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► **To cite this version:**

Josué Tonelli-Cueto. A Geometric Summation of the Geometric Series. The American Mathematical Monthly, 2022, pp.1-1. 10.1080/00029890.2022.2115825 . hal-03779492

HAL Id: hal-03779492

<https://inria.hal.science/hal-03779492>

Submitted on 16 Sep 2022

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This is an Accepted Manuscript version of the following article, accepted for publication in *The American Mathematical Monthly*:

Josué Tonelli-Cueto (2022) A Geometric Summation of the Geometric Series, *The American Mathematical Monthly*, DOI: 10.1080/00029890.2022.2115825

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A Geometric Summation of the Geometric Series

Let k be a finite field of size q . The n -dimensional projective space over k , $\mathbb{P}^n(k)$, is the set of orbits of $k^{n+1} \setminus \{0\}$ under the action of k^* given by scalar multiplication. Therefore $|\mathbb{P}^n(k)| = \frac{q^{n+1}-1}{q-1}$ since all orbits have the same size. Now, since the decomposition of $\mathbb{P}^n(k)$ into a disjoint union of affine spaces $\mathbb{P}^n(k) = A_0 \cup A_1 \cup \dots \cup A_n$ where

$$A_i := \{(x_0 : \dots : x_{i-1} : 1 : 0 : \dots : 0) \mid x_j \in k\}$$

is in bijection with the affine space k^i , then $|\mathbb{P}^n(k)| = \sum_{i=0}^n q^i$. Thus for all n and power of primes q ,

$$\sum_{i=0}^n q^i = \frac{q^{n+1} - 1}{q - 1}.$$

Since this identity of univariate rational functions over \mathbb{Q} is true for an infinite amount of numbers, it translates into an equality of rational functions. Hence

$$\sum_{i=0}^n X^i = \frac{X^{n+1} - 1}{X - 1}$$

with X indeterminate. Can this method be used to prove other identities by interpreting them geometrically using algebraic varieties over finite fields?

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