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ON THE CONTROLLABILITY OF NONLINEAR SYSTEMS WITH A PERIODIC DRIFT

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INTRODUCTION

This paper is devoted to the study of the controllability properties of control-affine systems,

$$\dot{x} = X^0(x) + \sum_{k=1}^m u_k X^k(x), \quad u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m, \quad (1)$$

where X^0, \dots, X^m are smooth vector fields on the ambient manifold \mathcal{M} , and where we impose specific restrictions on the drift X^0 and on the set U defining the constraints on the control. Such a system is said controllable if, for any two points in \mathcal{M} , there exists a measurable control valued in U that steers the first point to the second within some finite time.

Devising conditions on the vector fields (and the set U) for controllability is an old problem. A necessary condition is as follows: if the system is controllable, then it is bracket generating; this is a consequence of the orbit theorem [12]. While in general this condition does not ensure controllability, it becomes sufficient for systems without drift ($X^0 = 0$), see [9], though not for any set U : the control set (its convex hull, actually) has to be a neighbourhood of the origin. A more general condition than having no drift is to require that X^0 be Poisson stable; this assumption covers the important case when any solution of $\dot{x} = X^0(x)$ is periodic. The result, first stated in [2] (see also the textbooks [1, 5]), asserts that the system is controllable provided that it is bracket generating, that X^0 is positively Poisson stable, and that the convex hull of U is a neighbourhood of the origin. We focus here on the case when all solutions of $\dot{x} = X^0(x)$ are periodic, and U (assumed to be convex, without loss of generality, see explanations further) is *not* a neighborhood of zero, while still containing zero in its boundary, otherwise the property of the drift would be irrelevant.

In Section 1, we cast precisely the controllability problem we want to look at and recall the main definitions and results on the subject. Since our study is motivated by slow-fast control systems with one fast angle that are particular subcases of system (1), we devote a few lines to this particular situation. In Section 2, we give a result of (global) controllability in arbitrary time for systems with a periodic drift. To our knowledge this result is original and resorts to an *ad hoc* assumption on the transportation of the controlled vector fields by the drift. This assumption is very natural in the framework of systems with one fast angular variable, and allows to prove controllability with respect to the slow ones. Section 3, completed by a short appendix on time-varying linear controlled systems, addresses local controllability properties of system (1) along one particular closed trajectory of the drift and eventually provides an alternative local-to-global proof of Section 2 results.

1. STATEMENT OF THE PROBLEM

1.1. **Systems.** Our primary interest is for controllability properties of systems with one fast angular variable. The state manifold of such a system is a product $M \times \mathbb{S}^1$, with M a real analytic manifold of dimension $n > 0$ (n “slow variables”) and $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ the circle, parameterized by a “fast” angle. Denoting the state by $(I, \varphi) \in M \times \mathbb{S}^1$, the dynamics read:

$$\begin{aligned} \dot{I} &= \varepsilon \sum_{k=1}^m u_k F^k(I, \varphi), & u &= (u_1, \dots, u_m) \in U, \\ \dot{\varphi} &= \omega(I), \end{aligned} \tag{2}$$

where u_1, \dots, u_m is the control, U is a convex subset of \mathbb{R}^m , F^1, \dots, F^m are notations such that $(I, \varphi) \mapsto (F^k(I, \varphi), 0)$ is a real analytic vector field on $M \times \mathbb{S}^1$ whose projection on the second factor of the product is zero, and ω can be understood as a real analytic map $M \rightarrow \mathbb{R}$, or as defining a real analytic vector field $(I, \varphi) \mapsto (0, \omega(I))$ whose projection on the first factor is zero. In coordinates, these are real analytic maps respectively $O \times (\mathbb{R}/2\pi\mathbb{Z}) \rightarrow \mathbb{R}^n$ and $O \rightarrow \mathbb{R}$, with $O \subset \mathbb{R}^n$ a coordinate domain for the manifold M . The constant $\varepsilon > 0$ is not necessarily small, but it is helpful to call I a slow variable, and it is the case in space mechanics application (see, *e.g.*, [4] for an application to solar sailing). We also assume that the fast variable φ does not rest, *i.e.* that there is a positive constant ω_{\min} such that

$$\omega(I) > \omega_{\min} > 0 \quad \text{for all } I \text{ in } M. \tag{3}$$

It will be convenient to go back and forth from the specific form (2) to the more general affine control system (1) where X^0, \dots, X^m are $m + 1$ real analytic vector fields on a real analytic manifold \mathcal{M} of dimension d and the drift has the following property:

$$\text{for any } x \text{ in } \mathcal{M}, t \mapsto \exp(tX^0)(x) \text{ is periodic with minimal period } T(x). \tag{4}$$

This defines a function $T : \mathcal{M} \rightarrow (0, +\infty)$. It is clear that (2) is a special occurrence of (1) with $d = n + 1$, $\mathcal{M} = M \times \mathbb{S}^1$,

$$X^0 = \omega \frac{\partial}{\partial \varphi}, \quad X^k = \varepsilon F^k \frac{\partial}{\partial I}, \quad k \geq 1, \tag{5}$$

with ω and F^k as in (2). There is a slight abuse of notations but the reader will understand how (2) matches (1)-(5). The above X^0 satisfies assumption (4) with $T(x) = T(I, \varphi) = T(I) = 2\pi/\omega(I)$. We also make the following assumptions on the set U all over the paper:

$$U \text{ is a convex subset of } \mathbb{R}^m, \quad 0 \in U, \quad \text{and} \quad \text{Span } U = \mathbb{R}^m. \tag{6}$$

The last two assumptions in (6) above are extremely natural in view of what precedes: on the one hand, assumption (4) is about the behavior of the system for a zero control, that would be irrelevant if that control was not in the control set U or its convex hull, and on the other hand if $\text{Span } U$ was a subspace of \mathbb{R}^m of dimension $m' < m$, one could pick m' new control vector fields, constant linear combinations of X^1, \dots, X^m and use them as control vector fields in a system with m' controls satisfying the assumption. Assuming that U is convex is not restrictive when studying controllability in arbitrary time (*i.e.* the kind of controllability we introduced above, and study in Section 2) because our assumptions (see Theorems 8 and 9 in the next section) will imply that the family $\{X^0, X^1, \dots, X^m\}$ is bracket generating (see Proposition 7); then, Krener theorem (see [1, Theorem 8.1]) implies that the controllability properties of the original system are unchanged when one replaces

the control set by its convexification; when studying controllability on prescribed time (Section 3), convexity is an assumption that would have to be relaxed.

Remark 1. We saw that (2) is a particular case of a system (1) satisfying Assumption (4). Conversely, if (1) satisfies (4), taking a small transverse section O of a given periodic solution with nonzero period, the tubular neighbourhood of that periodic solution made of the union of periodic solutions through that section is diffeomorphic to $O \times \mathbb{S}^1$ and, on this product, system (1) can be written as

$$\dot{I} = \varepsilon \sum_{k=1}^m u_k F^k(I, \varphi), \quad \dot{\varphi} = \omega(I) + \varepsilon \sum_{k=1}^m u_k f^k(I, \varphi), \quad (7)$$

for some ω , F^k and f^k ; this is a more general form than (2), we prefer to omit the f^k -terms in (2) for simplicity, but the general results on systems of the type (1)-(4) can easily be translated to systems (7) with nonzero f^k terms. Of course, global decomposition as a product does not occur in general for a system (1) under assumption (4): the periodic orbits need not have a global section.

1.2. Definitions. A *solution* of (1) is a map $t \mapsto (u(t), x(t))$, defined on some interval, valued in $U \times \mathcal{M}$, where $u(\cdot)$ is integrable and $x(\cdot)$ absolutely continuous, and that satisfies equation (1). For such a solution on an interval $[0, t_f]$, one says that “the control $u(\cdot)$ steers $x(0)$ to $x(t_f)$ in time t_f ; indeed the solution is unique given $u(\cdot)$ and $x(0)$.”

The *reachable set from x_0 in time $t_f \geq 0$* is the set of points in \mathcal{M} that can be reached in forward time for x_0 for some control:

$$\mathcal{A}_{t_f}^U(x_0) = \{x(t_f), \text{ with } t \mapsto (x(t), u(t)) \text{ solution of (1), } x(0) = x_0\}. \quad (8)$$

It is of course implied that the interval of definition of the solution contains $[0, t_f]$. We keep the set U as a superscript to stress the constraint on the control. Obviously, $\mathcal{A}_{t_f}^V(x_0) \subset \mathcal{A}_{t_f}^U(x_0)$ if $V \subset U$. From this accessible set in a precise time, one defines the accessible set in arbitrary time:

$$\mathcal{A}^U(x_0) = \bigcup_{0 \leq t} \mathcal{A}_t^U(x_0). \quad (9)$$

It is customary to call $\mathcal{A}^U(x_0)$ simply “the accessible set from x_0 ” and to define:

Definition 1. System (1) with the constraint $u \in U$ on the control is completely controllable on \mathcal{M} if $\mathcal{A}^U(x) = \mathcal{M}$ for any x in \mathcal{M} .

Since System (2) is a special occurrence of (1) according to (5), one may obviously define $\mathcal{A}_{t_f}^U(I, \varphi)$ and $\mathcal{A}^U(I, \varphi)$ for System (2), for any (I, φ) in $M \times \mathbb{S}^1$. Since the zero control moves φ while leaving I unchanged, one readily sees that $\mathcal{A}^U(I, \varphi)$ does not depend on φ and is necessarily a product:

$$\mathcal{A}^U(I, \varphi) = \mathbf{A}^U(I) \times \mathbb{S}^1, \quad \text{with } \mathbf{A}^U(I) \text{ a subset of } M. \quad (10)$$

This may be understood as “System (2) is completely controllable on $M \times \mathbb{S}^1$ if and only if its slow variables are completely controllable on M ”. Let us call $\mathbf{A}^U(I)$ the accessible set from I with respect to the slow variables; one may define the one in precise time t_f :

$$\mathbf{A}_{t_f}^U(I_0, \varphi_0) = \{I(t_f), t \mapsto (I(t), \varphi(t), u(t)) \text{ solution of (2),} \\ \text{satisfying } I(0) = I_0, \varphi(0) = \varphi_0\}. \quad (11)$$

It is clear that the set $\mathbf{A}^U(I)$ defined by (10) satisfies, for any φ in \mathbb{S}^1 ,

$$\mathbf{A}^U(I) = \bigcup_{t \geq 0} \mathbf{A}_t^U(I, \varphi). \quad (12)$$

Clearly, $\mathbf{A}_{t_f}^U(I, \varphi)$ fails, in general, to have the product structure displayed in (10).

1.3. State of the art, Lie brackets. Let $V(\mathcal{M})$ be the set of real analytic vector fields on \mathcal{M} . The Lie bracket (we refer the reader to any advanced calculus or differential geometry textbook for its definition) of two elements of $V(\mathcal{M})$ also belongs to $V(\mathcal{M})$, this makes $V(\mathcal{M})$ a Lie algebra over the field \mathbb{R} . A family of analytic vector fields is a subset $\mathcal{F} \subset V(\mathcal{M})$. We will denote by $\mathcal{F}(x)$ the subset of $T_x\mathcal{M}$ made of the values at x of vector fields in \mathcal{F} and by $\text{Span } \mathcal{F}(x)$ the vector subspace of $T_x\mathcal{M}$ spanned by $\mathcal{F}(x)$: For a family of vector field \mathcal{F} on a manifold \mathcal{M} and x a point in \mathcal{M} , we denote by $\mathcal{F}(x)$ the vector subspace of $T_x\mathcal{M}$ spanned by the values at x of the vector fields belonging to \mathcal{F} :

$$\mathcal{F}(x) = \{X(x), X \in \mathcal{F}\} \subset T_x\mathcal{M}, \quad \text{Span } \mathcal{F}(x) = \text{Vect}_{\mathbb{R}}(\mathcal{F}(x)). \quad (13)$$

Accessibility. The term “accessible” is sometimes used to characterize systems whose accessible sets have a nonempty interior. Characterizations are well known; since they will be used in the sequel, let us briefly recall them. For this purpose, let us define the following families of vector fields, made of repeated Lie brackets of the vector fields X^0, X^1, \dots, X^m defining system (1):

$$\mathcal{C} = \left\{ [X^{i_k}, [X^{i_{k-1}}, [\dots [X^{i_2}, X^{i_1}] \dots]]] \right\}_{\substack{k \in \mathbb{N} \setminus \{0\} \\ (i_1, \dots, i_k) \in \{0, \dots, m\}^k}}, \quad (14)$$

$$\mathcal{C}_0 = \left\{ [X^{i_k}, [X^{i_{k-1}}, [\dots [X^{i_2}, X^{i_1}] \dots]]] \right\}_{\substack{k \in \mathbb{N} \setminus \{0\} \\ i_1 \in \{1, \dots, m\} \\ (i_2, \dots, i_k) \in \{0, \dots, m\}^{k-1}}}, \quad (15)$$

$$\mathcal{F}_0 = \{\text{ad}_{X^0}^j X^k, k \in \{1, \dots, m\}, j \in \mathbb{N}\}. \quad (16)$$

Definition 2 (Accessibility rank condition). Consider system (1) and assume that U satisfies condition (6). Let x be a point in \mathcal{M} .

- (a) System (1) satisfies the *accessibility rank condition* at x if and only if $\mathcal{C}(x) = T_x\mathcal{M}$.
One also says that “ $\{X^0, \dots, X^m\}$ is *bracket generating*” in this case.
- (b) System (1) satisfies the *strong accessibility rank condition* at x if and only if $\mathcal{C}_0(x) = T_x\mathcal{M}$.
- (c) System (1) satisfies the *ad-condition* at x if and only if $\mathcal{F}_0(x) = T_x\mathcal{M}$.

We simply say that these conditions are satisfied for the system if they hold at all x in \mathcal{M} .

Proposition 3 (Accessibility condition, Jurdjevic-Sussmann [13]). *Consider system (1) and assume that U satisfies condition (6). The topological interior of $\cup_{0 \leq t \leq t_f} \mathcal{A}_t^U(x)$ is nonempty for any x in \mathcal{M} and any positive t_f if and only if the accessibility condition is satisfied at any x in \mathcal{M} . The same holds for negative t_f replacing $0 \leq t \leq t_f$ with $t_f \leq t \leq 0$. The topological interior of $\mathcal{A}_{t_f}^U(x)$ is nonempty for any x in \mathcal{M} and any nonzero t_f if and only if the strong accessibility condition is satisfied at any x in \mathcal{M} .*

The original proofs can be found in [13]. These conditions are usually introduced as follows, associating to System (1) the following family of vector fields (also done in [1, chapter 8] or [5, chapter 3]):

$$\mathcal{G} = \{X^0 + u_1 X^1 + \dots + u_m X^m, (u_1, \dots, u_m) \in U\} \quad (17)$$

made of all the vector fields obtained by fixing, in (1), the control to a constant value that belongs to U , defining $\text{Lie } \mathcal{G}$ as the smallest Lie sub-algebra of $V(\mathcal{M})$ containing \mathcal{G} , the accessibility rank condition reads $\text{Lie } \mathcal{G}(x) = T_x\mathcal{M}$, and the strong accessibility rank condition reads $\mathcal{I}(x) = T_x\mathcal{M}$, where \mathcal{I} is the Lie ideal of $\text{Lie } \mathcal{G}$ generated by the family $\{X - Y, X \in \mathcal{G}, Y \in \mathcal{G}\}$. One easily sees that $\text{Lie } \mathcal{G}(x) = \mathcal{C}(x)$ and $\mathcal{I}(x) = \mathcal{C}_0(x)$ because U satisfies (6).

Controllability when $\text{conv } U$ is a neighbourhood of the origin. Let us review classical results based on Lie brackets, or more precisely on the *vector subspace* of the tangent space at all (or some) points spanned by the value at this point of vector fields obtained as Lie brackets of X^0, \dots, X^m . A well-known necessary condition for controllability is the following property.

Definition 4. The family of vector fields $\{X^0, \dots, X^m\}$ is said to be *bracket generating*¹ at $x \in \mathcal{M}$ if and only there are some Lie brackets, of any order, of these vector field whose value at x span the whole space $T_x\mathcal{M}$. It is said to be *bracket generating* on \mathcal{M} if the property is true for all x in \mathcal{M}

It is well known (for example according to the so-called ‘‘orbit theorem’’ due to Sussmann [12]) that this is necessary for controllability; we will always assume that this is satisfied, explicitly or through stronger assumptions. Accessibility (Theorem 3) is clearly weaker than controllability. One case where the assumptions of Theorem 3 also imply controllability is the one of driftless systems, namely systems (1) where X^0 is identically zero: in [9], it is proved that system (1), with X^0 assumed identically zero and U assumed to be a neighbourhood of the origin, is controllable if and only if $\{X^1, \dots, X^m\}$ is bracket generating. This is however very far from the situation of (2), where the drift is assumed to be non-vanishing. Another property of the drift that allows one to strengthen the conclusion of Theorem 3 into controllability is the Poisson stability (see, *e.g.*, [5], and [1] for a slightly different definition).

Definition 5. Let X^0 be a complete vector field on \mathcal{M} . A point $x \in \mathcal{M}$ is said to be (positively) *Poisson stable* for X^0 if there exists a sequence of positive times $(t_n)_n \rightarrow \infty$ such that $\exp(t_n X^0)(x) \rightarrow x$ when $n \rightarrow \infty$. If there is a dense subset of such points, the vector field itself is said to be Poisson stable.

It turns out that many physical dynamical systems, conservative in a loose sense, have this property; this makes the following result quite useful. (Here we have assumed that the control set is convex, one has otherwise to replace U by its convex hull in assumption (iii) below.)

Theorem 6 (Bonnard, 1981, [2]). *System (1) is controllable if*

- (i) *the vector field X^0 is Poisson stable,*
- (ii) *the family $\{X^0, X^1, \dots, X^m\}$ is bracket generating, and*
- (iii) *the convex control set U is a neighbourhood of 0 in \mathbb{R}^m .*

It is stated precisely in this form in the original reference (that mentions techniques due to [6]), and in the more recent textbook [5] (Chapter 4, theorem 5). It has been rather widely used, for instance to prove controllability prior to solving an optimal control problem, see *e.g.* [3]. The assumption that each solution is periodic is much more restrictive than Poisson stability, hence (i) is satisfied under our running assumptions. Condition (ii) is easy to check, it is indeed necessary for controllability in the analytic case. We precisely want to investigate the case when (iii) fails. Since (i) describes a behavior when the control is set to zero, it is irrelevant if U does not contain the origin, we study the case where it is on the boundary. Typically, U is included in a non trivial convex cone, strictly convex at the origin. In many other control systems, positivity constraints come naturally... here, we assume that 0 belongs to the border of U , and this defeats controllability in general under assumptions (i) and (ii) as evidenced by the following academic example.

¹In some references, one speaks of ‘‘satisfying the *Lie Algebra Rank Condition* (LARC)’’ rather than ‘‘being bracket generating’’.

Example 2. Take for (2): $\dot{I} = \varepsilon u$, $\dot{\varphi} = 1$, with $M = \mathbb{R}$, and take $U = [0, 1]$. Conditions (i) and (ii) are satisfied, (iii) is not, and the system is not controllable because, for instance, $I(t) \geq 0$ for $t > 0$ if $I(0) = 0$.

1.4. Further constructions, not based on Lie brackets. When U is not a neighbourhood of the origin, conditions based only on Lie brackets or even characterizations through linear subspaces of the tangent space cannot be relevant by themselves. We now present some constructions that, instead of testing whether some vector subspace is the total tangent space, end up testing whether some *cones* are the whole tangent space. For system (2), where we are interested in controllability of the slow variable I only, we define the following subsets of $T_I M$, taking advantage of the decomposition of the state-space as a product:

$$\mathbf{E}_\tau^U(I, \varphi) = \left\{ \sum_{k=1}^m u_k F_k(I, \varphi + \tau), u \in U \right\} \subset T_I M \quad (18)$$

and the union of these over the fast variable:

$$\begin{aligned} \mathbf{E}^U(I) &= \bigcup_{\tau \in [0, 2\pi]} \mathbf{E}_\tau^U(I, \varphi) \\ &= \left\{ \sum_{k=1}^m u_k F_k(I, \varphi), (u_1, \dots, u_m) \in U, \varphi \in \mathbb{S}^1 \right\} \subset T_I M. \end{aligned} \quad (19)$$

Note that while $\mathbf{E}_\tau^U(I, \varphi)$ is a convex subset of the vector subspace spanned by the value of the vector fields F_1, \dots, F_m at $(I, \varphi + \tau)$, $\mathbf{E}^U(I)$ is just a union of such sets that does not inherit any particular structure. In the sequel we use its convex hull $\text{conv}(\mathbf{E}^U(I))$.

Let us introduce similar constructions for the “general” system (1). Definitions do not benefit from a product structure or a very simple flow for the drift vector field, they are more abstract, but possibly more enlightening. The vector fields X^0, \dots, X^m and the bounded convex subset U of \mathbb{R}^m are these associated to (1).

$$E_\tau^U(x) = \left\{ \sum_{k=1}^m u_k (\exp(-\tau X^0)_* X^k)(x), u \in U \right\} \subset T_x \mathcal{M} \quad (20)$$

and, with $T(x)$ coming from (4),

$$\begin{aligned} E^U(x) &= \bigcup_{\tau \in [0, T(x)]} E_\tau^U(x) \\ &= \left\{ \sum_{k=1}^m u_k (\exp(-\tau X^0)_* X^k)(x), u \in U, \tau \in [0, T(x)] \right\} \subset T_x \mathcal{M} \end{aligned} \quad (21)$$

Because of periodicity, one may replace “ $\tau \in [0, T(x)]$ ” with “ $\tau \in \mathbb{R}$ ” without changing $E^U(x)$. Applying definitions (20)-(21) to (2) according to (5) does not yield $E_\tau^U(I, \varphi) = \mathbf{E}_{2\pi\tau/\omega(I)}^U(I, \varphi)$ or $E^U(I, \varphi) = \mathbf{E}^U(I)$ (the rescaling of τ is due to the fact that the subscript of E is a time modulo $T(x)$ while the subscript of \mathbf{E} is an angle modulo 2π), but rather $\mathbf{E}_{2\pi\tau/\omega(I)}^U(I, \varphi) \subset E_\tau^U(I, \varphi)$ and $\mathbf{E}^U(I) \subset E^U(I, \varphi)$, the left-hand side being the projections of the right-hand side on the first factor of the product $M \times \mathbb{S}^1$:

$$\mathbf{E}_\tau^U(I, \varphi) = \pi_1'(I, \varphi)(E_{2\pi\tau/\omega(I)}^U(I, \varphi)), \quad \mathbf{E}^U(I, \varphi) = \pi_1'(I, \varphi)(E^U(I, \varphi)). \quad (22)$$

In fact $\mathbf{E}^U(I, \varphi)$ does not depend on φ , and $E^U(\cdot)$, as well as the function $T(\cdot)$, is invariant under the flow of X^0 . Note also that the vector subspace of $T_x \mathcal{M}$ spanned by the vectors in $E_\tau^U(x)$, $\tau > 0$, can be characterized in terms of Lie brackets if all vector fields are real analytic, see Proposition 7 in the beginning of the next section.

2. CONTROLLABILITY IN ARBITRARY TIME

2.1. Preliminaries. Consider $m + 1$ vector fields X^0, \dots, X^m , at least infinitely differentiable, on a real analytic manifold \mathcal{M} (for instance $\mathcal{M} = \mathbb{R}^d$). Assume that X^0 is complete (i.e. $\exp(tX^0)(x)$ is well defined for all $x \in \mathcal{M}$ and $t \in \mathbb{R}$), and define on the one hand the family \mathcal{F}_0 of vector fields obtained by bracketing X^1, \dots, X^m with X^0 iteratively, see (16), and on the other hand families of vector fields obtained by transporting X^1, \dots, X^m along the flow of X^0 for some time:

$$\mathcal{F}_\alpha = \{\exp(-tX^0)_* X^k, k \in \{1, \dots, m\}, t \in \mathbb{R}, -\alpha < t < \alpha, \alpha > 0, \quad (23)$$

$$\mathcal{F}_\infty = \{\exp(-tX^0)_* X^k, k \in \{1, \dots, m\}, t \in \mathbb{R}\}. \quad (24)$$

Note that (16) does *not* match (23) with $\alpha = 0$ (while (24) does match (23) with $\alpha = +\infty$). It is obvious that, $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ if $0 < \alpha \leq \beta \leq \infty$. The relation between \mathcal{F}_0 and \mathcal{F}_α (with $\alpha > 0$) is less obvious and is given by the following Proposition. The meaning of ‘‘Span’’ is recalled in equation (13).

Proposition 7. *Let α be a positive number. One has $\text{Span } \mathcal{F}_0(x) \subset \text{Span } \mathcal{F}_\alpha(x) \subset \text{Span } \mathcal{F}_\infty(x)$ if X^0, X^1, \dots, X^m are smooth, and $\text{Span } \mathcal{F}_0(x) = \text{Span } \mathcal{F}_\alpha(x) = \text{Span } \mathcal{F}_\infty(x)$ if the vector fields are real analytic.*

Proof. For x in \mathcal{M} and p an element of $T_x^* \mathcal{M}$, let

$$a_k(t) = \langle p, \exp(-tX^0)_* X^k(x) \rangle, \quad t \in \mathbb{R}.$$

The map a_k is smooth and²

$$\frac{d^j a_k}{dt^j}(t) = \langle p, \exp(-tX^0)_* \text{ad}_{X^0}^j X^k(x) \rangle. \quad (25)$$

If p belongs to the annihilator of $\text{Span } \mathcal{F}_\alpha(x)$, then $a_k(t)$ is zero for all integer k in $\{1, \dots, m\}$ and t in $(-\alpha, \alpha)$; differentiating j times and then taking $t = 0$ and using (25) implies that $\langle p, \text{ad}_{X^0}^j X^k(x) \rangle = 0, k \in \{1, \dots, m\}, j \in \mathbb{N}$, hence p must belong to the annihilator of $\text{Span } \mathcal{F}_0(x)$. This proves $\text{Span } \mathcal{F}_0(x) \subset \text{Span } \mathcal{F}_\alpha(x)$, while $\text{Span } \mathcal{F}_\alpha(x) \subset \text{Span } \mathcal{F}_\infty(x)$ is obvious by definition. To prove the reverse inclusion, assume that p is in the annihilator of $\text{Span } \mathcal{F}_0(x)$. For each integer k in $\{1, \dots, m\}$, p vanishes on all vectors $\text{ad}_{X^0}^j X^k(x), j \in \mathbb{N}$, so according to (25) $\frac{d^j a_k}{dt^j}(0) = 0$ for all j . Since the map a_k is real analytic when the vector fields are real analytic, it must be identically zero on \mathbb{R} . This proves that $\text{Span } \mathcal{F}_\infty(x) \subset \text{Span } \mathcal{F}_0(x)$, hence $\text{Span } \mathcal{F}_0(x) = \text{Span } \mathcal{F}_\alpha(x) = \text{Span } \mathcal{F}_\infty(x)$. \square

2.2. Controllability results. Let us first state our controllability result about ‘‘controllability of the slow variables’’ for system (2).

Theorem 8. *Assume that all vector fields are real analytic, that U satisfies (6), and that*

$$\text{conv}(\mathbf{E}^U(I)) \text{ is a neighbourhood of } 0 \text{ in } T_I M \quad (26)$$

for all I in M . Then the system is controllable in projection to the slow variables, in the sense that $\mathbf{A}^U(I, \varphi) = M$ for each (I, φ) in $M \times \mathbb{S}^1$.

and in fact in all variables, see paragraph after (10)

This result is an immediate consequence of the analogous result below on the more general affine systems (1):

Theorem 9. *Assume that all vector fields are real analytic, that X^0 satisfies (4) (periodicity of all solutions), that U satisfies (6), and that*

$$\text{conv}(E^U(x)) + \mathbb{R} X^0(x) \text{ is a neighbourhood of } 0 \text{ in } T_x M \quad (27)$$

²This is because $\frac{d}{dt}(\exp(-tY)_* Z)(x) = (\exp(-tY)_*[Y, Z])(x)$ for any vector fields Y and Z .

for all x in \mathcal{M} , then the system is controllable in the sense that $\mathcal{A}^U(x) = \mathcal{M}$ for each x in \mathcal{M} .

Proof. For a family of vector fields \mathcal{F} , we denote by $\mathcal{A}_{\mathcal{F}}(x)$ the accessible set from x of this family of vector fields in positive (unspecified) time, *i.e.* the set of points that can be reached from x by following successively the flow of a finite number of vector fields in \mathcal{F} , each for a certain positive time. With \mathcal{G} the family of vector fields defined by (17), $\mathcal{A}_{\mathcal{G}}(x)$ is also the set of points that can be reached, for the control system (1), with piecewise constant controls; we are going to show that, under our assumptions, $\mathcal{A}_{\mathcal{G}}(x)$ is the whole manifold \mathcal{M} for any x in \mathcal{M} ; this obviously implies the Proposition.

Define the families \mathcal{G}_1 and \mathcal{G}_2 (with $\mathcal{G} \subset \mathcal{G}_1 \subset \mathcal{G}_2$) as follows:

$$\mathcal{G}_1 = \mathcal{G} \cup \{-X^0\}, \quad \mathcal{G}_2 = \{\exp(tX^0)_*X, X \in \mathcal{G}_1, t \in \mathbb{R}\}, \quad \mathcal{G}_3 = \text{cone}(\mathcal{G}_2), \quad (28)$$

where $\exp(tX^0)_*X$ denotes the push-forward of the vector field X by the diffeomorphism $\exp(tX^0)$ and $\text{cone}(\mathcal{G}_2)$ denotes the family made of all vector fields that are finite combinations of the form $\sum_k \lambda_k X_k$ with each X_k in \mathcal{G}_2 and each λ_k a positive number (conic combination). For all x , one has³

$$\mathcal{A}_{\mathcal{G}_1}(x) = \mathcal{A}_{\mathcal{G}}(x) \quad (29)$$

because on the one hand our assumption on the control set implies $X^0 \in \mathcal{G}$, and on the other hand, for any (I', φ') ,

$$\exp(-tX^0)((I', \varphi')) = \exp((-t + 2k\pi/\omega(I'))X^0)((I', \varphi'))$$

for all positive integers k , but for fixed t and I' , $-t + 2k\pi/\omega(I')$ is positive for k large enough. Since X^0 and $-X^0$ now belong to \mathcal{G}_1 , we have $\exp(tX^0)(x) \in \mathcal{A}_{\mathcal{G}}(x)$ for all x in \mathcal{M} and all t in \mathbb{R} , hence $\exp(tX^0)$ is according to [5, Chapter 3, Definition 5 and next lemma], a “normalizer” of the family \mathcal{G}_1 and, according to Theorem 9 in the same chapter of the same reference, this implies that³

$$\mathcal{A}_{\mathcal{G}_2}(x) \subset \overline{\mathcal{A}_{\mathcal{G}_1}(x)} \quad (30)$$

where the overline denotes topological closure (for the natural topology on \mathcal{M}). Now, [1, Corollary 8.2] or [5, chapter 3, Theorem 8(b)] tell us that³

$$\mathcal{A}_{\mathcal{G}_3}(x) \subset \overline{\mathcal{A}_{\mathcal{G}_2}(x)}. \quad (31)$$

Now, (27) implies (the notation $\mathcal{G}_3(x)$ is defined in (13)) that $\mathcal{G}_3(x) = T_x\mathcal{M}$ for all x in \mathcal{M} , and this in turn implies that $\overline{\mathcal{A}_{\mathcal{G}_3}(x)} = \mathcal{M}$. Together with (30)-(31), this implies $\overline{\mathcal{A}_{\mathcal{G}}(x)} = \mathcal{M}$, and finally $\mathcal{A}_{\mathcal{G}}(x) = \mathcal{M}$ from [1, Corollary 8.1] and the fact that, because of (27) and Proposition 7, the distribution is bracket-generating. \square

Remark 3. Theorem 8 is not simply Theorem 9 applied to system (2): both the hypothesis and the conclusion are different. Translated in terms of (1) according to (5), (26) amounts to the following (weaker than (27)):

$$\text{conv}(E^U(I, \varphi) + \mathbb{R}X^0(x)) \quad \text{is a neighbourhood of } 0 \text{ in } T_{(I, \varphi)}\mathcal{M} \quad (32)$$

Remark 4. Condition (27) implies the bracket generating property (Proposition 7), and in fact implies that $T_x\mathcal{M}$ is spanned by the value at x of vector fields $\text{ad}_{X^0}^j X^k$, $k \in \{1, \dots, m\}$, $j \in \mathbb{N}$ only, without any brackets between vector fields, like $[X^1, X^2]$ or $[X^1, [X^0, X^1]]$. This condition amounts to the controllability of the linearized system along the periodic solution running through x , see Section 3.

³ In the terminology of [1, Section 8.2], one could state (29), (30) and (31) as: $-X^0$ is compatible with \mathcal{G} , the vector fields in \mathcal{G}_2 are compatible with \mathcal{G}_1 , and the vector fields in \mathcal{G}_3 are compatible with \mathcal{G}_2 , respectively.

Comments on conditions (26) or (27). which is more difficult than computing Lie brackets (differentiation) and checking the rank of a family of vector fields (linear algebra), resorts to convex optimization and is discussed in [4]. Besides, let us point what kind of non-controllability occurs when these conditions fail.

Clearly, for a given I in M , (26) holds if and only if, for any $p \in T_I^*M$, it holds that

$$\left\langle p, \sum_{k=1}^m u_k F(I, \varphi) \right\rangle \geq 0, \quad (u_1, \dots, u_m) \in U, \varphi \in \mathbb{S}^1 \implies p = 0. \quad (33)$$

Obviously, the assumption of Theorem 8 fails if and only if, for at least one I in M , there is a nonzero $p \in T_I^*M$ such that

$$\left\langle p, \sum_{k=1}^m u_k F(I, \varphi) \right\rangle \geq 0, \quad (u_1, \dots, u_m) \in U, \varphi \in \mathbb{S}^1 \quad (34)$$

We do not claim that a converse of Theorem 8 holds: the above is not an obstruction to controllability. For instance there are cases where, for each I in M , there exists a nonzero $p \in T_I^*M$, depending on I , such that

$$\left\langle p, \sum_{k=1}^m u_k F(I, \varphi) \right\rangle = 0, \quad (u_1, \dots, u_m) \in U, \varphi \in \mathbb{S}^1, \quad (35)$$

even replacing U with \mathbb{R}^m , and such systems may very well be controllable, due to other phenomena than the ones studied in the present paper.

Let us examine the case where not only (34) but a strenghtened property holds. Namely assume that there exists a nonzero $p_0 \in T_{I_0}^*M$ such that the inequality in (34) is strict; then there is a whole neighbourhood $O \subset M$ of I_0 such that

$$\left\langle p_0, \sum_{k=1}^m u_k F(I, \varphi) \right\rangle > 0, \quad (u_1, \dots, u_m) \in U, \varphi \in \mathbb{S}^1 \quad (36)$$

for all I in O . actually holds; then there is a whole neighbourhood $O \subset M$ of I_0 where this inequality remains true. This is an obstruction to some form of controllability: the points I in O such that $\langle p_0, I \rangle < \langle p_0, I_0 \rangle$ (this makes sense in coordinates, identifying O to an open set of \mathbb{R}^d and $T_{I_0}^*M$ to \mathbb{R}^d) cannot be reached from I_0 by admissible controls without leaving O . This does not defeat controllability in the sense of Theorems 9 or 8 because there may be trajectories initiating from I_0 that leave O and reach I after re-entering O , but clearly defeats some sort of local controllability.

3. LOCAL CONTROLLABILITY IN PRESCRIBED TIME ALONG ONE TRAJECTORY OF THE DRIFT

This section is independent of the rest of the paper (except for Appendix A). Here, we consider the general system (1) and we do not make any periodicity assumption on the solutions of $\dot{x} = X^0(x)$; in particular we do not assume anything like condition (4). Instead, we consider an arbitrary solution $t \mapsto \bar{x}(t)$ of this differential equation on a compact interval $[0, t_f]$ (we use the notation t_f rather than T to avoid any confusion with the possible period in assumption (4)) and show that conditions similar to (27), but localized on this precise solution and with no periodicity assumption, imply that the reachable set in time t_f from $\bar{x}(0)$ with integrable controls valued in U (or even in εU , $0 < \varepsilon \leq 1$, if U is convex) is a neighbourhood of $\bar{x}(t_f)$. In other words, it is a sufficient condition for openness of the end-point mapping from the initial point in time t_f with U -valued controls to be open at the zero control.

Theorem 10. *Consider system (1) with U a bounded⁴ subset of \mathbb{R}^m containing the origin. Let $t \mapsto \bar{x}(t)$, $[0, t_f] \rightarrow \mathcal{M}$ be a solution of $\dot{x} = X^0(x)$. If, with E_t^U defined in (20), it holds that*

$$\text{conv} \left(\bigcup_{t \in [0, t_f]} E_t^U(\bar{x}(0)) \right) \text{ is a neighbourhood of } 0 \text{ in } T_{\bar{x}(0)}\mathcal{M}, \quad (37)$$

then the set $\mathcal{A}_{t_f}^{\varepsilon U}(\bar{x}(0))$ is a neighbourhood of $\bar{x}(t_f)$ for any $\varepsilon > 0$.

Lemma 11 below relates Condition (37) of this theorem to controllability of the control-constrained linear approximation of the control system (1) along the solution $\bar{x}(\cdot)$, with zero control. It is customary to write this linear approximation as

$$\dot{\delta x} = \frac{\partial X^0}{\partial x}(\bar{x}(t)) \delta x + \sum_1^m \delta u_k X^k(\bar{x}(t)), \quad (38)$$

where $(\delta u_1, \dots, \delta u_m)$ is a small variation of the control around zero and $\delta x(t)$ a small variation of the state around $\bar{x}(t)$. It is however more convenient, and more notationally correct, to write the equation for the small variations of $y(t) = \exp(-tX^0)(x(t))$ (where $x(\cdot)$ is a solution of (1)), rather than for the small variations of $x(t)$; indeed, $\delta y(t)$ is in the fixed tangent space $T_{\bar{x}(0)}\mathcal{M}$ for all t , while $\delta x(t)$ is in $T_{\bar{x}(t)}\mathcal{M}$, that changes with t . Differentiating $\dot{y} = \sum_k u_k (\exp(-tX^0)_* X^k)(y)$, the equation for δy reads $\dot{\delta y} = \sum_k \delta u_k (\exp(-tX^0)_* X^k)(\bar{x}(0))$; differentiating $x(t) = \exp(tX^0)(y(t))$ to recover δx from δy , and writing ξ instead of δy for simplicity, we get the following form for the linear approximation of (1) along $t \mapsto \bar{x}(t)$, clearly equivalent to (38), that we use in the sequel:

$$\begin{aligned} \dot{\xi} &= \sum_{k=1}^m \delta u_k (\exp(-tX^0)_* X^k)(\bar{x}(0)), \\ \delta x(t) &= \exp(tX^0)'(\bar{x}(0)) \xi(t), \end{aligned} \quad (39)$$

with initial condition $\xi(0) = 0$. Although it is not common (strictly speaking, a linear system has a linear space as state and control space), we may also constrain the control $\delta u = (\delta u_1, \dots, \delta u_m)$ of the linear system (38) to some subset V of \mathbb{R}^m , and define accessibility sets for this linear time-varying system, like we did in Section 1.2 for nonlinear time-invariant systems⁵:

$$\begin{aligned} \mathbf{A}_{0,t}^V(\delta x_0) &= \{ \delta x(t), \text{ with } s \mapsto (\delta x(s), \delta u(s)) \text{ solution of (39) on } [0, t], \\ &\quad \delta x(0) = \delta x_0, \delta u(s) \in V, \text{ a.e. } s \in [0, t] \}. \end{aligned} \quad (40)$$

denotes the accessible set at time t from δx_0 at time 0 for the time-varying linear system (39) (or (38)) with constraint $\delta u \in V$; this is consistent with (47) in Appendix A. The following result states that condition (37) is equivalent to some local controllability of the linear approximation.

Lemma 11. *For any subset U of \mathbb{R}^m , the linear constrained attainable set $\mathbf{A}_{0,t_f}^U(0)$ is a neighbourhood of the origin in $T_{\bar{x}(t_f)}\mathcal{M}$ if and only if Condition (37) holds.*

Proof. This follows from Proposition 12 (Condition (ii)) in Appendix A, applied with (38) playing the role of (46), and (39) the role of (49). \square

⁴If U is not bounded, it may be replaced by a bounded subset that satisfies condition (37).

⁵Dealing with a time-varying system, we need to keep track of both the initial and final times: it is no longer true that $\mathbf{A}_{t_0,t}^V(\delta x_0)$ depends only on (δx_0) and $t - t_0$. It however does not matter that the initial time is fixed to zero in (40) because we are dealing with the linearisation of a time-invariant system.

Remark 5 (on the proof of Theorem 10). In the case where U is a neighbourhood of the origin, the result is classical. Let \mathcal{E} be the usual *end-point mapping at time t_f starting from $\bar{x}(0)$* for the nonlinear system (1), *i.e.* \mathcal{E} maps a measurable control to the value at time t_f of the solution of (1) starting from $\bar{x}(0)$ with this control. It is usually defined $L^\infty([0, t_f], \mathbb{R}^m) \rightarrow \mathcal{M}$, but only the images of controls taking values in U belong to $\mathcal{A}_{t_f}^U(\bar{x}(0))$. Clearly, 0 meaning the zero control, one has $\mathcal{E}(0) = \bar{x}(t_f)$, and the derivative, or first order approximation, of \mathcal{E} at 0 is the end-point mapping of the linear tangent control system (38), which is onto $L^\infty([0, t_f], \mathbb{R}^m) \rightarrow T_{\bar{x}(t_f)}\mathcal{M}$ according to Lemma 11, hence \mathcal{E} is locally a submersion at 0, and in particular it is open, *i.e.* the image of any neighbourhood of 0 in $L^\infty([0, t_f], \mathbb{R}^m)$ is a neighbourhood of $\bar{x}(t_f)$. This proves the result if U is a neighbourhood of the origin in \mathbb{R}^m because, in that case, there is a neighbourhood of the zero control in $L^\infty([0, t_f], \mathbb{R}^m)$ that contains only controls taking values in U (for almost all times, but these may be replaced without changing the control as an element of $L^\infty([0, t_f], \mathbb{R}^m)$). The same holds with εU .

When the origin is on the boundary of U and not in its interior, the submersion property at the origin does not suffice because it is no longer true that controls close to the zero control in the L^∞ topology automatically satisfy the constraint of being valued in U .

Proof of Theorem 10. According to Lemma 11, Condition (37) implies that the accessible set $\mathbf{A}_{t_f}^U(0)$ of (39) in time t_f is a neighbourhood of the origin in $T_{\bar{x}(t_f)}\mathcal{M}$. Let e_0, \dots, e_d be the vertices of a convex polyhedron that is a neighbourhood of the origin and a subset of $\mathbf{A}_{t_f}^U(0)$. Since 0 is in the interior of the polyhedron generated by e_0, \dots, e_d , there exist $\lambda_0^0, \dots, \lambda_d^0$ with

$$\sum_{i=0}^d \lambda_i^0 e_i = 0, \quad \sum_{i=0}^d \lambda_i^0 = 1, \quad \text{and} \quad \lambda_i^0 > 0, \quad i \in \{0, \dots, d\}. \quad (41)$$

Since e_0, \dots, e_d are all in $\mathbf{A}_{t_f}^U(0)$, there also exist $\mathbf{u}_0, \dots, \mathbf{u}_d$, some integrable U -valued controls, defined on $[0, t_f]$, that steer the origin to e_0, \dots, e_d respectively, in time t_f , for the linear time-varying system (39) (or (38)). Define then the control $t \mapsto u^{\text{zer}}(t) = \sum_{i=0}^d \lambda_i^0 \mathbf{u}_i(t)$. Since U is convex, it takes values in U 's topological interior and, by linearity, steers the origin to the origin, still for the linear system (39) (or (38)). For any positive ε no larger than 1 (of course unrelated to the ε in (2)), still by linearity, the controls $\varepsilon \mathbf{u}_0, \dots, \varepsilon \mathbf{u}_d$ steer the origin to $\varepsilon e_0, \dots, \varepsilon e_d$ respectively, and the control $\varepsilon u^{\text{zer}}$ takes values in εU 's topological interior and steers the origin to the origin.

Now going back to the nonlinear system (1), let $\mathcal{E} : L^\infty([0, t_f], \mathbb{R}^m) \rightarrow \mathcal{M}$ be the end-point mapping at time t_f starting from $\bar{x}(0)$ for the nonlinear system (1), already mentioned in Remark 5. (To avoid problems due to non-complete vector fields, they may all be multiplied by some smooth cut-off functions equal to 1 in a neighborhood of $\bar{x}([0, t_f])$ to ensure that solutions do not explode before t_f , without changing the local property in this neighborhood, that we are going to prove.) All we need to prove is that the image by \mathcal{E} of controls valued in εU is a neighbourhood of $\bar{x}(t_f)$ for any $\varepsilon > 0$. It is well known that \mathcal{E} is continuously differentiable and that its derivative (linear approximation) at some control (here the zero control) is the end-point mapping of the linearized control system (39) (or (38)) at this control. Consider the continuously differentiable map $F : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathcal{M}$ defined by

$$F(\mu_1, \dots, \mu_d, \varepsilon) = \mathcal{E} \left(\varepsilon u^{\text{zer}} + \sum_{i=1}^d \mu_i (\mathbf{u}_i - \mathbf{u}_0) \right). \quad (42)$$

Since \mathcal{E} maps the zero control to $\bar{x}(t_f)$, and $\frac{\partial F}{\partial \varepsilon}(0, \dots, 0, 0)$ is the image of u^{zer} by the end-point mapping of the linear system (39) (or (38)) starting from zero, equal to zero, and, for j in $\{1, \dots, d\}$, $\frac{\partial F}{\partial \mu_j}(0, \dots, 0, 0)$ is the image of $u_j - u_0$ by the end-point mapping of the linear system (39) (or (38)) starting from zero, equal to $e_i - e_0$, one has

$$\begin{aligned} F(0, \dots, 0, 0) &= \bar{x}(t_f), \quad \frac{\partial F}{\partial \varepsilon}(0, \dots, 0, 0) = 0, \\ \text{and } \frac{\partial F}{\partial \mu_j}(0, \dots, 0, 0) &= e_i - e_0, \quad i \in \{1, \dots, d\}. \end{aligned} \quad (43)$$

The third relation in (43) implies that the Jacobian of F with respect to μ_1, \dots, μ_d at $(0, \dots, 0, 0)$ is invertible (the vectors $e_1 - e_0, \dots, e_d - e_0$ are linearly independent because e_0, e_1, \dots, e_d are affinely independent); then, from the first and third relations, the inverse function theorem yields a smooth map $\varepsilon \mapsto (\mu_1(\varepsilon), \dots, \mu_d(\varepsilon))$ such that

$$F(\mu_1(\varepsilon), \dots, \mu_d(\varepsilon), \varepsilon) \equiv \bar{x}(t_f). \quad (44)$$

The second relation in (43) implies that the derivative of that map with respect to ε at zero is zero, hence $|\mu_i(\varepsilon)| < C\varepsilon^2$ for some $C > 0$. By continuity, for ε small enough, the Jacobian of F with respect to μ_1, \dots, μ_d at $(\mu_1(\varepsilon), \dots, \mu_d(\varepsilon), \varepsilon)$ is also invertible, hence the map $(\mu_1, \dots, \mu_d) \mapsto F(\mu_1, \dots, \mu_d, \varepsilon)$ is open at $(\mu_1(\varepsilon), \dots, \mu_d(\varepsilon))$ for these small values of ε , meaning that:

$$\left. \begin{aligned} &\text{if } \varepsilon > 0 \text{ is small enough, } F(\Omega, \varepsilon) \text{ is a neighbourhood of } \bar{x}(t_f) \\ &\text{for any neighbourhood } \Omega \text{ of } (\mu_1(\varepsilon), \dots, \mu_d(\varepsilon)). \end{aligned} \right\} \quad (45)$$

Defining $\lambda_0(\cdot), \dots, \lambda_d(\cdot)$ and the control $u^\varepsilon : [t_0, t_f] \rightarrow \mathbb{R}^m$ by

$$\begin{aligned} \lambda_0(\varepsilon) &= \varepsilon \lambda_0^0 - \sum_{j=1}^d \mu_j(\varepsilon), \quad \lambda_i(\varepsilon) = \varepsilon \lambda_i^0 + \mu_i(\varepsilon), \quad 1 \leq i \leq d, \\ \text{and } u^\varepsilon(t) &= \sum_{k=0}^d \lambda_k(\varepsilon) u_k(t), \end{aligned}$$

equation (42) reads $F(\mu_1(\varepsilon), \dots, \mu_d(\varepsilon), \varepsilon) = \mathcal{E}(u^\varepsilon)$. The numbers $\lambda_i(\varepsilon)$, $0 \leq i \leq d$ are all positive if ε is small enough because $\lambda_i^0 > 0$, and we proved that $|\mu_i(\varepsilon)| < C\varepsilon^2$; they also satisfy $\sum_0^d \lambda_k(\varepsilon) = \varepsilon$; this implies that u^ε takes values in the interior of εU if ε is small enough. Taking ε small enough that the above holds and Ω a small enough neighbourhood of $(\mu_1(\varepsilon), \dots, \mu_d(\varepsilon))$ that the control $\varepsilon u^{\text{zer}} + \sum_{i=1}^d \mu_i(u_i - u_0)$ takes values in εU if (μ_1, \dots, μ_d) is in Ω , Property (45) implies that the image by \mathcal{E} of a set of controls valued in εU cover a neighbourhood of $\bar{x}(t_f)$, proving the theorem for small enough values of ε , hence also for large values because $\mathcal{A}_{t_f}^{\varepsilon U}(\bar{x}(0))$ increases with ε . (Since U is convex and contains 0, one has $\varepsilon U \subset \varepsilon' U$ if $\varepsilon < \varepsilon'$.) \square

APPENDIX A. CONTROLLABILITY OF TIME-VARYING LINEAR SYSTEMS WITH CONSTRAINED CONTROL

Consider a time-varying linear control system, with constraints on the control:

$$\dot{z} = A(t)z + B(t)v, \quad v \in V, \quad (46)$$

where the state z belongs to \mathbb{R}^d , the control v is constrained to stay in the control set $V \subset \mathbb{R}^m$ and $t \mapsto A(t)$ and $t \mapsto B(t)$ are real analytic (or sufficiently smooth) maps $\mathbb{R} \rightarrow \mathbb{R}^{d \times d}$ and $\mathbb{R} \rightarrow \mathbb{R}^{d \times m}$ respectively. Because of non-stationarity, the set

$$\mathbf{A}_{t_0, t_f}^V(z_0) = \{z(t_f), \text{ with } t \mapsto (z(t), v(t)) \text{ solution of (46), } z(t_0) = z_0\}. \quad (47)$$

of points that can be reached at time t_f starting from z_0 at time t_0 fully depends on both t_0 and t_f instead of depending on $t_f - t_0$ as in the case of time-invariant systems. It is customary to define the transition matrix as the map $(t_1, t_2) \mapsto \phi(t_2, t_1) \in \mathbb{R}^{d \times d}$ such that, for any $t_0 \in \mathbb{R}$ and $z_0 \in \mathbb{R}^d$, $t \mapsto \phi(t, t_0)z_0$ is the solution of $\dot{z} = A(t)z$ satisfying $z(t_0) = z_0$; it satisfies

$$\frac{\partial \phi}{\partial t_2}(t_2, t_1) = A(t_2)\phi(t_2, t_1), \quad \frac{\partial \phi}{\partial t_1}(t_2, t_1) = -\phi(t_2, t_1)A(t_1), \quad \phi(t, t) = I. \quad (48)$$

Note that one may always re-write (46) under the following form, for any t_0 :

$$\begin{aligned} \dot{\zeta}(t) &= \phi(t_0, t)B(t)v(t), \\ z(t) &= \phi(t, t_0)\zeta(t). \end{aligned} \quad (49)$$

This yields the general “formula” for the unique $z(\cdot)$, solution of (46) associated to a prescribed control $v(\cdot)$ and satisfying $z(t_0) = z_0$:

$$z(t) = \phi(t, t_0)z_0 + \int_{t_0}^t \phi(t, \tau)B(\tau)v(\tau)d\tau, \quad (50)$$

and also has the advantage that, if (46) is obtained as the linearization along a trajectory of a nonlinear system, $\zeta(t)$ lives in the tangent space at the original point of the trajectory for all time, while $z(t)$ is in a different tangent space at different times, see (38) and (39).

In the case of unconstrained controls ($V = \mathbb{R}^m$), we are in the fully linear situation where either the reachable set from any point in any time is contained in (in fact equal to) a proper affine subspace, so that controllability obviously fails, or the reachable set from any point is the whole state space and controllability holds. Criteria are recalled for instance in [7, Section 9.2], after results from [14] or [11]. The classical criterion from [14] states that, for any z_0 in \mathbb{R}^d and t_0, t_f in \mathbb{R} , the set $A_{t_0, t}^{\mathbb{R}^m}(z_0)$ is all \mathbb{R}^d if and only if “the rows of $s \mapsto \phi(t_0, s)B(s)$ are linearly independent on the interval $[t_0, t_f]$ ”, meaning, for $p \in (\mathbb{R}^d)^*$ (a line vector),

$$\left(p \phi(t_0, s) B(s) = 0 \text{ for all } s \text{ in } [t_0, t_f] \right) \Rightarrow p = 0. \quad (51)$$

If $A(\cdot)$ and $B(\cdot)$ are real analytic, (51) is equivalent to the columns of the matrices $(\frac{d}{dt} - A(t))^j B(t)$, $j \in \mathbb{N}$ having rank d for all t in $[t_0, t_f]$ (without the real analyticity assumption, the rank of these columns may vary); this time-varying extension of the Kalman criterion is given in [11].

The case where V is a proper subset of \mathbb{R}^m but V , or its convex hull, is a neighbourhood of the origin is not very different, locally, from the unconstrained case, as noticed in [10] and other references therein. Indeed, in that case, $A_{t_0, t}^V(0)$ is a neighbourhood of the origin if and only if (51) holds⁶.

The case where V , or its convex hull, are not a neighborhood of the origine is the purpose of the present paper. Controllability results for this case can be found in [10] and [8, Section 2.2]. The following proposition is the precise result we need in Section 3, characterizing when $A_{t_0, t_f}^V(0)$, the reachable set from the origin, is a neighbourhood of the origin in the state space. A very similar result is stated in [10], characterizing $\cup_{t \in (-\infty, t_f]} A_{t, t_f}^V(0)$. For the sake of completeness, we give a simple proof (indeed quite reminiscent of [10]) rather than referring to a proof of a somehow different statement.

⁶In [10], the question is more controllability *to* the origin than *from* the origin (is there a neighborhood of the origin made of points x such that $A_{t_0, t}^V(x) = 0$?), but this simply amounts to taking $t < t_0$ rather than $t > t_0$, or to changing A, B into $-A, -B$, and one easily checks that the criteria are true for A, B if and only if they are true for $-A, -B$.

Proposition 12. *Assume that V is a compact subset of \mathbb{R}^d containing zero, and that t_0, t_f are two real numbers. The accessible set $A_{t_0, t_f}^V(0)$ is a neighbourhood of the origin if one of the two following equivalent properties hold.*

(i) For $p \in (\mathbb{R}^d)^*$ (line vector),

$$\left(p \phi(t_0, s) B(s)v \geq 0 \text{ for all } v \in V, s \in [t_0, t_f] \right) \Rightarrow p = 0. \quad (52)$$

(ii) the convex hull of $\{\phi(t_0, s) B(s)v, s \in [t_0, t_f], v \in V\}$ is a neighbourhood of the origin in \mathbb{R}^d .

Proof. The two properties are equivalent by the separating hyperplane theorem. Let us prove that the first condition is necessary and sufficient. If it does not hold, there exists a nonzero p such that $p \phi(t_0, s) B(s)v \geq 0$ for any $v \in V$ and $s \in [t_0, t_f]$, and formula (50) obviously implies $p z(t_f) \geq 0$ at all time $t \geq t_0$ for any solution $(z(\cdot), v(\cdot))$ of (46) with $z(t_0) = 0$ and $v(\cdot)$ valued in V , implying that $A_{t_0, t_f}^V(0)$, contained in the half-space $\{z, pz \geq 0\}$, is not a neighbourhood of 0. To prove the converse, assume that $A_{t_0, t_f}^V(0)$ is not a neighbourhood of 0. It is convex according to [8, Chapter 2, Theorem 1A (appendix), p.164], hence there is a nonzero p^* (that is a separating hyperplane) such that any solution $(z(\cdot), v(\cdot))$ of (46) with $z(t_0) = 0$ and $v(\cdot)$ valued in V satisfies $p^* z(t_f) \geq 0$; it must also satisfy

$$p^* \phi(t_f, s) z(s) \geq 0, \quad s \in [t_0, t_f],$$

because $\phi(t_f, s) z(s)$ is the value at time t_f of the solution

$$\tau \mapsto \begin{cases} (z(\tau), v(\tau)) & \text{if } \tau \in [t_0, s) \\ (\phi(\tau, s) z(s), 0) & \text{if } \tau \in [s, t_f] \end{cases}$$

starting from zero, with control $v(\cdot)$ on $[t_0, s)$ and zero on $[s, t_f]$. Now, for any (constant) v in V and s in $[t_0, t_f]$, consider the control function on $[t_0, t_f]$ that is zero until time s and constant equal to v after time s , and $z(\cdot)$, starting from $z(t_0) = 0$ associated to this control. One has

$$\begin{aligned} p^* \phi(t_f, \tau) z(\tau) &= 0, \quad \tau \in [0, s], \\ \frac{d}{d\tau} (p^* \phi(t_f, \tau) z(\tau)) &= p^* \phi(t_f, \tau) B(\tau)v, \quad \tau \in [s, t_f] \end{aligned}$$

(at $\tau = s$, where $z(\cdot)$ is not differentiable, $\frac{d}{d\tau}$ means right-hand derivative), while, according to the above, $p^* \phi(t_f, \tau) z(\tau) \geq 0$ for all τ ; this implies $p^* \phi(t_f, s) B(s)v \geq 0$. Taking $p = p^* \phi(t_f, t_0)$ hence yields a nonzero p such that $p \phi(t_0, s) B(s)v \geq 0$ for all s in $[t_0, t_f]$ and all v in V , defeating (52). \square

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