Assumption 2. We also use the  $\mathbb{R}^n$ -valued functions

$$\hat{\mathcal{G}}(t) = E^{-1}\Psi(y_t) + \sum_{j=1}^q \epsilon_j E^{-1}\kappa(\gamma_{j,t}) \text{ and} \quad (16)$$

$$\hat{x}(t) = \hat{\mathcal{G}}(t) + \hat{z}(t) - E^{-1}(\hat{q}(t) - e^{-A^\top \tau} \hat{q}(t - \tau)), \quad (17)$$

where the dynamics for  $\hat{z}$  and  $\hat{q}$  are

$$\begin{cases} \dot{\hat{z}}(t) = A\hat{z}(t) + \hat{\Delta}_d(t) \\ \dot{\hat{q}}(t) = -A^\top \hat{q}(t) + C^\top C \hat{z}(t-h) \end{cases} \quad (18)$$

with the initial conditions  $\hat{z}(\ell) = \hat{q}(\ell) = 0$  for all  $\ell \leq 0$ , and

$$\hat{\Delta}_d(t) = \Delta(\hat{x}(t) + \hat{\xi}(t), t) - \Delta(\hat{\xi}(t), t). \quad (19)$$

In the next subsection, we will show that in terms of the variable (17), the state of (11), and the estimation error

$$\tilde{x} = x - \hat{x}, \quad (20)$$

we can construct positive constants  $\bar{c}_1$  and  $\bar{c}_2$  such that

$$|\tilde{x}(t)| \leq \bar{c}_1 e^{-\bar{c}_2 t} |\tilde{x}|_{[0, \tau+h]} \quad (21)$$

is satisfied for all  $t \geq \tau + h$  for all initial conditions for  $x(t)$ . The bound (21) will play an essential role in our observer design. However, since (17) contains  $\hat{\mathcal{G}}$  and so also the entries

of  $\epsilon = [\epsilon_1, \dots, \epsilon_q]$ , (17) it is not available for measurement. Therefore, we will instead use

$$\begin{aligned}\hat{x}_0(t) &= E^{-1}\Psi(y_t) \\ &\quad + \kappa^\sharp(t)(M^{-1}(t))^\top [\mathcal{V}(y_t) - C\hat{W}_0^d(t)]^\top \\ &\quad + \hat{z}_0(t) + E^{-1}[\hat{q}_0(t) - \hat{q}_0(t-\tau)] \\ \hat{\epsilon}(t) &= (\mathcal{V}(y_t) - C\hat{W}_0^d(t))M(t)^{-1}\end{aligned}\quad (22)$$

as our estimators, where

$$\begin{aligned}\hat{W}_0^d(t) &= [\hat{W}_0(t-\tau_1-h) - \hat{W}_0(t-T-\tau_1-h), \\ \dots, \hat{W}_0(t-\tau_q-h) - \hat{W}_0(t-T-\tau_q-h)]\end{aligned}\quad (23)$$

and where we use (7), (15), and the dynamic extensions

$$\begin{cases} \dot{\hat{z}}_0(t) = A\hat{z}_0(t) + \hat{\Delta}_{d0}(t) \\ \dot{\hat{q}}_0(t) = -A^\top \hat{q}_0(t) + C^\top C\hat{z}_0(t-h) \\ \dot{\hat{W}}_0(t) = A[\hat{z}_0(t) - E^{-1}(\hat{q}_0(t) - e^{-A^\top \tau} \hat{q}_0(t-\tau))] \\ \quad + \hat{\Delta}_{d0}(t) \end{cases}\quad (24)$$

with the initial conditions  $\hat{z}_0(\ell) = \hat{q}_0(\ell) = \hat{W}_0(\ell) = 0$  for all  $\ell \leq 0$ , and where

$$\hat{\Delta}_{d0}(t) = \Delta(\hat{x}_0(t) + \hat{\xi}(t), t) - \Delta(\hat{\xi}(t), t).\quad (25)$$

We prove the following:

*Theorem 1:* If Assumptions 1-3 be satisfied, then we can find constants  $c_1 > 0$  and  $c_2 > 0$  such that

$$\begin{aligned} &|((\hat{\epsilon}(t) - \epsilon)M(t), \hat{x}_0(t) - x(t))| \\ &\leq c_1 e^{-c_2 t} |(\tilde{x}, \hat{x}_0 - x)|_{[0, T+\tau+\tau_q+h]}\end{aligned}\quad (26)$$

is satisfied for all  $t \geq 2(T+\tau+\tau_q+h)$  and all initial states  $x(0) \in \mathbb{R}^n$  for (11). Also, if Assumptions 1-2 are satisfied and if  $\Delta$  is the zero function, then the equalities

$$x(t) = E^{-1}\Psi(y_t) + \kappa^\sharp(t) (\mathcal{V}(y_t)M(t)^{-1})^\top \quad (27)$$

$$\text{and } [\epsilon_1 \dots \epsilon_q] = \mathcal{V}(y_t)M(t)^{-1} \quad (28)$$

hold for all  $t \geq \tau_q + T + \tau + h$  and all initial states  $x(0) \in \mathbb{R}^n$  for (11).

*Remark 1:* Since  $M(t)$  is known and invertible for all  $t$ ,  $(\hat{\epsilon}(t) - \epsilon)M(t)$  and  $\hat{\epsilon}(t) - \epsilon$  in (26) contain the same information. We use  $(\hat{\epsilon}(t) - \epsilon)M(t)$  in (26) instead of  $\hat{\epsilon}(t) - \epsilon$  to ensure existence of the constant  $c_1$  without requiring the function  $M_*(t) = M(t)^{-1}$  to be bounded.  $\square$

#### D. Proof of Theorem 1

Using variation of parameters shows that

$$\begin{aligned} e^{A(s-h-t)}x(t) &= \sum_{i=1}^q \epsilon_i \int_{s-h}^t e^{A(s-h-m)} \gamma_i(m) dm \\ &\quad + x(s-h) + \int_{s-h}^t e^{A(s-h-m)} \Delta_d(m) dm \end{aligned}\quad (29)$$

for all  $s \geq h$  and  $t \geq s-h$ . By left multiplying (29) by  $e^{A^\top(s-t)}C^\top C$ , and then integrating the resulting equality over the interval  $[t-\tau, t]$  when  $t \geq \tau$ , we obtain

$$\begin{aligned} Ex(t) &= E\hat{G}(t) \\ &\quad + \int_{t-\tau}^t e^{A^\top(\ell-t)}C^\top C \left[ \int_{\ell-h}^t e^{A(\ell-h-m)} \Delta_d(m) dm \right] d\ell \end{aligned}\quad (30)$$

for all  $t \geq \tau+h$ , with  $E$  defined in (2) and  $\hat{G}$  is from (16).

Recalling the invertibility of  $E$  is (2), it follows that

$$\begin{aligned} x(t) &= \hat{G}(t) \\ &\quad + E^{-1} \int_{t-\tau}^t e^{A^\top(\ell-t)} C_a \left[ \int_{\ell-h}^t e^{A(\ell-h-m)} \Delta_d(m) dm \right] d\ell \end{aligned}\quad (31)$$

for all  $t \geq \tau+h$ , where  $C_a = C^\top C$ . Also, using the variation of parameters formula, in follows from (17)-(18) that

$$\begin{aligned} \hat{x}(t) &= \hat{G}(t) \\ &\quad + E^{-1} \int_{t-\tau}^t e^{A^\top(\ell-t)} C_a \left[ \int_{\ell-h}^t e^{A(\ell-h-m)} \hat{\Delta}_d(m) dm \right] d\ell \end{aligned}\quad (32)$$

for all  $t \geq \tau+h$ . The formula (32) follows by using variation of parameters to write the double integral in (32) as

$$\begin{aligned} &\int_{t-\tau}^t e^{A^\top(\ell-t)} C_a e^{A(\ell-h-t)} \left[ \int_{\ell-h}^t e^{A(t-m)} \hat{\Delta}_d(m) dm \right] d\ell \\ &= \int_{t-\tau}^t C^\sharp(\ell-t) e^{-A\ell} [\hat{z}(t) - e^{A(t-\ell+h)} \hat{z}(\ell-h)] d\ell \\ &= E\hat{z}(t) - \int_{t-\tau}^t e^{A^\top(\ell-t)} C^\top C \hat{z}(\ell-h) d\ell \\ &= E\hat{z}(t) - (\hat{q}(t) - e^{-A^\top \tau} \hat{q}(t-\tau)), \end{aligned}\quad (33)$$

where  $C^\sharp(r) = e^{A^\top r} C^\top C e^{Ar}$ . Recalling (12), (19), and our choices of  $\bar{\beta}$  in (6) and  $\tilde{x}$  in (20), it follows from subtracting (32) from (31) that  $|\tilde{x}(t)| \leq \rho |\tilde{x}|_{[t-\tau-h, t]}$  for all  $t \geq \tau+h$ , where  $\rho = \bar{\beta} L_\Delta \in [0, 1)$  (by Assumption 3). Hence, the existence of the required constants  $\bar{c}_1$  and  $\bar{c}_2$  to satisfy (21) follows from applying [18, Lemma 1] to the function  $w(t) = |\tilde{x}(t+\tau+h)|$ . Also, when  $\Delta = 0$ , equality (31) gives

$$x(t) = E^{-1}\Psi(y_t) + \sum_{i=1}^q \epsilon_i E^{-1} \kappa(\gamma_{i,t}) \quad (34)$$

for all  $t \geq \tau+h$ , since  $\Delta_d = 0$  in that case.

Also, we deduce from (11) and our choice (17) of  $\hat{x}$  that  $x(t) = \hat{x}(t) + \tilde{x}(t)$  and therefore also

$$\begin{aligned} x(t) &= \tilde{x}(t) + \hat{G}(t) + \hat{z}(t) \\ &\quad - E^{-1}(\hat{q}(t) - e^{-A^\top \tau} \hat{q}(t-\tau)) \end{aligned}\quad (35)$$

for all  $t \geq \tau$ , so our choice (16) of  $\hat{G}$  gives

$$\begin{aligned} \dot{\hat{x}}(t) &= AE^{-1}\Psi(y_t) + \sum_{i=1}^q \epsilon_i AE^{-1} \kappa(\gamma_{i,t}) \\ &\quad + \sum_{i=1}^q \epsilon_i \gamma_i(t) + H(t), \end{aligned}\quad (36)$$

where

$$H(t) = A[\tilde{x}(t) + \hat{z}(t) - E^{-1}(\hat{q}(t) - e^{-A^\top \tau} \hat{q}(t-\tau))] + \Delta_d(t).\quad (37)$$

By reorganizing terms in (36), then integrating the result on  $[t-T-h, t-h]$  where  $T > 0$  is the constant in (4), we get

$$\begin{aligned} &\epsilon_1 \int_{t-h-T}^{t-h} [AE^{-1} \kappa(\gamma_{1,m}) + \gamma_1(m)] dm \\ &\quad + \dots + \epsilon_q \int_{t-T-h}^{t-h} [AE^{-1} \kappa(\gamma_{q,m}) + \gamma_q(m)] dm \\ &= x(t-h) - x(t-T-h) - AE^{-1} \int_{t-T-h}^{t-h} \Psi(y_m) dm \\ &\quad - \int_{t-T-h}^{t-h} H(\ell) d\ell \end{aligned}\quad (38)$$

for all  $t \geq T+h$ . By left multiplying (38) by  $C$ , we obtain

$$\begin{aligned} [\epsilon_1 \dots \epsilon_q] W(t) &= y(t) - y(t-T) \\ &\quad - CAE^{-1} \int_{t-T-h}^{t-h} \Psi(y_m) dm - C \int_{t-T-h}^{t-h} H(\ell) d\ell, \end{aligned}\quad (39)$$

with  $W$  defined in (4). We deduce that

$$\begin{aligned} [\epsilon_1 \quad \dots \quad \epsilon_q]M(t) &= \mathcal{V}(y_t) \\ -C \left[ \int_{t-T-\tau_1-h}^{t-\tau_1-h} H(\ell)d\ell, \dots, \int_{t-T-\tau_q-h}^{t-\tau_q-h} H(\ell)d\ell \right] \end{aligned} \quad (40)$$

with  $\mathcal{V}$  defined in (15). Using Assumption 2, we can right multiply (40) by  $M(t)^{-1}$  to obtain

$$\begin{aligned} [\epsilon_1 \quad \dots \quad \epsilon_q] &= \mathcal{V}(y_t)M(t)^{-1} \\ -C \left[ \int_{t-T-\tau_1-h}^{t-\tau_1-h} H(\ell)d\ell, \dots, \int_{t-T-\tau_q-h}^{t-\tau_q-h} H(\ell)d\ell \right] &M(t)^{-1} \end{aligned} \quad (41)$$

which agrees with (28) when  $\Delta$  is the zero function, because (31)-(32) give  $\hat{q}(t) = \hat{z}(t) = \hat{x}(t) = x(t) - \hat{x}(t) = 0$  for all  $t \geq \tau + h$  when  $\Delta$  is the zero function. Hence, we can assume in the sequel that  $L_\Delta > 0$ . Also, our formulas for  $\hat{x}_0$  from (22) and  $H$  from (37), and the argument that led to (33) (with  $\hat{\Delta}_d$  replaced by  $\hat{\Delta}_{do}$ ), give

$$\begin{aligned} H(t) &= A\tilde{x}(t) + \Delta_d(t) + \\ AE^{-1} \int_{t-\tau}^t C^\#(\ell-t) \left[ \int_{\ell-h}^t e^{A(t-h-m)} \hat{\Delta}_d(m) dm \right] d\ell \end{aligned} \quad (42)$$

and

$$\begin{aligned} [\hat{\epsilon}_1(t) \quad \dots \quad \hat{\epsilon}_q(t)] &= \mathcal{V}(y_t)M(t)^{-1} \\ -C \left[ \int_{t-T-\tau_1-h}^{t-\tau_1-h} \hat{H}(\ell)d\ell, \dots, \int_{t-T-\tau_q-h}^{t-\tau_q-h} \hat{H}(\ell)d\ell \right] &M(t)^{-1} \end{aligned} \quad (43)$$

and so also

$$\begin{aligned} \hat{x}_0(t) &= \left\{ E^{-1}\Psi(y_t) + \kappa^\#(t)(M^{-1}(t))^\top \left[ \mathcal{V}(y_t) \right. \right. \\ &\quad \left. \left. - \left[ \int_{t-T-\tau_1-h}^{t-\tau_1-h} C\hat{H}(\ell)d\ell, \dots, \int_{t-T-\tau_q-h}^{t-\tau_q-h} C\hat{H}(\ell)d\ell \right]^\top \right] \right\} \\ &+ E^{-1} \int_{t-\tau}^t C^\#(\ell-t) \left[ \int_{\ell-h}^t e^{A(t-h-m)} \hat{\Delta}_{d0}(m) dm \right] d\ell \end{aligned} \quad (44)$$

where

$$\begin{aligned} \hat{H}(t) &= \hat{\Delta}_{do}(t) + \\ AE^{-1} \int_{t-\tau}^t C^\#(\ell-t) \left[ \int_{\ell-h}^t e^{A(t-h-m)} \hat{\Delta}_{d0}(m) dm \right] d\ell \end{aligned} \quad (45)$$

and  $\hat{\Delta}_{do}$  is defined in (25). Also, our formulas (16) for  $\hat{G}$  and (41) for the row vector  $\epsilon$  give  $\hat{G}(t) = E^{-1}\Psi(y_t) + \kappa^\#(t)\epsilon^\top$  and therefore also

$$\begin{aligned} \hat{G}(t) &= E^{-1}\Psi(y_t) + \kappa^\#(t)(M(t)^{-1})^\top \left[ \mathcal{V}(y_t) \right. \\ &\quad \left. - \left[ \int_{t-T-\tau_1-h}^{t-\tau_1-h} CH(\ell)d\ell, \dots, \int_{t-T-\tau_q-h}^{t-\tau_q-h} CH(\ell)d\ell \right]^\top \right], \end{aligned} \quad (46)$$

and (33) and (37) and (45) give

$$\begin{aligned} |CH(\ell) - C\hat{H}(\ell)| &\leq L_\Delta |C| |x(\ell) - \hat{x}_0(\ell)| \\ + |CA| |\tilde{x}(\ell)| &+ L_\Delta \bar{\beta} |CA| |x(\ell) - \hat{x}_0(\ell)| \end{aligned} \quad (47)$$

for all  $\ell \in [t-T-\tau_q+h, t-\tau_1]$  and  $t \geq T+\tau_q$ . Hence, our formulas (31), (41), (44), and (42) give

$$\begin{aligned} |x(t) - \hat{x}_0(t)| &\leq \sqrt{q} |\kappa_*|_\infty |CA| \bar{\beta} T L_\Delta |x - \hat{x}_0|_{[t-T-\tau_q-\tau-h, t]} \\ &+ \sqrt{q} T |\kappa_*|_\infty |CA| |\tilde{x}|_{[t-\tau-h, t]} \\ &+ \sqrt{q} T |\kappa_*|_\infty |C| L_\Delta |x - \hat{x}_0|_{[t-T-\tau_q-h, t]} \\ &+ \bar{\beta} L_\Delta |x - \hat{x}_0|_{[t-\tau-h, t]} \end{aligned} \quad (48)$$

for all  $t \geq T + \tau + \tau_q + h$ , where the function  $\kappa_*$  is defined in Assumption 3, and where the first three terms in the sum on the right side of (48) came from upper bounding the difference between  $\hat{G}(t)$  in (31) and the terms in curly braces in (44) (by substituting (46) into (31) and using (47)), and where the last term in (48) came from upper bounding the difference between the double integrals in (31) and (44).

Therefore, with the choices

$$\begin{aligned} \bar{\beta}_1^\# &= L_\Delta (\bar{\beta} [\sqrt{q} |\kappa_*|_\infty |CA| T + 1] + |\kappa_*|_\infty |C| \sqrt{q} T) \\ \text{and } \bar{\beta}_2^\# &= \sqrt{q} T |\kappa_*|_\infty |CA|, \end{aligned} \quad (49)$$

we obtain

$$\begin{aligned} |x(t) - \hat{x}_0(t)| &\leq \bar{\beta}_1^\# |x - \hat{x}_0|_{[t-T-\tau_q-\tau-h, t]} \\ &+ \bar{\beta}_2^\# |\tilde{x}|_{[t-\tau-h, t]} \end{aligned} \quad (50)$$

for all  $t \geq T + \tau + \tau_q + h$ . Since Assumption 3 gives  $\bar{\beta}_1^\# \in (0, 1)$ , we can reapply [18, Lemma 1] (with  $w(t) = |x(t + T + \tau + \tau_q + h) - \hat{x}_0(t + T + \tau + \tau_q + h)|$ ) to obtain positive constants  $\bar{c}_3$ ,  $\bar{c}_4$ , and  $\bar{c}_5$  such that

$$\begin{aligned} |x(t) - \hat{x}_0(t)| &\leq \bar{c}_3 e^{-\bar{c}_4 t} |x - \hat{x}_0|_{[0, T + \tau + \tau_q + h]} \\ &+ \bar{c}_5 |\tilde{x}|_{[0, t]} \end{aligned} \quad (51)$$

for all  $t \geq \tau + T + \tau_q + h$ , with  $\bar{c}_4 = \ln(\bar{\beta}_1^\#)/(T + \tau + \tau_q + h)$ . By combining (21) and (51) and using standard small-gain arguments, we obtain positive constants  $\bar{c}_6$  and  $\bar{c}_7$  such that

$$\begin{aligned} |(\tilde{x}(t), \hat{x}_0(t) - x(t))| &\leq \bar{c}_6 e^{-\bar{c}_7 t} |(\tilde{x}, \hat{x}_0 - x)|_{[0, T + \tau + \tau_q + h]} \end{aligned} \quad (52)$$

for all  $t \geq T + \tau + \tau_q + h$ . Also, using (42) and (45), we can use our formulas for  $\epsilon$  and  $\hat{\epsilon}$  in (41) and (43) to find a constant  $\bar{c}_8 > 0$  such that  $|(\hat{\epsilon}(t) - \epsilon)M(t)| \leq \bar{c}_8 |(\tilde{x}, \hat{x}_0 - x)|_{[t-T-\tau-\tau_q-h, t]}$  for all  $t \geq T + \tau + \tau_q + h$ . Hence, the existence of the required constants  $c_1$  and  $c_2$  in the theorem follows from taking the maximum on both sides of (52) to upper bound  $|\hat{\epsilon}(t) - \epsilon|$  for  $t \geq 2(T + \tau + \tau_q + h)$ , and then adding the result to (52). This concludes the proof.

*Remark 2:* Our proof of Theorem 1 shows that  $c_2 \rightarrow +\infty$  as  $L_\Delta \rightarrow 0$  (since  $\ln(\bar{\beta}_1^\#) \rightarrow +\infty$  as  $L_\Delta \rightarrow 0$ ), which is the almost fixed time property. Using linear growth properties of the dynamics and Gronwall's inequality, we can bound the supremum on the right side of (26) by a multiple of  $|x(0)| + |\epsilon|$ . In practice, bounds on  $x(0)$  and the  $\epsilon_j$ 's may be known. Therefore, we obtain upper bounds on the left side of (26) in terms of  $|x(0)| + |\epsilon|$  for all times  $t \geq 2(T + \tau + \tau_q + h)$ .

We can replace the requirement that  $\kappa_*$  is bounded and (8) by the two requirements that  $\kappa^\#$  is bounded and  $L_\Delta \bar{\beta} < 1$ , and then we instead obtain an input-to-state stability (or ISS) conclusion with respect to the approximation error  $\hat{\epsilon}(t) - \epsilon$ . In fact, with these replacements, the part of the proof of the theorem through (44) except with the terms in curly braces in (44) replaced by  $E^{-1}\Psi(y_t) + \hat{\epsilon}_1(t)E^{-1}\kappa(\gamma_{1,t}) + \dots + \hat{\epsilon}_q(t)E^{-1}\kappa(\gamma_{q,t})$  provides positive constants  $c_1$ ,  $c_2$ , and  $c_3$  such that

$$\begin{aligned} |\hat{x}_0(t) - x(t)| &\leq \\ c_1 e^{-c_2 t} |(\hat{x}_0 - x)|_{[0, T + \tau + \tau_q]} &+ c_3 |\epsilon - \hat{\epsilon}|_{[0, t]} \end{aligned} \quad (53)$$

for all  $t \geq T + \tau + \tau_q + h$ . Also, Theorem 1 remains true by a similar proof if we replace the estimator  $\hat{\epsilon}$  from (22) by

$$\hat{\epsilon}(t) = \text{sat}_{\bar{\epsilon}, \underline{\epsilon}}\{(\mathcal{V}(y_t) - C\hat{W}_0^d(t))M(t)^{-1}\}, \quad (54)$$

where  $\text{sat}_{\bar{\epsilon}, \underline{\epsilon}}(v)$  has the  $i$ th entry

$$\text{sat}_{\bar{\epsilon}_i, \underline{\epsilon}_i}(v_i) = \max\{\underline{\epsilon}_i, \min\{v_i, \bar{\epsilon}_i\}\} \quad (55)$$

for  $i = 1, \dots, q$  and using known bounds  $\epsilon_i \in [-\underline{\epsilon}_i, \bar{\epsilon}_i]$  on the  $\epsilon_i$ 's. This can be interpreted to mean that smaller known intervals  $[-\underline{\epsilon}_i, \bar{\epsilon}_i]$  containing the unknown  $\epsilon_i$ 's produce smaller overshoots in the ISS estimate (53).

### III. ALTERNATIVE EXPRESSIONS WITHOUT INTEGRALS

The integrals in (4), (13), and (14) can be replaced by states of delayed dynamical extensions. This replacement can facilitate checking our assumptions, while also making it possible to provide expressions for the observer values that do not contain any integrals. To see how this can be done in the special case where  $\Delta = 0$ , we first show that for  $i = 1, \dots, q$ , we have  $W_i(t) = V_i(t) + W_i(0)$ , where  $V_i$  is the first component of the state of the dynamic extension

$$\begin{cases} \dot{V}_i(t) = C[AE^{-1}(EL_{1i}(t-h) - L_{2i}(t-h) \\ + e^{-A^\top \tau} L_{2i}(t-\tau-h)) + \gamma_i(t-h) \\ - AE^{-1}(EL_{1i}(t-T-h) \\ - L_{2i}(t-T-h) \\ + e^{-A^\top \tau} L_{2i}(t-T-\tau-h)) \\ - \gamma_i(t-T-h)] \\ \dot{L}_{1i}(t) = AL_{1i}(t-h) + \gamma_i(t) \\ \dot{L}_{2i}(t) = -A^\top L_{2i}(t) + C^\top CL_{1i}(t-h) \end{cases} \quad (56)$$

and where the initial states are  $V_i(0) = 0$  and  $L_{1i}(s) = L_{2i}(s) = 0$  for all  $s \in [-T - \tau - h, 0]$ . To obtain the preceding formula for  $W_i$ , first note that for each  $i$  and  $t \geq 0$ , we have

$$\begin{aligned} & \kappa_i(\gamma_i, t) \\ &= \int_{-\tau}^0 e^{A^\top r} C^\top C \int_{r-h}^0 e^{A(r-h-m)} \gamma_i(m+t) dm dr \\ &= \int_{t-\tau}^t C^\#(r-t) e^{-Ah} \left[ \int_{r-h}^t e^{A(t-m)} \gamma_i(m) dm \right] dr \\ &= \int_{t-\tau}^t C^\#(r-t) e^{-Ah} [L_{1i}(t) - e^{(t-r)A} L_{1i}(r-h)] dr \\ &= EL_{1i}(t) - \int_{t-\tau}^t e^{-A^\top(t-r)} C^\top CL_{1i}(r-h) dr \\ &= EL_{1i}(t) - (L_{2i}(t) - e^{-A^\top \tau} L_{2i}(t-\tau)), \end{aligned} \quad (57)$$

where  $C^\#(r) = e^{A^\top r} C^\top C e^{Ar}$  as before, and where the third and last equalities were obtained by applying the method of variation of parameters to the  $L_{1i}$  and then the  $L_{2i}$  dynamics from (56). Therefore, the formulas for the  $W_i$ 's in (4) give  $\dot{W}_i = \dot{V}_i$  for all  $i$ , so the Fundamental Theorem of Calculus gives the required formula  $W_i(t) = V_i(t) + W_i(0)$ .

Similar arguments give

$$\Psi(y_t) = L_3(t) - e^{-A^\top \tau} L_3(t-\tau), \quad (58)$$

and

$$\begin{aligned} \mu(t) &= y(t) - y(t-T) \\ &- CAE^{-1}(L_4(t-h) - L_4(t-T-h)), \end{aligned} \quad (59)$$

where  $L_3$  and  $L_4$  are solutions of the dynamical extension

$$\begin{cases} \dot{L}_3(t) = -A^\top L_3(t) + C^\top y(t) \\ \dot{L}_4(t) = L_3(t) - e^{-A^\top \tau} L_3(t-\tau) \end{cases} \quad (60)$$

with initial functions 0, by applying the method of variation of parameters to the  $L_3$  dynamics in (60) to eliminate the integral from the  $\Psi$  formula in (13), and then using the result to eliminate the integral in the formula for  $\mu$  in (14). Combined with the formulas (57), this makes it possible to eliminate the integrals from the assumptions and conclusions of Theorem 1 when  $\Delta = 0$ . Similar but more complicated dynamical extensions allow us to eliminate the integrals for nonzero  $\Delta$  choices as well.

### IV. ILLUSTRATIONS

We provide two examples to illustrate the ease with which our assumptions can be checked. First, we revisit the main example from [4] where we can choose  $\Delta = 0$ , by showing how Theorem 1 makes it possible to identify the weighting coefficients  $\epsilon_i$  in an unknown uncertainty that was not present in [4]. Then, we provide a generalization of the example from [16] that allows measurement delays and nonzero state dependent terms  $\Delta$  which were not allowed in [16].

#### A. Motor Example

In [4], the unperturbed dynamics of a single-link robotic manipulator coupled to a DC motor with a nonrigid joint after a change in coordinates produced the dynamics (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{K}{J_1} & -\frac{F_1}{J_1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{K^2}{N^2 J_1 J_2} & 0 & -\frac{K}{N^2 J_2} & -\frac{F_2}{J_2} & 1 \\ 0 & 0 & 0 & -\frac{K_b K_t}{J_2 \mathcal{L}} & -\frac{R}{\mathcal{L}} \end{bmatrix}, \quad (61)$$

$$\varphi(C\xi, u) = \begin{bmatrix} 0 \\ \frac{mg_* d}{J_1} [1 - \cos(\xi_1)] \\ 0 \\ 0 \\ \frac{K_t K}{J_1 J_2 N \mathcal{L}} u \end{bmatrix},$$

and  $C = [1 \ 0 \ 0 \ 0 \ 0]$ ,

and each  $\gamma_i$  and  $\Delta$  being the zero function, where the positive constants  $J_1$  and  $J_2$  are the inertias, the positive constants  $F_1$  and  $F_2$  are the viscous friction constants,  $K$  is the positive spring constant,  $K_t$  is the positive torque constant,  $K_b$  is the positive back EMF constant,  $R$  and  $\mathcal{L}$  are the armature resistance and inductance respectively and are positive constants,  $m$  is the constant positive link mass, the positive constant  $d$  is the position of the link's center of gravity,  $N > 0$  is the positive gear ratio, and  $g_* > 0$  is the constant gravity acceleration.

However, here we study a more general situation for the preceding dynamics where there is also an added uncertainty  $\delta(t) = \epsilon_1 \gamma_1(t) + \dots + \epsilon_3 \gamma_3(t)$  on its right side, where each entry of  $\gamma_i : \mathbb{R} \rightarrow \mathbb{R}^5$  is  $e^{-(t-\kappa_i)^2} / (2w_i^2)$  for suitable positive constants  $\kappa_i$  and  $w_i$  for  $i = 1, 2, 3$ , which are commonly used in basis functions in neural network expansions. Since

$\Delta = 0$ , it remains to check Assumptions 1-2, with  $n = 5$  and  $q = 3$ . These assumptions hold for many choices of the parameters. For instance, by choosing all of the constants in  $A$  in (61) to be 1 as was done in [4], and  $\kappa_i = w_i = i$  for  $i = 1, 2, 3$  in the basis functions  $\gamma_i$ , and with  $\tau = T = 1$  and  $\tau_1 = 0$ ,  $\tau_2 = 0.2$ , and  $\tau_3 = 0.75$ , we used Mathematica to check that Assumptions 1-2 hold for all measurement delays  $h \in [0, 0.25]$ . This extends the motor example from [4] to cover more realistic cases with measurement delays and additive uncertainties whose weights  $\epsilon_i$  we can identify in finite time.

### B. Nonzero Nonlinearity $\Delta$

Next we consider the system (1) with  $n = 2$ ,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } C = [1 \quad 0] \quad (62)$$

with  $\varphi = 0$  and  $q = 1$ . Moreover, we assume that  $\gamma_1(\ell) = [0, 1]^\top$  for all  $\ell$ . Then Assumption 1 is satisfied. Let us check that Assumption 2 is satisfied as well, and then we find conditions on the uniform global Lipschitz constant  $L_\Delta$  on an additive state-dependent uncertainty on the right side of the dynamics such that Assumption 3 is satisfied.

Recalling that

$$e^{A\ell} = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} \quad (63)$$

for all  $\ell \in \mathbb{R}$ , simple calculations show that (63) gives

$$\kappa(\gamma_1, t) = \begin{bmatrix} -\frac{h\tau^2}{2} - \frac{h^2\tau}{2} - \frac{\tau^3}{6} \\ \frac{\tau^4}{8} + \frac{h\tau^3}{3} + \frac{h^2\tau^2}{4} \end{bmatrix}. \quad (64)$$

It then readily follows that

$$\begin{aligned} & \int_{t-T-h}^{t-h} C[AE^{-1}\kappa(\gamma_1, m) + \gamma_1(m)]dm \\ &= T \left( \frac{1}{2}\tau + h \right) \neq 0. \end{aligned} \quad (65)$$

Thus Assumption 2 holds for any  $T > 0$ ,  $h > 0$ , and  $\tau > 0$ . Also, with the preceding parameters,  $T = 2$ , and  $\tau = 0.5$ , and with  $h = 0$ , we used the Mathematica program to check that condition (8) is equivalent to  $L_\Delta < 0.44619$ . If instead  $h = 0.125$  and the other parameters are kept the same, then we found that the bound on  $L_\Delta$  from (8) is  $L_\Delta < 0.295463$ . Finally, with  $h = 0.25$  and all of other parameters kept the same, we found that our condition (8) is  $L_\Delta < 0.212663$ . This illustrates a tradeoff, where larger delays  $h$  produced smaller bounds on the allowable growth rates  $L_\Delta$  on  $\Delta$ .

## V. CONCLUSIONS

We provided new fixed and almost finite-time observers for unknown model parameters and unknown states. Our assumptions used a simple invertibility criterion, and they cover artificial neural network expansions. When the unmeasured state enters affinely, our observers are fixed time converging. This affineness holds for a broad class of dynamics (e.g., motor dynamics [4], and vibrating membranes [2]). When the nonlinearity depends on the unmeasured state, we proved an analog of the almost finite-time work [6]. This made it possible to identify model parameters, which was not possible using [6], which was confined to state observers. Our

assumptions place conditions on the allowable measurement delays, convergence times, and growth rates for the nonlinear function of the unmeasured state. We aim to relax these assumptions to compensate for arbitrarily long measurement delays, and to extend [6] to compute basins of attraction for systems with nonlinearities that grow quadratically. We also aim to develop reduced order versions, to build on works like [3] which did not identify model parameters.

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