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ON THE CONTROLLABILITY OF NONLINEAR SYSTEMS WITH A PERIODIC DRIFT

J.-B. CAILLAU, L. DELL'ELCE, A. HERASIMENKA, AND J.-B. POMET

ABSTRACT. Sufficient and necessary conditions are established for controllability of affine control systems, motivated by these with a drift all of whose solutions are periodic. In contrast with previously known results, these conditions encompass the case of a control set whose convex hull contains the origin but is not a neighborhood of the origin. The condition is expressed by means of pushforwards along the flow of the drift, rather than in terms of Lie brackets. It turns out that this also amounts to local controllability of a time-varying linear approximation with constrained controls. Global and local results are given, as well as a few illustrative examples.

Keywords. Nonlinear controllability, control constraints, solar sails.

AMS subject classification (2020). 93B05, 93B03, 95C15.

INTRODUCTION

This paper is devoted to controllability properties of affine control systems, defined by a drift vector field and $m \geq 1$ control vector fields. We consider either local controllability around a solution with zero control (solution of the drift vector field) or global controllability in the case where all such solutions are periodic; in both cases, the control is constrained to a subset U of \mathbb{R}^m that contains zero, but whose convex hull, unlike in most known controllability results, is not necessarily a neighborhood of zero. A particular case of such systems are solar sails, which are a core motivation for this study, see references [9, 8]¹ by the authors. Governed by solar radiation pressure, solar sails are only capable of generating forces (controls) contained in a convex cone of revolution around the direction of the incoming light; this is a typical case where zero (vertex of the cone) is in the boundary of the control set (part of the cone).

Devising conditions on the vector fields (and the set U) for controllability is quite an old problem. Most known sufficient conditions for controllability assume that the Lie brackets of the vector fields defining the system span the tangent space at all points (bracket generating property), and prove controllability under additional conditions concerning in general the drift vector field, as recalled in Section 1.4, based on [17, 2, 21, 13, 10, 1], but these results always require —at least— that zero belongs to the interior of the convex hull of U ; this assumption is crucial in the proofs, and allows for a decoupling between this condition on U and conditions on the vector fields. In the present paper, where zero is allowed to belong to the boundary of $\text{conv } U$, novel results are based on *ad hoc* constructions involving the transportation along the flow of the drift vector field of some subsets of the tangent space defined from the control vector fields *and* the set U .

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¹A preliminary version of some results stated here can be found in [9, Section II].

In Section 1, we cast precisely the controllability problem we want to look at and recall the main definitions and results on the subject; since our study is motivated by slow-fast control systems with one fast angle, we devote a few lines to this particular situation. All results are stated in Section 2; they include a sufficient condition for global controllability in unprescribed time under the periodic solutions assumption on the drift, particularly adapted to the slow-fast context mentioned above, sufficient conditions for local controllability in prescribed time along trajectories—periodic or not—of the drift, that give, in the periodic case, a more precise *local* picture (local controllability in one revolution), and then some necessary conditions that say when the shape of the control set forbids controllability; proofs are in that same section, completed by a short appendix on time-varying linear control systems. Section 3 is devoted to comments on the results, more general than the ones given along in Section 2, and to two academic examples, one illustrating our results, and the other one exploring a case where controllability holds while none of our sufficient conditions is met.

1. STATEMENT OF THE PROBLEM

1.1. **Systems.** We consider an affine control system

$$\dot{x} = X^0(x) + \sum_{k=1}^m u_k X^k(x), \quad u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m, \quad (1)$$

where the set U defines the constraints on the control and X^0, \dots, X^m are $m+1$ smooth (C^∞) vector fields on a smooth connected (second countable) manifold \mathcal{M} of dimension d . We assume that the vector field X^0 is complete.

Assumptions on the control set U . All over the paper, we assume that

$$0 \in U \text{ and } \text{Span } U = \mathbb{R}^m. \quad (2)$$

The first point is natural: results rely in part on assumptions on the behavior of the differential equation $\dot{x} = X^0(x)$ obtained for the zero control, that should hence be permitted. The second point is not restrictive: if $\text{Span } U$ was a subspace of \mathbb{R}^m of dimension $m' < m$, one could pick m' new control vector fields, constant linear combinations of X^1, \dots, X^m and use them as control vector fields in a system with m' controls satisfying the assumption.

Periodic orbits of the drift, angular variable. The following property of the drift is assumed in some results of the paper on global controllability:

$$\begin{aligned} X^0 \text{ has no equilibrium point, and, for any } x \text{ in } \mathcal{M}, \\ t \mapsto \exp(tX^0)(x) \text{ is periodic with minimal period } T(x). \end{aligned} \quad (3)$$

This defines a function $T : \mathcal{M} \rightarrow (0, +\infty)$, as smooth as the vector field X^0 . The set of periodic orbits has the structure of a smooth manifold of dimension $d-1$. Indeed, X^0 with property (3) induces the following free action of the compact Lie group $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ on \mathcal{M} : $(\theta, x) \mapsto \exp((T(x)/2\pi)\theta X^0)(x)$, whose orbits are the periodic orbits of X^0 . The orbit space is the quotient $M = \mathcal{M}/\mathbb{S}^1$; it is a smooth manifold because the action is free and proper (free from the absence of equilibria, proper because the group is compact; see *e.g.* [7, Theorem 1.95] for the smooth manifold structure). Without using group action, the property is also clear by taking coordinates on a small transverse section of one periodic orbit as local coordinates on the quotient. This defines a smooth fibration

$$\pi : \mathcal{M} \rightarrow M \quad (4)$$

where, for each x in \mathcal{M} , $\pi^{-1}(\pi(x))$ is the periodic orbit of X^0 passing through x . By fibration we mean a fibre bundle over M that is, by definition, locally trivial: M

can be covered with open sets O such that $\pi^{-1}(O) \simeq O \times \mathbb{S}^1$. On such an open set, denoting I the current point in O and φ the current point in the fibre \mathbb{S}^1 , system (1) can be written as

$$\begin{aligned} \dot{I} &= \sum_{k=1}^m u_k F^k(I, \varphi), \\ \dot{\varphi} &= \omega(I) + \sum_{k=1}^m u_k f^k(I, \varphi), \end{aligned} \quad u = (u_1, \dots, u_m) \in U, \quad (5)$$

for some ω , F^k and f^k whose precise definition is left to the reader. Note that $\omega(I)$ naturally comes out as $\omega(I, \varphi)$, but one may remove the dependence on φ via a change of variables $(I, \varphi) \mapsto (I, a(I, \varphi))$; Equation (5) assumes that such a change of variables has been performed. In the applications to control of solar sails recalled in the introduction, the system is in this product form in a wide zone of interest of the state manifold, using classical coordinates from celestial mechanics; the variable I is usually slow compared to the angle φ , *i.e.* U is usually “small”. In this paper, when system (1) is assumed to satisfy (3), we sometimes use the product form (5) because it allows for a simpler and more concrete statement of results.

1.2. Accessible sets and controllability, definitions. A solution of (1) is a map $t \mapsto (x(t), u(t))$, defined on some interval, valued in $\mathcal{M} \times U$, where $u(\cdot)$ is essentially bounded and $x(\cdot)$ absolutely continuous, and that satisfies equation (1) for almost all time t . For such a solution on an interval $[0, t_f]$, $t_f \geq 0$, one says that “the control $u(\cdot)$ steers $x(0)$ to $x(t_f)$ in time t_f ”; indeed the solution is unique given $u(\cdot)$ and $x(0)$. One may also say that the same control steers $x(t_f)$ to $x(0)$ in time $-t_f$ because uniqueness holds also backwards.

Remark 1. In the situation where (3) is satisfied and defines π as in (4), the following fact is true: if there is a control steering an initial point x to a final point y in some time τ , there is also a control steering any point x' in the same orbit as x ($\pi(x') = \pi(x)$) to any point y' in the same orbit as y ($\pi(y') = \pi(y)$), in time less than $\tau + T(x) + T(y)$. Indeed, the control consists, for some t_1 in $[0, T(x))$ and t_2 in $[0, T(y))$, in applying a zero control on $[0, t_1]$ and $[t_1 + \tau, t_1 + \tau + t_2]$ and the original control on $[t_1, t_1 + \tau]$.

The reachable set from x_0 in time $t_f \in \mathbb{R}$ is the set of points in \mathcal{M} that can be reached in forward or backward time t_f from x_0 for some control:

$$\mathcal{A}_{t_f}^U(x_0) = \{x(t_f), \text{ with } t \mapsto (x(t), u(t)) \text{ solution of (1) on } [0, t_f], x(0) = x_0\}. \quad (6)$$

It is of course assumed that the interval of definition of the solution contains $[0, t_f]$. If $t_f < 0$, $[0, t_f]$ should be understood as $[t_f, 0]$, and it is clear that:

$$y \in \mathcal{A}_t^U(x) \Leftrightarrow x \in \mathcal{A}_{-t}^U(y). \quad (7)$$

We keep the set U as a superscript to stress the constraint on the control. Obviously, $\mathcal{A}_{t_f}^V(x) \subset \mathcal{A}_{t_f}^U(x)$ if $V \subset U$. From accessible sets in prescribed time, we define

$$\mathcal{A}^U(x) = \bigcup_{0 \leq t} \mathcal{A}_t^U(x), \quad \mathcal{A}_-^U(x) = \bigcup_{t \leq 0} \mathcal{A}_t^U(x), \quad (8)$$

the accessible sets in unprescribed forward and backward time ($\mathcal{A}^U(x)$ is sometimes called simply “the accessible set from x ”, but we remain interested in properties in prescribed time), and our corresponding notion of global controllability. Below, $\pi : \mathcal{M} \rightarrow M$ is a fibration, with M a manifold of arbitrary dimension $n < d$; it need not be constructed as in Section 1.1 from X^0 , not assumed to satisfy (3) here.

Definition 2 (global controllability). **I.** System (1) is *globally controllable* if and only if $\mathcal{A}^U(x) = \mathcal{M}$ for all x in \mathcal{M} . **II.** It is *globally controllable with respect to π* if and only if $\pi(\mathcal{A}^U(x)) = M$ for all x in \mathcal{M} .

Remark 3. Global controllability obviously implies global controllability with respect to π . The converse does not hold: take any non controllable system and $M = \{0\}$, $n = 0$, π constant. However, under the periodicity assumption (3) and if π is the fibration (4) induced by the periodic orbits, global controllability with respect to π does imply global controllability according to Remark 1.

Controllability *with respect to π* could also be called *partial controllability*, a familiar notion where only the partial information $\pi(x)$ on the final point is specified rather than x itself. In [6], it is called *output controllability*, as π can be viewed as an n -dimensional “output” of the system, specified at final time. In [15] or [3, Section 7.2], the notion of *configuration controllability* for mechanical systems is similar: roughly speaking, $\pi : TQ \rightarrow Q$ is defined by $\pi(q, \dot{q}) = q$, where q (position) is the current point in the configuration manifold $M = Q$ and (q, \dot{q}) (position-velocity) the one in the state manifold $\mathcal{M} = TQ$, tangent bundle of Q ; there is however a slight difference in the choice of initial conditions (zero velocity initial conditions are privileged in [3, 15]).

Let us now pursue with *local* controllability, either around a particular solution, with a specified final time t_f (Définition 4), or around a particular *periodic* solution, defined for all time, under the periodicity assumption (3), where final time is unspecified (Définition 7). In both cases, the particular solution is one of $\dot{x} = X^0(x)$ (or $\dot{I} = 0, \dot{\varphi} = \omega(I)$ in the product form (5)); definitions for a solution $t \mapsto (\bar{x}(t), \bar{u}(t))$ of (1), with nonzero control, are similar and results may be extended, but it is not the point of this paper, due to original motivations.

Definition 4 (local controllability around a solution). Let $t \mapsto \bar{x}(t)$ be a particular solution of $\dot{x} = X^0(x)$, defined on $[0, t_f]$ for some $t_f > 0$.

I. System (1) is *locally controllable around the solution $\bar{x}(\cdot)$* if and only if the set

$$\begin{aligned} \{x(t_f), \text{ with } t \mapsto (x(t), u(t)) \text{ solution of (1) on } [0, t_f] \\ \text{such that } x(0) = \bar{x}(0) \text{ and } x(t) \in \Omega \text{ for all } t \text{ in } [0, t_f]\} \end{aligned} \quad (9)$$

is a neighborhood of $\bar{x}(t_f)$ for any neighborhood Ω of $\bar{x}([0, t_f])$ in \mathcal{M} .

II. If $\pi : \mathcal{M} \rightarrow M$ is some fibration with M a manifold of dimension $n < d$, system (1) is *locally controllable with respect to π around the solution $\bar{x}(\cdot)$* if and only if the set

$$\begin{aligned} \{\pi(x(t_f)), \text{ with } t \mapsto (x(t), u(t)) \text{ solution of (1) on } [0, t_f] \\ \text{such that } x(0) = \bar{x}(0) \text{ and } \pi(x(t)) \in \Omega' \text{ for all } t \text{ in } [0, t_f]\} \end{aligned} \quad (10)$$

is a neighborhood of $\pi(\bar{x}(t_f))$ in M for any neighborhood Ω' of $\bar{x}([0, t_f])$ in \mathcal{M} .

Remark 5. Theorems 15 and 19 give sufficient conditions, when U is convex and compact, for a stronger property: “for any $\varepsilon > 0$, $\mathcal{A}_{t_f}^{\varepsilon U}(\bar{x}(0))$ is a neighborhood of $\bar{x}(t_f)$ ”. This does imply local controllability around $\bar{x}(\cdot)$ because, given Ω as in the definition, for ε small enough, any solution of system (1) with initial condition $\bar{x}(0)$ and control constrained in εU stays in Ω from continuous dependence on the control (since U is compact, the controls in εU are small) and are solution also of the system with constraints in U (since U is convex, $\varepsilon U \subset U$ if $\varepsilon \in [0, 1]$).

Remark 6. While Definition 2 of global controllability is classical, local controllability is a moving notion in the literature, so that Definitions 4 or 7 agree with only part of the literature. Two remarks are in order.

In our setting, local controllability is at the same time weaker (it only deals with some neighborhood of the end point of a trajectory) and stronger (trajectories are required to remain close to the reference one) than global controllability.

We believe that a “local” property of a control system at some locus should only depend on the description of this system on arbitrary small neighborhoods of that locus. In this sense, Definitions 4 and 7 are local with respect to the state only: system (1) is locally controllable around the solution $\bar{x}(\cdot)$ if and only if any system $\dot{x} = f(x, u)$ such that $f(x, u) = X^0(x) + \sum u_k X^k(x)$ for (x, u) in $\mathcal{O} \times U$, with \mathcal{O} an open neighborhood of $\bar{x}([0, t_f])$, is locally controllable around the solution $\bar{x}(\cdot)$.

We finally give a weaker notion, specific to periodic solutions, where final time is not prescribed and control is not required to be small. Weaker is preferred because we will only give a necessary condition for this controllability property.

Definition 7 (local orbital controllability around a periodic solution). Assume that the drift vector field $X^0(x)$ satisfies (3), and consider system (1) in product form (5) locally around a periodic solution $t \mapsto (\bar{I}, \bar{\varphi} + \omega(\bar{I})t)$, simply called “the periodic solution \bar{I} ” because the phase $\bar{\varphi}$ does not matter. system (5), or (1), is *locally controllable around the periodic solution \bar{I}* if and only if the set

$$\{I(t_f), t_f \in [0, +\infty) \text{ and } t \mapsto (I(t), \varphi(t), u(t)) \text{ solution of (5) on } [0, t_f] \text{ such that } I(0) = \bar{I} \text{ and } I(t) \in \Omega \text{ for all } t \text{ in } [0, t_f]\} \quad (11)$$

is a neighborhood of \bar{I} in M for any neighborhood Ω of \bar{I} in M .

This last definition is set up in the product form (5) for convenience, but could be formulated otherwise, with π the fibration (4) induced by (3): system (1) is *locally controllable with respect to π around the periodic solution $\bar{x}(\cdot)$* if and only if the set

$$\{\pi(x(t_f)), \text{ with } t_f \in [0, +\infty) \text{ and } t \mapsto (x(t), u(t)) \text{ solution of (1) on } [0, t_f] \text{ such that } \pi(x(0)) = \pi(\bar{x}(0)) \text{ and } \pi(x(t)) \in \Omega \text{ for all } t \text{ in } [0, t_f]\}$$

is a neighborhood of $\pi(\bar{x}(t_f))$ in M for any neighborhood Ω of $\bar{x}([0, \bar{T}])$ in M .

1.3. Some families of vector fields. Let $V(\mathcal{M})$ be the set of smooth (*i.e.* either C^∞ or C^ω) vector fields on \mathcal{M} ; the Lie bracket of two elements of $V(\mathcal{M})$ (defined in any advanced calculus textbook) is also in $V(\mathcal{M})$, making $V(\mathcal{M})$ a Lie algebra over the field \mathbb{R} . A family of vector fields is a subset $\mathcal{F} \subset V(\mathcal{M})$; we denote by $\mathcal{F}(x)$ the subset of $T_x\mathcal{M}$ made of the values at x of vector fields in \mathcal{F} :

$$\mathcal{F}(x) = \{X(x), X \in \mathcal{F}\} \subset T_x\mathcal{M}, \quad (12)$$

and by $\text{Span } \mathcal{F}(x)$ the vector subspace of $T_x\mathcal{M}$ spanned by $\mathcal{F}(x)$. Finally, we denote by $\mathcal{A}_{\mathcal{F}}(x)$ the *accessible set from x of this family*, *i.e.* the set of points that can be reached from x by following successively the flow of a finite number of vector fields in \mathcal{F} , each for a certain positive time; this notion differs from $\mathcal{A}^U(x)$ in (8), attached to the control system (1), that allows arbitrary time-varying controls.

Let us define two families of vector fields from X^0, \dots, X^m defining system (1): \mathcal{L} , made of all iterated Lie brackets and \mathcal{F}_0 , of a selection of them. They are not needed to state or prove our results; \mathcal{L} is used in the present introductory section and \mathcal{F}_0 in Section 3 when discussing our sufficient conditions, and their implications in terms of Lie brackets in the real analytic case.

$$\mathcal{L} = \left\{ [X^{i_k}, [X^{i_{k-1}}, [\dots [X^{i_2}, X^{i_1}] \dots]]] \right\}_{\substack{k \in \mathbb{N} \setminus \{0\} \\ (i_1, \dots, i_k) \in \{0, \dots, m\}^k}}, \quad (13)$$

$$\mathcal{F}_0 = \{\text{ad}_{X^0}^j X^k, k \in \{1, \dots, m\}, j \in \mathbb{N}\}. \quad (14)$$

Definition 8 (Bracket generating property). The control system (1), or the family of vector fields $\{X^0, \dots, X^m\}$, is called *bracket generating* at point x if and only if $\text{Span } \mathcal{L}(x) = T_x \mathcal{M}$. It is called bracket generating if this is true for all x in \mathcal{M} .

Being bracket generating is also called the Lie Algebra Rank Condition (LARC), see [20] in the context of local controllability at an equilibrium. In most of the literature (e.g. [1, chapter 8] or [10, chapter 3]), one proves controllability properties by piecewise constant controls, *i.e.* analyzes $\mathcal{A}_G(x)$ for the family of vector fields

$$\mathcal{G} = \{X^0 + u_1 X^1 + \dots + u_m X^m, (u_1, \dots, u_m) \in U\}; \quad (15)$$

see first paragraph of Section 1.3 for a definition of $\mathcal{A}_G(x)$, which is also the set of points that can be reached, for the control system (1), with piecewise U -valued constant controls, to the effect that $\mathcal{A}_G(x) \subset \mathcal{A}^U(x)$. If U satisfies (2), \mathcal{G} is bracket generating if and only if $\{X^0, \dots, X^m\}$ is.

1.4. Controllability when $\text{conv } U$ is a neighborhood of the origin, state of the art. It is natural to assume that the family $\{X^0, \dots, X^m\}$ is bracket generating because this property is (for example according to the “orbit Theorem” due to Sussmann [20]) necessary for controllability if the vector fields are real analytic, which is a reasonable framework although the results of the present paper do not require it. Bracket generating is not however sufficient, it rather implies a weaker property, where the accessible set from a point has a nonempty topological interior, see the work of Sussmann and Jurdjevic [21] or Krener [13].

One case where the bracket generating property, assuming that U is a neighborhood of the origin, implies controllability is the one of driftless systems, namely systems (1) with $X^0 = 0$, see *e.g.* the work of Lobry [17]. This is however far from assumption (3), where the drift is assumed to be non-vanishing. It turns out that this periodicity assumption, or the more general property of Poisson stability also allows one to strengthen the weaker property from [13, 21] into controllability. Recall that, for a complete vector field X^0 on \mathcal{M} , a point $x \in \mathcal{M}$ is said to be *Poisson stable* for X^0 if there exists a sequence of positive times $t_n \rightarrow \infty$ such that $\exp(t_n X^0)(x) \rightarrow x$ when $n \rightarrow \infty$; X^0 is said to be Poisson stable if there is a dense subset of such points. It turns out that many physical dynamical systems have this property; this makes the following result quite useful.

Theorem 9 (Bonnard, 1981, [2]). *System (1) is globally controllable if*

- (i) *the vector field X^0 is Poisson stable,*
- (ii) *the family $\{X^0, X^1, \dots, X^m\}$ is bracket generating, and*
- (iii) *$\text{conv } U$ is a neighborhood of 0 in \mathbb{R}^m .*

It is stated in this form in the textbook [10] (Chapter 4, Theorem 5) or in the original reference, that mentions techniques due to [11]. It has been widely used, for instance to prove controllability prior to solving an optimal control problem, like in [4, 16] for the controlled Kepler problem. Assumption (iii) is essential to Theorem 9, as evidenced by a very simple academic example of the form (1) with $\mathcal{M} = \mathbb{R} \times \mathbb{S}^1$ and one scalar control u ($m = 1$): $\dot{I} = u$, $\dot{\varphi} = 1$, $u \in U = [0, 1]$; conditions (i) and (ii) are satisfied, (iii) is not, and controllability cannot hold because $I \in \mathbb{R}$ cannot decrease. Our Theorem 13 can be seen as a generalisation of Theorem 9 to the case where (iii) does not hold. There we keep (i) (and even the stronger (3)) but we have to replace (ii) by a stronger condition, involving not only the vector fields but also the set U . Next section introduces the necessary tools.

1.5. Further constructions, not based on Lie brackets. Classical controllability results are based on the vector subspace of the tangent space at some points spanned by some Lie brackets of X^0, \dots, X^m . When the convex hull of U is not

a neighborhood of the origin, conditions based on these *linear subspaces* of the tangent space, and checking whether it is the whole tangent space, cannot be relevant. We present instead a construction that involves both the vector fields and the control set U , based on transporting subsets of the tangent space along flows.

We recall that, for a diffeomorphism φ and a vector field X on \mathcal{M} , the pushforward φ_*X is defined according to

$$\varphi_*(X)(y) := \varphi'(\varphi^{-1}(y)) \cdot X(\varphi^{-1}(y)), \quad y \in \mathcal{M}. \quad (16)$$

To the vector fields X^0, \dots, X^m and the set U defining system (1), we associate, for any x in \mathcal{M} and any real number τ , or real numbers $t_1 < t_2$, the following subsets² of $T_x\mathcal{M}$:

$$E_\tau^U(x) = \left\{ \sum_{k=1}^m u_k (\exp(-\tau X^0)_* X^k)(x), u \in U \right\} \subset T_x\mathcal{M}, \quad (17)$$

$$E_{[t_1, t_2]}^U(x) = \bigcup_{\tau \in [t_1, t_2]} E_\tau^U(x) \subset T_x\mathcal{M}. \quad (18)$$

and, when the periodicity assumption (3) holds and defines $T(x)$,

$$E^U(x) = E_{[0, T(x)]}^U(x) = \bigcup_{\tau \in [0, T(x)]} E_\tau^U(x) \subset T_x\mathcal{M} \quad (19)$$

Note that $E^U(\cdot)$ and the function $T(\cdot)$, are invariant under the flow of X^0 .

Remark 10. In the particular case when the fibration π is the one associated with the orbits of the periodic drift, in the coordinates (I, φ) from Equation (5), one may write (17), (18) and (19) in a simpler form because the flow of X^0 is explicit: $\exp(tX^0)(I, \varphi) = (I, \varphi + \omega(I)t)$, so that $(\exp(-\tau X^0)_* Z)(I, \varphi) = Z(I, \varphi + \omega(I)\tau)$ for any vector field Z . In particular, with $x = (I, \varphi)$, denoting by $\pi'(x)$ the derivative (tangent map) of the fibration π at point x (*i.e.* the linear projection on the first factor in $T_x\mathcal{M} = T_I M \times \mathbb{R}$), the projection of $E^U(x)$ has a simple expression:

$$\begin{aligned} \pi'(I, \varphi)(E^U(I, \varphi)) &= \check{E}^U(I) \subset T_I M \quad \text{with} \\ \check{E}^U(I) &= \left\{ \sum_{k=1}^m u_k F^k(I, \varphi), \quad (u_1, \dots, u_m) \in U, \varphi \in \mathbb{S}^1 \right\}. \end{aligned} \quad (20)$$

Invariance of $E^U(\cdot)$ by the flow of X^0 implies that $E^U(I, \varphi)$ is independent on φ .

The subsets $E_{[t_1, t_2]}^U(x)$ or $E^U(x)$ of $T_x\mathcal{M}$ depend both on the vector fields and on the control set U . The vector subspace of $T_x\mathcal{M}$ spanned by them is however independent on U ; next proposition gives a characterization of that vector space in terms of Lie brackets if all vector fields are real analytic.

Proposition 11. *If the vector fields X^0, \dots, X^m are real analytic, one has, with \mathcal{F}_0 defined in (14) and $E_{[t_1, t_2]}^U$ defined in (18), for any x and any $t_1 < t_2$,*

$$\text{Span } E_{[t_1, t_2]}^U(x) = \text{Span } \mathcal{F}_0(x). \quad (21)$$

Let us give a short proof, although it is classical: introducing these ideas in controllability dates back from the work of Hermann and Krener in the early 1960s,

²These subsets are related to control variations in the proof of the Pontryagin Maximum Principle. For instance, the condition of [1, Lemma 12.6] is in terms of $E_{[t_1, t_2]}^U(x)$ (with $\tilde{u} \equiv 0$, $\mathcal{T} = [t_1, t_2]$, $g_{\tau, u}(x) = \sum_{k=1}^m u_k (\exp(-\tau X^0)_* X^k)(x)$). Our Theorem 15 can be viewed as a consequence of that lemma in the simpler case where the reference control is smooth.

see [10, Chapter 2, Theorem 6], similar to Proposition 11. It uses the following definitions (and subsequent lemma). Assuming that X^0 is complete, set

$$\mathcal{F}_{(\alpha_1, \alpha_2)} = \{\exp(-tX^0)_* X^k, k \in \{1, \dots, m\}, t \in \mathbb{R}, \alpha_1 < t < \alpha_2\}, \alpha_1 < \alpha_2, \quad (22)$$

$$\mathcal{F}_\infty = \{\exp(-tX^0)_* X^k, k \in \{1, \dots, m\}, t \in \mathbb{R}\}. \quad (23)$$

Lemma 12. *Let $\alpha_1 < \alpha_2$ be some numbers. In general (i.e. if X^0, X^1, \dots, X^m are C^∞), one has $\text{Span } \mathcal{F}_0(x) \subset \text{Span } \mathcal{F}_{(\alpha_1, \alpha_2)}(x) \subset \text{Span } \mathcal{F}_\infty(x)$. If the vector fields are real analytic, one has $\text{Span } \mathcal{F}_0(x) = \text{Span } \mathcal{F}_{(\alpha_1, \alpha_2)}(x) = \text{Span } \mathcal{F}_\infty(x)$.*

Proof. For x in \mathcal{M} , k in $\{1, \dots, m\}$ and p an element of $T_x^* \mathcal{M}$, define the smooth map $a_k : \mathbb{R} \rightarrow \mathbb{R}$ by $a_k(t) = \langle p, \exp(-tX^0)_* X^k(x) \rangle$. Since $\frac{d}{dt}(\exp(-tY)_* Z)(x) = (\exp(-tY)_*[Y, Z])(x)$ for any vector fields Y and Z , one has, for all t ,

$$\frac{d^j a_k}{dt^j}(t) = \langle p, \exp(-tX^0)_* \text{ad}_{X^0}^j X^k(x) \rangle. \quad (24)$$

If p is in the annihilator of $\text{Span } \mathcal{F}_{(\alpha_1, \alpha_2)}(x)$, then $a_k(t) = 0$ for all k in $\{1, \dots, m\}$ and t in (α_1, α_2) ; differentiating j times, taking $t = 0$ and using (24) implies $\langle p, \text{ad}_{X^0}^j X^k(x) \rangle = 0$, $k \in \{1, \dots, m\}$, $j \in \mathbb{N}$, hence p in the annihilator of $\text{Span } \mathcal{F}_0(x)$. This proves $\text{Span } \mathcal{F}_0(x) \subset \text{Span } \mathcal{F}_{(\alpha_1, \alpha_2)}(x)$, while $\text{Span } \mathcal{F}_{(\alpha_1, \alpha_2)}(x) \subset \text{Span } \mathcal{F}_\infty(x)$ is obvious by definition. To prove the reverse inclusion, assume p in the annihilator of $\text{Span } \mathcal{F}_0(x)$. For each k in $\{1, \dots, m\}$, p vanishes on all vectors $\text{ad}_{X^0}^j X^k(x)$, $j \in \mathbb{N}$, and (24) implies $\frac{d^j a_k}{dt^j}(0) = 0$ for all j . Since the map a_k is now real analytic, it must be identically zero on \mathbb{R} . This proves $\text{Span } \mathcal{F}_\infty(x) \subset \text{Span } \mathcal{F}_0(x)$, hence $\text{Span } \mathcal{F}_0(x) = \text{Span } \mathcal{F}_{(\alpha_1, \alpha_2)}(x) = \text{Span } \mathcal{F}_\infty(x)$. \square

Proof of Proposition 11. Apply Lemma 12 with $\alpha_1 = t_1 \pm \varepsilon$, $\alpha_2 = t_2 \pm \varepsilon$, taking into account the fact that, for any $\varepsilon > 0$,

$$\text{Span } \mathcal{F}_{(t_1+\varepsilon, t_2-\varepsilon)} \subset \text{Span } \bigcup_{\tau \in [t_1, t_2]} E_\tau^U(x) \subset \text{Span } \mathcal{F}_{(t_1-\varepsilon, t_2+\varepsilon)}. \quad \square$$

2. GLOBAL AND LOCAL CONTROLLABILITY RESULTS

All results are concerned with system (1) (or with system (5) in product form if X^0 satisfies (3)). A few remarks are in order on the assumptions in these results.

Smoothness. All vector fields are assumed smooth, i.e. infinitely differentiable, although the reader may check that all proofs work if smooth solely means differentiable with a locally Lipschitz continuous first derivative.

Periodicity of solutions of the drift. This property (3) of X^0 , discussed in Section 1.1, is assumed in Theorem 13, that was a primary motivation of this work, but is *not* a running assumption. Also, we indicate when $\pi : \mathcal{M} \rightarrow M$ is the fibration induced by (3), the base M being the manifold of orbits of X^0 ; it is otherwise an arbitrary fibration, unrelated to X^0 , with M of arbitrary dimension $n \leq d$.

The control set U . It is always assumed to satisfy (2). We also need U to be convex and compact in Sections 2.2 and 2.3 (Theorems 15 and 19, see Remark 17).

Rank of the control vector fields. In Section 2.4, we need the control vector fields to be linearly independent to give local *necessary* conditions (Theorems 20 and 21, see also Remark 22). This is not needed anywhere else in the paper, we do not even need to assume $m \leq d$ in (1).

Notation: for a subset S of a real vector space E , its *conic hull* is smallest convex cone containing S , i.e. the set of *conic combinations* of elements of S :

$$\text{cone } S = \{ \lambda_1 x_1 + \dots + \lambda_k x_k, x_i \in S, \lambda_i \in [0, +\infty), i \in \{1 \dots k\}, k \in \mathbb{N} \}. \quad (25)$$

2.1. Global controllability under the periodicity assumption on the drift.

Let us now state our generalization of the now classical Theorem 9 on global controllability. Here, the fibration π is the one induced by the periodic drift with fibre \mathbb{S}^1 , as in (4). Global controllability is as in Definition 2; as noticed in Remark 3, it is equivalent, in the present case, to “global controllability with respect to π ”.

Theorem 13. *Assume that X^0 satisfies (3) (periodicity of all solutions), inducing a fibration π as in (4), that U satisfies (2), and that*

$$\pi'(x)(\text{cone } E^U(x)) = T_{\pi(x)}M \quad \text{for all } x \text{ in } \mathcal{M}. \quad (26)$$

Then system (1) is globally controllable, and a fortiori globally controllable with respect to π . Under the same conditions, global controllability holds as well in backward time, i.e. $\mathcal{A}_-^U(x) = \mathcal{M}$ for all x in \mathcal{M} .

Using the local product form (5) and the notation \check{E} from (20), Condition (26) for all x in \mathcal{M} equates to:

$$\text{cone } \check{E}^U(I) = T_I M \quad \text{for all } I \text{ in } M. \quad (27)$$

Proof. We prove controllability in forward time; backward time follows upon changing each vector field X^i into $-X^i$, preserving all assumptions of the theorem. The accessible set $\mathcal{A}_{\mathcal{F}}(x)$ of a family \mathcal{F} of vector fields was defined in the first paragraph of Section 1.3. The family \mathcal{G} was defined in (15), and we noted there that it satisfies $\mathcal{A}_{\mathcal{G}}(x) \subset \mathcal{A}^U(x)$ for all x . The conclusion of the theorem is then obviously equivalent to $\mathcal{A}_{\mathcal{G}}(x) = \mathcal{M}$ for all x in \mathcal{M} , that we are going to prove.

Define, from \mathcal{G} , the families \mathcal{G}_1 and \mathcal{G}_2 by

$$\mathcal{G}_1 = \mathcal{G} \cup \{-X^0\} \quad \text{and} \quad \mathcal{G}_2 = \{\exp(tX^0)_*X, X \in \mathcal{G}_1, t \in \mathbb{R}\}, \quad (28)$$

where the push-forward notation $\exp(tX^0)_*X$ was recalled in (16). Obviously, $\mathcal{G} \subset \mathcal{G}_1 \subset \mathcal{G}_2$. Let us first prove that, for all x ,

$$\mathcal{A}_{\mathcal{G}}(x) = \mathcal{A}_{\mathcal{G}_1}(x) = \mathcal{A}_{\mathcal{G}_2}(x). \quad (29)$$

The family \mathcal{G} contains X^0 because $0 \in U$, see (2); hence $\exp(tX^0)(x) \in \mathcal{A}_{\mathcal{G}}(x)$ for any x in \mathcal{M} and $t \geq 0$; it is also true if $t < 0$ because (3) implies $\exp(tX^0)(x) = \exp((t + kT)X^0)(x)$, where the integer k can be taken large enough that $t + kT(x) \geq 0$; the reverse inclusion being clear, this proves the first equality. Turning to the second one, consider some y in $\mathcal{A}_{\mathcal{G}_2}(x)$; by definition, $y = \exp(s_1 Y^1) \circ \dots \circ \exp(s_N Y^N)(x)$ for some positive integer N , and some positive real numbers s_j and vector fields Y^j , $j \in \{1, \dots, N\}$, belonging to \mathcal{G}_2 , hence such that, for some real number t_j and some $Z^j \in \mathcal{G}_1$, $Y^j = \exp(t_j X^0)_* Z^j$; since the latter implies $\exp(s_j Y^j) = \exp(t_j X^0) \circ \exp(s_j Z^j) \circ \exp(-t_j X^0)$, and both X^0 and $-X^0$ belong to \mathcal{G}_1 , the above yields y as the image of x by a composition of $3N$ flows in positive time of vector fields belonging to \mathcal{G}_1 , hence $y \in \mathcal{A}_{\mathcal{G}_1}(x)$, which proves the second equality in (29), the reverse inclusion being, again, obvious; this ends the proof of (29).

For any x in \mathcal{M} , (26) implies $\text{cone } \mathcal{G}_2(x) = T_x \mathcal{M}$, hence there are $d+1$ vector fields Y^0, \dots, Y^d in \mathcal{G}_2 such that $Y^0(x), \dots, Y^d(x)$ generate $T_x \mathcal{M}$ by conic combinations. Let $\phi : (-\alpha, \alpha)^d \rightarrow \mathcal{M}$ (with $\alpha > 0$ small enough so that the flows are defined) be the smooth map given by $\phi(s_0, \dots, s_d) = \exp(s_0 Y^0) \circ \dots \circ \exp(s_d Y^d)(x)$, and apply Lemma 14 (see below) with $k = d + 1$, $W^k = (-\alpha, \alpha)^{d+1}$, $N = \mathcal{M}$ and $G = \phi$. Since $\phi'(0)(\mathbb{R}_+^k) = T_{\phi(0)} \mathcal{M}$ (this follows from the fact that $\partial \phi / \partial s_i(0) = Y^i(x)$, $i \in \{0 \dots d\}$, and the conic combination property of $Y^0 \dots Y^d$ stated above), the lemma tells us that $\phi(W_+^k)$ is a neighborhood of x . Since $\phi(W_+^k) = \phi([0, \alpha)^{d+1})$ is contained in $\mathcal{A}_{\mathcal{G}_2}(x) = \mathcal{A}_{\mathcal{G}}(x)$ (see (28)), we have shown the following property, valid also for $\mathcal{A}_{-\mathcal{G}}(x)$ by changing each X^i into $-X^i$:

$$\text{for all } x \text{ in } \mathcal{M}, \mathcal{A}_{\mathcal{G}}(x) \text{ and } \mathcal{A}_{-\mathcal{G}}(x) \text{ are neighborhoods of } x \text{ in } \mathcal{M}. \quad (30)$$

Any $\mathcal{A}_G(x)$ is open because, for y in $\mathcal{A}_G(x)$, applying (30) to $\mathcal{A}_G(y)$, the latter is a neighborhood of y , contained in $\mathcal{A}_G(x)$ by definition. Any $\mathcal{A}_G(x)$ is also closed: picking y in its topological closure, $\mathcal{A}_{-G}(y)$ is a neighborhood of y according to (30), so $\mathcal{A}_{-G}(y)$ intersects $\mathcal{A}_G(x)$ and y must belong to $\mathcal{A}_G(x)$. Being an open, closed, nonempty subset of the connected manifold \mathcal{M} , $\mathcal{A}_{-G}(y)$ is equal to \mathcal{M} . \square

The following lemma is used in the above proof and again in the proof of Theorem 15, Section 2.2. We denote by \mathbb{R}_+^k the positive orthant in \mathbb{R}^k :

$$\mathbb{R}_+^k = \{(w_1, \dots, w_k) \in \mathbb{R}^k, w_i \geq 0, i = 1 \dots k\}. \quad (31)$$

Lemma 14 ([1, Lemma 12.4]). *Let k be a positive integer, W^k be an open neighborhood of the origin in \mathbb{R}^k and $W_+^k = W^k \cap \mathbb{R}_+^k$. If $G: W^k \rightarrow N$ is a smooth map, with N a smooth manifold, such that $G'(0) \cdot W_+^k = T_{G(0)}N$, then $G(V \cap W_+^k)$ is a neighborhood of $G(0)$ in N for any neighborhood V of the origin in \mathbb{R}^k .*

Proof. This is stated and proved in [1] under minimal assumptions, namely G Lipschitz continuous everywhere and differentiable at zero, based on Brouwer fixed point theorem. If G is continuously differentiable, the implicit function theorem gives a smooth curve $\alpha \mapsto \gamma(\alpha)$ in W_+^k , defined for small non-negative α , such that $\gamma(0) = 0$, $\gamma(\alpha)$ in the interior of W_+^k if $\alpha > 0$, $G(\gamma(\alpha)) = G(0)$ for all $\alpha \geq 0$ and $\text{Rank } G'(\gamma(\alpha)) = \dim N$ for all α , so that the standard inverse function theorem may be applied locally around any $\gamma(\alpha)$, $\alpha > 0$. \square

2.2. Local controllability around a particular solution. Let us consider

$$t \mapsto \bar{x}(t), \text{ solution of } \dot{x} = X^0(x), \text{ defined on the time-interval } [0, t_f] \quad (32)$$

(the real number t_f may be negative, $[0, t_f]$ should then be understood as $[t_f, 0]$), and define the following subset of $T_{\bar{x}(0)}\mathcal{M}$, attached to the solution $\bar{x}(\cdot)$:

$$\bar{E} = E_{[0, t_f]}^U(\bar{x}(0)), \quad (33)$$

with $E_{[0, t_f]}^U(x)$ defined in (18). Next theorem gives, in terms of \bar{E} , sufficient conditions for local controllability along $\bar{x}(\cdot)$. See Definition 4 for local controllability (with respect to π or not); Remark 5 explains why inclusions (35) and (37) for arbitrarily small $\varepsilon > 0$ (small control) imply local controllability. Part I is a particular case of Part II, with $M = \mathcal{M}$ and $\pi = \text{Id}$; it is stated independently for readability.

Theorem 15. *Consider system (1) with U a convex compact subset of \mathbb{R}^m containing the origin, and let $\bar{x}(\cdot)$ and \bar{E} be as described above.*

I. *If*

$$\text{cone } \bar{E} = T_{\bar{x}(0)}\mathcal{M}, \quad (34)$$

then system (1) is locally controllable around $\bar{x}(\cdot)$ and, more precisely,

$$\mathcal{A}_{t_f}^{\varepsilon U}(\bar{x}(0)) \text{ is a neighborhood of } \bar{x}(t_f) \text{ in } \mathcal{M} \text{ for any } \varepsilon > 0. \quad (35)$$

II. *Let $\pi: \mathcal{M} \rightarrow M$ be a smooth fibration³ with M of dimension $n \leq d$. If*

$$(\pi \circ \exp(t_f X^0))'(\bar{x}(0))(\text{cone } \bar{E}) = T_{\pi(\bar{x}(t_f))}M, \quad (36)$$

then system (1) is locally controllable around $\bar{x}(\cdot)$ with respect to π ; more precisely,

$$\pi(\mathcal{A}_{t_f}^{\varepsilon U}(\bar{x}(0))) \text{ is a neighborhood of } \pi(\bar{x}(t_f)) \text{ in } M \text{ for any } \varepsilon > 0. \quad (37)$$

³It is not necessarily related to π defined in (4) from Assumption (3). For the present (local) purpose, a submersion from a neighborhood of $\bar{x}(t_f)$ onto an open subset of \mathbb{R}^n would be sufficient.

Remark 16. The short notation \bar{E} used in (33) because $\bar{x}(\cdot)$ and t_f are fixed in the theorem should not hide the dependence of this set on the initial condition and more interestingly on t_f . It is clear from (18) that $E_{[0,t]}^U(x)$ grows with $t > 0$ for fixed x ; hence, if one follows the same solution (assumed defined on $[0, +\infty)$), either Condition (34) is satisfied for all t_f or it is not satisfied for small values of t_f , but is satisfied, if ever, for t_f larger than a certain value. Note that it is not necessarily true for (36), unless π is invariant by the flow of X^0 ($\pi \circ \exp(tX^0) = \pi$), which is indeed the case when π is associated to periodic orbits of X^0 as in (3) and (4).

Remark 17 (assumptions on U). Theorems 15 and 19, unlike Theorem 13, assume U convex and compact. Compactness is natural; it a priori avoids unbounded controls and allows εU to be small for small $\varepsilon > 0$. Convexity is more restrictive, it is technically needed in the proofs, and could probably be relaxed. If U is not convex, the same conclusions hold, replacing $\mathcal{A}_{t_f}^{\varepsilon U}(\bar{x}(0))$ with its topological closure.

Before proving Theorem 15, let us recall that the linear approximation of the control system (1) along the solution $\bar{x}(\cdot)$ is the following linear time-varying system:

$$\begin{aligned} \dot{\xi} &= \sum_{k=1}^m \delta u_k (\exp(-tX^0)_* X^k)(\bar{x}(0)), \\ \delta x(t) &= \exp(tX^0)'(\bar{x}(0)) \xi(t), \end{aligned} \quad (38)$$

with initial condition $\xi(0) = 0$, where $\delta u = (\delta u_1, \dots, \delta u_m)$ is a small variation of the control around zero and $\delta x(t)$ the corresponding small variation of the state around $\bar{x}(t)$ (in other words, $t \mapsto (\bar{x}(t) + \delta x(t), \delta u(t))$ is a solution of (1)). Note that (38) is the intrinsic way of writing $\dot{\delta x} = \frac{\partial X^0}{\partial x}(\bar{x}(t)) \delta x + \sum_1^m \delta u_k X^k(\bar{x}(t))$, because ξ is an element of the tangent space $T_{\bar{x}(0)}$ for all t , contrary to δx . One may constrain the control $\delta u = (\delta u_1, \dots, \delta u_m)$ to some $U \subset \mathbb{R}^m$ and define, for this linear control-constrained time-varying system, the accessible set for initial time zero, initial state x_0 and final time t as the following subset of $T_{\bar{x}(t)}\mathcal{M}$, see also (54) in Appendix A:

$$\begin{aligned} \mathbf{A}_{0,t}^U(\delta x_0) &= \{\delta x(t), \text{ with } s \mapsto (\xi(s), \delta u(s), \delta x(s)) \text{ solution of (38) on } [0, t], \\ &\quad \delta x(0) = \delta x_0, \delta u(s) \in U \text{ a.e. } s \in [0, t]\}. \end{aligned} \quad (39)$$

The following result states that Condition (34) is equivalent to controllability of the constrained linear approximation around $\bar{x}(\cdot)$ (obviously it also applies to (26) or (27)).

Lemma 18. **I.** *The linear constrained attainable set $\mathbf{A}_{0,t_f}^U(0)$ is a neighborhood of the origin in $T_{\pi(\bar{x}(t_f))}M$ if and only if Condition (34) holds.*

II. *The set $\pi'(\bar{x}(t_f))(\mathbf{A}_{0,t_f}^U(0))$ is a neighborhood of the origin in $T_{\pi(\bar{x}(t_f))}M$ if and only if Condition (36) holds.*

Proof. This follows from Proposition 25 for point I and Proposition 26 for point II in the Appendix, applied with $t_0 = 0$, with (38) playing the role of (56). In coordinates, the k^{th} column of $B(s)$ is the coordinate vector of $X^k(\bar{x}(s))$, and the k^{th} column of $\phi(0, s)B(s)$ is the coordinate vector of $(\exp(-sX^0)_* X^k)(\bar{x}(0))$. \square

Proof of Theorem 15. We give a proof of point II. For a proof of point I (particular case of point II) without reference to the projection π , replace M with \mathcal{M} , n with d , π with Id , $(\pi \circ \exp(t_f X^0))'(\bar{x}(0))(\bar{E})$ with \bar{E} , and $\pi'(\bar{x}(t_f))(\mathbf{A}_{0,t_f}^U(0))$ with $\mathbf{A}_{0,t_f}^U(0)$.

According to Lemma 18, point II, Condition (36) implies that $\pi'(\bar{x}(t_f))(\mathbf{A}_{0,t_f}^U(0))$, projection of the accessible set of the linearized system (38) in time t_f , is a neighborhood of the origin in $T_{\pi(\bar{x}(t_f))}M$. Let e_0, \dots, e_n be the vertices of a convex polyhedron in $T_{\pi(\bar{x}(t_f))}M$ that is both contained in $\pi'(\bar{x}(t_f))(\mathbf{A}_{0,t_f}^U(0))$ and a neighborhood

of the origin. Since e_0, \dots, e_n are in $\pi'(\bar{x}(t_f))(A_{0,t_f}^U(0))$, there are some $\hat{e}_0, \dots, \hat{e}_n$ in $A_{0,t_f}^U(0)$ such that $e_i = \pi'(\bar{x}(t_f))(\hat{e}_i)$, and there also exist $\mathbf{u}_0, \dots, \mathbf{u}_n$, some integrable U -valued controls, defined on $[0, t_f]$, that steer the origin to $\hat{e}_0, \dots, \hat{e}_n$ respectively, in time t_f , for the linear time-varying system (38).

Going back to the nonlinear system (1), let $\mathcal{E} : L^\infty([0, t_f], \mathbb{R}^m) \rightarrow \mathcal{M}$ be the end-point mapping at time t_f starting from $\bar{x}(0)$ for the nonlinear system (1). To avoid problems due to non-complete vector fields and differential equations, multiply all vector fields X^0, \dots, X^m by some smooth cut-off function equal to 1 in a neighborhood of $\bar{x}([0, t_f])$ to ensure that solutions all exist until t_f (and in fact $+\infty$), without changing the local properties in this neighborhood, that we are proving. It is well known [10, 1] that \mathcal{E} is continuously differentiable and that its derivative at some control is the end-point mapping, with initial point the origin, of the linearized control system (38) at this control. Consider the continuously differentiable map $G : \mathbb{R}^{n+1} \rightarrow M$ defined by

$$G(\lambda) = G(\lambda_0, \dots, \lambda_n) = \pi \circ \mathcal{E} \left(\sum_{i=0}^n \lambda_i \mathbf{u}_i \right). \quad (40)$$

Since \mathcal{E} maps the zero control to $\bar{x}(t_f)$, and $\frac{\partial G}{\partial \lambda_i}(0, \dots, 0)$, $i \in \{0, \dots, n\}$, is the projection to M of the image by the end-point mapping of the linear system (38) of \mathbf{u}_i , these images being \hat{e}_i and their projections e_i , one has

$$G(0) = \pi(\bar{x}(t_f)), \quad \frac{\partial G}{\partial \lambda_i}(0) = e_i, \quad i \in \{0, \dots, n\}. \quad (41)$$

The second relation implies that the linear map $G'(0) : \mathbb{R}^{n+1} \rightarrow T_{\pi(\bar{x}(t_f))}M$ satisfies

$$G'(0)(\mathbb{R}_+^{n+1}) = T_{\pi(\bar{x}(t_f))}M \quad (42)$$

(\mathbb{R}_+^{n+1} is defined in (31)) because $\text{cone}\{e_0, \dots, e_n\} = T_{\pi(\bar{x}(t_f))}M$. Now, applying Lemma 14 with $k = n+1$, $N = M$, $W_k = \mathbb{R}^{n+1}$ and $V_\varepsilon = \{\lambda \in \mathbb{R}^{n+1}, \lambda_0 + \dots + \lambda_n \leq \varepsilon\}$, one obtains that $G(\mathbb{R}_+^{n+1} \cap V_\varepsilon)$ is a neighborhood of $\pi(\bar{x}(t_f))$ for any ε , and this does yield (37) because, since U is convex and contains the origin, the control $\sum_{i=0}^n \lambda_i \mathbf{u}_i$ takes values in εU if each \mathbf{u}_i takes values in U and $(\lambda_0, \dots, \lambda_n)$ is in $\mathbb{R}_+^{n+1} \cap V_\varepsilon$. \square

2.3. Application to periodic solutions of the drift. Theorem 15 applies, in particular, to periodic solutions of the drift vector field under condition (3), yielding the following interesting complement to Theorem 13.

Theorem 19. *Assume that the vector field X^0 satisfies the periodicity assumption (3), and that U is convex compact and satisfies conditions (2).*

- I. *If $\text{cone } E^U(x) = T_x M$, then $\mathcal{A}_{T(x)}^{\varepsilon U}(x)$ is a neighborhood of x in \mathcal{M} for all $\varepsilon > 0$.*
- II. *Let π be as in Theorem 13. If $\pi'(x)(\text{cone } E^U(x)) = T_{\pi(x)}M$, $\pi(\mathcal{A}_{T(x)}^{\varepsilon U}(x))$ is a neighborhood of $\pi(x)$ in M for all $\varepsilon > 0$.*

The same conclusions hold for $\mathcal{A}_{-T(x)}^{\varepsilon U}(x)$ and $\pi(\mathcal{A}_{-T(x)}^{\varepsilon U}(x))$ (backward time) under the same assumptions.

Proof. Direct consequence of Theorem 15, with $t_f = T(x)$, $\bar{x}(0) = \bar{x}(t_f) = x$. \square

The above result is more precise (locally) than Theorem 13 in two respects: points in a neighborhood of the initial orbit may be reached over one period without leaving some neighborhood of the initial orbit while Theorem 13 gives no estimate on final time, and allows trajectories going arbitrarily “far”. Since final time is prescribed, Remark 3 does not apply and parts I and II have no reason to be

equivalent. Global controllability can be recovered from Theorem 19 as seen in the proof below, provided U is assumed compact and convex.

Alternative proof of Theorem 13 (under the assumption that U is convex).

According to Theorem 19, part II, $\pi(\mathcal{A}_{T(y)}^U(y))$ is a neighborhood of $\pi(y)$ in M for any y in \mathcal{M} , and so is $\pi(\mathcal{A}_{-T(y)}^U(y))$. For x in M , let us show that $\mathcal{A}^U(x)$ is both open and closed in M ; this implies the theorem by connectedness of M . First, consider y in $\mathcal{A}^U(x)$, so that $\mathcal{A}^U(x)$ contains $\mathcal{A}_{T(y)}^U(y)$, and $\pi(\mathcal{A}^U(x))$ contains $\pi(\mathcal{A}_{T(y)}^U(y))$, that was just pointed out as a neighborhood of $\pi(y)$, hence $\pi(\mathcal{A}^U(x))$ is a neighborhood of $\pi(y)$, and, using Remark 1 (Section 1.1), $\mathcal{A}^U(x) = \pi^{-1}(\pi(\mathcal{A}^U(x)))$ is a neighborhood of y ; openness is proven. Now suppose that some y is in $\overline{\mathcal{A}^U(x)}$, hence $\pi(y)$ in $\overline{\pi(\mathcal{A}^U(x))}$. We pointed out that $\pi(\mathcal{A}_{-T(y)}^U(y))$ is a neighborhood of $\pi(y)$; hence it intersects $\pi(\mathcal{A}^U(x))$, *i.e.* there is some z such that $z \in \mathcal{A}_{-T(y)}^U(y)$ and $z' \in \mathcal{A}^U(x)$ such that $\pi(z) = \pi(z')$, but, according to Remark 1 again, $z' \in \mathcal{A}^U(x)$ then implies $z \in \mathcal{A}^U(x)$; $z \in \mathcal{A}^U(x)$ and $y \in \mathcal{A}_{T(y)}^U(z)$ do imply $y \in \mathcal{A}^U(x)$; this proves closedness and ends the proof of the theorem. \square

2.4. Obstructions to local controllability. Theorems 13, 15 and 19 state sufficient conditions (take *e.g.* (26), (27) or (34)) that require that some convex cone C be the whole ambient vector space (some tangent space). These sufficient conditions are not necessary, see *e.g.* Example 24, hence C being contained in a closed half-space is not an obstruction to controllability; it turns out that one gets an obstruction by requiring that \overline{C} (topological closure) be contained in a closed half-space and intersect the separating hyperplane only at the origin. Let us recall that, for a convex cone C in a vector space V , the *polar cone* of C is the cone $C^\circ = \{p \in V^*, \langle p, v \rangle \leq 0, v \in C\}$; a co-vector p in V^* is in the topological interior of C° if and only if $\langle p, v \rangle < 0$ for all v in $\overline{C} \setminus \{0\}$. Hence the property that a cone has a topological closure that intersects some hyperplane only at the origin is equivalent to its polar cone having nonempty interior.

Theorem 20 gives an obstruction to local controllability in the case of Theorem 15, Theorem 13 an obstruction to the weaker local *orbital* controllability in the periodic case. The latter is stated in product coordinates $(I, \varphi) \in M \times \mathbb{S}^1$ for simplicity, like Definition 7, but may also be formulated in terms of $x \in \mathcal{M}$ and the projection π ; global controllability (conclusion of Theorem 13) could still hold, see Remark 6 and Example 23. We do not state the proof of Theorem 20, to be deduced from the other proof, *mutatis mutandis*.

Theorem 20. Consider system (1), and $t_f, \bar{x}(\cdot), \bar{E}$ as in (32) and (33).

I. If **(i)** the interior in $T_{\bar{x}(0)}^* \mathcal{M}$ of the polar cone to cone \bar{E} is nonempty, and **(ii)** the vectors $X^1(\bar{x}(t)), \dots, X^m(\bar{x}(t))$ then system (1) is not locally controllable around the solution $\bar{x}(\cdot)$.

II. Let $\pi : \mathcal{M} \rightarrow M$ be as in Theorem 15, Part II. If **(i)** the interior in $T_{\pi(\bar{x}(t_f))}^* M$ of the polar cone to $(\pi \circ \exp(t_f X^0))'(\bar{x}(0))(\text{cone } \bar{E})$ is nonempty, and **(ii)** the vectors $\pi'(\bar{x}(t)) X^1(\bar{x}(t)), \dots, \pi'(\bar{x}(t)) X^m(\bar{x}(t))$ are linearly independent in $T_{\pi(\bar{x}(t))} M$, then system (1) is not locally controllable with respect to π around $\bar{x}(\cdot)$.

Theorem 21. Consider system (5) (or equivalently system (1) under assumption (3) on the drift vector field). If, for some \bar{I} in $O \subset M$, **(i)** the polar cone to cone $\check{E}^U(\bar{I})$ has a nonempty interior, and **(ii)** the vectors $F^1(\bar{I}, \varphi), \dots, F^m(\bar{I}, \varphi)$ are linearly independent in $T_{\bar{I}} M$ for all φ in \mathbb{S}^1 , then system (5) is not locally orbitally controllable around the periodic solution \bar{I} (see Definition 7).

Proof of Theorem 21. Let $K = \{u \in \text{cone } U, u_1^2 + \dots + u_m^2 = 1\}$. Fix \bar{p} in the interior of the polar cone to $\check{E}^U(\bar{I})$. Let $\alpha < 0$ be its maximum of the smooth function $(\varphi, u) \mapsto \langle \bar{p}, \sum_{k=1}^m u_k F^k(\bar{I}, \varphi) \rangle$ on the compact $\mathbb{S}^1 \times K$, where it remains negative by construction. Using some coordinates on an open set $O \subset M$ around \bar{I} , there is a neighborhood O' of \bar{I} in O such that, for all (I, φ, u) in $O' \times \mathbb{S}^1 \times K$, $\langle \bar{p}, \sum_{k=1}^m u_k F^k(I, \varphi) \rangle < \alpha/2 < 0$. From (5), this means that $\frac{d}{dt} \langle \bar{p}, I \rangle < 0$ for any (I, φ) in $O' \times \mathbb{S}^1$ if u is in U ; $(\langle \bar{p}, I \rangle)$ makes sense in coordinates; in fact we have locally built a function $h: O' \rightarrow \mathbb{R}$ such that $dh(\bar{I}) = \bar{p}$ and $\frac{d}{dt} h(I) < 0$ locally), and this implies that $\langle \bar{p}, I(t_f) \rangle \leq \langle \bar{p}, I(0) \rangle$ for any $t_f \geq 0$ and any solution such that $I(t)$ stays in O' ; this defeats local controllability, taking $\Omega = O'$ in (11). \square

Remark 22. In Theorems 21 and 20, we assume (ii) that the control vector fields are linearly independent at all points of the trajectory of interest, and it is important in the proof to deduce that the property on $\text{cone}(\check{E}^U(\bar{I}))$ extends to $\text{cone}(\check{E}^U(I))$ in some neighborhood. Let us show that without this assumption, the set of points where the property on $\text{cone}(\check{E}^U(I))$ holds is not open in general. Consider a system of the form (5) with $m = n = 2$, $M = \mathbb{R}^2$, $U = [0, 1] \times [0, 1]$,

$$F^1 = \partial/\partial I_1 + \sin \varphi \partial/\partial I_2 \quad \text{and} \quad F^2 = I_2 \partial/\partial I_1.$$

One easily checks that $\text{cone}(\check{E}^U(I_1, I_2))$ is $\{(v_1, v_2), v_1 \geq 0, |v_2| \leq v_1\} \subset T_I M = \mathbb{R}^2$ (with $\check{E}^U(I_1, I_2) = \{(v_1, v_2), v_1 \in [0, 1 + I_2], |v_2| \leq \min(1, v_1)\}$) if $I_2 \geq 0$, and is all $T_I M$ if $I_2 < 0$ (with $\check{E}^U(I_1, I_2) = \{(v_1, v_2), v_1 \in [I_2, 1], |v_2| \leq \min(1, I_2 + v_1)\}$). The polar cone is $\{(p_1, p_2), p_1 \leq 0, |p_2| \leq |p_1|\}$ in the first case and $\{0\}$ in the second case. Theorem 21 does not apply at the periodic solution $\bar{I} = (0, 0)$ because point (i) is satisfied but point (ii) is not (F^2 vanishes), and indeed the proof cannot be extended because no $p \neq 0$ can be (in coordinates) in the polar cone to $\check{E}^U(I)$ for I in a neighborhood of \bar{I} ; one may indeed check by hand that local controllability holds at this periodic solution on this specific example (but this is incidental: local controllability does not hold if F^2 is $I_1 \partial/\partial I_1$ instead of $I_2 \partial/\partial I_1$).

3. COMMENTS, EXAMPLES AND COUNTER-EXAMPLES

On constructively checking the conditions of the theorems. Conditions (26), (27), (34), (36), or the sufficient condition in Theorem 19, involve both the vector fields and the control set U and are much more difficult to check than the rank of a family of vector fields. This point is discussed by the authors in [8, 9], where the condition is formulated in terms of convex optimization and conveniently solved, in the case of solar sails, using sum of square positivity criteria on trigonometric polynomials (this last point cannot be generalized to arbitrary periodic systems and sets U), yielding minimum requirements on the characteristics of a solar sail to provide controllability around a particular orbit.

Implications of our conditions in terms of Lie brackets. Let us assume for a while that the vector fields X^0, \dots, X^m are real analytic. The sufficient conditions ((26), (27), (34), (36)...) from Theorems 13, 15 and 19 then have the following consequences according to Proposition 11:

- condition (26) implies $\text{Span}(\mathcal{F}_0 \cup \{X^0\})(x) = T_x \mathcal{M}$ (for all x);
- condition (34) implies $\text{Span } \mathcal{F}_0(\bar{x}(0)) = T_{\bar{x}(0)} \mathcal{M}$
(and $\text{Span } \mathcal{F}_0(\bar{x}(t)) = T_{\bar{x}(t)} \mathcal{M}$ for all t by invariance);
- condition (36) implies $(\pi \circ \exp(t_f X^0))'(\bar{x}(0)) (\text{Span } \mathcal{F}_0(\bar{x}(0))) = T_{\pi(\bar{x}(t_f))} M$.

These bracket conditions on \mathcal{F}_0 are necessary for some sufficient conditions to hold, but they are by no means sufficient by themselves, as evidenced by Example 23: the system there always satisfies these Lie bracket conditions but may be controllable

or not according to the value of parameters (this could not happen if the control set U was a neighborhood of the origin).

It is important to note that these bracket conditions are quite more restrictive than the classical bracket generating condition discussed in Section 1.4 because \mathcal{F}_0 discards all Lie brackets like $[X^1, [X^0, X^1]]$ or $[X^1, X^2]$ involving control vector fields more than once, compare (13), (14) and Definition 8. In fact, full rank of \mathcal{F}_0 amounts to controllability of the linear approximation; see e.g. the discussion in [5, Section 3.2] around equation (3.35) for the case where U is a neighborhood of the origin, or (38) and Lemma 18 in the present paper for the case where U is not. Studying controllability of systems that are bracket generating but do not satisfy our condition on \mathcal{F}_0 , while the control is constrained to a set U that is not a neighborhood of the origin, is outside the scope of this paper. Example 24 displays some systems in this category for which we prove controllability by ad-hoc methods, whose generalisation could be a topic of further investigation.

Example 23. $M = \mathbb{R}^3$, $I = (I_1, I_2, I_3)$. Two controls ($m = 2$).

$$\begin{pmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{I}_3 \end{pmatrix} = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(u_1 \begin{pmatrix} \cos \theta \\ 0 \\ \sin \theta \end{pmatrix} + u_2 \begin{pmatrix} -\sin \theta \\ 0 \\ \cos \theta \end{pmatrix} \right) \quad (43)$$

$$\dot{\varphi} = 1,$$

$$(u_1, u_2) \in U = \{(u_1, u_2) \in \mathbb{R}^2, u_1^2 + u_2^2 \leq 1, |u_2| \leq u_1 \tan \alpha\},$$

where θ, α are fixed parameters, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $0 < \alpha < \frac{\pi}{2}$, $0 < |\alpha| + |\theta| < \frac{\pi}{2}$. This system is of the form (5), with F^1, F^2 that depend on φ only, and also of the form (1) with obvious vector fields X^0, X^1, X^2 on $\mathcal{M} = M \times \mathbb{S}^1$. One has $E^U(I, \varphi) = \check{E}^U(I) \times \{0\}$ where the set $\check{E}^U(I)$, defined by (20), is obtained by rotating around the I_3 axis the image by $(u_1, u_2) \mapsto u_1 F^1(0) + u_2 F^2(0)$ of the domain U pictured in Figure 1-(a) :

$$\begin{aligned} \check{E}^U(I) = \{ & (v_1, v_2, v_3) \in T_I M = \mathbb{R}^3, v_1^2 + v_2^2 + v_3^2 \leq 1 \\ & \text{and } \sqrt{v_1^2 + v_2^2} \tan(\theta - \alpha) \leq v_3 \leq \sqrt{v_1^2 + v_2^2} \tan(\theta + \alpha) \}. \end{aligned} \quad (44)$$

If $|\theta| < \alpha$, $\check{E}^U(I)$ has the aspect shown in Figure 1-(b), its convex hull is a

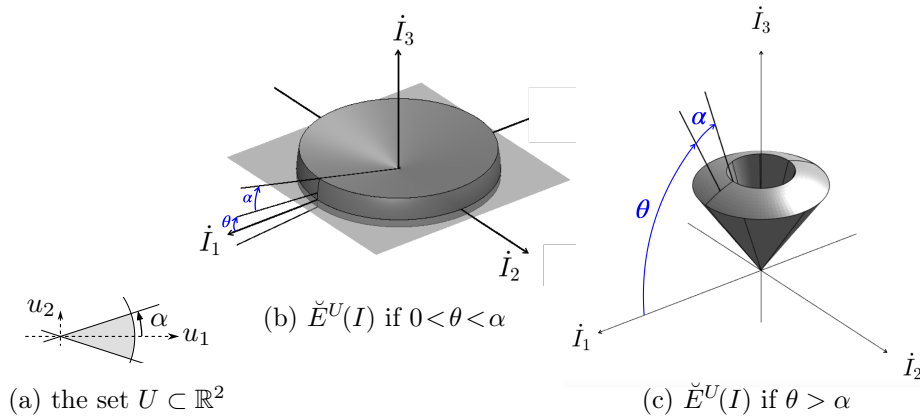


FIGURE 1. Control set U , and set $\check{E}^U(I)$ for system (43).

neighborhood of the origin, Theorem 13 yields global controllability and Theorem 19 local controllability over one period. If $\theta > \alpha$, $\check{E}^U(I)$ has the aspect shown in Figure 1-(c); its convex hull is clearly contained in the upper half space, with a

symmetric picture contained in the lower half space if $\theta < -\alpha$; since strict convexity at the vertex of the cone implies a dual cone with nonempty interior, Theorem 21 yields non controllability if $|\theta| > \alpha$. If $|\theta| = \alpha$, cone $\check{E}^U(I)$ is a full half space; one sees by inspection that the system is non controllable (\dot{I}_3 has a fixed sign), but Theorem 21 does not apply (the dual cone to cone $\check{E}^U(I)$ is a half line).

From the point of view of the remarks on Lie brackets made just before this example, it is clear that $\mathcal{F}_0(I, \varphi)$ spans $T_I M$ at all points (use $F^1, F^2, \partial F^1/\partial\varphi, \partial F^2/\partial\varphi$), for any value of the parameters: we clearly see that full rank of \mathcal{F}_0 is not, here, sufficient for controllability, that occurs for some values of the parameters only.

Let us now enrich this academic example by making θ a component of the state rather than a fixed parameter, $\alpha \in [0, \pi/4]$ being still a constant parameter:

$$\begin{aligned} \begin{pmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{I}_3 \end{pmatrix} &= \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(u_1 \begin{pmatrix} \cos \theta \\ 0 \\ \sin \theta \end{pmatrix} + u_2 \begin{pmatrix} -\sin \theta \\ 0 \\ \cos \theta \end{pmatrix} \right) \\ \dot{\theta} &= u_3 \cos^2 2\theta, \\ \dot{\varphi} &= 1 \end{aligned} \tag{45}$$

$$u \in U = \{(u_1, u_2, u_3) \in \mathbb{R}^3, u_1 \in [0, 1], |u_2| \leq u_1 \tan \alpha, u_3 \in [-1, 1]\}.$$

$$(I_1, I_2, I_3, \theta, \varphi) \in M \times \mathbb{S}^1, M = \mathbb{R}^3 \times \left(-\frac{\pi}{4}, \frac{\pi}{4}\right).$$

The factor $\cos^2 2\theta$ in $\dot{\theta}$ prevents solutions from leaving the state space $\mathbb{R}^3 \times (-\frac{\pi}{4}, \frac{\pi}{4}) \times \mathbb{S}^1$ in finite time with admissible controls. If $\alpha \geq \pi/4$, Condition (27) holds everywhere and conclusions are the same as in the case $|\theta| < \alpha$ for system (43). If $\alpha < \pi/4$, Condition (27) holds in the region $R_1 = \{(I_1, I_2, I_3, \theta, \varphi), |\theta| < \alpha\}$ and fails in the regions $R_0 = \{(I_1, I_2, I_3, \theta, \varphi), \theta = \pm\alpha\}$ and $R_2 = \{(I_1, I_2, I_3, \theta, \varphi), |\theta| > \alpha\}$.

Theorem 13 does not apply globally, Theorem 19 yields local controllability in one period in region R_1 , Theorem 21 tells us that the system is not locally controllable around its periodic solutions in region R_2 , and we see by inspection that this local controllability does not hold either in R_0 . In fact, although local controllability does not hold in half of the space and Theorem 13 does not predict controllability, this system is indeed globally controllable if $\alpha < \pi/4$. Indeed, starting from some $(I_1^0, I_2^0, I_3^0, \theta^0, \varphi^0)$ to reach $(I_1^f, I_2^f, I_3^f, \theta^f, \varphi^f)$, one may always use the control u_3 to reach the region R_1 (for instance go to $(I_1^0, I_2^0, I_3^0, 0, \varphi^0)$), then use the controls u_1, u_2 to reach I_1^f, I_2^f, I_3^f (possible as in (43) with $\theta = 0$) with constant θ and then use again u_3 to steer θ from 0 to θ^f . In general, it is hard to tell in advance if a system where condition (26) holds in some regions and fails on some others will be globally controllable or not.

Example 24. Consider the following system of the form (5) with scalar control ($m = 1, \mathcal{M} = M \times \mathbb{S}^1, M = \mathbb{R}^3, d = 4$):

$$\begin{aligned} \begin{pmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{I}_3 \end{pmatrix} &= u \left(\cos \varphi \begin{pmatrix} 1 \\ 0 \\ -I_2/2 \end{pmatrix} + \sin \varphi \begin{pmatrix} 0 \\ 1 \\ I_1/2 \end{pmatrix} \right), \\ \dot{\varphi} &= 1, \end{aligned} \tag{46} \quad u \in U \subset \mathbb{R}.$$

The set U will be either $[-1, 1]$ or $[0, 1]$. The system also reads $\dot{x} = X^0(x) + uX^1(x)$, with $x = (I_1, I_2, I_3, \varphi)$ and the vector fields X^0 and X^1 defined by:

$$X^0 = \frac{\partial}{\partial \varphi}, \quad X^1 = \cos \varphi \left(\frac{\partial}{\partial I_1} - \frac{1}{2} I_2 \frac{\partial}{\partial I_3} \right) + \sin \varphi \left(\frac{\partial}{\partial I_2} + \frac{1}{2} I_1 \frac{\partial}{\partial I_3} \right). \tag{47}$$

With reference to the second paragraph of this Section 3, the vector fields $X^0, X^1, [X^0, X^1]$ and $[X^1, [X^0, X^1]]$ are linearly independent, hence the bracket generating

condition is satisfied (this implies global controllability from Theorem 9 if $U = [-1, 1]$), while the rank of \mathcal{F}_0 is only 2: it is spanned by X^1 and $[X^0, X^1]$ and vanishes on the differential form $\frac{1}{2}I_2 dI_1 - \frac{1}{2}I_1 dI_2 + dI_3$ and $d\varphi$. We are therefore in the case where our sufficient conditions cannot hold. Indeed, a simple computation from (19), (17) and (20) yields, $E^U(I, \varphi) = \check{E}^U(I) \times \{0\}$ with

$$\check{E}^U(I) = \{(v_1, v_2, v_3) \in T_I M = \mathbb{R}^3, \frac{1}{2}I_2 v_1 - \frac{1}{2}I_1 v_2 + v_3 = 0 \text{ and } v_1^2 + v_2^2 \leq 1\}, \quad (48)$$

whether U is $[-1, 1]$ or $[0, 1]$ (in the former case, it is ‘‘covered twice’’), so that cone $\check{E}^U(I)$ is a proper vector subspace of $T_I M$ and none of the sufficient conditions in Theorems 13, 19 or 15 hold true, while the polar cone of $\check{E}^U(I)$ is a line in $T_I^* M$, hence has an empty interior, and one may not either apply Theorem 20 or 21 to infer non-controllability. No result from the present paper applies here.

Let us however investigate local controllability in the spirit of the conclusions of Theorem 19, by ad-hoc techniques outside the scope of the results of the paper. We are going to prove the following three properties for system (46) (recall that the state x is (I, φ) , with $I \in \mathbb{R}^3$, $\pi(I, \varphi) = I$, and the period $T(x)$ is $T(I) = 2\pi$):

- (a) $\pi(\mathcal{A}_{T(I)}^{[-\varepsilon, \varepsilon]}(I, \varphi))$ is a neighborhood of I ,
- (b) $\pi(\mathcal{A}_{T(I)}^{[0, \varepsilon]}(I, \varphi))$ is not a neighborhood of I ,
- (c) $\pi(\mathcal{A}_{2T(I)}^{[0, \varepsilon]}(I, \varphi))$ is a neighborhood of I ,

for any ε , $0 < \varepsilon \leq 1$. Point (a) is in addition of the global controllability mentioned above when $U = [-1, 1]$, while points (b) and (c) concerning the case $U = [0, 1]$, more in the spirit of the paper, display an interesting behavior where local controllability is obtained avec two periods but not over one. Before establishing them, note that the general solution of (46) starting from $(I_1^0, I_2^0, I_3^0, \varphi^0)$ at time zero is given by $\varphi(t) = \varphi^0 + t$ and, with complex notations for (I_1, I_2) ,

$$\begin{aligned} I_1(t) + i I_2(t) &= I_1^0 + i I_2^0 + \int_0^t u(s) e^{i(\varphi^0 + s)} ds, \\ I_3(t) &= I_3^0 + \frac{1}{2} \iint_{0 \leq r \leq s \leq t} u(s) u(r) \sin(s - r) dr ds \\ &\quad + \frac{1}{2} \Im \left((I_1^0 - i I_2^0) (I_1(t) + i I_2(t)) \right), \end{aligned} \quad (49)$$

The very last term is zero if $(I_1(t), I_2(t)) = (I_1^0, I_2^0)$. For intuition behind equations, recall that $\dot{I}_3 = \frac{1}{2}(I_1 \dot{I}_2 - I_2 \dot{I}_1)$ infers that the variation $I_3(t_2) - I_3(t_1)$ is the area swept from the origin by the plane curve $t \mapsto (I_1(t), I_2(t))$ on the interval $[t_1, t_2]$ and φ is (if u is positive) the polar angle of the velocity vector on that curve. It turns out that the area swept from the origin by a closed curve generated by a non-negative control over a time interval of length at most 2π cannot be negative:

$$\left(u(\cdot) \geq 0 \text{ and } I_1(2\pi) = I_1(0) \text{ and } I_2(2\pi) = I_2(0) \right) \Rightarrow I_3(2\pi) \geq I_3(0). \quad (50)$$

To prove this, first decompose the integral in the expression for $I_3(2\pi)$ in (49):

$$\iint_{0 \leq r \leq s \leq 2\pi} = \iint_{0 \leq r \leq s \leq \pi} + \iint_{\pi \leq r \leq s \leq 2\pi} + \iint_{(s,r) \in [\pi, 2\pi] \times [0, \pi]}. \quad (51)$$

From (49), $(I_1(2\pi), I_2(2\pi)) = (I_1^0, I_2^0)$ implies $\int_0^{2\pi} u(s) \sin(s - r) ds = 0$ for any r , hence $\iint_{(s,r) \in [0, 2\pi] \times [0, \pi]} u(s) u(r) \sin(s - r) dr ds = 0$, implying that the last term in (51) is equal to $-\iint_{(s,r) \in [0, \pi] \times [0, \pi]} \sin(s - r) u(s) u(r) ds dr$, which is zero by symmetry around the axis $r = s$; hence the integral giving $I_3(2\pi)$ in (49) reduces to the region $\{0 \leq r \leq s \leq \pi\} \cup \{\pi \leq r \leq s \leq 2\pi\}$, where $u(s) u(r) \sin(s - r)$ is non negative; this proves (50). Point (b) obviously follows.

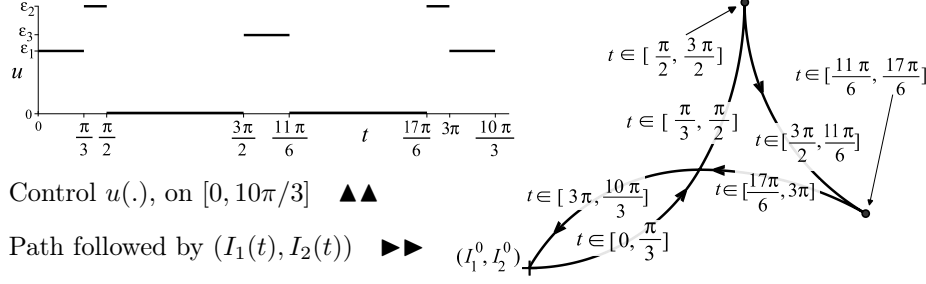


FIGURE 2. *Left*: the control used for Point (a) in Example 24; the constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are tuned according to (52). *Right*: the closed curve $t \mapsto (I_1(t), I_2(t))$ on the interval $[0, \frac{4\pi}{3}]$, assuming $\varphi(0) = 0$ (if not, rotate the curve around (I_1^0, I_2^0) by an angle $\varphi(0)$).

Let us now turn to Point (c). Consider the piecewise constant control depicted in Figure 2 (left) for t in $[0, \frac{10\pi}{3}]$, continued by zero on $[\frac{10\pi}{3}, 4\pi]$. The constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are chosen satisfying

$$\varepsilon_1 = \sqrt{\frac{(2\pi + 3)\sqrt{3} - 5\pi}{2\pi - 3\sqrt{3}}} \varepsilon_2, \quad \varepsilon_3 = (\sqrt{3} - 1) \varepsilon_2, \quad 0 < \varepsilon_1 < \varepsilon_3 < \varepsilon_2 < \varepsilon. \quad (52)$$

The second relation is needed to have $(I_1(\frac{10\pi}{3}), I_2(\frac{10\pi}{3})) = (I_1^0, I_2^0)$, *i.e.* a closed curve in the I_1, I_2 plane; the first relation implies that $I_3(\frac{10\pi}{3}) = I_3^0$ (or, for the closed planar curve displayed in Figure 2, that resembles a goldfish, that the area of the “body”, counted positively because run counter clockwise, is equal to the area of the “tail”, counted negatively because run clockwise). On the time interval $[0, \frac{\pi}{3}]$, the control is constant equal to ε_1 , that belongs to the topological interior of $[0, \varepsilon]$; the linear approximation on that interval hence reads $\delta \dot{I} = A(t)\delta I + B(t)\delta u$, with

$$A(t) = \varepsilon_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\sin \varphi(t) & \cos \varphi(t) & 0 \end{pmatrix} \quad \text{and} \quad B(t) = \varepsilon_1 \begin{pmatrix} \cos \varphi(t) \\ \sin \varphi(t) \\ I_1(t) \sin \varphi(t) - I_2(t) \cos \varphi(t) \end{pmatrix}.$$

The vectors $B(t)$, $(\frac{d}{dt} - A(t))B(t)$, $(\frac{d}{dt} - A(t))^2 B(t)$ are always a basis of \mathbb{R}^3 , as seen in the following formulas, where the t argument is omitted in the right-hand sides (φ should be $\varphi(t)$, or $\varphi^0 + t$):

$$\left(\frac{d}{dt} - A(t)\right)B(t) = \varepsilon_1^2 \begin{pmatrix} -\sin \varphi \\ \cos \varphi \\ I_1 \cos \varphi + I_2 \sin \varphi \end{pmatrix}, \quad \left(\frac{d}{dt} - A(t)\right)^2 B(t) = -\varepsilon_1^2 B(t) + \begin{pmatrix} 0 \\ 0 \\ 2\varepsilon_1^3 \end{pmatrix}.$$

Controllability of the linear approximation follows, according to [19] (see also our remarks after Equation (58)). This implies that $\pi(\mathcal{A}_{\pi/3}^{[0, \varepsilon]}(I_1^0, I_2^0, I_3^0, \varphi^0))$ is a neighborhood of $(I_1(\frac{\pi}{3}), I_2(\frac{\pi}{3}), I_3(\frac{\pi}{3}))$ according to the classical linear test for controllability (classical implicit function theorem applied to the end-point mapping) because the control set $U = [0, \varepsilon]$ is a neighborhood of the reference control around which linearization is performed. This implies Point (c) because $\mathcal{A}_{2T(I)}^{[0, \varepsilon]}(I^0, \varphi^0)$ contains the image of $\mathcal{A}_{\pi/3}^{[0, \varepsilon]}(I^0, \varphi^0)$ by the time-varying flow corresponding to the reference control from Figure 2 on the time-interval $[\pi/3, 4\pi]$.

Point (a) is easier than Point (c), negative controls being allowed: skip the two intervals or length π on which $u \equiv 0$ in Figure 2, left, and replace ε_3 with $-\varepsilon_3$; the obtained control on $[0, 4\pi/3]$ yields the same curve on Figure 2, right, without the stops at each cusp, but with the same linearized controllability.

APPENDIX A. CONTROLLABILITY OF TIME-VARYING
 LINEAR SYSTEMS WITH CONSTRAINED CONTROL

Consider a time-varying linear control system, with constraints on the control:

$$\dot{z} = A(t)z + B(t)v, \quad v \in V, \quad (53)$$

where the state z belongs to \mathbb{R}^d , V is a compact subset of \mathbb{R}^m , and $t \mapsto A(t)$ and $t \mapsto B(t)$ are smooth maps $\mathbb{R} \rightarrow \mathbb{R}^{d \times d}$ and $\mathbb{R} \rightarrow \mathbb{R}^{d \times m}$ respectively. This appendix is devoted to giving conditions for

$$\mathbf{A}_{t_0, t_f}^V(z_0) = \{z(t_f), \text{ with } t \mapsto (z(t), v(t)) \text{ solution of (53), } z(t_0) = z_0\} \quad (54)$$

to be a neighborhood of the origin. It is known that $\mathbf{A}_{t_0, t_f}^V(z_0)$ is convex, at least if V is compact, whether V is convex or not, see e.g. [14, Chapter 2, Theorem 1A (appendix), p.164].

It is customary to define the transition matrix (of $A(\cdot)$) as the map $(t_1, t_2) \mapsto \phi(t_2, t_1) \in \mathbb{R}^{d \times d}$ such that, for any $t_0 \in \mathbb{R}$ and $z_0 \in \mathbb{R}^d$, $t \mapsto z(t) = \phi(t, t_0)z_0$ is the solution of $\dot{z} = A(t)z$, $z(t_0) = z_0$; it satisfies

$$\frac{\partial \phi}{\partial t_2}(t_2, t_1) = A(t_2) \phi(t_2, t_1), \quad \frac{\partial \phi}{\partial t_1}(t_2, t_1) = -\phi(t_2, t_1) A(t_1), \quad \phi(t, t) = I. \quad (55)$$

The change of variables $\zeta = \phi(t_0, t)z$ (for any t_0) “kills” the term Az in (53), yielding another formulation of system (53):

$$\begin{aligned} \dot{\zeta}(t) &= \phi(t_0, t)B(t)v(t), \\ z(t) &= \phi(t, t_0)\zeta(t), \end{aligned} \quad (56)$$

usually used to derive the general formula for the unique $z(\cdot)$, solution of (53) associated to a prescribed control $v(\cdot)$ and satisfying $z(t_0) = z_0$:

$$z(t) = \phi(t, t_0)z_0 + \int_{t_0}^t \phi(t, \tau)B(\tau)v(\tau)d\tau. \quad (57)$$

If the convex hull of V is a neighborhood of the origin (in particular if $V = \mathbb{R}^m$), one has the following characterization, after [12, Section 9.2], [22], [19]: $\mathbf{A}_{t_0, t}^V(0)$ is a neighborhood of the origin if and only if, for $p \in (\mathbb{R}^d)^*$ (a line vector),

$$\left(p \phi(t_0, s)B(s) = 0 \text{ for all } s \text{ in } [t_0, t_f] \right) \Rightarrow p = 0. \quad (58)$$

This property is stated in [12] as “the rows of $s \mapsto \phi(t_0, s)B(s)$ are linearly independent on the interval $[t_0, t_f]$ ”, and is equivalent to

$$\text{Span} \{ \phi(t_0, s)B(s)v, s \in [t_0, t_f], v \in V \} = \mathbb{R}^d. \quad (59)$$

The case where the convex hull of V is *not* a neighborhood of the origin, the one of interest here, is treated in [18], and sketched in [14, Section 2.2]. Let us state the precise results needed in Section 2.2 (it is difficult to refer to [18], that characterizes $\cup_{t \in (-\infty, t_f]} \mathbf{A}_{t, t_f}^V(0)$ rather than $\mathbf{A}_{t_0, t_f}^V(0)$ for a fixed t_0).

Proposition 25. *Assume that V is a compact subset of \mathbb{R}^m containing zero, and that t_0, t_f are two real numbers. The accessible set $\mathbf{A}_{t_0, t_f}^V(0)$ is a neighborhood of the origin if and only if*

$$\text{cone} \{ \phi(t_0, s)B(s)v, s \in [t_0, t_f], v \in V \} = \mathbb{R}^d. \quad (60)$$

We also need the following characterization of when some projection of the accessible set is a neighborhood of the origin in the projection of the state space (controllability with respect to a projection, see Definition 4, part II).

Proposition 26. *Assume that V is a compact subset of \mathbb{R}^m containing zero, and that t_0, t_f are two real numbers. Let n be a positive integer no larger than d , and P a surjective linear map $\mathbb{R}^d \rightarrow \mathbb{R}^n$ (or a $n \times d$ matrix with rank n). The set $PA_{t_0, t_f}^V(0)$ (image of $A_{t_0, t_f}^V(0)$ by P) is a neighborhood of the origin in \mathbb{R}^n if and only if*

$$\text{cone}\{P\phi(t_f, s)B(s)v, s \in [t_0, t_f], v \in V\} = \mathbb{R}^n. \quad (61)$$

Note that, by the separating hyperplane theorem, (60) and (61) are equivalent to

$$(p\phi(t_0, s)B(s)v \geq 0 \text{ for all } v \in V, s \in [t_0, t_f]) \Rightarrow p = 0 \quad (62)$$

$$\text{and } (pP\phi(t_f, s)B(s)v \geq 0 \text{ for all } v \in V, s \in [t_0, t_f]) \Rightarrow p = 0 \quad (63)$$

respectively, $p \in (\mathbb{R}^d)^*$, $p \in (\mathbb{R}^n)^*$ in (62), (63). Conditions are given under this form in the literature, rather than (60) or (61).

Proposition 25 is a particular case of Proposition 26: take $M = \mathcal{M}$, $n = d$, and $P = I$ (identity map), and factor out the linear isomorphism $\phi(t_0, t_f)$ in (61) to “replace t_f with t_0 ”. When $n < d$, the controllability property characterized in Proposition 26 is strictly weaker than the one in Proposition 25. Let us end this appendix with a short proof of Proposition 26.

Proof of Proposition 26. Let us prove necessity first. If (61) does not hold, there exists a nonzero p such that $pP\phi(t_f, s)B(s)v \geq 0$ for any $v \in V$ and $s \in [t_0, t_f]$, and formula (57), with $z_0 = 0$, obviously implies $pPz(t_f) \geq 0$ for any solution, implying that $PA_{t_0, t_f}^V(0)$ is contained in the half-space $\{y \in \mathbb{R}^n, py \geq 0\}$, hence is not a neighborhood of 0.

To prove the converse, assume that $PA_{t_0, t_f}^V(0)$ is *not* a neighborhood of 0 in \mathbb{R}^n . Since $PA_{t_0, t_f}^V(0)$ is convex (this was recalled at the beginning of this appendix, see [14, Chapter 2, Theorem 1A (appendix), p. 164]), it is contained in the half space $\{y \in \mathbb{R}^n, py \geq 0\}$, for some nonzero $p \in (\mathbb{R}^n)^* \setminus \{0\}$, *i.e.* any solution $(z(\cdot), v(\cdot))$ of (53) with $z(t_0) = 0$ and $v(\cdot)$ valued in V satisfies $pPz(t_f) \geq 0$. It is not too difficult to see that, for the same p , one must then have $pP\phi(t_f, s)B(s)v \geq 0$ for all v in V and all s in $[t_0, t_f]$; this defeats (63), hence (61). \square

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