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Finite-Time Observers for Parameters and State Variables of Nonlinear Systems [★]

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Abstract: We provide a new class of finite-time observers for continuous-time nonlinear systems that are affine in the unmeasured state variable and that also contain unknown constant parameters. Our method provides exact values of the state variables and of the constant parameters, in a fixed finite time that is independent of the initial values. We also provide analogs where there is a known delay in the measurements, and for discrete-time systems. Our approach does not rely on an adaptive controller, and is suitable for computing exact values of constant weights in artificial neural network representations of additive uncertainties in the dynamics. Our example illustrates the ease with which our assumptions can be verified in practice.

Keywords: Delay, estimation, finite-time, nonlinear systems

1. INTRODUCTION

We continue our search (which we began in Ahmed et al. (2019), Mazenc et al. (2018), Mazenc et al. (2020), Mazenc et al. (2021), Mazenc and Malisoff (2022), and Mazenc et al. (2022)) for finite-time observers for nonlinear systems that satisfy suitable structural conditions. While our prior works provided arbitrarily fast convergence of the observers for states of perturbed systems (which gave fixed finite-time convergence in the special case where the uncertainties are zero), here we consider a more complex problem where in addition to needing finite-time state observers, we must identify unknown model parameters in finite time. Our work is motivated by the ubiquity of unmeasured states in applications, and the usefulness of observers to substitute for unmeasured states in controllers; see, e.g., Borri et al. (2017); Buccella et al. (2014); Cacace et al. (2010, 2014, 2015); Dinh et al. (2015); Engel and Kreisselmeier (2002); Katz et al. (2021); Mazenc et al. (2015); Parikh et al. (2017); Sauvage et al. (2007).

One natural setting where unknown constant model parameters arise is in artificial neural network expansions, where an unknown time-varying function is expressed as a linear combination of known basis functions with unknown constant weights, and where one then aims to find values of the unknown constant coefficients in the linear combination in order to identify the unknown uncertainty. One basic approach to estimating unknown model parameters is adaptive control, which usually entails a persistency of excitation (or PE) condition (but see Malisoff et al. (2017), where coefficients in an artificial neural network expansion were identified using an adaptive controller and

barrier Lyapunov functions, in the context of curvature estimation in planar curve tracking that arises in marine robotic applications, but without requiring any PE conditions). However, adaptive control does not in general give fixed finite-time parameter identification, and is usually not amenable to the estimation of unmeasured states.

This motivates this work, which addresses the preceding challenges, in three significant settings. First, we consider cases in Section 2, where there are no measurement delays and where a simple invertibility criterion makes it possible to construct unknown state and parameter values in a fixed finite time that is independent of the initial state. Then, in Section 3, we provide an analog under measurement delays, assuming a delayed analog of the usual invertibility condition on the Gramian. In Section 4, we provide an analog for discrete time systems, where the invertibility of the Gramian is replaced by a positivity condition that again ensures fixed time identification of unknown parameters and fixed finite-time convergence of the observer. We close in Section 5 with our conjectures and research plans.

We use standard notation, which we simplify when no confusion would arise from the context. The dimensions of our Euclidean spaces are arbitrary, unless we indicate otherwise. Also, we set $g_t(s) = g(t+s)$ for functions g and all s and t such that $t+s$ is in the domain of g .

2. SYSTEMS WITHOUT MEASUREMENT DELAYS

2.1 Studied system

We consider the system

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + \varphi(C\xi(t), u(t)) + \epsilon_1\gamma_1(t) + \\ \quad \dots + \epsilon_q\gamma_q(t) \\ y_\xi(t) = C\xi(t) \end{cases} \quad (1)$$

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where ξ is valued in \mathbb{R}^n , y_ξ is valued in \mathbb{R} , the real constants ϵ_j are unknown, φ is a Lipschitz continuous nonlinear function, u is a known input, $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{1 \times n}$ are constant real matrices, and the functions $\gamma_j : \mathbb{R} \rightarrow \mathbb{R}^n$ are locally bounded and piecewise continuous. Our structure (1) where the dynamics are affine in the unmeasured state ξ is motivated by significant applications whose models lead to dynamics of this form, e.g., manipulator dynamics and nonholonomic systems in chained form after changes in variables (as in Mazenc et al. (2020) and Mazenc et al. (2021)), vibrating membranes (as in Ahmed et al. (2019) and Mazenc et al. (2018)), and more. Our first assumption is the following standard observability condition:

Assumption 1: The pair (A, C) is observable. \square

Let us introduce a constant $\tau > 0$, the matrix

$$E = \int_{-\tau}^0 e^{A^\top s} C^\top C e^{As} ds \quad (2)$$

which is invertible because of Assumption 1, and the functional

$$\kappa(\phi) = \int_{-\tau}^0 e^{A^\top r} C^\top C \int_r^0 e^{A(r-\ell)} \phi(\ell) d\ell dr. \quad (3)$$

We also use the function $W : \mathbb{R} \rightarrow \mathbb{R}^q$ that is defined by

$$\begin{aligned} W(t) &= [W_1(t), \dots, W_q(t)]^\top, \text{ where} \\ W_i(t) &= \int_{t-T}^t C [AE^{-1} \kappa(\gamma_{i,m}) + \gamma_i(m)] dm \\ &\text{for } i = 1, \dots, q, \end{aligned} \quad (4)$$

where $T > 0$ is a constant, and where $\gamma_{i,m}$ is defined by $\gamma_{i,m}(s) = \gamma_i(m+s)$ for all real m and s and $i = 1, \dots, q$.

Our second assumption is as follows (but see Section 2.4 for ways to express the W_i 's in terms of states of dynamical extensions instead of using integrals, and thereby eliminate the need to compute integrals to check our assumptions):

Assumption 2. There are q nonnegative constants τ_1, \dots, τ_q such that $\tau_i < \tau_j$ for pairs (i, j) with $j > i$ and a constant $\bar{T} > T + \tau$ such that the matrix

$$M(t) = [W(t - \tau_1) \quad \dots \quad W(t - \tau_q)] \quad (5)$$

is invertible for all $t \in [\tau_q + T + \tau, \bar{T} + \tau_q]$. \square

2.2 Simplification

We first propose a simplification. We use the dynamic extension

$$\dot{\hat{\xi}}(t) = A\hat{\xi}(t) + \varphi(y_\xi(t), u(t)). \quad (6)$$

Next, let

$$x(t) = \xi(t) - \hat{\xi}(t). \quad (7)$$

Then we obtain the system

$$\begin{cases} \dot{x}(t) = Ax(t) + \epsilon_1 \gamma_1(t) + \dots + \epsilon_q \gamma_q(t) \\ y(t) = Cx(t) \end{cases} \quad (8)$$

where y is a known measurement since $y(t) = y_\xi(t) - C\hat{\xi}(t)$, $y_\xi(t)$, and $C\hat{\xi}(t)$ are all known.

Then, to find $\xi(t)$ and the constants ϵ_j , we can focus our attention on the system (8). Hence, our new objective is to find x and the constants ϵ_j in finite time.

2.3 Fixed Time Observer

In terms of our assumptions and notation from the preceding subsection, we next introduce the functionals

$$\Psi(y_t) = \int_{t-\tau}^t e^{A^\top(s-t)} C^\top y(s) ds, \quad (9)$$

$$\mu(t) = y(t) - y(t-T) - CAE^{-1} \int_{t-T}^t \Psi(y_m) dm, \quad (10)$$

$$\mathcal{V}(y_t) = [\mu(t - \tau_1) \quad \dots \quad \mu(t - \tau_q)] \quad (11)$$

for values τ_1, \dots, τ_q that satisfy our requirements from Assumption 2, and

$$\kappa^\sharp(t) = [E^{-1} \kappa(\gamma_{1,t}) \quad \dots \quad E^{-1} \kappa(\gamma_{q,t})]. \quad (12)$$

We prove the following, which provides exact state values for all $t \geq \tau_q + T + \tau$ in the special case where the inverse $M(t)^{-1}$ of the matrix $M(t)$ exists for all $t \geq \tau_q + T + \tau$:

Theorem 1. Let Assumptions 1-2 be satisfied. Then, for all $t \in [\tau_q + T + \tau, \bar{T}]$, the equalities

$$x(t) = E^{-1} \Psi(y_t) + \kappa^\sharp(t) (\mathcal{V}(y_t) M(t)^{-1})^\top \quad (13)$$

and

$$[\epsilon_1 \quad \dots \quad \epsilon_q] = \mathcal{V}(y_t) M(t)^{-1} \quad (14)$$

are satisfied.

Proof. Simple calculations give

$$\begin{aligned} e^{A(s-t)} x(t) &= x(s) + \epsilon_1 \int_s^t e^{A(s-m)} \gamma_1(m) dm \\ &\quad + \dots + \epsilon_q \int_s^t e^{A(s-m)} \gamma_q(m) dm \end{aligned} \quad (15)$$

for all $t \geq 0$ and $s \geq 0$. By left multiplying (15) by C , we obtain

$$\begin{aligned} Ce^{A(s-t)} x(t) &= y(s) + \epsilon_1 C \int_s^t e^{A(s-m)} \gamma_1(m) dm \\ &\quad + \dots + \epsilon_q C \int_s^t e^{A(s-m)} \gamma_q(m) dm. \end{aligned} \quad (16)$$

By left multiplying (16) by $e^{A^\top(s-t)} C^\top$, we obtain

$$\begin{aligned} e^{A^\top(s-t)} C^\top Ce^{A(s-t)} x(t) &= e^{A^\top(s-t)} C^\top y(s) \\ &\quad + \epsilon_1 e^{A^\top(s-t)} C^\top C \int_s^t e^{A(s-m)} \gamma_1(m) dm \\ &\quad + \dots + \epsilon_q e^{A^\top(s-t)} C^\top C \int_s^t e^{A(s-m)} \gamma_q(m) dm. \end{aligned} \quad (17)$$

Then, by integrating the equality (17) over the interval $[t - \tau, t]$ with $t \geq \tau$, we obtain

$$Ex(t) = \Psi(y_t) + \epsilon_1 \kappa(\gamma_{1,t}) + \dots + \epsilon_q \kappa(\gamma_{q,t}), \quad (18)$$

with E defined in (2), Ψ from (9), and κ from (3).

Recalling that the observability of (A, C) implies that E is invertible, it follows that

$$\begin{aligned} x(t) &= E^{-1} \Psi(y_t) + \epsilon_1 E^{-1} \kappa(\gamma_{1,t}) \\ &\quad + \dots + \epsilon_q E^{-1} \kappa(\gamma_{q,t}). \end{aligned} \quad (19)$$

Then, we deduce from (8) that

$$\begin{aligned} \dot{x}(t) &= AE^{-1} \Psi(y_t) + \epsilon_1 AE^{-1} \kappa(\gamma_{1,t}) \\ &\quad + \dots + \epsilon_q AE^{-1} \kappa(\gamma_{q,t}) + \epsilon_1 \gamma_1(t) + \dots + \epsilon_q \gamma_q(t). \end{aligned} \quad (20)$$

By reorganizing the terms, we obtain

$$\begin{aligned} & \epsilon_1[AE^{-1}\kappa(\gamma_{1,t}) + \gamma_1(t)] + \dots + \epsilon_q[AE^{-1}\kappa(\gamma_{q,t}) + \gamma_q(t)] \\ & = \dot{x}(t) - AE^{-1}\Psi(y_t). \end{aligned} \quad (21)$$

Then we integrate (21) over $[t-T, t]$ where $T > 0$ is the constant present in (4) to obtain

$$\begin{aligned} & \epsilon_1 \int_{t-T}^t [AE^{-1}\kappa(\gamma_{1,m}) + \gamma_1(m)] dm \\ & + \dots + \epsilon_q \int_{t-T}^t [AE^{-1}\kappa(\gamma_{q,m}) + \gamma_q(m)] dm \\ & = x(t) - x(t-T) - AE^{-1} \int_{t-T}^t \Psi(y_m) dm. \end{aligned} \quad (22)$$

By left multiplying (22) by C , we obtain

$$\begin{aligned} \sum_{i=1}^q \epsilon_i W_i(t) & = [\epsilon_1 \quad \dots \quad \epsilon_q] W(t) \\ & = y(t) - y(t-T) \\ & \quad - CAE^{-1} \int_{t-T}^t \Psi(y_m) dm, \end{aligned} \quad (23)$$

with W defined in (4). We deduce that

$$[\epsilon_1 \quad \dots \quad \epsilon_q] M(t) = \mathcal{V}(y_t) \quad (24)$$

with \mathcal{V} defined in (11). It follows from Assumption 2 that (14) is satisfied. This combined with (19) yields the equality (13). This concludes the proof.

2.4 Alternative Expressions without Integrals

The integrals in (4), (9), (10) can be replaced by states of delayed dynamical extensions. This replacement can facilitate checking our assumptions, while also making it possible to provide expressions for the observer values that do not contain any integrals. To see how this can be done, we first show that for $i = 1, \dots, q$, we have $W_i(t) = V_i(t) + W_i(0)$, where V_i is the first component of the state of the dynamic extension

$$\begin{cases} \dot{V}_i(t) = C[AE^{-1}(EL_{1i}(t) - L_{2i}(t) \\ \quad + e^{-A^\top \tau} L_{2i}(t - \tau)) + \gamma_i(t) \\ \quad - AE^{-1}(EL_{1i}(t - T) - L_{2i}(t - T) \\ \quad + e^{-A^\top \tau} L_{2i}(t - T - \tau)) - \gamma_i(t - T)] \\ \dot{L}_{1i}(t) = AL_{1i}(t) + \gamma_i(t) \\ \dot{L}_{2i}(t) = -A^\top L_{2i}(t) + C^\top CL_{1i}(t) \end{cases} \quad (25)$$

and where the initial states are $V_i(0) = 0$ and $L_{1i}(s) = L_{2i}(s) = 0$ for all $s \in [-T - \tau, 0]$. To obtain the preceding formula for W_i , first note that for each i and $t \geq 0$, we have

$$\begin{aligned} & \kappa_i(\gamma_{i,t}) \\ & = \int_{-\tau}^0 e^{A^\top r} C^\top C \int_r^0 e^{A(r-m)} \gamma_i(m+t) dm dr \\ & = \int_{t-\tau}^t e^{A^\top(r-t)} C^\top C e^{A(r-t)} \left[\int_r^t e^{A(t-m)} \gamma_i(m) dm \right] dr \\ & = \int_{t-\tau}^t e^{A^\top(r-t)} C^\top C e^{A(r-t)} [L_{1i}(t) - e^{(t-r)A} L_{1i}(r)] dr \\ & = EL_{1i}(t) - \int_{t-\tau}^t e^{-A^\top(t-r)} C^\top CL_{1i}(r) dr \\ & = EL_{1i}(t) - (L_{2i}(t) - e^{-A^\top \tau} L_{2i}(t - \tau)), \end{aligned} \quad (26)$$

where the third and last equalities were obtained by applying the method of variation of parameters to the

L_{1i} and then the L_{2i} dynamics from (25). Therefore, the formulas for the W_i 's in (4) give

$$\dot{W}_i = \dot{V}_i \quad (27)$$

for all i , so the Fundamental Theorem of Calculus gives the required formula $W_i(t) = V_i(t) + W_i(0)$.

Similar arguments give

$$\Psi(y_t) = L_3(t) - e^{-A^\top \tau} L_3(t - \tau), \quad (28)$$

and

$$\mu(t) = y(t) - y(t-T) - CAE^{-1}(L_4(t) - L_4(t-T)), \quad (29)$$

where L_3 and L_4 are solutions of the dynamical extension

$$\begin{cases} \dot{L}_3(t) = -A^\top L_3(t) + C^\top y(t) \\ \dot{L}_4(t) = L_3(t) - e^{-A^\top \tau} L_3(t - \tau) \end{cases} \quad (30)$$

with initial states 0, by applying the method of variation of parameters to the L_3 dynamics in (30) to eliminate the integral from the Ψ formula in (9), and then using the result to eliminate the integral in the formula for μ in (10). Combined with the formulas (26), this makes it possible to eliminate the integrals from the assumptions and conclusions of Theorem 1.

2.5 Illustrations

First we consider the system (1) with $n = 2$,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and } C = [1 \quad 0] \quad (31)$$

with $\varphi = 0$ and $q = 1$. Moreover, we assume that $\tau = 1$ and $\gamma_1(\ell) = [0, 1]^\top$ for all ℓ . Then Assumption 1 is satisfied. Let us check that Assumption 2 is satisfied as well.

Recalling that

$$e^{A\ell} = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} \quad (32)$$

for all $\ell \in \mathbb{R}$, we obtain

$$E = \int_{-\tau}^0 \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} ds = \int_{-\tau}^0 \begin{bmatrix} 1 & s \\ s & s^2 \end{bmatrix} ds. \quad (33)$$

Consequently,

$$E = \begin{bmatrix} \tau & -\frac{\tau^2}{2} \\ -\frac{\tau^2}{2} & \frac{\tau^3}{3} \end{bmatrix}, \quad (34)$$

and therefore also

$$E^{-1} = \frac{12}{\tau^3} \begin{bmatrix} \frac{\tau^2}{3} & \frac{\tau}{2} \\ \frac{\tau}{2} & 1 \end{bmatrix} \text{ and } CE^{-1} = \begin{bmatrix} \frac{4}{\tau} & \frac{6}{\tau^2} \end{bmatrix}. \quad (35)$$

Also, by the formula (32), we have

$$\begin{aligned}
\kappa(\gamma_{1,t}) &= \int_{t-\tau}^t e^{A^\top(s-t)} C^\top C \int_s^t e^{A(s-m)} \gamma_1(m) dm ds \\
&= \int_{t-\tau}^t \begin{bmatrix} 1 & 0 \\ s-t & 0 \end{bmatrix} \int_s^t \begin{bmatrix} s-m \\ 1 \end{bmatrix} dm ds \\
&= \int_{t-\tau}^t \begin{bmatrix} 1 & 0 \\ s-t & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2}(t-s)^2 \\ t-s \end{bmatrix} ds \\
&= \frac{1}{2} \int_{t-\tau}^t \begin{bmatrix} -(t-s)^2 \\ (t-s)^3 \end{bmatrix} ds \\
&= -\frac{1}{2} \int_{-\tau}^0 \begin{bmatrix} s^2 \\ s^3 \end{bmatrix} ds = \begin{bmatrix} -\frac{\tau^3}{6} \\ \frac{\tau^4}{8} \end{bmatrix}.
\end{aligned} \tag{36}$$

Consequently

$$\begin{aligned}
&\int_{t-\tau}^t C [AE^{-1} \kappa(\gamma_{1,m}) + \gamma_1(m)] dm \\
&= \frac{12}{\tau^3} \int_{t-\tau}^t \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\tau^2}{3} & \frac{\tau}{2} \\ \frac{\tau}{2} & 1 \end{bmatrix} dm \begin{bmatrix} -\frac{\tau^3}{6} \\ \frac{\tau^4}{8} \end{bmatrix} \\
&= \frac{12}{\tau^2} \begin{bmatrix} \frac{\tau}{2} & 1 \end{bmatrix} \begin{bmatrix} -\frac{\tau^3}{6} \\ \frac{\tau^4}{8} \end{bmatrix} \\
&= 12 \begin{bmatrix} -\frac{\tau}{2} \frac{\tau}{6} + \frac{\tau^2}{8} \end{bmatrix} \neq 0.
\end{aligned} \tag{37}$$

Thus Assumption 2 is satisfied for any choice of \bar{T} .

More generally, starting with a system of the form

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + \varphi(C\xi(t), u(t)) + \delta(t) \\ y_\xi(t) = C\xi(t) \end{cases} \tag{38}$$

with an unknown continuous function δ representing uncertainty that we wish to identify, it is natural to represent δ in an artificial neural network expansion of the form

$$\delta(t) = \epsilon_1 \gamma_1(t) + \dots + \epsilon_q \gamma_q(t) \tag{39}$$

with known basis functions γ_i for $i = 1, \dots, q$, where the corresponding constants ϵ_i must be identified. Standard methods for identifying the ϵ_i 's include adaptive control and least squares methods. However, these methods would not provide the fixed convergence time for the state observers and parameter identifiers that we provide here. Hence, we believe that our approach can benefit applications that use artificial neural network expansions. \square

3. SYSTEMS WITH DELAYED MEASUREMENTS

We consider the system

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + B(t, u(t)) + \epsilon\gamma(t) \\ y_\xi(t) = C\xi(t-h) \end{cases} \tag{40}$$

where ξ is valued in \mathbb{R}^n , where u is a known input, y_ξ is the output valued in \mathbb{R} , and B is piecewise continuous and locally bounded with respect to t and continuous with respect to u . We assume that the constant delay $h \geq 0$ is known and that the function $\gamma : [0, +\infty) \rightarrow \mathbb{R}^n$ is

piecewise-continuous and locally bounded. To simplify, we consider the case where only one parameter is unknown. We conjecture that an extension where there are an arbitrary number of unknown parameters can be obtained.

As in Section 2.2, we can assume without loss of generality that the $B(t, u(t))$ is not present. Thus we study:

$$\begin{cases} \dot{x}(t) = Ax(t) + \epsilon\gamma(t) \\ y(t) = Cx(t-h) \end{cases} \tag{41}$$

Let us introduce the matrix

$$E_h = \int_{-\tau}^0 e^{A^\top(s-h)} C^\top C e^{A(s-h)} ds, \tag{42}$$

the function

$$\begin{aligned}
\theta(t) &= \int_{t-\tau}^t e^{A^\top(s-t-h)} C^\top C \left[\int_{s-h}^s e^{A(s-\ell-h)} \gamma(\ell) d\ell \right. \\
&\quad \left. + \int_s^t e^{A(s-m-h)} \gamma(m) dm \right] ds,
\end{aligned} \tag{43}$$

and the function

$$\rho(t) = \int_{t-\tau}^t \theta(m-h)^\top (E_h^{-1})^\top C^\top C E_h^{-1} \theta(m-h) dm \tag{44}$$

where $\tau > 0$ is a constant, and where the invertibility of E_h will follow from our observability assumption on (A, C) .

For a given constant $\bar{T} > 2(\tau + h)$, we assume:

Assumption 3: For all $t \in [2(\tau + h), \bar{T}]$, we have $\rho(t) \neq 0$.

In terms of the functions

$$\begin{aligned}
\epsilon_\#(t, y_t) &= \frac{1}{\rho(t)} \int_{t-\tau}^t \theta(m-h)^\top (E_h^{-1})^\top C^\top y(m) dm \\
&\quad - \frac{1}{\rho(t)} \int_{t-\tau}^t \theta(m-h)^\top (E_h^{-1})^\top C^\top C E_h^{-1} \mathcal{G}(m) dm
\end{aligned} \tag{45}$$

and

$$\mathcal{G}(m) = \int_{m-h-\tau}^{m-h} e^{A^\top(s-m)} C^\top y(s) ds, \tag{46}$$

we prove the following which provides exact state measurements for all $t \geq 2(\tau + h)$ in the special case where $\rho(t) \neq 0$ for all $t \geq 2(\tau + h)$ (but see Remark 1 for ways to eliminate the integrals from the estimates and the assumptions):

Theorem 2. Let (41) satisfy Assumptions 1 and 3. Then

$$\begin{aligned}
x(t) &= E_h^{-1} \int_{t-\tau}^t e^{A^\top(s-t-h)} C^\top y(s) ds \\
&\quad + \epsilon_\#(t, y_t) E_h^{-1} \theta(t)
\end{aligned} \tag{47}$$

and

$$\epsilon = \epsilon_\#(t, y_t) \tag{48}$$

hold for all $t \in [2(\tau + h), \bar{T}]$.

Proof. One can easily prove that

$$y(t) = C e^{-Ah} x(t) - \epsilon C \int_{t-h}^t e^{A(t-\ell-h)} \gamma(\ell) d\ell \tag{49}$$

for all $t \geq h$. On the other hand,

$$\begin{aligned}
C e^{A(s-t-h)} x(t) &= \\
C e^{-Ah} x(s) + \epsilon C \int_s^t e^{A(s-m-h)} \gamma(m) dm
\end{aligned} \tag{50}$$

for all t and s . Combining (49) and (50), we obtain

$$\begin{aligned}
C e^{A(s-t-h)} x(t) &= \\
&= y(s) + \epsilon C \left[\int_{s-h}^s e^{A(s-\ell-h)} \gamma(\ell) d\ell \right. \\
&\quad \left. + \int_s^t e^{A(s-m-h)} \gamma(m) dm \right]
\end{aligned} \tag{51}$$

when $t \geq s \geq h$. As an immediate consequence,

$$\int_{t-\tau}^t e^{A^\top(s-t-h)} C^\top C e^{A(s-t-h)} ds x(t) = \int_{t-\tau}^t e^{A^\top(s-t-h)} C^\top y(s) ds + \epsilon \theta(t) \quad (52)$$

for all $t \geq \tau + h$ with θ defined in (43). It follows that

$$E_h x(t) = \int_{t-\tau}^t e^{A^\top(s-t-h)} C^\top y(s) ds + \epsilon \theta(t) \quad (53)$$

with E_h defined in (42). Assumption 1 ensures that

$$x(t) = E_h^{-1} \int_{t-\tau}^t e^{A^\top(s-t-h)} C^\top y(s) ds + \epsilon E_h^{-1} \theta(t) \quad (54)$$

for all $t \geq h + \tau$. We deduce that

$$y(t) = Cx(t-h) = CE_h^{-1} \int_{t-h-\tau}^{t-h} e^{A^\top(s-t)} C^\top y(s) ds + \epsilon CE_h^{-1} \theta(t-h). \quad (55)$$

It follows that with the choice $C^\# = C^\top C$, we have

$$\begin{aligned} & \epsilon \theta(t-h)^\top (E_h^{-1})^\top C^\# E_h^{-1} \theta(t-h) \\ &= \theta(t-h)^\top (E_h^{-1})^\top C^\top y(t) \\ & - \theta(t-h)^\top (E_h^{-1})^\top C^\# E_h^{-1} \int_{t-h-\tau}^{t-h} e^{A^\top(s-t)} C^\top y(s) ds. \end{aligned} \quad (56)$$

By integrating (56) over $[t-\tau, t]$ with $t \geq 2\tau + h$, it follows from our choices of ρ in (44) and \mathcal{G} in (46) that

$$\begin{aligned} \epsilon \rho(t) &= \int_{t-\tau}^t \theta(m-h)^\top (E_h^{-1})^\top C^\top y(m) dm \\ & - \int_{t-\tau}^t \theta(m-h)^\top (E_h^{-1})^\top C^\top CE_h^{-1} \mathcal{G}(m) dm. \end{aligned} \quad (57)$$

Then Assumption 3 ensures that the equality

$$\begin{aligned} \epsilon &= \frac{1}{\rho(t)} \int_{t-\tau}^t \theta(m-h)^\top (E_h^{-1})^\top C^\top y(m) dm \\ & - \frac{1}{\rho(t)} \int_{t-\tau}^t \theta(m-h)^\top (E_h^{-1})^\top C^\top CE_h^{-1} \mathcal{G}(m) dm \end{aligned} \quad (58)$$

is satisfied when $t \leq \bar{T}$. The theorem now follows by combining (54) and (58).

Remark 1. Using the method of variation of parameters as we did in the case where there were no measurement delays in Section 2.4, we can eliminate the integrals from Theorem 2. This facilitates checking the assumptions and computing the observers from Theorem 2.

4. DISCRETE-TIME SYSTEMS

Consider the system

$$\begin{cases} \xi_{k+1} = A\xi_k + \varphi(y_{\xi,k}, u_k) + \epsilon\gamma_k \\ y_{\xi,k} = C\xi_k \end{cases} \quad (59)$$

where ξ is valued in \mathbb{R}^n , u is the input, y_ξ is the real valued output, $y_{\xi,k}$ is the output at discrete time k and similarly for the other subscripts k , and for simplicity we again only consider one unknown constant $\epsilon \in \mathbb{R}$ as in the preceding section. Our goal is to find ξ_k and ϵ in finite time. We use

$$F = \sum_{\ell=-n}^{-1} (A^\top)^\ell C^\top C A^\ell \quad (60)$$

which will be invertible under Assumption 4 below, and the function

$$\begin{aligned} & \beta(\gamma_{k-n}, \dots, \gamma_{k-1}) = \\ & CF^{-1} \sum_{\ell=k-n}^{k-1} (A^\top)^{\ell-k} C^\top C [A^{\ell-k} \gamma_{k-1} + \dots + A^{-1} \gamma_\ell]. \end{aligned} \quad (61)$$

Our assumptions in this section are:

Assumption 4: The matrix A is invertible and the pair (A, C) is observable. \square

Assumption 5: There are integers $r > 0$ and $R > n + r$ such that

$$\sum_{i=k-r}^k \beta(\gamma_{i-n}, \dots, \gamma_{i-1})^2 > 0 \quad (62)$$

holds for each integer $k \in [n + r, R]$. \square

As in Section 2.2, we use the dynamic extension

$$\hat{\xi}_{k+1} = A\hat{\xi}_k + \varphi(y_{\xi,k}, u_k) \quad (63)$$

and $x_k = \xi_k - \hat{\xi}_k$, to obtain

$$\begin{cases} x_{k+1} = Ax_k + \epsilon\gamma_k \\ y_k = Cx_k \end{cases} \quad (64)$$

with y_k valued in \mathbb{R} . In terms of the functions

$$\alpha(y_{k-n}, \dots, y_{k-1}) = CF^{-1} \sum_{\ell=k-n}^{k-1} (A^\top)^{\ell-k} C^\top y_\ell, \quad (65)$$

$$\begin{aligned} \epsilon_\Delta(y_{k-n-r}, \dots, y_k, \gamma_{k-n-r}, \dots, \gamma_{k-1}) &= \\ & \frac{1}{\sum_{i=k-r}^k \beta(\gamma_{i-n}, \dots, \gamma_{i-1})^2} \left(\sum_{i=k-r}^k \beta(\gamma_{i-n}, \dots, \gamma_{i-1}) y_i \right. \\ & \left. - \sum_{i=k-r}^k \beta(\gamma_{i-n}, \dots, \gamma_{i-1}) \alpha(y_{i-n}, \dots, y_{i-1}) \right), \end{aligned} \quad (66)$$

and

$$\mathcal{M}_k = \sum_{\ell=k-n}^{k-1} (A^\top)^{\ell-k} C^\top C [A^{\ell-k} \gamma_{k-1} + \dots + A^{-1} \gamma_\ell], \quad (67)$$

we then have the following discrete time result, which gives exact state values for all $k \geq n + r$ in the special case where (62) holds for all $k \geq n + r$:

Theorem 3. Let (64) satisfy Assumptions 4-5. Choose r and R satisfying the assumptions. Then

$$\begin{aligned} x_k &= F^{-1} \sum_{\ell=k-n}^{k-1} (A^\top)^{\ell-k} C^\top y_\ell \\ & + \epsilon_\Delta(y_{k-n-r}, \dots, y_k, \gamma_{k-n-r}, \dots, \gamma_{k-1}) F^{-1} \mathcal{M}_k \end{aligned} \quad (68)$$

and

$$\epsilon = \epsilon_\Delta(y_{k-n-r}, \dots, y_k, \gamma_{k-n-r}, \dots, \gamma_{k-1}) \quad (69)$$

hold for all $k \in [n + r, R]$.

Proof. For all integers $\ell \geq 0$, we can prove that

$$x_k = A^{k-\ell} x_\ell + \epsilon[\gamma_{k-1} + A\gamma_{k-2} + \dots + A^{k-\ell-1} \gamma_\ell] \quad (70)$$

for all $k > \ell$, by induction on k . Thus,

$$x_\ell = A^{\ell-k} x_k - \epsilon[A^{\ell-k} \gamma_{k-1} + \dots + A^{-1} \gamma_\ell]. \quad (71)$$

By left multiplying this equality by C , we obtain

$$y_\ell = CA^{\ell-k} x_k - \epsilon C[A^{\ell-k} \gamma_{k-1} + \dots + A^{-1} \gamma_\ell]. \quad (72)$$

As an immediate consequence, we get

$$\begin{aligned} (A^\top)^{\ell-k} C^\top y_\ell &= (A^\top)^{\ell-k} C^\top C A^{\ell-k} x_k \\ & - \epsilon (A^\top)^{\ell-k} C^\top C [A^{\ell-k} \gamma_{k-1} + \dots + A^{-1} \gamma_\ell]. \end{aligned} \quad (73)$$

It follows that for all integers $k > n$, we have

$$\begin{aligned} & \sum_{\ell=k-n}^{k-1} (A^\top)^{\ell-k} C^\top y_\ell \\ &= Fx_k - \epsilon \sum_{\ell=k-n}^{k-1} (A^\top)^{\ell-k} C^\top C [A^{\ell-k} \gamma_{k-1} + \dots + A^{-1} \gamma_\ell] \end{aligned} \quad (74)$$

which gives

$$\begin{aligned} x_k &= F^{-1} \sum_{\ell=k-n}^{k-1} (A^\top)^{\ell-k} C^\top y_\ell \\ &+ \epsilon F^{-1} \sum_{\ell=k-n}^{k-1} (A^\top)^{\ell-k} C^\top C [A^{\ell-k} \gamma_{k-1} + \dots + A^{-1} \gamma_\ell]. \end{aligned} \quad (75)$$

This gives $y_k = \alpha(y_{k-n}, \dots, y_{k-1}) + \epsilon \beta(\gamma_{k-n}, \dots, \gamma_{k-1})$, with β defined in (61) and α defined in (65). It follows that

$$\begin{aligned} \beta(\gamma_{k-n}, \dots, \gamma_{k-1}) y_k &= \epsilon \beta(\gamma_{k-n}, \dots, \gamma_{k-1})^2 \\ &+ \beta(\gamma_{k-n}, \dots, \gamma_{k-1}) \alpha(y_{k-n}, \dots, y_{k-1}). \end{aligned} \quad (76)$$

Then

$$\begin{aligned} \epsilon \sum_{i=k-r}^k \beta(\gamma_{i-n}, \dots, \gamma_{i-1})^2 &= \sum_{i=k-r}^k \beta(\gamma_{i-n}, \dots, \gamma_{i-1}) y_i \\ - \sum_{i=k-r}^k \beta(\gamma_{i-n}, \dots, \gamma_{i-1}) \alpha(y_{i-n}, \dots, y_{i-1}). \end{aligned} \quad (77)$$

The result follows from (75), (77), and Assumption 5.

5. CONCLUSIONS

We provided new fixed finite-time observers for the simultaneous computation of unknown model parameters and unknown states. Our structural conditions can represent artificial neural network expansions of unknown time-varying functions. A key assumption was that the unmeasured state enters the dynamics in an affine way. While this assumption holds for a broad class of significant dynamics (e.g., the motor dynamics in Mazenc et al. (2021), and the dynamics of vibrating membranes in Mazenc et al. (2018)), it would be of interest to relax this assumption. The work Mazenc et al. (2022) uses a small-gain approach to obtain fixed time finite-time observers for unperturbed nonlinear systems that are not required to be affine in the unmeasured state, but Mazenc et al. (2022) is not amenable to the fixed time calculation of unknown model parameters. We hope to combine the present paper with small-gain methods from Mazenc et al. (2022) to relax our requirement that the dynamics are affine in the unmeasured state and to allow sampling. We also aim to develop reduced order versions, to build on works like Mazenc et al. (2020) which provided fixed time state estimators but not parameter identification.

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