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# Strong and weak Leonov functions for global boundedness of state periodic systems

Johannes Schiffer and Denis Efimov

**Abstract**—We present new results for global boundedness of state periodic systems. Thereby, we address both the case of systems, whose dynamics is periodic with respect to a *part* of the state vector and the case of systems, whose dynamics is periodic with respect to *all* state variables. Both classes of systems are of high relevance in diverse and timely applications. To derive the results the notion of *strong* Leonov functions is introduced. The main results are complemented by a number of relaxations based on the concept of *weak* Leonov functions.

## I. INTRODUCTION

Stability is one of the most fundamental properties in the analysis, design and engineering of dynamical systems [1]–[6]. It is, hence, not surprising that the development of methods and tools to assess stability is a highly active research domain in the control systems community and beyond. As of today, the prevalent theory underpinning many modern approaches, *e.g.*, [5], [7]–[9], is Lyapunov theory [3]. The importance of Lyapunov-based stability methods rests upon the fact that they permit to assess boundedness and convergence properties of a system’s solutions without explicit computation of the latter.

Different kinds of stability properties may be of relevance in characterizing the behavior of dynamical systems [2], [4]. The classical notion of stability concerns properties of a single isolated equilibrium point [3], [4]. However, many dynamical systems appearing in (engineering) applications exhibit multiple equilibrium points. Paramount examples comprise biological systems [10]–[12], power systems and microgrids [13]–[15] as well as complex networks of coupled oscillators [16]–[19]. In recent years the classical Lyapunov theory has been extended to the case of systems with several disjoint invariant sets [10], [16], [20]–[23]. A characterization of the input-to-state stability (ISS) property in the presence of multiple equilibria is provided in [24], [25] and extended to cascaded systems in [26] to delay systems in [27], [28] as well as to the integral ISS (iISS) property in [29], see also the overview given in [30]. Additional related results on robust stability analysis are given in [11], [31], [32].

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In the present work, we focus on two special classes of systems, which inherently exhibit multiple equilibria. The first class are systems, whose dynamics is periodic with respect to a *part* of the state vector. The second class are systems, whose dynamics is periodic with respect to *all* state variables. In both cases, the state evolves in  $\mathbb{R}^n$ . Applications in which the former type of dynamics arises are power systems and microgrids [15], [33], [34], the forced nonlinear pendulum [11] and phase-locked loops [20], [35], [36]. Paramount examples for the latter type of dynamics are complex networks of coupled oscillators [16], [18] and, in particular, Kuramoto oscillators [17], [37].

For both classes of systems, a direct application of the stability results reviewed above is highly challenging. This originates from the fact that the periodicity of the dynamics significantly complicates the construction of a Lyapunov function. More precisely, an application of the approach presented in [24], [25] to a state periodic system would require to construct a Lyapunov function, which respects the system periodicity, *i.e.*, it also needs to verify the definiteness requirements with respect to the equilibria (or invariant sets) as well as of its time derivative along the system’s solutions. This also applies to the periodic Lyapunov functional method [38]. These stringent requirements can often not be met in applications and, consequently, the majority of stability results for state periodic systems are limited to *local* stability claims, see [17], [34].

In this context, usually the main difficulty resides in establishing *global boundedness* of solutions, *i.e.*, Lagrange stability [1]. For systems with infinitely many equilibria, the property of boundedness of trajectories is the natural counterpart to stability of a single equilibrium point in the classical Lyapunov theory. Once the existence of compact forward invariant sets for the system’s solutions is established, there are a variety of methods available to assess their asymptotic behavior. These comprise LaSalle’s invariance principle [4], contraction [39], [40] and convergence [41], or differential positivity [42]. Hence, our main focus in the present paper is on the provision of relaxed conditions to infer global boundedness of solutions.

The path we take to address the above objective is to explicitly exploit the periodicity of the system dynamics in order to relax the (definiteness and periodicity) requirements compared to standard Lyapunov functions, while still ensuring boundedness of solutions. This approach was pioneered by Leonov [43] and Noldus [44] for systems whose dynamics are periodic with respect to a *scalar* state variable.

The original derivations of Leonov and Noldus critically rely on the restriction that the periodic variable is *scalar*. In such a setting, the individual cells are only connected via equilibria, which is instrumental for the proofs in [43], [44]. Yet, obviously, this restriction significantly limits the application of these results to networks and interconnected systems with periodic dynamics.

Motivated by the successful application of the original cell structure approach to establish global properties of a synchronous machine infinite bus system [45], [46] as well as first extensions of the ISS theory to state periodic systems [47]–[49], in our previous work [50], [51] a *multivariable* cell structure framework is derived. The derivation is built upon the concept of a Leonov function and it yields sufficient conditions for boundedness of solutions of systems whose dynamics are periodic with respect to *several* state variables.

In a nutshell, a Leonov function is sign-indefinite with respect to the periodic states and radially unbounded with respect to the nonperiodic states. Moreover, it is negative definite with respect to the distance to a set, which separates the equilibria of the system. This last requirement is a major difference compared to the functions employed in [43], [44] and is key to establish the main result in the multivariable case, which - as already pointed out in [44] - is far from being trivial. First applications of these results to establish almost global stability properties of power systems and microgrids have been reported in [52], [53]. More recently, also the iISS theory has been adapted to state periodic systems [54].

In continuation of the results given in [50], [51], the main contributions of the present paper are three-fold.

- 1) The notion of a strong Leonov function (sLeF) is introduced, based on which sufficient conditions for global boundedness of systems, whose dynamics are periodic with respect to a *part* of the state vector are presented. Compared to our previous, related work [50], [51], the required properties of an sLeF are refined. These refinements are instrumental for improving the proof of the main result of [50], [51], which in turn is instrumental for the derivations in item 2) below.
- 2) Based on the refinements in item 1), we introduce several relaxations of our main result. These are formalized via the concept of a weak Leonov function (wLeF).
- 3) The properties established in items 1) and 2) are extended to the case of systems, whose dynamics are periodic with respect to the *full* state vector. This type of systems has not been considered previously neither in [43], [44] nor in [50], [51].

The remainder of the paper is structured as follows. Preliminaries are given in Section II. Sufficient conditions for boundedness of systems, whose dynamics are periodic with respect to part of the state vector are presented in Section III based on the notions of sLeFs and wLeFs. Their counterparts for systems with purely periodic dynamics are given in Section IV. The paper is wrapped-up in Section V, in which also an outlook on future work is provided.

## II. PRELIMINARIES

Denote the sets of real and integer numbers by  $\mathbb{R}$  and  $\mathbb{Z}$ , respectively, and let  $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$  as well as  $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$ . Let  $\mathbb{Z}^k$  denote a vector of  $k > 0$  integers.

Let the map  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be of class  $\mathcal{C}^1$ ,  $f(0) = 0$ , and consider a nonlinear system of the following form:

$$\dot{x}(t) = f(x(t)) \quad \forall t \geq 0 \quad (\text{II.1})$$

with the state  $x(t) \in \mathbb{R}^n$ . We denote by  $X(t, x)$  the uniquely defined solution of (II.1) at time  $t$  fulfilling  $X(0, x) = x$ .

A set  $S \subset \mathbb{R}^n$  is invariant for the system (II.1) if  $X(t, x) \in S$  for all  $t \in \mathbb{R}$  and for all  $x \in S$ . For  $x \in \mathbb{R}^n$  the point  $y \in \mathbb{R}^n$  belongs to its  $\omega$ -limit ( $\alpha$ -limit) set if there is a sequence  $t_i$ ,  $\lim_{i \rightarrow \infty} t_i = +\infty$ , such that  $\lim_{i \rightarrow +\infty} X(t_i, x) = y$  ( $\lim_{i \rightarrow +\infty} X(-t_i, x; 0) = y$ ). For any  $x \in \mathbb{R}^n$  its  $\alpha$ - and  $\omega$ -limit sets are invariant [55]. We define the distance from a point  $x \in \mathbb{R}^n$  to the set  $S \subset \mathbb{R}^n$  as  $|x|_S = \inf_{a \in S} |x - a|$ , where  $|x| = |x|_0$  for  $x \in \mathbb{R}^n$  is the usual Euclidean norm of a vector  $x \in \mathbb{R}^n$ . Furthermore, we introduce the vector norm

$$|x|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

A continuous function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{K}$  if  $\alpha(0) = 0$  and the function is strictly increasing. The function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}_\infty$  if  $\alpha \in \mathcal{K}$  and its supremum is infinity. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{KL}$  if  $\beta(\cdot, t) \in \mathcal{K}_\infty$  for each fixed  $t \in \mathbb{R}_+$  and  $\lim_{t \rightarrow \infty} \beta(s, t) = 0$  for each fixed  $s \in \mathbb{R}_+$ .

## III. BOUNDEDNESS OF DYNAMICAL SYSTEMS THAT ARE PERIODIC WITH RESPECT TO PART OF THE STATE VARIABLES

In this section, we make the following assumption on the dynamics (II.1).

*Assumption III.1.* Let  $x = (z, \theta) \in \mathbb{R}^n$ , where  $z \in \mathbb{R}^k$  and  $\theta \in \mathbb{R}^q$  are two subsets of the state vector,  $n = k + q$ ,  $k > 0$  and  $q > 0$ . The vector field  $f$  in (II.1) is  $2\pi$ -periodic with respect to  $\theta$ .

In other words, we suppose that the dynamics (II.1) are periodic with respect to the *part* of the state vector denoted by  $\theta$ . Thus, the system (II.1) can be embedded into a manifold  $M = \mathbb{R}^k \times \mathbb{S}^q$  by a simple projection of the variables  $\theta(t)$  on the sphere  $\mathbb{S}^q$ .

### A. Strong Leonov functions and main result

We introduce the notion of a sLeF based on our previous work [51]. For our subsequent derivations, it is convenient to introduce some auxiliary sets. Let  $\mathcal{W} \subset (\mathbb{R}^k \times \mathcal{S})$ , where  $\mathcal{S}$  is a  $q^{\text{th}}$  sphere (a set topologically equivalent to  $\mathbb{S}^q$ ), such that for some  $\underline{c} \in (0, 2\pi)$  and  $\bar{c} \in (\pi, 2\pi)$ ,  $\underline{c} < \bar{c}$ ,

$$\{x \in \mathbb{R}^n : \underline{c} \leq |\theta|_\infty \leq \bar{c}\} \subseteq \mathcal{W},$$

$$\mathcal{U} = \bigcup_{r \in \mathbb{Z}_+} \mathcal{U}_r, \quad (\text{III.1})$$

$$\mathcal{U}_r = \{x = (z, \theta) \in \mathbb{R}^n : |\theta|_\infty = 2r\pi, f(x) = 0\}.$$

Thus, the set  $\mathcal{U}$  includes all equilibria obtained by shifting the one at the origin using the property that, by assumption,  $f$  is  $2\pi$ -periodic in  $\theta$  (due to continuity of  $f$  there are also other equilibria of (II.1) not included in  $\mathcal{U}$ ).

Compared to [51], in (IV.1) the properties of the set  $\mathcal{W}$  in (IV.1) are made more precise. With this refinement we can present an improved proof of the main result of [51], which paves the path for the subsequently derived relaxations. To this end, we employ the following definition of a sLeF.

**Definition III.2.** A  $\mathcal{C}^1$  function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a *strong* Leonov function (sLeF) for (II.1) with Assumption III.1 if there exist functions  $\alpha \in \mathcal{K}_\infty$ ,  $\psi \in \mathcal{K}$ , and a continuous function  $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\lambda(0) = 0$  and  $\lambda(s)s > 0$  for all  $s \neq 0$  as well as a scalar  $g \geq 0$ , such that for all  $x = (z, \theta) \in \mathbb{R}^n$

$$\begin{aligned} \alpha(|z|) - \psi(|\theta|) - g &\leq V(x) \quad \forall x \in \mathbb{R}^n, \\ \inf_{x \in \mathcal{W}} V(x) &> 0, \quad \sup_{x \in \mathcal{U}} V(x) \leq 0 \end{aligned} \quad (\text{III.2})$$

and the following dissipation holds:

$$DV(x)f(x) \leq -\lambda(V(x)). \quad (\text{III.3})$$

We are not asking the function  $V$  to be  $2\pi$ -periodic in  $\theta$ . Hence, any analysis involving such a sLeF has to be carried out in  $\mathbb{R}^n$ . This observation guides the derivations of the following result.

**Theorem III.3.** *If for the system (II.1) under Assumption III.1 there exists a sLeF as in Definition III.2, then for all  $x_0 \in \mathbb{R}^n$  the corresponding trajectories  $X(t, x_0)$  are bounded for all  $t \geq 0$ .*

*Proof.* Denote the set of negative values of  $V$  as

$$\Omega = \{x \in \mathbb{R}^n : V(x) \leq 0\}.$$

From the definition of the set  $\Omega$  and the lower bound of the function  $V$  in (III.2) we obtain:

$$\alpha(|z|) \leq g + \psi(|\theta|) \quad \forall x = (z, \theta) \in \Omega.$$

Recall the sets  $\mathcal{U}$  and  $\mathcal{W}$  defined in (IV.1), and that  $|\theta| \leq q|\theta|_\infty$ . Then from the definition of a sLeF, see (III.2), the set  $\mathcal{U}$  belongs to  $\Omega$ , while  $\mathcal{W} \notin \Omega$ . Furthermore, the set  $\Omega$  can be decomposed into two parts:

$$\Omega = \Omega' \cup \Omega''.$$

The first of these sets,

$$\Omega' \subset \{x \in \mathbb{R}^n : |z| \leq \alpha^{-1}(\tilde{g}), |\theta|_\infty < \underline{c}\}$$

is a compact set containing the origin, see the definition of the set  $\mathcal{W}$  in (IV.1), *i.e.*,  $\mathcal{U}_0 \subset \Omega'$ , with  $g \leq \tilde{g} \leq g + \psi(q\underline{c})$  (the constant  $\tilde{g}$  evaluates the upper bound on  $|z|$  after which the growing terms in  $z$  will force  $V$  to take positive values for any  $|\theta|_\infty < \underline{c}$ ). The second set is defined as

$$\Omega'' \subset \{x \in \mathbb{R}^n : |\theta|_\infty > \bar{c}\}.$$

Both these sets are non-empty since  $0 \in \Omega'$  and  $\mathcal{U}_r \in \Omega''$  for  $r \geq 1$  due to (III.2). Furthermore, the set  $\Omega''$  may be

unbounded. In addition, the subsets  $\Omega'$  and  $\Omega''$  are separated by the set  $\mathcal{W}$ .

Denote  $V(t) = V(X(t, x_0))$  for any  $x_0 \in \mathbb{R}^n$ , then under the conditions of the theorem we have:

$$\dot{V}(t) + \lambda(V(t)) \leq 0 \quad \forall t \geq 0.$$

Thus, clearly,  $V(t)$  is strictly decreasing while  $X(t, x_0) \in \mathbb{R}^n \setminus \Omega$ , *i.e.*, while  $V(t) > 0$ . Therefore, if  $x_0 \in \Omega$  then  $X(t, x_0) \in \Omega$  for all  $t \geq 0$  and the set  $\Omega$  is forward invariant for (II.1). Conversely, if  $x_0 \in \mathbb{R}^n \setminus \Omega$  then there exists  $0 \leq T_{x_0} \leq +\infty$  such that  $X(T_{x_0}, x_0) \in \Omega$  and, by invariance, the trajectory remains in this set for all  $t \geq T_{x_0}$ . Thus, the set  $\Omega$  is globally attractive and forward invariant. Note that this property does not imply any kind of stability since  $\Omega$  may be unbounded (it could also be interpreted as *instability* of a set containing  $\mathcal{W}$  with the domain of repulsion  $\mathbb{R}^n \setminus \Omega$ ).

To establish stability, we exploit the *periodicity* of (II.1). Denote by  $j = [j_1, \dots, j_q] \in \mathbb{Z}^q$  a multi-index vector. Introduce the new variable  $x_j = x - \begin{bmatrix} 0_k \\ 2\pi j \end{bmatrix}$  and the function  $V_j(x) = V\left(x - \begin{bmatrix} 0_k \\ 2\pi j \end{bmatrix}\right)$  for any such multi-index vector  $j$  (*i.e.*,  $V_0(x) = V(x)$ ), where  $0_k$  is the zero vector of dimension  $k$ . Hence

$$V_j(x) = V_j\left(x_j + \begin{bmatrix} 0_k \\ 2\pi j \end{bmatrix}\right) = V(x_j).$$

Furthermore, by  $2\pi$ -periodicity of  $f$  in  $\theta$ ,

$$\dot{x}_j(t) = f(x(t)) = f(x_j(t)) \quad \forall t \geq 0.$$

Therefore, taking into account the properties substantiated for (II.1) and  $V$ , we obtain that the set  $\{x_j \in \mathbb{R}^n : V(x_j) \leq 0\}$  is globally attractive and forward invariant, which in the original coordinates  $x$  implies these properties for the set

$$\begin{aligned} \Omega_j &= \{x \in \mathbb{R}^n : V_j(x) \leq 0\} \\ &= \left\{x \in \mathbb{R}^n : V\left(x - \begin{bmatrix} 0_k \\ 2\pi j \end{bmatrix}\right) \leq 0\right\}. \end{aligned}$$

Using similar arguments as for  $\Omega$  it is possible to show that  $\Omega_j = \Omega'_j \cup \Omega''_j$ , where

$$\begin{aligned} \Omega'_j &\subset \{x \in \mathbb{R}^n : |z| \leq \alpha^{-1}(\tilde{g}), |\theta - 2\pi j|_\infty < \underline{c}\}, \\ \Omega''_j &\subset \{x \in \mathbb{R}^n : |\theta - 2\pi j|_\infty > \bar{c}\}. \end{aligned}$$

Since the sets  $\Omega_j$  are forward invariant, any trajectory  $X(t, x_0)$  has to belong asymptotically to the intersection of these sets:

$$\bigcap_{j \in \mathbb{Z}^q} \Omega_j = \bigcap_{j \in \mathbb{Z}^q} (\Omega'_j \cup \Omega''_j).$$

Next, we show that  $\bigcap_{j \in \mathbb{Z}^q} \Omega_j$  is composed of compact isolated subsets.

At first, we note that since  $\underline{c} < 2\pi$ , then  $\bigcap_{j \in \mathbb{Z}^q} \Omega'_j = \emptyset$ . Furthermore, by definition  $\bigcap_{j \in \mathbb{Z}^q} \Omega''_j = \emptyset$  due to the fact that  $\Omega''_j \subset \mathbb{R}^n \setminus \Upsilon_j$  with  $\Upsilon_j = \{x \in \mathbb{R}^n : |\theta - 2\pi j|_\infty \leq \bar{c}\}$  and  $\bar{c} > \pi$ , then  $\bigcap_{j \in \mathbb{Z}^q} \Omega''_j \subset \bigcap_{j \in \mathbb{Z}^q} \mathbb{R}^n \setminus \Upsilon_j = \mathbb{R}^n \setminus \bigcup_j \Upsilon_j = \mathbb{R}^n \setminus \mathbb{R}^n = \emptyset$ .

Recall that  $\Omega_j = \Omega'_j \cup \Omega''_j$  with  $j \in \mathbb{Z}^q$  and  $\bigcap_{j \in \mathbb{Z}^q} \Omega'_j = \bigcap_{j \in \mathbb{Z}^q} \Omega''_j = \emptyset$ . Then,

$$\begin{aligned} \bigcap_{j \in \mathbb{Z}^q} \Omega_j &= \bigcap_{j \in \mathbb{Z}^q} (\Omega'_j \cup \Omega''_j) = \bigcap_{j \in \mathbb{Z}^q} \Omega'_j \cup \bigcap_{j \in \mathbb{Z}^q} \Omega''_j \\ &\cup \bigcup_{j \in \mathbb{Z}^q} \left( \Omega'_j \cap \bigcap_{k \in \mathbb{Z}^q, k \neq j} \Omega''_k \right) \\ &\cup \bigcup_{i, j \in \mathbb{Z}^q, i \neq j} \left( \Omega'_j \cap \Omega'_i \cap \bigcap_{k \in \mathbb{Z}^q, k \neq j, k \neq i} \Omega''_k \right) \\ &\cup \bigcup_{i, j, s \in \mathbb{Z}^q, i \neq j, s \neq j, i \neq s} \left( \Omega'_j \cap \Omega'_i \cap \Omega'_s \cap \bigcap_{k \in \mathbb{Z}^q, k \neq j, k \neq i, k \neq s} \Omega''_k \right) \\ &\cup \dots \end{aligned}$$

If  $\Omega'_j \cap \Omega'_k = \emptyset$  for  $j, k \in \mathbb{Z}^q$  and  $j \neq k$ , then all intersections in the second and other lines disappear (they all are empty sets). This is in particular the case if  $\underline{c} \in (0, \pi)$ , see (IV.1).

If  $\underline{c} \in [\pi, 2\pi)$ , then some of the elementary intersections  $\Omega'_j \cap \Omega'_i$  or  $\Omega'_j \cap \Omega'_i \cap \Omega'_s$ , etc., as above, may not be empty. Yet, then they belong to  $\Omega'_j$  (by construction, as  $\Omega'_j$  is the first element of the intersections) and they form disjoint (positively separated) sets in  $\Omega'_j$ . To clarify the latter claim consider two terms:  $T_1 = \Omega'_j \cap \bigcap_{k \in \mathbb{Z}^q, k \neq j} \Omega''_k$  and  $T_2 = \Omega'_j \cap \Omega'_i \cap \bigcap_{k \in \mathbb{Z}^q, k \neq j, k \neq i} \Omega''_k$  for any  $i, j \in \mathbb{Z}^q$  and  $i \neq j$ , then  $T_1 \cap T_2 = \emptyset$  since  $\Omega'_i \cap \Omega'_j = \emptyset$  by definition and  $T_1 \subset \Omega''_i$ , while  $T_2 \subset \Omega'_i$  (and there is a non-zero distance between sets  $\Omega'_i$  and  $\Omega''_i$ ). The same verification can be repeated for any other terms.

Hence,

$$\bigcap_{j \in \mathbb{Z}^q} \Omega_j = \bigcap_{j \in \mathbb{Z}^q} \Omega'_j \cup \bigcap_{j \in \mathbb{Z}^q} \Omega''_j \cup \bigcup_{j \in \mathbb{Z}^q} (\Omega'_j \cap \mathcal{R}_j) = \bigcup_{j \in \mathbb{Z}^q} \mathcal{R}_j,$$

where  $\mathcal{R}_j \subseteq \Omega'_j$  is composed by the disjoint sets representing all non-empty intersections of  $\Omega'_j$  with other  $\Omega'_k$  and  $\Omega''_i$  for  $i, j, k \in \mathbb{Z}^q$ ,  $k \neq j$  and  $i \neq j$ .

Thus, the compact sets in  $\bigcap_{j \in \mathbb{Z}^q} \Omega_j$  form a kind of cell cover of  $\{x \in \mathbb{R}^n : |z| \leq \alpha^{-1}(\bar{g})\}$ . Recall that for all multi-index vectors  $j$  the corresponding sets  $\Omega_j$  are globally attracting and forward invariant. Take any  $x_0 \in \mathbb{R}^n$ , then  $X(t, x_0)$  asymptotically enters and stays in a cell  $\mathcal{R}_j$  for some  $j$ . Hence, for any  $x_0 \in \mathbb{R}^n$  the corresponding solution  $X(t, x_0)$  is bounded for all  $t \geq 0$ .  $\square$

We have the following corollary to Theorem III.3. For its presentation, we introduce the equilibrium set of the dynamics (II.1), i.e.,

$$\mathcal{D} = \{x \in \mathbb{R}^n \mid f(x) = 0\}. \quad (\text{III.4})$$

Clearly,  $\mathcal{U} \subseteq \mathcal{D}$ .

**Corollary III.4.** *If for the system (II.1) under Assumption III.1 there exists a sLeF as in Definition III.2 with*

$$V(0) \leq 0, \quad \inf_{x \in \mathcal{U} \setminus \mathcal{U}_0} V(x) \leq 0, \quad \inf_{x \in \mathcal{W}} V(x) > 0, \quad (\text{III.5})$$

and with the following dissipation inequality

$$DV(x)f(x) < 0 \quad \forall x \in \mathbb{R}^n \setminus \{\mathcal{D} \cup \Omega\}, \quad (\text{III.6})$$

then for all  $x_0 \in \mathbb{R}^n$  the corresponding trajectories  $X(t, x_0)$  are bounded for all  $t \geq 0$ .

*Proof.* The set

$$\Omega = \{x \in \mathbb{R}^n : V(x) \leq 0\}$$

has the same structure as in Theorem III.3. That is,

$$\Omega = \Omega' \cup \Omega'',$$

where, due to (III.5),  $\mathcal{U}_0 \subseteq \Omega'$  (and it is compact by definition) and there is at least one  $k \in \mathbb{Z}^+$ , such that  $\mathcal{U}_k \subseteq \Omega''$ . Thus, both  $\Omega'$  and  $\Omega''$  are non-empty, but  $\Omega''$  may be unbounded. Clearly,

$$DV(x)f(x) = 0 \quad \forall x \in \mathcal{D}.$$

Hence, from (IV.5) it follows that  $\Omega$  is forward invariant and the set  $\Omega \cup \mathcal{D}$  is globally attractive. Recall that  $\mathcal{D}$  only contains equilibria of (II.1). This together with the fact that, due to (IV.5),  $V$  is strictly decreasing for all  $x \in \mathbb{R}^n \setminus \{\mathcal{D} \cup \Omega\}$ , implies that any trajectory that enters  $\mathcal{D}$  cannot move inside  $\mathcal{D}$  and, consequently, is bounded (even if  $\mathcal{D}$  itself may contain an unbounded continuum of equilibria). Next, the proof of Theorem III.3 can be repeated since asymptotically all trajectories have to be in  $\mathcal{D} \cup \bigcap_{j \in \mathbb{Z}^q} (\Omega'_j \cup \Omega''_j)$ .  $\square$

## B. Further relaxations via weak Leonov functions

In this section, we consider the following special case of (II.1) under Assumption III.1

$$\begin{aligned} \dot{\theta} &= p(x), \\ \dot{z} &= g(z, h(\theta)), \end{aligned} \quad (\text{III.7})$$

with the maps  $p : \mathbb{R}^n \rightarrow \mathbb{R}^q$ ,  $p \in \mathcal{C}^1$ ,  $p(0) = 0$ ,  $g : \mathbb{R}^{k+w} \rightarrow \mathbb{R}^k$ ,  $g(0, 0) = 0$ , and  $h : \mathbb{R}^q \rightarrow \mathbb{R}^w$ ,  $h \in \mathcal{C}^1$ ,  $h(0) = 0$ .

Since,  $h \in \mathcal{C}^1$  and, under Assumption III.1,  $h$  is periodic in  $\theta$ , there exists a constant  $\bar{h}$ , such that

$$\bar{h} > \sup_{\theta \in \mathbb{R}^q} |h(\theta)|. \quad (\text{III.8})$$

In addition, we make the following assumption.

*Assumption III.5.* The dynamics  $\dot{z} = g(z, h(\theta))$  are input-to-state stable (ISS) with respect to the input  $h(\theta)$ .

Under Assumption III.5, there exist functions  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}_\infty$ , such that for all  $z_0 \in \mathbb{R}^k$  [5]

$$|z(t)| \leq \beta(|z_0|, t) + \gamma(\bar{h}), \quad t \geq 0, \quad (\text{III.9})$$

which implies that  $z$  is bounded.

Next the notion of a wLeF is introduced as follows.

**Definition III.6.** A  $\mathcal{C}^1$  function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a *weak Leonov function* (wLeF) for (III.7) with Assumptions III.1 and III.5 if there exist functions  $\alpha_i \in \mathcal{K}_\infty$ ,  $\psi_i \in \mathcal{K}$ ,  $i = 1, 2$ , and a continuous function  $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\lambda(0) = 0$  and  $\lambda(s)s > 0$  for all  $s \neq 0$  as well as constants  $g_i \geq 0$ ,  $i = 1, 2$ , such that

$$\alpha_2(\gamma(\bar{h})) + g_2 < \psi_2^*, \quad \psi_2^* = \sup_{s \in \mathbb{R}_{>0}} \psi_2(s), \quad (\text{III.10})$$



where  $\bar{h}$  is given in (III.8) and  $\gamma$  is given in (III.9), it holds for all  $x = (z, \theta) \in \mathbb{R}^n$  that

$$\alpha_1(|z|) - \psi_1(|\theta|) - g_1 \leq V(x) \leq \alpha_2(|z|) - \psi_2(|\theta|) + g_2$$

$$\inf_{x \in \mathcal{W}} V(x) > 0, \quad \sup_{x \in \mathcal{U}} V(x) \leq 0, \quad (\text{III.11})$$

and

$$DV(x)f(x) \leq -\lambda(V(x)) \quad (\text{III.12})$$

is satisfied for either one of these three options:

- $\forall x \in \Omega_\epsilon \setminus \Omega$ ,
- $\forall x \in \Omega'_\epsilon \setminus \Omega'$  and  $\alpha_2(g(\bar{h})) + g_2 - \psi_2(c) < 0$ ,  $0 < c < \underline{c}$ ,
- $\forall x \in \Omega''_\epsilon \setminus \Omega''$ ,

where<sup>1</sup>

$$\Omega_\epsilon = \Omega'_\epsilon \cup \Omega''_\epsilon = \{x \in \mathbb{R}^n : V(x) \leq \epsilon\}$$

for some sufficiently small  $\epsilon > 0$ .

With Definition III.6, we relax the conditions of Theorem III.3, which may be important in applications. In addition, the result below corrects some flaws in its previous counterparts [51, Corollaries 2 and 3].

**Theorem III.7.** *If for the system (III.7) under Assumptions III.1 and III.5 there exists a wLeF as in Definition III.6, then for all  $x_0 \in \mathbb{R}^n$  the corresponding trajectories  $X(t, x_0)$  are bounded for all  $t \geq 0$ .*

*Proof.* Under the imposed restrictions, the solutions  $X(t, x_0)$  are defined for all  $x_0 \in \mathbb{R}^n$  and all  $t \geq 0$  (i.e., the system (III.7) is forward complete). Indeed,  $z(t)$  is bounded due to (III.9), while  $p$  is a continuous periodic function of  $\theta$ , with bounded  $z$ . Hence,  $p(X(t, x_0)) \leq p^*_{|z_0|, \bar{h}}$  for some finite  $p^*_{|z_0|, \bar{h}} \in \mathbb{R}_{>0}$ .

According to (IV.7),  $\Omega = \Omega' \cup \Omega''$  as in Theorem III.3, where  $\Omega'$  is a bounded set. Also, from (IV.7), for all  $x_0 \in \mathbb{R}^n$  while  $V(X(t, x_0)) > 0$ , we have

$$\psi_2(|\theta(t)|) \leq \alpha_2(\beta(|z_0|, t) + \gamma(\bar{h})) + g_2, \quad (\text{III.13})$$

which in light of (III.10) together with the fact that  $\beta \in \mathcal{KL}$  shows that for any  $x_0 = (z_0, \theta_0)$  the set  $\mathcal{V} \subset \{x \in \mathbb{R}^n : V(x) > 0\}$ , which contains the solutions  $X(t, x_0)$  under the constraint  $V(t) > 0$ , is a bounded set for  $t \rightarrow \infty$ .

*Case a).* Since (IV.8) is verified on  $\Omega_\epsilon \setminus \Omega$ , then  $\Omega$  is locally attractive and forward invariant. For any  $x_0 \in \mathbb{R}^n$  there are two options. First,  $X(t, x_0)$  stays in  $\mathcal{V}$  for all  $t \geq 0$  and, consequently, the trajectory is bounded (i.e., the set  $\mathcal{V}$  contains forward invariant subsets for (III.7)). Alternatively, at some point in time  $0 \leq T_{x_0} \leq +\infty$ ,  $X(t, x_0)$  crosses to  $\Omega$  and, by invariance, the trajectory remains in  $\Omega$  for all  $t \geq T_{x_0}$ . In this case, boundedness of the trajectory has to be analyzed following the arguments based on the periodicity of the system (III.7) from Theorem III.3.

<sup>1</sup>Recall from the proof of Theorem III.3 that  $\Omega = \{x \in \mathbb{R}^n : V(x) \leq 0\}$ . Hence, by definition  $\Omega \subset \Omega_\epsilon$  and since  $\Omega = \Omega' \cup \Omega''$ , then  $\Omega_\epsilon = \Omega'_\epsilon \cup \Omega''_\epsilon$ , where  $\Omega' \subset \Omega'_\epsilon$  and  $\Omega'' \subset \Omega''_\epsilon$  are separated sets for a sufficiently small  $\epsilon > 0$ .

With  $V_j(x)$  as in the proof of Theorem III.3, we have the sets

$$\Omega_j = \Omega'_j \cup \Omega''_j = \{x \in \mathbb{R}^n : V_j(x) \leq 0\},$$

$$\mathcal{V}_j \subset \{x \in \mathbb{R}^n : V_j(x) > 0\}. \quad (\text{III.14})$$

By using the same arguments as for  $\Omega$  and  $\mathcal{V}$  it is possible to show that  $\Omega'_j$  and  $\mathcal{V}_j$  are bounded sets and that  $\Omega_j$  is locally attractive and forward invariant and that  $\mathcal{V}_j$  may contain forward invariant subsets.

From the above considerations, it follows that for each  $j \in \mathbb{Z}^q$ , all trajectories of (III.7) asymptotically belong to the union

$$(\Omega'_j \cup \Omega''_j) \cup \mathcal{V}_j = \Omega'_j \cup \Omega''_j \cup \mathcal{V}_j, \quad (\text{III.15})$$

which implies that asymptotically all trajectories belong to the intersection of the sets in (III.15), i.e., to

$$\bigcap_{j \in \mathbb{Z}^q} (\Omega'_j \cup \Omega''_j \cup \mathcal{V}_j).$$

For each  $j \in \mathbb{Z}^q$ , if a trajectory is asymptotically settled either in  $\Omega'_j$  or in  $\mathcal{V}_j$ , then it has to be bounded by boundedness of the respective set. If the trajectory is in  $\Omega''_j$ , then the periodicity arguments can be used by repeating the same consideration for other indexes  $j \in \mathbb{Z}^q$  and we recall that  $\bigcap_j \Omega''_j = \emptyset$ . This shows that no trajectory can be in  $\Omega''_j$  for all  $j \in \mathbb{Z}^q$ , which implies that any trajectory is captured by some  $\mathcal{V}_j$  or  $\Omega'_j$ . Hence, it is bounded.

*Case b).* In this case the set  $\Omega'$  is forward invariant and locally attractive and the properties of  $\Omega''$  are not specified by the conditions of the theorem. Since  $\alpha_2(g(\bar{h})) + g_2 - \psi_2(c) < 0$ , with (III.10) we have asymptotically that any  $X(t, x_0) \in \mathcal{V}$  satisfies

$$|\theta|_\infty \leq |\theta| \leq \psi_2^{-1}(\alpha_2(\gamma(\bar{h})) + g_2) < c.$$

This implies that the set  $\Omega' \cup \mathcal{V}$  has similar properties to  $\Omega'$  in Theorem III.3. Most importantly, since

$$\mathcal{V} \subset \{x \in \mathbb{R}^n : |\theta|_\infty < c\}, \quad \Omega'' \subset \{x \in \mathbb{R}^n : |\theta|_\infty > \bar{c}\},$$

$\mathcal{V}$  and  $\Omega''$  are separated sets, which means that there can be no trajectory cycling between  $\mathcal{V}$  and  $\Omega''$ . The rest of the proof repeats the arguments of case a).

*Case c).* In this case the set  $\Omega''$  is forward invariant and locally attractive and the properties of  $\Omega'$  are not specified by the conditions of the theorem. For any  $x_0 \in \mathbb{R}^n$  there are two options. First,  $X(t, x_0)$  stays in  $\mathcal{V} \cup \Omega'$  for all  $t \geq 0$  and, consequently, the trajectory is bounded (i.e., the set  $\mathcal{V} \cup \Omega'$  contains forward invariant subsets for (III.7)). Alternatively, at some point in time  $0 \leq T_{x_0} \leq +\infty$ ,  $X(t, x_0)$  crosses to  $\Omega''$  and, by invariance, the trajectory remains in  $\Omega''$  for all  $t \geq T_{x_0}$ . If the trajectory is in  $\Omega''$ , then the periodicity arguments can be used by repeating the same consideration for other indexes  $j \in \mathbb{Z}^q$  and we recall that  $\bigcap_j \Omega''_j = \emptyset$ . This shows that no trajectory can be in  $\Omega''_j$  for all  $j \in \mathbb{Z}^q$ , which implies that any trajectory is captured by some  $\mathcal{V}_j$  or  $\Omega'_j$ . Hence, it is bounded. The same reasoning applies for the special case  $\Omega' = \emptyset$  (which only requires a slight redefinition of the set  $\mathcal{U}$ ).

□

#### IV. BOUNDEDNESS OF DYNAMICAL SYSTEMS THAT ARE PERIODIC WITH RESPECT TO ALL STATE VARIABLES

Next, we focus on systems, whose dynamics are periodic with respect to *all* state variables. This is formalized in the assumption below.

*Assumption IV.1.* The vector field  $f$  in (II.1) is  $2\pi$ -periodic, i.e.,  $f(x + 2\pi) = f(x)$  for all  $x \in \mathbb{R}^n$ .

Clearly, Assumption IV.1 implies that  $x = \theta \in \mathbb{R}^n$ .

##### A. Strong Leonov functions and main result

As in the previous section, it is convenient to introduce the auxiliary sets  $\mathcal{W}$  and  $\mathcal{U}$ , which under Assumption IV.1 become

$$\begin{aligned} \{x \in \mathbb{R}^n : \underline{c} \leq |x|_\infty \leq \bar{c}\} &\subseteq \mathcal{W}, \\ \mathcal{U} &= \bigcup_{r \in \mathbb{Z}_+} \mathcal{U}_r, \\ \mathcal{U}_r &= \{x \in \mathbb{R}^n : |x|_\infty = 2r\pi, f(x) = 0\}. \end{aligned} \quad (\text{IV.1})$$

We employ the following definition.

**Definition IV.2.** We say that a  $\mathcal{C}^1$  function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a *strong Leonov function* (sLeF) for (II.1) with Assumption IV.1 if

$$\inf_{x \in \mathcal{W}} V(x) > 0, \quad \sup_{x \in \mathcal{U}} V(x) \leq 0 \quad (\text{IV.2})$$

and there exists a continuous function  $\lambda : \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $\lambda(0) = 0$  and  $\lambda(s)s > 0$  for all  $s \neq 0$ , such that

$$\frac{\partial V(x)}{\partial x} f(x) \leq -\lambda(V(x)) \quad \forall x \in \mathbb{R}^n. \quad (\text{IV.3})$$

As in Definition III.2, we are not asking the function  $V$  to be  $2\pi$ -periodic, hence, any analysis involving such a sLeF has to be carried out in  $\mathbb{R}^n$ . Moreover, compared to Definition III.2 no requirements on lower bounds of  $V(x)$  are made, since  $x = \theta$ , i.e., the dynamics (II.1) are assumed to be periodic with respect to *all* state variables  $x = \theta$ .

**Theorem IV.3.** *If for the system (II.1) under Assumption III.1 there exists a sLeF as in Definition IV.2, then for all  $x_0 \in \mathbb{R}^n$  the corresponding trajectories  $X(t, x_0)$  are bounded for all  $t \geq 0$ .*

*Proof.* Denote the set of negative values of  $V$  as

$$\Omega = \{x \in \mathbb{R}^n : V(x) \leq 0\}.$$

Recall the sets  $\mathcal{U}$  and  $\mathcal{W}$  defined in (IV.1). Then from the definition of a sLeF, see (IV.2), the set  $\mathcal{U}$  belongs to  $\Omega$ , while  $\mathcal{W} \notin \Omega$ . Furthermore, the set  $\Omega$  can be decomposed into two parts:

$$\Omega = \Omega' \cup \Omega''.$$

The first of these sets,

$$\Omega' \subset \{x \in \mathbb{R}^n : |x|_\infty < \underline{c}\}$$

is a compact set containing the origin, see the definition of the set  $\mathcal{W}$  in (IV.1), i.e.  $\mathcal{U}_0 \subset \Omega'$ . The second set is defined as

$$\Omega'' \subset \{x \in \mathbb{R}^n : |x|_\infty > \bar{c}\}.$$

Both these sets are non-empty since  $0 \in \Omega'$  and  $\mathcal{U}_r \in \Omega''$  for  $r \geq 1$  due to (IV.2). Furthermore, the set  $\Omega''$  is unbounded. In addition, the subsets  $\Omega'$  and  $\Omega''$  are separated by the set  $\mathcal{W}$ .

Denote  $V(t) = V(X(t, x_0))$  for any  $x_0 \in \mathbb{R}^n$ , then under the conditions of the theorem we have:

$$\dot{V}(t) + \lambda(V(t)) \leq 0 \quad \forall t \geq 0.$$

Thus, clearly,  $V(t)$  is strictly decreasing while  $X(t, x_0) \in \mathbb{R}^n \setminus \Omega$ , i.e. while  $V(t) > 0$ . Therefore, if  $x_0 \in \Omega$  then  $X(t, x_0) \in \Omega$  for all  $t \geq 0$  and the set  $\Omega$  is forward invariant for (II.1). Conversely, if  $x_0 \in \mathbb{R}^n \setminus \Omega$  then there exists  $0 \leq T_{x_0} \leq +\infty$  such that  $X(T_{x_0}, x_0) \in \Omega$  and, by invariance, the trajectory remains in this set for all  $t \geq T_{x_0}$ . Thus, the set  $\Omega$  is globally attractive and forward invariant. Note that this property does not imply any kind of stability since  $\Omega$  may be unbounded (it could also be interpreted as *instability* of a set containing  $\mathcal{W}$  with the domain of repulsion  $\mathbb{R}^n \setminus \Omega$ ).

From here on, the claim is completed by following the steps of the proof of Theorem III.3.  $\square$

We have the following corollary to Theorem IV.3. For its presentation, we recall the equilibrium set  $\mathcal{D}$  of the dynamics (II.1) defined in (III.4).

**Corollary IV.4.** *If for the system (II.1) under Assumption IV.1 there exists a sLeF as in Definition IV.2 with*

$$V(0) \leq 0, \quad \inf_{x \in \mathcal{U} \setminus \mathcal{U}_0} V(x) \leq 0, \quad \inf_{x \in \mathcal{W}} V(x) > 0, \quad (\text{IV.4})$$

*and with the following dissipation inequality*

$$DV(x)f(x) < 0 \quad \forall x \in \mathbb{R}^n \setminus \{\mathcal{D} \cup \Omega\}, \quad (\text{IV.5})$$

*then for all  $x_0 \in \mathbb{R}^n$  the corresponding trajectories  $X(t, x_0)$  are bounded for all  $t \geq 0$ .*

*Proof.* The claim is established by following the steps of the proof of Corollary III.4.  $\square$

##### B. Further relaxations via weak Leonov functions

With the aim of relaxing the conditions of Theorem IV.3, we introduce the following notion.

**Definition IV.5.** Consider a  $\mathcal{C}^1$  function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  and let

$$\Xi := \{x \in \mathbb{R}^n : V(x) > 0\}, \quad \Omega := \mathbb{R}^n \setminus \Xi. \quad (\text{IV.6})$$

Then,  $V$  is a *weak Leonov function* (wLeF) for (II.1) with Assumption IV.1 if there exist constants  $\underline{c} \in (0, 2\pi)$  and  $\bar{c} \in (\pi, 2\pi)$ ,  $\underline{c} < \bar{c}$ , such that

$$\{x \in \mathbb{R}^n : \underline{c} < |x|_\infty < \bar{c}\} \subset \Xi \subset \{x \in \mathbb{R}^n : |x|_\infty < 2\pi\}, \quad (\text{IV.7})$$

and

$$\frac{\partial V(x)}{\partial x} f(x) \leq 0 \quad \forall x \in \partial\Xi, \quad (\text{IV.8})$$

where  $\partial\Xi$  denotes the boundary of  $\Xi$ .

The function  $V$  takes positive values on the compact set  $\Xi$  and nonpositive values on the set  $\Omega$ . Thus, roughly speaking, the function  $V$  is negative definite with respect to the distance to the set  $\Xi$  (it takes positive values on the set  $\Xi$  and negative values on its complement  $\Omega$ ). Compared to a sLeF, see Definition IV.2, in the case of a wLeF  $\Omega$  does not necessarily contain the origin. Also, its time derivative on the interior of the set  $\Xi$  is not further specified. Yet, since  $V$  is a  $C^1$  function and  $\Xi$  is a bounded set, we know that  $\sup_{x \in \Xi} V(x) < +\infty$ .

**Theorem IV.6.** *If for the system (II.1) under Assumption IV.1 there exists a wLeF as in Definition IV.5, then for all  $x_0 \in \mathbb{R}^n$  the corresponding trajectories  $X(t, x_0)$  are bounded for all  $t \geq 0$ .*

*Proof.* In general, the set  $\Omega$  defined in (IV.6) can be decomposed into two parts:

$$\Omega = \Omega' \cup \Omega''.$$

The first of these sets,

$$\Omega' \subset \{x \in \mathbb{R}^n : |x|_\infty < \underline{c}\},$$

maybe empty (if  $0 \in \Xi$ ). If it is non-empty, then  $\Omega'$  is a compact set for some  $\underline{c} \in (0, 2\pi)$ , where the constant  $\underline{c}$  is determined by the properties of the set  $\Xi$  stated in (IV.7). The second set is defined as

$$\Omega'' \subset \{x \in \mathbb{R}^n : |x|_\infty > \bar{c}\},$$

where again the constant  $\bar{c}$  is determined by the properties of the set  $\Xi$ . Furthermore, the set  $\Omega''$  is non-empty and unbounded. In addition, if  $\Omega'$  is non-empty, the subsets  $\Omega'$  and  $\Omega''$  are separated by the set  $\Xi$ .

If  $x_0 \in \Omega$ , then under the conditions of the corollary  $X(t, x_0) \in \Omega$  for all  $t \geq 0$  since

$$\sup_{x \in \Omega} V(x) \leq 0, \quad \inf_{x \in \Xi} V(x) \geq 0, \quad \Xi = \mathbb{R}^n \setminus \Omega$$

and

$$\frac{\partial V(x)}{\partial x} f(x) \leq 0 \quad \forall x \in \partial \Xi.$$

Thus, the set  $\Omega$  is forward invariant for (II.1).

Conversely, suppose that  $x_0 \in \Xi = \mathbb{R}^n \setminus \Omega$ . Then, the behavior of  $V(t) = V(X(t, x_0))$  in the interior of  $\Xi$  is not specified by the conditions of the theorem. Thus, there are two possibilities. Either  $X(t, x_0)$  stays in the compact set  $\Xi$  for all  $t \geq 0$  or at some point in time  $0 \leq T_{x_0} \leq +\infty$  it crosses to  $\Omega$  and, by invariance, the trajectory remains in  $\Omega$  for all  $t \geq T_{x_0}$ . Note that these properties do not imply any kind of stability since  $\Omega$  is unbounded.

To establish stability, we exploit the *periodicity* of (II.1). Denote by  $j = [j_1, \dots, j_n] \in \mathbb{Z}^n$  a multi-index vector. Introduce the new variable  $x_j = x - 2\pi j$  and the function  $V_j(x) = V(x - 2\pi j)$  for any such multi-index vector  $j$  (i.e.,  $V_0(x) = V(x)$ ). Hence

$$V_j(x) = V_j(x_j + 2\pi j) = V(x_j).$$

Furthermore, by  $2\pi$ -periodicity of  $f$ ,

$$\dot{x}_j(t) = f(x(t)) = f(x_j(t)) \quad \forall t \geq 0.$$

Therefore, taking into account the properties substantiated for (II.1) and  $V$ , we obtain that the set

$$\begin{aligned} \Omega_j &= \{x \in \mathbb{R}^n : V_j(x) \leq 0\} \\ &= \{x \in \mathbb{R}^n : V(x - 2\pi j) \leq 0\} \end{aligned}$$

is forward invariant. By letting

$$\begin{aligned} &\{x \in \mathbb{R}^n : \underline{c} < |x - 2\pi j|_\infty < \bar{c}\} \subset \Xi_j \\ &= \{x \in \mathbb{R}^n : V_j(x) > 0\} \subset \{x \in \mathbb{R}^n : |x - 2\pi j|_\infty < 2\pi\} \end{aligned}$$

and using similar arguments as for  $\Omega$  it is possible to show that  $\Omega_j = \Omega'_j \cup \Omega''_j$ , where

$$\begin{aligned} \Omega'_j &\subset \{x \in \mathbb{R}^n : |x - 2\pi j|_\infty < \underline{c}\}, \\ \Omega''_j &\subset \{x \in \mathbb{R}^n : |x - 2\pi j|_\infty > \bar{c}\}. \end{aligned}$$

In order to show that under the conditions of the corollary, all solutions of (II.1) are bounded, we make the following observations. As it has been shown above for any  $x_0 \in \mathbb{R}^n$  the respective trajectory  $X(t, x_0)$  has at most three possibilities: 1) If  $\Omega'_j \neq \emptyset$ , it reaches  $\Omega'_j$  for some  $j \in \mathbb{Z}^n$ . Then it has to stay there by invariance of  $\Omega'_j$ , which also implies boundedness of the trajectory due to compactness of  $\Omega'_j$ . 2) It stays in an invariant subset of  $\Xi_j$ . Then the trajectory is also bounded due to the same property of  $\Xi_j$ . 3) It is captured by  $\Omega''_j$ , which is unbounded and this case has to be analyzed. Note that  $\bigcap_{j \in \mathbb{Z}^n} \Omega''_j = \emptyset$  (the proof is given in Theorem IV.3). Thus, it has to be in an intersection of  $\Omega''_j$  with  $\Omega''_k$  or  $\Xi_k$  for some  $j, k \in \mathbb{Z}^n$  with  $k \neq j$ . This implies boundedness of the trajectory  $X(t, x_0)$ .  $\square$

## V. CONCLUSIONS

We have addressed the problem of establishing global boundedness of solutions for two classes of state periodic systems - namely, systems, whose dynamics are periodic either with respect to a part of the state vector or with respect to the full state vector. Both classes of systems are frequently encountered in diverse applications, such as AC power systems or networks of oscillators. Due to their periodicity, such systems always exhibit multiple equilibria. This significantly complicates the derivation of *global* system properties.

This challenge has been tackled by introducing the notion of sLeFs, based on which we have provided sufficient conditions for global boundedness of solutions. Several relaxations of this main result have been derived via the concept of wLeFs.

Our current work is geared towards the application of the presented methodical framework to assess global stability properties of prominent engineering examples. These comprise, in particular, low-inertia power systems with dynamical power line models and networks of coupled Kuramoto oscillators.

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