A Geometric Summation of the Geometric Series
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A Geometric Summation of the Geometric Series

Let $k$ be a finite field of size $q$. The $n$-dimensional projective space over $k$, $\mathbb{P}^n(k)$, is the set of orbits of $k^{n+1}\setminus\{0\}$ under the action of $k^*$ given by scalar multiplication. Therefore $|\mathbb{P}^n(k)| = \frac{q^{n+1}-1}{q-1}$ since all orbits have the same size. Now, since the decomposition of $\mathbb{P}^n(k)$ into a disjoint union of affine spaces $\mathbb{P}^n(k) = A_0 \cup A_1 \cup \ldots \cup A_n$ where

$$A_i := \{(x_0 : \ldots : x_{i-1} : 1 : 0 : \ldots : 0) \mid x_j \in k\}$$

is in bijection with the affine space $k^i$, then $|\mathbb{P}^n(k)| = \sum_{i=0}^{n} q^i$. Thus for all $n$ and power of primes $q$,

$$\sum_{i=0}^{n} q^i = \frac{q^{n+1} - 1}{q - 1}.$$ 

Since this identity of univariate rational functions over $\mathbb{Q}$ is true for an infinite amount of numbers, it translates into an equality of rational functions. Hence

$$\sum_{i=0}^{n} X^i = \frac{X^{n+1} - 1}{X - 1}$$

with $X$ indeterminate. Can this method be used to prove other identities by interpreting them geometrically using algebraic varieties over finite fields?

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