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# ON THE CONTROLLABILITY OF NONLINEAR SYSTEMS WITH A PERIODIC DRIFT

J.-B. CAILLAU\*, L. DELL'ELCE†, A. HERASIMENKA\*, AND J.-B. POMET†

**Abstract.** Sufficient and necessary conditions are established for controllability of affine control systems where the control is constrained to a set whose convex hull contains the origin but is not necessarily, in contrast with previously known results, a neighborhood of the origin. Part of the results, in particular these on global controllability, are specific to systems where all solutions with zero control (drift) are periodic. These conditions are expressed by means of pushforwards along the flow of the drift, rather than in terms of Lie brackets; it turns out that they amount to local controllability of a time-varying linear approximation with constrained controls. Global and local results are given, as well as a few illustrative examples.

**Key words.** Nonlinear controllability, control constraints, solar sails

**MSC codes.** 93B05, 93B03, 95C15

**Introduction.** This paper is devoted to controllability properties of affine control systems, defined by a drift vector field and  $m \geq 1$  control vector fields. We consider either local controllability around a solution with zero control (solution of the drift vector field) or global controllability in the case where all such solutions are periodic; in both cases, the control is constrained to a subset  $U$  of  $\mathbb{R}^m$  that contains zero, but whose convex hull, unlike in most known controllability results, is not necessarily a neighborhood of zero. Our original motivation for this study was the control of spacecrafts propelled by solar sails, see references [9, 10]<sup>1</sup> Governed by solar radiation pressure, solar sails are only capable of generating forces contained in a subset  $U$  of a convex cone of revolution around the direction of the incoming light. Zero is the vertex of this cone and is contained in  $U$ , but is obviously not in the interior of the convex hull of  $U$ . If we consider an unperturbed controlled Kepler problem in the region of negative total energy, all solutions of the drift are periodic.

Devising conditions on the vector fields (and the set  $U$ ) for controllability is quite an old problem. Most known sufficient conditions for controllability assume that the Lie brackets of the vector fields defining the system span the tangent space at all points (bracket generating property), and prove controllability under additional conditions concerning in general the drift vector field, as recalled in Section 1.4, based on [19, 2, 25, 15, 11, 1]. Nevertheless, these results always require —at least— that zero belongs to the interior of the convex hull of  $U$ . This assumption is crucial in the proofs, and allows for a decoupling between this condition on  $U$  and conditions on the vector fields. In the present paper, where zero is allowed to belong to the boundary of  $\text{conv } U$ , novel results are based on *ad hoc* constructions involving the transportation along the flow of the drift vector field of some subsets of the tangent space defined from the control vector fields *and* the set  $U$ .

In Section 1, we cast precisely the controllability problem we want to look at, and recall the main definitions and results on the subject. All results and proofs are stated in Section 2, completed by a short appendix on time-varying linear control systems.

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<sup>1</sup>A preliminary version of some results stated in Section 2.1 can be found in [9, Section II]. by the authors.

Section 2.1 is devoted to sufficient conditions for global controllability in unprescribed time under the periodicity assumption on the drift. The given proofs generalize the well-known ones in [2] (for instance), relying on augmentation of families of vector fields (exposed into details, *e.g.*, in the textbook [11]). In Section 2.2, we give sufficient conditions for local controllability in prescribed time along trajectories of the drift, periodic or not, via a different method of proof, based on linear approximation as a control-constrained time-varying linear system. Section 2.3 uses the latter results in the periodic case, yielding a more precise *local* picture (local controllability in one revolution) than Section 2.1. Section 2.4 gives necessary conditions for controllability: some possible “shapes” of  $U$  forbid controllability. Section 3 is devoted to comments on the results, more general than the ones given along in Section 2, and to two academic examples, one illustrating our results, and the other one exploring a case where controllability holds while none of our sufficient conditions is met.

## 1. Statement of the problem.

**1.1. Systems.** We consider an affine control system

$$(1) \quad \dot{x} = X^0(x) + \sum_{k=1}^m u_k X^k(x), \quad u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m,$$

where the set  $U$  defines the constraints on the control and  $X^0, \dots, X^m$  are  $m+1$  smooth ( $C^\infty$ ) vector fields on a smooth connected (Hausdorff, second countable) manifold  $\mathcal{M}$  of dimension  $d$ . We do not require  $m \leq d$  although it usually is the case. We assume that the vector field  $X^0$  is complete.

*Assumptions on the control set  $U$ .* All over the paper, we assume that

$$(2) \quad 0 \in U \quad \text{and} \quad \text{Span } U = \mathbb{R}^m.$$

The first point is natural: our results partially rely on assumptions on the behavior of the differential equation  $\dot{x} = X^0(x)$  obtained for the zero control, that should hence be permitted. The second point ( $\text{Span } U = \mathbb{R}^m$ ) is not restrictive: if  $\text{Span } U$  was a subspace of  $\mathbb{R}^m$  of dimension  $m' < m$ , one could pick  $m'$  new control vector fields, constant linear combinations of  $X^1, \dots, X^m$  and use them as control vector fields in a system with  $m'$  controls satisfying the assumption.

*Periodic orbits of the drift, angular variable.* Let us denote by  $\exp(tX)$  the flow at time  $t$  of a smooth vector field  $X$ . The following property of the drift is assumed in some results on global controllability, and in the rest of this subsection.

$$(3) \quad \begin{aligned} &X^0 \text{ has no equilibrium point, and, for any } x \text{ in } \mathcal{M}, \\ &t \mapsto \exp(tX^0)(x) \text{ is periodic with minimal period } T(x). \end{aligned}$$

Let  $M$  be the set of periodic orbits and  $\pi: \mathcal{M} \rightarrow M$  the projection that maps any  $x$  in  $\mathcal{M}$  to its orbit, so that  $\pi^{-1}(\pi(x)) = \{\exp(tX^0)(x), t \in \mathbb{R}\}$ .

**PROPOSITION 1.** *Assuming (3), the set  $M$  of periodic orbits of  $X^0$  is a smooth manifold of dimension  $d - 1$  and the projection  $\pi: \mathcal{M} \rightarrow M$  is a (locally trivial) fibration with fiber  $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ , i.e., any  $\bar{I} \in M$  has a neighborhood  $O$  in  $M$  such that  $\pi^{-1}(O)$  is diffeomorphic to  $O \times \mathbb{S}^1$ .*

*Proof.* The manifold structure of  $M$  is obtained as follows<sup>2</sup>: any orbit  $\bar{I}$  has (many) smooth transverse sections  $\Sigma$  (smooth submanifold of  $\mathcal{M}$  of dimension  $d - 1$ ),

<sup>2</sup>Alternatively,  $M$  is a manifold as the quotient of  $\mathcal{M}$  by the smooth free action  $(\varphi, x) \mapsto \exp((T(x)/2\pi)\varphi X^0)(x)$  of the compact Lie group  $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$  on  $\mathcal{M}$ ; see *e.g.*, [7, Theorem 1.95].

small enough to intersect any orbit at most once; let  $O$  be the set of orbits that intersect  $\Sigma$ , and  $\rho: O \rightarrow \Sigma$  the one-to-one map that sends such an orbit to the point where it intersects  $\Sigma$ ;  $O$  provides a neighborhood of  $\bar{I}$  in  $M$ , that inherits the manifold structure of  $\Sigma$ . Concerning the fiber bundle structure, for an arbitrary  $\bar{I}$ , take  $O$  as above and define  $\Phi: O \times \mathbb{S}^1 \rightarrow \mathcal{M}$  by  $\Phi(I, \varphi) = \exp\left(\frac{T(\rho(I))}{2\pi}\varphi X^0\right)(\rho(I))$ , where we note that (3) defines  $T(\cdot)$  as smooth as  $X^0$ .  $\square$

On such an open set  $O \times \mathbb{S}^1$ , using a diffeomorphism  $\Phi: O \times \mathbb{S}^1 \rightarrow \pi^{-1}(O)$ , and decomposing on  $T_{(I, \varphi)}(O \times \mathbb{S}^1) = T_I O \oplus T_\varphi \mathbb{S}^1 = T_I O \oplus \mathbb{R}$ , one has

$$(4) \quad X^0 = \Phi_* \left( \omega \frac{\partial}{\partial \varphi} \right) \quad \text{and} \quad X^i = \Phi_* \left( f^i \frac{\partial}{\partial \varphi} + F^i \right) = F^i, \quad i = 1, \dots, m$$

where  $\omega: M \times \mathbb{S}^1 \rightarrow \mathbb{R}$  and each  $f^k: M \times \mathbb{S}^1 \rightarrow \mathbb{R}$  are smooth functions, and each  $F^k$  is a vector field on  $O$  smoothly depending on  $\varphi \in \mathbb{S}^1$  (or a vector field on  $O \times \mathbb{S}^1$  with zero component on  $T\mathbb{S}^1$ ), and the pushforward notation  $\Phi_*$  is, as usual, defined by

$$(5) \quad \Phi_*(X)(y) := \Phi'(\varphi^{-1}(y)) \cdot X(\Phi^{-1}(y)), \quad y \in \mathcal{M}.$$

for any diffeomorphism  $\Phi$  and any vector field  $X$  on a manifold  $\mathcal{M}$ . Then System (1) can be written as follows to take advantage of the product structure

$$(6) \quad \begin{aligned} \dot{I} &= \sum_{k=1}^m u_k F^k(I, \varphi), \\ \dot{\varphi} &= \omega(I) + \sum_{k=1}^m u_k f^k(I, \varphi), \end{aligned} \quad u = (u_1, \dots, u_m) \in U.$$

Note that  $\omega(I)$  naturally comes out as  $\omega(I, \varphi)$ ; composing  $\Phi$  with a well chosen diffeomorphism of  $O \times \mathbb{S}^1$  of the form  $(I, \varphi) \mapsto (I, a(I, \varphi))$  removes the dependence on  $\varphi$ , so that  $\omega$  can be taken as a function from  $M$  to  $\mathbb{R}$ . This product form allows for a simpler and more concrete statement of results or proofs when System (1) satisfies (3), see for instance (16)–(19).

In the applications to control of solar sails mentioned in the introduction, and at least without perturbations to a Newtonian potential,  $\mathcal{M}$  (of dimension 6) being the phase space restricted to negative total energy, Condition (3) is met. It is customary to use a product form like (6) where  $I$  contains five independent first integrals of the Kepler problem, and  $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$  is an angle indicating position on an ellipse. It is well known in celestial mechanics that, as we saw under the mere Condition (3), the fibration  $\mathcal{M} \rightarrow M$  is defined globally, but  $\mathcal{M}$  is not diffeomorphic to  $M \times \mathbb{S}^1$ ; this is why we do not assume  $O = M$  in (4)–(6), in order to keep an equivalence with the general form (1) under Assumption (3). Our illustrative examples in Section 3 and Remark 21 are however globally in the product form (6) for simplicity.

**1.2. Accessible sets and controllability, definitions.** A solution of (1) is a map  $t \mapsto (x(t), u(t))$ , defined on some interval, valued in  $\mathcal{M} \times U$ , where  $u(\cdot)$  is locally essentially bounded and  $x(\cdot)$  is absolutely continuous, that satisfies (1) for almost all time  $t$ . For such a solution on an interval  $[0, t_f]$ ,  $t_f \geq 0$ , one says that the control  $u(\cdot)$  steers  $x(0)$  to  $x(t_f)$  in time  $t_f$ ; indeed the solution is unique given  $u(\cdot)$  and  $x(0)$ . One may also say that the same control steers  $x(t_f)$  to  $x(0)$  in time  $-t_f$  because uniqueness holds also backwards.

*Remark 2.* In the situation where (3) is satisfied, the following fact is true, with  $\pi$  as in [Proposition 1](#): if there is a control steering an initial point  $x$  to a final point  $y$  in some time  $\tau$ , then there is also a control steering any point  $x'$  in the same orbit as  $x$  ( $\pi(x') = \pi(x)$ ) to any point  $y'$  in the same orbit as  $y$  ( $\pi(y') = \pi(y)$ ), in time less than  $\tau + T(x) + T(y)$ . Indeed, the control consists, for some  $t_1$  in  $[0, T(x))$  and  $t_2$  in  $[0, T(y))$ , in applying a zero control on  $[0, t_1]$  and  $[t_1 + \tau, t_1 + \tau + t_2]$  and the original control on  $[t_1, t_1 + \tau]$ .

The reachable set from  $x_0$  in time  $t_f \in \mathbb{R}$  is the set of points in  $\mathcal{M}$  that can be reached in forward or backward time  $t_f$  from  $x_0$  for some control:

$$(7) \quad \mathcal{A}_{t_f}^U(x_0) = \{x(t_f), \text{ with } t \mapsto (x(t), u(t)) \text{ solution of (1) on } [0, t_f], x(0) = x_0\}.$$

We keep the set  $U$  as a superscript to stress the constraint on the control. Obviously, for two subsets  $V_1$  and  $V_2$  of  $U$ ,  $\mathcal{A}_{t_f}^{V_1}(x) \subset \mathcal{A}_{t_f}^{V_2}(x)$  if  $V_1 \subset V_2$ . It is of course assumed that the interval of definition of the solution contains  $[0, t_f]$ . If  $t_f < 0$ ,  $[0, t_f]$  should be understood as  $[t_f, 0]$ , and it is clear that  $y$  belongs to  $\mathcal{A}_t^U(x)$  if and only if  $x$  belongs to  $\mathcal{A}_{-t}^U(y)$ . From accessible sets in prescribed time, we define

$$(9) \quad \mathcal{A}^U(x) = \bigcup_{0 \leq t} \mathcal{A}_t^U(x) \quad \text{and} \quad \mathcal{A}_-^U(x) = \bigcup_{t \leq 0} \mathcal{A}_t^U(x),$$

the accessible sets in unprescribed forward and backward time ( $\mathcal{A}^U(x)$  is sometimes called simply “the accessible set from  $x$ ”).

**DEFINITION 3** (Global controllability). I. *System (1) is **globally controllable** if and only if  $\mathcal{A}^U(x) = \mathcal{M}$  for all  $x$  in  $\mathcal{M}$ .* II. *Let  $\pi : \mathcal{M} \rightarrow M$  be some fibration<sup>3</sup>, with  $M$  a manifold of arbitrary dimension  $n < d$ . System (1) is **globally controllable with respect to  $\pi$**  if and only if  $\pi(\mathcal{A}^U(x)) = M$  for all  $x$  in  $\mathcal{M}$ .*

*Remark 4.* Global controllability (I) obviously implies global controllability with respect to  $\pi$  (II). The converse does not hold: take any non controllable system and  $M = \{0\}$ ,  $n = 0$ ,  $\pi$  constant. However, under the periodicity Assumption (3) and if  $\pi$  is the fibration induced by the periodic orbits (cf. [Proposition 1](#)), global controllability with respect to  $\pi$  does imply global controllability according to [Remark 2](#).

Controllability *with respect to  $\pi$*  could also be called *partial controllability*, a familiar notion where only the partial information  $\pi(x)$  on the final point (at final time) is specified rather than  $x$  itself. In [6, 14, 20] it is called *output controllability*, as  $\pi$  can be viewed as an  $n$ -dimensional output of the system, specified at final time. In [17] or [3, Section 7.2], the notion of *configuration controllability* for mechanical systems is similar: roughly speaking,  $\pi : TQ \rightarrow Q$  is defined by  $\pi(q, \dot{q}) = q$ , where  $q$  (position) is the current point in the configuration manifold  $M = Q$  and  $(q, \dot{q})$  (position-velocity) the one in the state manifold  $\mathcal{M} = TQ$ , tangent bundle of  $Q$ ; there is however a slight difference in the choice of initial conditions (zero velocity initial conditions are privileged in [3, 17]).

Let us now pursue with *local* controllability, either around a particular solution, with a specified final time, or around a particular *periodic* solution, with unprescribed final time. In both cases, the particular solution is one of  $\dot{x} = X^0(x)$ . The given definitions would be similar for a solution  $t \mapsto (\bar{x}(t), \bar{u}(t))$  of (1) with a nonzero

<sup>3</sup>This  $\pi$  need not be related to the construction from [Section 1.1](#) ( $X^0$  is even not assumed to satisfy (3)); being a fibration could be replaced by the weaker requirement of being a submersion.

control, and the results of the paper could be extended, but it does not meet the original motivations of the paper. The following notation is introduced to make Definition 5 concise. Let

$$(10) \quad \mathcal{B}_{t_f}^{U,\Omega}(x_0) = \{x(t_f), \text{ with } t \mapsto (x(t), u(t)) \text{ solution of (1) on } [0, t_f] \\ \text{such that } x(0) = x_0 \text{ and } x(t) \in \Omega \text{ for all } t \text{ in } [0, t_f]\},$$

with  $U, x_0, t_f$  as above and  $\Omega$  some open subset of  $\mathcal{M}$ . Compared to the accessible sets  $\mathcal{A}$  defined above, we added the requirement that the trajectories  $x(\cdot)$  have to stay in  $\Omega$ . Thus,  $\mathcal{A}_{t_f}^U(x_0) = \mathcal{B}_{t_f}^{U,\mathcal{M}}(x_0)$ .

DEFINITION 5 (Local controllability). *Let  $t \mapsto \bar{x}(t)$  be a particular solution of  $\dot{x} = X^0(x)$ , defined on  $[0, t_f]$  for some  $t_f > 0$ , and let  $\Gamma = \bar{x}([0, t_f])$ .*

- I. *System (1) is **locally controllable around the solution  $\bar{x}(\cdot)$**  if and only if the set  $\mathcal{B}_{t_f}^{U,\Omega}(\bar{x}(0))$  is a neighborhood of  $\bar{x}(t_f)$  for any neighborhood  $\Omega$  of  $\Gamma$  in  $\mathcal{M}$ .*
- II. *Let  $\pi : \mathcal{M} \rightarrow M$  be as in Definition 3-II. System (1) is **locally controllable with respect to  $\pi$  around the solution  $\bar{x}(\cdot)$**  if and only if the set*

$$\pi\left(\mathcal{B}_{t_f}^{U,\pi^{-1}(\Omega')}(\bar{x}(0))\right)$$

*is a neighborhood of  $\pi(\bar{x}(t_f))$  in  $M$  for any neighborhood  $\Omega'$  of  $\pi(\Gamma)$  in  $M$ .*

- III. *Assume that the drift vector field  $X^0(x)$  satisfies (3), let  $\pi : \mathcal{M} \rightarrow M$  be the fibration with base  $M$  equal to the respective orbit space, and let  $\bar{x}(\cdot)$  be one of these periodic orbits,  $t_f > T(\bar{x}(0))$ .*

*System (6) is **locally orbitally controllable around the periodic solution  $\bar{x}(\cdot)$**  if and only if the set*

$$\bigcup_{t \in [0, +\infty)} \pi\left(\mathcal{B}_t^{U,\pi^{-1}(\Omega')}(\bar{x}(0))\right)$$

*is a neighborhood of  $\pi(\bar{x}(0))$  in  $M$  for any neighborhood  $\Omega'$  of  $\pi(\bar{x}(0))$  in  $M$ .*

In part III, since (3) holds, the definition may be re-formulated using the product form (6) locally on some open  $O \subset M$  containing  $\bar{I} = \pi(\bar{x}(0))$  (in which the solution  $t \mapsto \bar{x}(t)$  reads  $t \mapsto (\bar{I}, \bar{\varphi} + \omega(\bar{I})t)$ ): System (6), or (1), is *locally controllable around the periodic solution  $\bar{I}$*  (it does not depend on  $\bar{\varphi}$ ) if and only if the set

$$(11) \quad \{I(t_f), t_f \in [0, +\infty) \text{ and } t \mapsto (I(t), \varphi(t), u(t)) \text{ solution of (6) on } [0, t_f] \text{ such} \\ \text{that } I(0) = \bar{I} \text{ and } I(t) \in \Omega' \text{ for all } t \text{ in } [0, t_f]\}$$

is a neighborhood of  $\bar{I}$  in  $M$  for any neighborhood  $\Omega'$  of  $\bar{I}$  in  $M$ .

*Remark 6.* We give necessary conditions for these properties in Section 2.4; while Theorems 13 and 17 in Sections 2.2 and 2.3 give sufficient conditions, when  $U$  is convex and compact, for stronger local controllability properties. These properties require, for instance, instead of local controllability around  $\bar{x}(\cdot)$  as in Definition 5-I above, that “for any  $\varepsilon > 0$ ,  $\mathcal{A}_{t_f}^{\varepsilon U}(\bar{x}(0))$  is a neighborhood of  $\bar{x}(t_f)$ ”. This implies the above “local controllability around  $\bar{x}(\cdot)$ ” because, for any neighborhood  $\Omega$  of  $\Gamma$ , one has  $\mathcal{A}_{t_f}^{\varepsilon U}(x_0) \subset \mathcal{B}_{t_f}^{U,\Omega}(x_0)$  for  $\varepsilon > 0$  small enough, from continuous dependence of the solutions on the control, and the fact that  $\varepsilon U \subset U$  if  $\varepsilon \in [0, 1]$ .

*Remark 7.* While Definition 3 of global controllability is classical, local controllability is a moving notion in the literature, so that Definition 5 agrees with only part

of the literature. Two remarks are in order.

- In our setting, local controllability is at the same time weaker (it only deals with reaching some neighborhood of the end point of a trajectory) and stronger (trajectories are required to remain close to the reference one) than global controllability. This is illustrated at the end of [Example 24](#).
- We believe that a “local” property of a control system at some locus should only depend on the description of this system on arbitrary small neighborhoods of that locus. In this sense, [Definition 5](#) is local with respect to the state only: System (1) is locally controllable around the solution  $\bar{x}(\cdot)$  if and only if any system  $\dot{x} = f(x, u)$  such that  $f(x, u) = X^0(x) + \sum u_k X^k(x)$  for  $(x, u)$  in  $\mathcal{O} \times U$ , with  $\mathcal{O}$  an open neighborhood of  $\bar{x}([0, t_f])$ , is locally controllable around the solution  $\bar{x}(\cdot)$ .

**1.3. Some families of vector fields.** Let  $V(\mathcal{M})$  be the set of smooth (*i.e.* either  $C^\infty$  or  $C^\omega$ ) vector fields on  $\mathcal{M}$ ; the Lie bracket of two elements of  $V(\mathcal{M})$  (defined in any advanced calculus textbook) is also in  $V(\mathcal{M})$ , making  $V(\mathcal{M})$  a Lie algebra over the field  $\mathbb{R}$ . A family of vector fields is a subset  $\mathcal{F} \subset V(\mathcal{M})$ ; we denote by  $\mathcal{F}(x)$  the subset of  $T_x\mathcal{M}$  made of the values at  $x$  of vector fields in  $\mathcal{F}$ :

$$(12) \quad \mathcal{F}(x) = \{X(x), X \in \mathcal{F}\} \subset T_x\mathcal{M},$$

and by  $\text{Span } \mathcal{F}(x)$  the vector subspace of  $T_x\mathcal{M}$  spanned by  $\mathcal{F}(x)$ . Finally, we denote by  $\mathcal{A}_{\mathcal{F}}(x)$  the *accessible set from  $x$  of this family*, *i.e.* the set of points that can be reached from  $x$  by following successively the flow of a finite number of vector fields in  $\mathcal{F}$ , each for a certain positive time; this notion differs from  $\mathcal{A}^U(x)$  in (9), attached to the control system (1), that allows arbitrary  $L_{\text{loc}}^\infty$  controls.

Let us define two families of vector fields from  $X^0, \dots, X^m$  defining System (1):

$$(13) \quad \mathcal{L} = \left\{ [X^{i_k}, [X^{i_{k-1}}, [\dots [X^{i_2}, X^{i_1}] \dots]]], \right. \\ \left. (i_1, \dots, i_k) \in \{0, \dots, m\}^k, k \in \mathbb{N} \setminus \{0\} \right\},$$

$$(14) \quad \mathcal{F}_0 = \{\text{ad}_{X^0}^j X^k, k \in \{1, \dots, m\}, j \in \mathbb{N}\}.$$

Recall that  $\text{ad}_X^j Y$  is defined by  $\text{ad}_X^0 Y = Y$ , and  $\text{ad}_X^j Y = [X, \text{ad}_X^{j-1} Y]$ .  $\mathcal{L}$  is used in [Section 1.4](#) to discuss literature, and  $\mathcal{F}_0$  in [Section 3](#) to discuss our results.

**DEFINITION 8** (Bracket generating property). *The control system (1), or the family of vector fields  $\{X^0, \dots, X^m\}$ , is called bracket generating at point  $x$  if and only if  $\text{Span } \mathcal{L}(x) = T_x\mathcal{M}$ , and bracket generating if this is true for all  $x$  in  $\mathcal{M}$ .*

Being bracket generating is also called the Lie Algebra Rank Condition (LARC), see [\[23\]](#). In most of the literature (*e.g.* [\[1, chapter 8\]](#) or [\[11, chapter 3\]](#)), one proves controllability properties by piecewise constant controls, *i.e.* analyzes  $\mathcal{A}_{\mathcal{G}}(x)$  for the family of vector fields

$$(15) \quad \mathcal{G} = \{X^0 + u_1 X^1 + \dots + u_m X^m, (u_1, \dots, u_m) \in U\}.$$

See the lines after (12) for a definition of  $\mathcal{A}_{\mathcal{G}}(x)$ , which is also the set of points that can be reached, for the Control system (1), with piecewise constant  $U$ -valued controls, to the effect that  $\mathcal{A}_{\mathcal{G}}(x) \subset \mathcal{A}^U(x)$ . If  $U$  satisfies (2), then  $\text{Span}\{X^0, \dots, X^m\}(x) = \text{Span } \mathcal{G}(x)$  for all  $x$ , so that  $\mathcal{G}$  is bracket generating if and only if  $\{X^0, \dots, X^m\}$  is.

**1.4. Controllability when  $\text{conv } U$  is a neighborhood of the origin, state of the art.** It is natural to assume that the family  $\{X^0, \dots, X^m\}$  is bracket generating because this property is (for example according to the Sussman’s Orbit Theorem [\[23\]](#)) necessary for controllability if the vector fields are real analytic, which



is a reasonable framework although the results of the present paper do not require it. Bracket generating is not however sufficient, it rather implies a weaker property, where the accessible set from a point has a nonempty topological interior, see the work of Sussmann and Jurdjevic [25] or Krener [15].

One case where the bracket generating property, assuming that  $U$  is a neighborhood of the origin, implies controllability is the one of driftless systems, namely Systems (1) with  $X^0 = 0$ , see, *e.g.*, the work of Lobry [19]. This is however far from Assumption (3), where the drift is assumed to be non-vanishing. It turns out that controllability is also obtained under this periodicity assumption, or the more general property of Poisson stability. Recall that, for a complete vector field  $X^0$  on  $\mathcal{M}$ , a point  $x \in \mathcal{M}$  is said to be *Poisson stable* for  $X^0$  if there exists a sequence of positive times  $t_n \rightarrow \infty$  such that  $\exp(t_n X^0)(x) \rightarrow x$  when  $n \rightarrow \infty$ ;  $X^0$  is said to be Poisson stable if there is a dense subset of such points. It turns out that many physical dynamical systems have this property; this makes the following result quite useful.

**THEOREM 9** (Bonnard, 1981, [2]). *System (1) is globally controllable if*  
(i) *the vector field  $X^0$  is Poisson stable,*  
(ii) *the family  $\{X^0, X^1, \dots, X^m\}$  is bracket generating, and*  
(iii)  *$\text{conv } U$  is a neighborhood of 0 in  $\mathbb{R}^m$ .*

It is stated in this form in the textbook [11] (Chapter 4, Theorem 5) or in the original reference, that mentions techniques due to [12]. It has been widely used, for instance to prove controllability prior to solving an optimal control problem, like in [4, 18] for the controlled Kepler problem. Assumption (iii) is essential to Theorem 9, as evidenced by a very simple academic example of the form (1) with  $\mathcal{M} = \mathbb{R} \times \mathbb{S}^1$  and one scalar control  $u$  ( $m = 1$ ):  $\dot{I} = u$ ,  $\dot{\varphi} = 1$ ,  $u \in U = [0, 1]$ ; conditions (i) and (ii) are satisfied, (iii) is not, and controllability cannot hold because  $I \in \mathbb{R}$  cannot decrease. Our Theorem 11 can be seen as a generalization of Theorem 9 to the case where (iii) does not hold. There we keep (i) (and even the stronger Condition (3)) but we have to replace (ii) by a stronger condition, involving not only the vector fields but also the set  $U$ . Next section introduces the necessary tools.

**1.5. Further constructions, not based on Lie brackets.** To the vector fields  $X^0, \dots, X^m$  and the set  $U$  defining System (1), we associate, for any  $x$  in  $\mathcal{M}$  and any real numbers  $\tau$ , or  $t_1 < t_2$ , the following subsets<sup>4</sup> of  $T_x \mathcal{M}$ , where the pushforward notation “ $*$ ” was recalled in (5):

$$(16) \quad E_\tau^U(x) = \left\{ \sum_{k=1}^m u_k (\exp(-\tau X^0)_* X^k)(x), u \in U \right\} \subset T_x \mathcal{M},$$

$$(17) \quad E_{[t_1, t_2]}^U(x) = \bigcup_{\tau \in [t_1, t_2]} E_\tau^U(x) \subset T_x \mathcal{M},$$

and, when the periodicity assumption (3) holds and defines  $T(x)$ ,

$$(18) \quad E^U(x) = E_{[0, T(x)]}^U(x) = \bigcup_{\tau \in [0, T(x)]} E_\tau^U(x) \subset T_x \mathcal{M}.$$

Note that  $E^U(\cdot)$  and the function  $T(\cdot)$ , are invariant under the flow of  $X^0$ .

<sup>4</sup>These subsets are related to control variations in the proof of the Pontryagin Maximum Principle. For instance, the condition of [1, Lemma 12.6] is in terms of  $E_{[t_1, t_2]}^U(x)$  (with  $\tilde{u} \equiv 0$ ,  $\mathcal{T} = [t_1, t_2]$ ,  $g_{\tau, u}(x) = \sum_{k=1}^m u_k (\exp(-\tau X^0)_* X^k)(x)$ ). Our Theorem 13 can be viewed as a consequence of that lemma in the simpler case where the reference control is smooth.



*Remark 10.* In the particular case when the fibration  $\pi$  is the one associated with the orbits of the periodic drift, using the product presentation  $(I, \varphi)$  from Equations (4)–(6), with  $\varphi \in \mathbb{S}^1$  and  $I \in O \subset M$ , one may simplify the expressions in (16), (17) and (18) because the construction leads to  $X^0 = \omega \partial / \partial \varphi$ , hence the explicit expression of the flow of  $X^0$ :  $\exp(tX^0)(I, \varphi) = (I, \varphi + \omega(I)t)$ , so that  $(\exp(-\tau X^0)_* Z)(I, \varphi) = Z(I, \varphi + \omega(I)\tau) - \tau Z\omega(I, \varphi + \omega(I)\tau) \partial / \partial \varphi$  for any vector field  $Z$  ( $Z\omega$  stands for the Lie derivative along  $Z$  of  $\omega : O \times \mathbb{S}^1 \rightarrow \mathbb{R}$ ). In particular, with  $x = (I, \varphi)$ , denoting by  $\pi'(x)$  the derivative (tangent map) of the fibration  $\pi$  at point  $x$  (i.e. the linear projection on the first factor in  $T_x \mathcal{M} = T_I M \times \mathbb{R}$ ), the projection of  $E^U(x)$  has a simple expression:

$$(19) \quad \begin{aligned} \pi'(I, \varphi)(E^U(I, \varphi)) &= \check{E}^U(I) \subset T_I M \quad \text{with} \\ \check{E}^U(I) &= \left\{ \sum_{k=1}^m u_k F^k(I, \varphi), \quad u \in U, \varphi \in \mathbb{S}^1 \right\}. \end{aligned}$$

Here, invariance of  $E^U(\cdot)$  by the flow of  $X^0$  implies that  $\check{E}^U$  does not depend on  $\varphi$ .

**2. Global and local controllability results.** All results are concerned with System (1) (or with System (6) in product form if  $X^0$  satisfies (3)). A few remarks are in order on the assumptions in these results.

*Smoothness.* All vector fields are assumed to be smooth, i.e. infinitely differentiable, although the reader may check that all proofs work if smooth is replaced by differentiable with a locally Lipschitz continuous first derivative.

*Periodicity of solutions of the drift.* This Property (3) of  $X^0$ , discussed in Section 1.1, is assumed in Theorem 11, that was a primary motivation of this work, but is *not* a running assumption. Also, we indicate when  $\pi : \mathcal{M} \rightarrow M$  is the fibration induced by (3), with base  $M$  equal to the respective orbit space; it is otherwise an arbitrary fibration, unrelated to  $X^0$ , with  $M$  of arbitrary dimension  $n \leq d$ .

*The control set  $U$ .* It is always assumed to satisfy (2). We also need  $U$  to be convex and compact in Sections 2.2 and 2.3 (Theorems 13 and 17, see Remark 15).

*Notation.* For a subset  $S$  of a real vector space  $E$ , its *conic hull* is the smallest convex cone containing  $S$ , i.e. the set of *conic combinations* of elements of  $S$ :

$$(20) \quad \text{cone } S = \{ \lambda_1 x_1 + \dots + \lambda_k x_k, \ x_i \in S, \ \lambda_i \in [0, +\infty), \ i \in \{1 \dots k\}, \ k \in \mathbb{N} \}.$$

**2.1. Global controllability under the periodicity assumption on the drift.** Let us now state our generalization of the now classical Theorem 9 on global controllability. Here, the fibration  $\pi$  is the one induced by the periodic drift, see Proposition 1. Global controllability is as in Definition 3; as noticed in Remark 4, it is equivalent, in the present case, to “global controllability with respect to  $\pi$ ”.

**THEOREM 11.** *Assume that  $U$  satisfies (2), that  $X^0$  satisfies (3) (periodicity of all solutions), inducing a fibration  $\pi$  as in Proposition 1, and that*

$$(21) \quad \pi'(x)(\text{cone } E^U(x)) = T_{\pi(x)} M \quad \text{for all } x \text{ in } \mathcal{M}.$$

*Then System (1) is globally controllable, and a fortiori globally controllable with respect to  $\pi$ . Under the same conditions, global controllability holds as well in backward time, i.e.  $\mathcal{A}_-^U(x) = \mathcal{M}$  for all  $x$  in  $\mathcal{M}$ .*

Using the local product form (6) and the notation  $\check{E}$  from (19), Condition (21) for all  $x$  in  $\mathcal{M}$  equates to the following relation for all  $I$ :

$$(22) \quad \text{cone } \check{E}^U(I) = T_I M \quad \text{for all } I \text{ in } M.$$

*Proof.* We prove controllability in forward time; backward time follows upon changing each vector field  $X^i$  into  $-X^i$ , preserving all assumptions of the theorem. The accessible set  $\mathcal{A}_{\mathcal{F}}(x)$  of a family  $\mathcal{F}$  of vector fields was defined in the first paragraph of [Section 1.3](#). The family  $\mathcal{G}$  was defined in [\(15\)](#), and we noted there that it satisfies  $\mathcal{A}_{\mathcal{G}}(x) \subset \mathcal{A}^U(x)$  for all  $x$ . The conclusion of the theorem is then obviously implied by  $\mathcal{A}_{\mathcal{G}}(x) = \mathcal{M}$  for all  $x$  in  $\mathcal{M}$ , that we are going to prove.

Define, from  $\mathcal{G}$ , the families  $\mathcal{G}_1$  and  $\mathcal{G}_2$  by

$$(23) \quad \mathcal{G}_1 = \mathcal{G} \cup \{-X^0\} \quad \text{and} \quad \mathcal{G}_2 = \{\exp(tX^0)_*X, X \in \mathcal{G}_1, t \in \mathbb{R}\},$$

where the push-forward notation  $\exp(tX^0)_*X$  was recalled in [\(5\)](#). Obviously,  $\mathcal{G} \subset \mathcal{G}_1 \subset \mathcal{G}_2$ . Let us first prove that, for all  $x$ ,

$$(24) \quad \mathcal{A}_{\mathcal{G}}(x) = \mathcal{A}_{\mathcal{G}_1}(x) = \mathcal{A}_{\mathcal{G}_2}(x).$$

The family  $\mathcal{G}$  contains  $X^0$  because  $0 \in U$ , see [\(2\)](#); hence  $\exp(tX^0)(x) \in \mathcal{A}_{\mathcal{G}}(x)$  for any  $x$  in  $\mathcal{M}$  and  $t \geq 0$ ; it is also true if  $t < 0$  because [\(3\)](#) implies  $\exp(tX^0)(x) = \exp((t+kT)X^0)(x)$ , where the integer  $k$  can be taken large enough that  $t+kT(x) \geq 0$ ; the reverse inclusion being clear, this proves the first equality. Turning to the second one, consider some  $y$  in  $\mathcal{A}_{\mathcal{G}_2}(x)$ . By definition,  $y = \exp(s_1Y^1) \circ \dots \circ \exp(s_NY^N)(x)$  for some positive integer  $N$ , and some positive real numbers  $s_j$  and vector fields  $Y^j$ ,  $j \in \{1, \dots, N\}$ , belonging to  $\mathcal{G}_2$ , such that, for some real number  $t_j$  and some  $Z^j \in \mathcal{G}_1$ ,  $Y^j = \exp(t_jX^0)_*Z^j$ . Since the latter implies  $\exp(s_jY^j) = \exp(t_jX^0) \circ \exp(s_1Z^1) \circ \exp(-t_jX^0)$ , and both  $X^0$  and  $-X^0$  belong to  $\mathcal{G}_1$ , the above yields  $y$  as the image of  $x$  by a composition of  $3N$  flows in positive time of vector fields belonging to  $\mathcal{G}_1$ . Hence  $y \in \mathcal{A}_{\mathcal{G}_1}(x)$ , which proves the second equality in [\(24\)](#), the reverse inclusion being, again, obvious. This ends the proof of [\(24\)](#).

For any  $x$  in  $\mathcal{M}$ , [\(21\)](#) implies  $\text{cone } \mathcal{G}_2(x) = T_x\mathcal{M}$ , hence there are  $d+1$  vector fields  $Y^0, \dots, Y^d$  in  $\mathcal{G}_2$  such that  $Y^0(x), \dots, Y^d(x)$  generate  $T_x\mathcal{M}$  by conic combinations. Let  $\phi: (-\alpha, \alpha)^d \rightarrow \mathcal{M}$  (with  $\alpha > 0$  small enough so that the flows are defined) be the smooth map given by  $\phi(s_0, \dots, s_d) = \exp(s_0Y^0) \circ \dots \circ \exp(s_dY^d)(x)$ , and apply [Lemma 12](#) (see below) with  $k = d+1$ ,  $W^k = (-\alpha, \alpha)^{d+1}$ ,  $N = \mathcal{M}$  and  $G = \phi$ . Since  $\phi'(0)(\mathbb{R}_+^k) = T_{\phi(0)}\mathcal{M}$  (this follows from the fact that  $\partial\phi/\partial s_i(0) = Y^i(x)$ ,  $i \in \{0 \dots d\}$ , and the conic combination property of  $Y^0 \dots Y^d$  stated above), the lemma tells us that  $\phi(W_+^k)$  is a neighborhood of  $x$ . Since  $\phi(W_+^k) = \phi([0, \alpha)^{d+1})$  is contained in  $\mathcal{A}_{\mathcal{G}_2}(x) = \mathcal{A}_{\mathcal{G}}(x)$  (see [\(23\)](#)), we have shown the following property, valid also for  $\mathcal{A}_{-\mathcal{G}}(x)$  by changing each  $X^i$  into  $-X^i$ :

$$(25) \quad \text{for all } x \text{ in } \mathcal{M}, \mathcal{A}_{\mathcal{G}}(x) \text{ and } \mathcal{A}_{-\mathcal{G}}(x) \text{ are neighborhoods of } x \text{ in } \mathcal{M}.$$

Any  $\mathcal{A}_{\mathcal{G}}(x)$  is open because, for  $y$  in  $\mathcal{A}_{\mathcal{G}}(x)$ , applying [\(25\)](#) to  $\mathcal{A}_{\mathcal{G}}(y)$ , the latter is a neighborhood of  $y$ , contained in  $\mathcal{A}_{\mathcal{G}}(x)$  by definition. Any  $\mathcal{A}_{\mathcal{G}}(x)$  is also closed: picking  $y$  in its topological closure,  $\mathcal{A}_{-\mathcal{G}}(y)$  is a neighborhood of  $y$  according to [\(25\)](#), so  $\mathcal{A}_{-\mathcal{G}}(y)$  intersects  $\mathcal{A}_{\mathcal{G}}(x)$  and  $y$  must belong to  $\mathcal{A}_{\mathcal{G}}(x)$ . Being an open, closed, nonempty subset of the connected manifold  $\mathcal{M}$ ,  $\mathcal{A}_{-\mathcal{G}}(y)$  is equal to  $\mathcal{M}$ .  $\square$

The following lemma is used in the above proof, and in the proof of [Theorem 13](#).

LEMMA 12 (Open mapping lemma, see [\[24, Theorem 2.1.3\]](#), [\[1, Lemma 12.4\]](#), or [\[8, Theorem 5\]](#)). *With  $k$  a positive integer, let  $\mathbb{R}_+^k$  be the positive orthant in  $\mathbb{R}^k$ :*

$$(26) \quad \mathbb{R}_+^k = \{(w_1, \dots, w_k) \in \mathbb{R}^k, w_i \geq 0, i = 1 \dots k\},$$

$W^k$  an open neighborhood of  $0 \in \mathbb{R}^k$ , and  $W_+^k = W^k \cap \mathbb{R}_+^k$ . If  $G: W^k \rightarrow N$  is a smooth map, with  $N$  a smooth manifold, such that  $G'(0) \cdot \mathbb{R}_+^k = T_{G(0)}N$ , then  $G(V \cap W_+^k)$  is a neighborhood of  $G(0)$  in  $N$  for any neighborhood  $V$  of the origin in  $\mathbb{R}^k$ .

We state the smooth case because it is what we need. The proof in [1, 24] uses Brouwer's fixed point theorem under the much weaker assumption that  $G$  is continuous and differentiable at zero (Lipschitz continuous in [1]). In [8],  $\mathbb{R}^k$  and  $N$  are replaced by Banach spaces under a condition slightly weaker than continuous differentiability; that result also implies Lemma 12.

## 2.2. Local controllability around a particular solution. Let us consider

$$(27) \quad t \mapsto \bar{x}(t), \text{ solution of } \dot{x} = X^0(x), \text{ defined on the time-interval } [0, t_f]$$

(the real number  $t_f$  may be negative,  $[0, t_f]$  should then be understood as  $[t_f, 0]$ ), and define the following subset of  $T_{\bar{x}(0)}\mathcal{M}$ , attached to the solution  $\bar{x}(\cdot)$ :

$$(28) \quad \bar{E} = E_{[0, t_f]}^U(\bar{x}(0)),$$

with  $E_{[0, t_f]}^U(x)$  defined in (17). Next theorem gives, in terms of  $\bar{E}$ , sufficient conditions for local controllability along  $\bar{x}(\cdot)$ .

Local controllability (with respect to  $\pi$  or not) is introduced in Definition 5, and Remark 6 explains why Conditions (30) and (32) are stronger. Part I of Definition 5 is a particular case of Part II, with  $M = \mathcal{M}$  and  $\pi = \text{Id}$ ; it is stated independently for readability ( $\pi = \text{Id}$  yields the simpler sufficient Condition (29)).

**THEOREM 13.** *Consider System (1) with  $U$  a convex compact subset of  $\mathbb{R}^m$  containing the origin, and let  $\bar{x}(\cdot)$  and  $\bar{E}$  be as described above.*

I. *If*

$$(29) \quad \text{cone } \bar{E} = T_{\bar{x}(0)}\mathcal{M},$$

*then System (1) is locally controllable around  $\bar{x}(\cdot)$  and, more precisely,*

$$(30) \quad \mathcal{A}_{t_f}^{\varepsilon U}(\bar{x}(0)) \text{ is a neighborhood of } \bar{x}(t_f) \text{ in } \mathcal{M} \text{ for any } \varepsilon > 0.$$

II. *Let  $\pi: \mathcal{M} \rightarrow M$  be a smooth fibration with  $M$  of dimension  $n \leq d$ . If*

$$(31) \quad (\pi \circ \exp(t_f X^0))'(\bar{x}(0))(\text{cone } \bar{E}) = T_{\pi(\bar{x}(t_f))}M,$$

*then System (1) is locally controllable around  $\bar{x}(\cdot)$  with respect to  $\pi$ ; more precisely,*

$$(32) \quad \pi(\mathcal{A}_{t_f}^{\varepsilon U}(\bar{x}(0))) \text{ is a neighborhood of } \pi(\bar{x}(t_f)) \text{ in } M \text{ for any } \varepsilon > 0.$$

**Remark 14.** Since  $\bar{x}(\cdot)$  and  $t_f$  are fixed in the theorem, we use the short notation  $\bar{E}$ ; this should not hide the dependence of this set on the initial condition and more interestingly on  $t_f$ . It is clear from (17) that  $E_{[0, t]}^U(x)$  grows with  $t > 0$  for fixed  $x$ ; hence, if one follows the same solution (assumed to be defined on  $[0, +\infty)$ ), either Condition (29) is satisfied for all  $t_f$  or it is not satisfied for small values of  $t_f$ , but is satisfied, if ever, for  $t_f$  larger than a certain value. Note that it is not necessarily true for (31), unless  $\pi$  is invariant by the flow of  $X^0$  ( $\pi \circ \exp(tX^0) = \pi$ ), which is indeed the case when (3) holds and  $\pi$  is associated to periodic orbits of  $X^0$ .

*Remark 15* (assumptions on  $U$ ). **Theorems 13** and **17**, unlike **Theorem 11**, assume  $U$  convex and compact. Compactness is natural; it a priori avoids unbounded controls and allows  $\varepsilon U$  to be small for small  $\varepsilon > 0$ . Convexity is more restrictive, it is technically needed in the proofs, and could probably be relaxed. If  $U$  is not convex, the same conclusions hold, replacing  $\mathcal{A}_{t_f}^{\varepsilon U}(\bar{x}(0))$  with its topological closure.

Before proving **Theorem 13**, let us recall that the linear approximation of the control system (1) along the solution  $\bar{x}(\cdot)$  is the following linear time-varying system:

$$(33) \quad \begin{aligned} \dot{\xi} &= \sum_{k=1}^m \delta u_k (\exp(-tX^0)_* X^k)(\bar{x}(0)), \\ \delta x(t) &= \exp(tX^0)'(\bar{x}(0)) \xi(t), \end{aligned}$$

with initial condition  $\xi(0) = 0$ , where  $\delta u = (\delta u_1, \dots, \delta u_m)$  is a small variation of the control around zero and  $\delta x(t)$  the corresponding small variation of the state around  $\bar{x}(t)$  (in other words,  $t \mapsto (\bar{x}(t) + \delta x(t), \delta u(t))$  is a solution of (1)). Note that (33) is the intrinsic way of writing  $\dot{\delta x} = \frac{\partial X^0}{\partial x}(\bar{x}(t)) \delta x + \sum_1^m \delta u_k X^k(\bar{x}(t))$ , because  $\xi$  is an element of the tangent space  $T_{\bar{x}(0)}$  for all  $t$ , contrary to  $\delta x$ . One may constrain the control  $\delta u = (\delta u_1, \dots, \delta u_m)$  to some  $U \subset \mathbb{R}^m$  and define, for this linear control-constrained time-varying system, the accessible set for initial time zero, initial state  $x_0$  and final time  $t$  as the following subset of  $T_{\bar{x}(t)}\mathcal{M}$ , see also (52) in **Appendix A**:

$$(34) \quad \begin{aligned} \mathbf{A}_{0,t}^U(\delta x_0) &= \{\delta x(t), \text{ with } s \mapsto (\xi(s), \delta u(s), \delta x(s)) \text{ solution of (33) on } [0, t], \\ &\quad \delta x(0) = \delta x_0, \delta u([0, t]) \subset U\}. \end{aligned}$$

The following result states that Condition (29) is equivalent to controllability of the constrained linear approximation around  $\bar{x}(\cdot)$  (obviously it also applies to (21) and (22)).

- LEMMA 16.** **I.** *The linear constrained attainable set  $\mathbf{A}_{0,t_f}^U(0)$  is a neighborhood of the origin in  $T_{\bar{x}(t_f)}\mathcal{M}$  if and only if Condition (29) holds.*
- II.** *The set  $\pi'(\bar{x}(t_f))(\mathbf{A}_{0,t_f}^U(0))$  is a neighborhood of the origin in  $T_{\pi(\bar{x}(t_f))}M$  if and only if Condition (31) holds.*

*Proof.* This follows from **Proposition 26** for point I and **Proposition 27** for point II in the Appendix, applied with  $t_0 = 0$ , with (33) playing the role of (54). In coordinates, the  $k^{\text{th}}$  column of  $B(s)$  is the coordinate vector of  $X^k(\bar{x}(s))$ , and the  $k^{\text{th}}$  column of  $\phi(0, s)B(s)$  is the coordinate vector of  $(\exp(-sX^0)_* X^k)(\bar{x}(0))$ .  $\square$

*Proof of Theorem 13.* We give a proof of point II. For a proof of point I (particular case of point II) without reference to the projection  $\pi$ , replace  $M$  with  $\mathcal{M}$ ,  $n$  with  $d$ ,  $\pi$  with  $Id$ ,  $(\pi \circ \exp(t_f X^0))'(\bar{x}(0))(\bar{E})$  with  $\bar{E}$ , and  $\pi'(\bar{x}(t_f))(\mathbf{A}_{0,t_f}^U(0))$  with  $\mathbf{A}_{0,t_f}^U(0)$ . Also, according to **Remark 6**, Conditions (30) and (32) for arbitrarily small  $\varepsilon > 0$  imply local controllability, “with respect to  $\pi$ ” or not; hence we now prove only (32).

According to **Lemma 16**, point II, Condition (31) implies that  $\pi'(\bar{x}(t_f))(\mathbf{A}_{0,t_f}^U(0))$ , projection of the accessible set of the linearized system (33) in time  $t_f$ , is a neighborhood of the origin in  $T_{\pi(\bar{x}(t_f))}M$ . Let  $e_0, \dots, e_n$  be the vertices of a convex polyhedron in  $T_{\pi(\bar{x}(t_f))}M$  that is both contained in  $\pi'(\bar{x}(t_f))(\mathbf{A}_{0,t_f}^U(0))$  and a neighborhood of the origin. Since  $e_0, \dots, e_n$  are in  $\pi'(\bar{x}(t_f))(\mathbf{A}_{0,t_f}^U(0))$ , there are some  $\hat{e}_0, \dots, \hat{e}_n$  in  $\mathbf{A}_{0,t_f}^U(0)$  such that  $e_i = \pi'(\bar{x}(t_f))(\hat{e}_i)$ , and there also exist  $u_0, \dots, u_n$ , some  $U$ -valued  $L_{\text{loc}}^\infty$ -controls, defined on  $[0, t_f]$ , that steer the origin to  $\hat{e}_0, \dots, \hat{e}_n$  respectively, in time  $t_f$ , for the linear time-varying system (33).

Going back to the nonlinear system (1), and possibly multiplying the vector fields  $X^0, \dots, X^m$  by some smooth cut-off function equal to 1 in a neighborhood of  $\bar{x}([0, t_f])$  to guarantee that solutions all exist until  $t_f$ , and in fact until  $+\infty$ , let  $\mathcal{E} : L^\infty([0, t_f], \mathbb{R}^m) \rightarrow \mathcal{M}$  be the end-point mapping at time  $t_f$  starting from  $\bar{x}(0)$  for the nonlinear system (1). Let also  $\mathcal{E}^{\text{lin}}$  be the end-point mapping at time  $t_f$  starting from the origin for the linear system (33). It is well known [11, 1] that  $\mathcal{E}$  is continuously differentiable and that its derivative at the zero control is  $\mathcal{E}^{\text{lin}}$ . Consider the continuously differentiable map  $G : \mathbb{R}^{n+1} \rightarrow M$  defined by

$$(35) \quad G(\lambda) = G(\lambda_0, \dots, \lambda_n) = \pi \circ \mathcal{E} \left( \sum_{i=0}^n \lambda_i \mathbf{u}_i \right).$$

Since  $\mathcal{E}$  maps the zero control to  $\bar{x}(t_f)$ , and  $\frac{\partial G}{\partial \lambda_i}(0, \dots, 0)$ ,  $i \in \{0, \dots, n\}$ , is equal to  $\pi'(\bar{x}(t_f))(\mathcal{E}^{\text{lin}}(\mathbf{u}_i))$  where, by construction,  $\mathcal{E}^{\text{lin}}(\mathbf{u}_i) = \hat{e}_i$  and  $\pi'(\bar{x}(t_f))(\hat{e}_i) = e_i$ , one has

$$(36) \quad G(0, \dots, 0) = \pi(\bar{x}(t_f)), \quad \frac{\partial G}{\partial \lambda_i}(0, \dots, 0) = e_i, \quad i \in \{0, \dots, n\}.$$

The second relation implies that the linear map  $G'(0) : \mathbb{R}^{n+1} \rightarrow T_{\pi(\bar{x}(t_f))}M$  satisfies

$$(37) \quad G'(0)(\mathbb{R}_+^{n+1}) = T_{\pi(\bar{x}(t_f))}M$$

( $\mathbb{R}_+^{n+1}$  is defined in (26)) because  $\text{cone}\{e_0, \dots, e_n\} = T_{\pi(\bar{x}(t_f))}M$ . Now, applying Lemma 12 with  $k = n+1$ ,  $N = M$ ,  $W_k = \mathbb{R}^{n+1}$  and  $V_\varepsilon = \{\lambda \in \mathbb{R}^{n+1}, \lambda_0 + \dots + \lambda_n \leq \varepsilon\}$ , one obtains that  $G(\mathbb{R}_+^{n+1} \cap V_\varepsilon)$  is a neighborhood of  $\pi(\bar{x}(t_f))$  for any  $\varepsilon$ , and this yields (32) because, since  $U$  is convex and contains the origin, the control  $\sum_{i=0}^n \lambda_i \mathbf{u}_i$  takes values in  $\varepsilon U$  if each  $\mathbf{u}_i$  takes values in  $U$ , and  $(\lambda_0, \dots, \lambda_n)$  is in  $\mathbb{R}_+^{n+1} \cap V_\varepsilon$ .  $\square$

**2.3. Application to periodic solutions of the drift.** Theorem 13 applies, in particular, to periodic solutions of the drift vector field under condition (3), yielding, as a direct consequence (take  $t_f = T(x)$ ,  $\bar{x}(0) = \bar{x}(t_f) = x$ ) the following interesting complement to Theorem 11.

**THEOREM 17.** *Assume that the vector field  $X^0$  satisfies the periodicity Assumption (3), and that  $U$  is convex compact and satisfies Conditions (2).*

- I.** *If  $\text{cone } E^U(x) = T_x \mathcal{M}$ , then  $\mathcal{A}_{T(x)}^{\varepsilon U}(x)$  is a neighborhood of  $x$  in  $\mathcal{M}$  for all  $\varepsilon > 0$ .*
- II.** *Let  $\pi$  be as in Theorem 11. If  $\pi'(x)(\text{cone } E^U(x)) = T_{\pi(x)}M$ , then  $\pi(\mathcal{A}_{T(x)}^{\varepsilon U}(x))$  is a neighborhood of  $\pi(x)$  in  $M$  for all  $\varepsilon > 0$ .*

*The same conclusions hold for  $\mathcal{A}_{-T(x)}^{\varepsilon U}(x)$  and  $\pi(\mathcal{A}_{-T(x)}^{\varepsilon U}(x))$  (backward time) under the same assumptions.*

This theorem is more precise (locally) than Theorem 11 in two ways: here, points in a neighborhood of the initial orbit may be reached over one period without leaving some neighborhood of the initial orbit while Theorem 11 gives no estimate on final time, and allows trajectories to go arbitrarily far. Since the final time is prescribed, Remark 4 does not apply and parts I and II have no reason to be equivalent. Finally, global controllability can be recovered from Theorem 17 as seen in the proof below.

*Alternative proof of Theorem 11 (under the assumption that  $U$  is convex).*  
According to Theorem 17, part II,  $\pi(\mathcal{A}_{T(y)}^U(y))$  is a neighborhood of  $\pi(y)$  in  $M$  for any  $y$  in  $\mathcal{M}$ , and so is  $\pi(\mathcal{A}_{-T(y)}^U(y))$ . For  $x$  in  $\mathcal{M}$ , let us show that  $\mathcal{A}^U(x)$  is both open

and closed in  $\mathcal{M}$ ; this implies the theorem by connectedness of  $\mathcal{M}$ . First, consider  $y$  in  $\mathcal{A}^U(x)$ , so that  $\mathcal{A}^U(x)$  contains  $\mathcal{A}_{T(y)}^U(y)$ , and  $\pi(\mathcal{A}^U(x))$  contains  $\pi(\mathcal{A}_{T(y)}^U(y))$ , that was just pointed out as a neighborhood of  $\pi(y)$ , hence  $\pi(\mathcal{A}^U(x))$  is a neighborhood of  $\pi(y)$ , and, using [Remark 2 \(Section 1.1\)](#),  $\mathcal{A}^U(x) = \pi^{-1}(\pi(\mathcal{A}^U(x)))$  is a neighborhood of  $y$ ; openness is proven. Now suppose that some  $y$  is in  $\overline{\mathcal{A}^U(x)}$ , and  $\pi(y)$  in  $\overline{\pi(\mathcal{A}^U(x))}$ . We pointed out that  $\pi(\mathcal{A}_{-T(y)}^U(y))$  is a neighborhood of  $\pi(y)$ ; hence it intersects  $\pi(\mathcal{A}^U(x))$ , *i.e.* there is some  $z$  such that  $z \in \mathcal{A}_{-T(y)}^U(y)$  and  $z' \in \mathcal{A}^U(x)$  such that  $\pi(z) = \pi(z')$ , but, according to [Remark 2](#) again,  $z' \in \mathcal{A}^U(x)$  then implies  $z \in \mathcal{A}^U(x)$ ;  $z \in \mathcal{A}^U(x)$  and  $y \in \mathcal{A}_{T(y)}^U(z)$  do imply  $y \in \mathcal{A}^U(x)$ ; this proves closedness and ends the proof of the theorem.  $\square$

**2.4. Obstructions to local controllability.** [Theorems 11, 13 and 17](#) state sufficient conditions (see [\(21\)](#), [\(22\)](#), [\(29\)](#), [\(31\)](#), and the condition in [Theorem 17](#)) that require some convex cone to be the whole tangent space. Let us generically call this cone  $C$ . The sufficient conditions are not necessary, see, *e.g.*, [Example 25](#), hence  $C$  being contained in a closed half-space is not an obstruction to controllability. This section explains that one gets an obstruction by requiring that  $\overline{C}$  (topological closure) is contained in a closed half-space and intersects the separating hyperplane (border of the half space) only at the origin. Let us recall that, for a cone  $C$  in a vector space  $V$ , the *polar cone* of  $C$  is the cone  $C^\circ = \{p \in V^*, \langle p, v \rangle \leq 0, v \in C\}$ ; a co-vector  $p$  in  $V^*$  is in the topological interior of  $C^\circ$  if and only if  $\langle p, v \rangle < 0$  for all  $v$  in  $\overline{C} \setminus \{0\}$ . Therefore, the property that a cone has a topological closure that intersects some hyperplane only at the origin is equivalent to its polar cone having nonempty interior.

[Proposition 18](#) gives an obstruction to local controllability in the case of [Theorem 13](#), [Proposition 19](#) an obstruction to the weaker local *orbital* controllability in the periodic case. Note that these obstructions to local controllability are *not* obstructions to global controllability because controllability could be obtained by “going far and coming back”. This is a well know fact, noted in [Remark 7](#) and illustrated in [Example 24](#) (on the second System [\(43\)](#), see the last paragraph of the example). For conciseness, we do not give the proof of [Proposition 18](#), to be deduced from the other proof, *mutatis mutandis*.

PROPOSITION 18. Consider System [\(1\)](#), and  $t_f$ ,  $\bar{x}(\cdot)$ ,  $\bar{E}$  as in [\(27\)](#) and [\(28\)](#).

- I. If **(i)** the interior in  $T_{\bar{x}(0)}^* \mathcal{M}$  of the polar cone to cone  $\bar{E}$  is nonempty, and **(ii)** the vectors  $X^1(\bar{x}(t)), \dots, X^m(\bar{x}(t))$  are linearly independent in  $T_{\bar{x}(t)} \mathcal{M}$  for all  $t$  in  $[0, t_f]$ , then System [\(1\)](#) is not locally controllable around the solution  $\bar{x}(\cdot)$ .
- II. Let  $\pi : \mathcal{M} \rightarrow M$  be as in [Theorem 13](#), Part II. If **(i)** the interior in  $T_{\pi(\bar{x}(t_f))}^* M$  of the polar cone to  $(\pi \circ \exp(t_f X^0))'(\bar{x}(0))(\text{cone } \bar{E})$  is nonempty, and **(ii)** the vectors  $\pi'(\bar{x}(t)) X^1(\bar{x}(t)), \dots, \pi'(\bar{x}(t)) X^m(\bar{x}(t))$  are linearly independent in  $T_{\pi(\bar{x}(t))} M$ , then System [\(1\)](#) is not locally controllable with respect to  $\pi$  around  $\bar{x}(\cdot)$ .

PROPOSITION 19. Consider System [\(1\)](#) under Assumption [\(3\)](#) on  $X^0$  and let  $\bar{x}(\cdot)$  be a particular periodic solution of  $\dot{x} = X^0(x)$  with  $\bar{I} = \pi(\bar{x}(t))$ ,  $t \in \mathbb{R}$ . If

- (i)** the polar cone to  $\text{cone}(\pi'(\bar{x}(0))E^U(\bar{x}(0)))$  has a nonempty interior in  $T_{\bar{I}} M$ , and
  - (ii)** the vectors  $\pi'(\bar{x}(t))X^1(\bar{x}(t)), \dots, \pi'(\bar{x}(t))X^m(\bar{x}(t))$  are linearly independent in  $T_{\bar{I}} M$  for all  $t$  in  $[0, T(x)]$ ,
- then System [\(1\)](#) is not locally orbitally controllable around the periodic solution  $\bar{x}(\cdot)$ .

In point (i) ([Proposition 19](#)),  $\bar{x}(0)$  may be replaced by  $\bar{x}(t)$ , for any other  $t$ , because  $\pi'(\bar{x}(t))E^U(\bar{x}(t))$  does not depend on  $t$ , see [\(18\)](#)–[\(19\)](#); this is consistent with the defi-



inition of “locally orbitally controllability” that does not take time or initial condition on the orbit into account.

*Remark 20* (Re-formulation of [Proposition 19](#) in the product form (6)). Since Condition (3) is assumed and the property is local around a periodic orbit  $\bar{I}$  (with  $\pi(\bar{x}(t)) = \bar{I}$ ,  $t \in \mathbb{R}$ ), one may re-formulate [Proposition 19](#) as follows, in the product form (6),  $(I, \varphi) \in O \times \mathbb{S}^1$  (one may not take  $O = M$  in general).

If, for some  $\bar{I}$  in  $O$ ,

- (i) the polar cone to cone  $\check{E}^U(\bar{I})$  has a nonempty interior, and
  - (ii)  $F^1(\bar{I}, \varphi), \dots, F^m(\bar{I}, \varphi)$  are linearly independent in  $T_{\bar{I}}M$  for all  $\varphi$  in  $\mathbb{S}^1$ ,
- then System (6) is not locally orbitally controllable around the periodic solution  $\bar{I}$ .

The equivalence between the two formulations is easily deduced from (4), with  $x = \Phi(I, \varphi)$ . For local orbital controllability in this setting, see (11) after [Definition 5-III](#).

*Proof of Proposition 19.* We prove the re-formulation. Let  $K = \{u \in \text{cone } U, u_1^2 + \dots + u_m^2 = 1\}$ . Fix  $\bar{p}$  in the interior of the polar cone to cone  $\check{E}^U(\bar{I})$ . Let  $\alpha < 0$  be the maximum of the smooth function  $(\varphi, u) \mapsto \langle \bar{p}, \sum_{k=1}^m u_k F^k(\bar{I}, \varphi) \rangle$  on the compact  $\mathbb{S}^1 \times K$ , where it remains negative from points (i) and (ii). Using some coordinates on an open set  $O' \subset O$  around  $\bar{I}$ , one may define the smooth function  $h : O' \rightarrow \mathbb{R}$  by  $h(I) = \langle \bar{p}, I \rangle$ , and also give a meaning to  $\langle \bar{p}, v \rangle$  if  $v \in T_I O$ ,  $I \in O'$ . Then the above implies that there is a neighborhood  $O''$  of  $\bar{I}$  in  $O'$  such that  $\langle \bar{p}, \sum_{k=1}^m u_k F^k(I, \varphi) \rangle < \alpha/2 < 0$  for all  $(I, \varphi, u)$  in  $O'' \times \mathbb{S}^1 \times K$ , hence  $\langle \bar{p}, \sum_{k=1}^m u_k F^k(I, \varphi) \rangle \leq 0$  for all  $(I, \varphi, u)$  in  $O'' \times \mathbb{S}^1 \times U$ . This means that, if  $t \mapsto (I(t), \varphi(t), u(t))$  is a solution of (6) defined on  $[0, t_f]$  such that  $I(0) = \bar{I}$  and  $I(t) \in O''$  for all  $t$  in  $[0, t_f]$ , then  $\frac{d}{dt} h(I(t)) \leq 0$  for all  $t$ , hence  $h(I(t_f)) \leq h(\bar{I})$ . This defeats local orbital controllability, taking  $\Omega = O''$  in (11), because  $I(t_f)$  is constrained to belong to  $\{J \in O'', h(J) \leq h(\bar{I})\}$ , which is not a neighborhood of  $\bar{I}$  because  $h$  is nondegenerate at  $\bar{I}$ , where its derivative is  $\bar{p} \neq 0$ .  $\square$

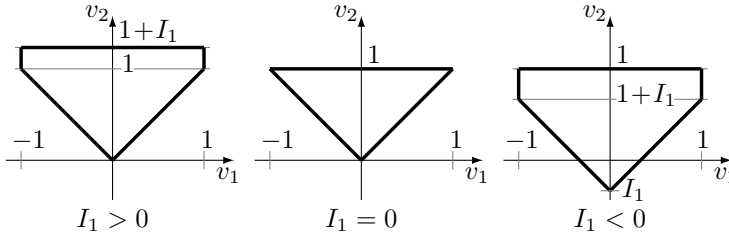


FIG. 1. For the example in [Remark 21](#), the set  $\check{E}^U(I_1, I_2) \subset T_{(I_1, I_2)}M$  defined in (19) reads  $\{(u_1 \sin \varphi, u_1 + u_2 I_1), (u_1, u_2) \in [0, 1]^2, \varphi \in \mathbb{S}^1\}$  in the coordinates  $v_1, v_2$  associated to  $I_1, I_2$ . This yields the above representation of  $\check{E}^U(I_1, I_2)$ , depending on the sign of  $I_1$ .

*Remark 21.* In [Propositions 18](#) and [19](#), it may seem odd that the control vector fields have to be linearly independent, but it is important in the proof above: if not, at the third line of the proof,  $\langle \bar{p}, \sum_{k=1}^m u_k F^k(\bar{I}, \varphi) \rangle$  could very well vanish at some points in  $\mathbb{S}^1 \times K$ , and one could even not prove that the interior of the polar cone to cone  $\check{E}^U(I)$  is nonempty for  $I$  close enough to  $\bar{I}$ . Let us show that, indeed, Assumption (ii) cannot be removed from [Proposition 19](#) or [18](#). As a counterexample, consider the system of the form (6) with  $m=n=2$ ,  $M=\mathbb{R}^2$ ,  $U=[0, 1] \times [0, 1]$ ,

$$F^1 = \partial/\partial I_2 + \sin \varphi \partial/\partial I_1 \quad \text{and} \quad F^2 = I_1 \partial/\partial I_2.$$

Comparing [Figure 1](#), the polar cone at the origin to  $\check{E}^U(I_1, I_2)$  is  $\{0\}$  if  $I_1 < 0$  and contains, for instance,  $(0, -1)$  in its interior if  $I_1 \geq 0$ . Hence, at  $\bar{I} = (0, 0)$ ,



- Assumption (i) of Proposition 19 is met, but (ii) is not, because  $F_2(0,0) = 0$ ,
- local controllability holds (to reach a neighborhood, first go to close-by points where  $I_1 < 0$ ), so that the conclusion of Proposition 19 does not hold,
- the topological interior of the polar cone at the origin to  $\tilde{E}^U(I)$  is non-empty at  $I = (0,0)$  but empty at some arbitrarily close points  $(I_1, I_2)$  with  $I_1 < 0$ .

### 3. Comments, examples and counter-examples.

*On constructively checking the conditions of the theorems.* Conditions (21), (22), (29), (31), or the sufficient condition in Theorem 17, involve both the vector fields and the control set  $U$  and are much more difficult to check than the rank of a family of vector fields. This point is discussed by the authors in [10] (or in the earlier reference [9]), where the condition is formulated in terms of convex optimization and conveniently solved, in the case of solar sails, using sum of square positivity criteria on trigonometric polynomials (this last point cannot be generalized to arbitrary periodic systems and sets  $U$ ), yielding minimum requirements on the characteristics of a solar sail to provide controllability around a particular orbit.

*Implications of our conditions in terms of Lie brackets.* The sufficient conditions given in Sections 2.1 to 2.3 are based on the *conic hull* of the subsets of  $T_x\mathcal{M}$ , for some or all  $x$  in  $\mathcal{M}$ , defined in (16), (17) and (18) being the whole tangent space or not. Classical controllability results are not based on cones, but rather on some *vector subspaces* of the tangent space, generally spanned by collections of Lie Brackets, see Section 1.4. Obviously, two different subsets of the tangent space may span the same vector subspace while generating different convex cones, so that the information contained in the convex hull is finer. For instance, cone  $E_{[t_1, t_2]}^U(x)$  (or cone  $E_\tau^U$ , or cone  $E^U$ ) a priori depends on  $U$  (see Example 24 below), whereas  $\text{Span } E_{[t_1, t_2]}^U(x)$  does not because, under Assumption (2), one has  $\text{Span } E_{[t_1, t_2]}^U(x) = \text{Span } E_{[t_1, t_2]}^{\mathbb{R}^m}(x)$ . It is however interesting to characterize  $\text{Span } E_{[t_1, t_2]}^U(x)$ , both for the sake of comparison with known results, and because a *necessary* condition for cone  $E_{[t_1, t_2]}^U(x) = T_x\mathcal{M}$  is  $\text{Span } E_{[t_1, t_2]}^U(x) = T_x\mathcal{M}$ ; we will use this fact in examples to show that our sufficient conditions are *not* met.

PROPOSITION 22. *With  $\mathcal{F}_0$  defined in (14), one has, for any  $x$  and any  $t_1 < t_2$ ,  $\text{Span } E_{[t_1, t_2]}^U(x) \supset \text{Span } \mathcal{F}_0(x)$ , and, under the additionnal assumption that the vector fields  $X^0, \dots, X^m$  are real analytic,  $\text{Span } E_{[t_1, t_2]}^U(x) = \text{Span } \mathcal{F}_0(x)$ .*

Let us give a short proof, although it is classical: introducing properties of families of *analytic* vector fields in control dates back to the work of Hermann and Krener in the early 1960s, see [11, Chapter 2, Theorem 6]. Assuming that  $X^0$  is complete, set

$$(38) \quad \mathcal{F}_{(\alpha_1, \alpha_2)} = \{\exp(-tX^0)_* X^k, k \in \{1, \dots, m\}, t \in \mathbb{R}, \alpha_1 < t < \alpha_2\}, \quad \alpha_1 < \alpha_2,$$

$$(39) \quad \mathcal{F}_\infty = \{\exp(-tX^0)_* X^k, k \in \{1, \dots, m\}, t \in \mathbb{R}\}.$$

LEMMA 23. *One has  $\text{Span } \mathcal{F}_0(x) \subset \text{Span } \mathcal{F}_{(\alpha_1, \alpha_2)}(x) \subset \text{Span } \mathcal{F}_\infty(x)$ , for any  $\alpha_1 < \alpha_2$ , and even, under the additionnal assumption that the vector fields  $X^0, \dots, X^m$  are real analytic,  $\text{Span } \mathcal{F}_0(x) = \text{Span } \mathcal{F}_{(\alpha_1, \alpha_2)}(x) = \text{Span } \mathcal{F}_\infty(x)$ .*

*Proof.* For  $x$  in  $\mathcal{M}$ ,  $k$  in  $\{1, \dots, m\}$  and  $p$  an element of  $T_x^*\mathcal{M}$ , define the smooth map  $a_k : \mathbb{R} \rightarrow \mathbb{R}$  by  $a_k(t) = \langle p, \exp(-tX^0)_* X^k(x) \rangle$ . Since  $\frac{d}{dt}(\exp(-tY)_* Z)(x) = (\exp(-tY)_*[Y, Z])(x)$  for any vector fields  $Y$  and  $Z$ , one has, for all  $t$ ,

$$(40) \quad \frac{d^j a_k}{dt^j}(t) = \left\langle p, \exp(-tX^0)_* \text{ad}_{X^0}^j X^k(x) \right\rangle.$$

If  $p$  is in the annihilator of  $\text{Span } \mathcal{F}_{(\alpha_1, \alpha_2)}(x)$ , then  $a_k(t) = 0$  for all  $k$  in  $\{1, \dots, m\}$  and  $t$  in  $(\alpha_1, \alpha_2)$ ; differentiating  $j$  times, taking  $t = 0$  and using (40) implies

$$\langle p, \text{ad}_{X^0}^j X^k(x) \rangle = 0, \quad k \in \{1, \dots, m\}, \quad j \in \mathbb{N},$$

hence  $p$  in the annihilator of  $\text{Span } \mathcal{F}_0(x)$ . This proves  $\text{Span } \mathcal{F}_0(x) \subset \text{Span } \mathcal{F}_{(\alpha_1, \alpha_2)}(x)$ , while  $\text{Span } \mathcal{F}_{(\alpha_1, \alpha_2)}(x) \subset \text{Span } \mathcal{F}_\infty(x)$  is obvious by definition. To prove the reverse inclusion, assume  $p$  in the annihilator of  $\text{Span } \mathcal{F}_0(x)$ . For each  $k$  in  $\{1, \dots, m\}$ ,  $p$  vanishes on all vectors  $\text{ad}_{X^0}^j X^k(x)$ ,  $j \in \mathbb{N}$ , and (40) implies  $\frac{d^j a_k}{dt^j}(0) = 0$  for all  $j$ . Since the map  $a_k$  is now real analytic, it must be identically zero on  $\mathbb{R}$ . This proves  $\text{Span } \mathcal{F}_\infty(x) \subset \text{Span } \mathcal{F}_0(x)$ , hence  $\text{Span } \mathcal{F}_0(x) = \text{Span } \mathcal{F}_{(\alpha_1, \alpha_2)}(x) = \text{Span } \mathcal{F}_\infty(x)$ .  $\square$

*Proof of Proposition 22.* Apply Lemma 23 with  $\alpha_1 = t_1 \pm \varepsilon$ ,  $\alpha_2 = t_2 \pm \varepsilon$ , taking into account the fact that, for any  $\varepsilon > 0$ ,

$$\text{Span } \mathcal{F}_{(t_1+\varepsilon, t_2-\varepsilon)} \subset \text{Span } \bigcup_{\tau \in [t_1, t_2]} E_\tau^U(x) \subset \text{Span } \mathcal{F}_{(t_1-\varepsilon, t_2+\varepsilon)}. \quad \square$$

The sufficient conditions ((21), (22), (29), (31)...) from Theorems 11, 13 and 17 now have, if the vector fields  $X^0, \dots, X^m$  are real analytic (it is more intricate in the  $C^\infty$  case), the following consequences according to Proposition 22:

- Condition (21) implies  $\text{Span}(\mathcal{F}_0 \cup \{X^0\})(x) = T_x \mathcal{M}$  (for all  $x$ );
- Condition (29) implies  $\text{Span } \mathcal{F}_0(\bar{x}(0)) = T_{\bar{x}(0)} \mathcal{M}$   
(and  $\text{Span } \mathcal{F}_0(\bar{x}(t)) = T_{\bar{x}(t)} \mathcal{M}$  for all  $t$  by invariance);
- Condition (31) implies  $(\pi \circ \exp(t_f X^0))'(\bar{x}(0))(\text{Span } \mathcal{F}_0(\bar{x}(0))) = T_{\pi(\bar{x}(t_f))} M$ .

These bracket conditions on  $\mathcal{F}_0$  appear in some sufficient conditions for controllability, but are by no means sufficient by themselves, as evidenced by Example 24: the system there always satisfies these Lie bracket conditions but may be controllable or not according to the value of parameters that change the shape of  $U$ .

It is important to note that these bracket conditions are quite more restrictive than the classical bracket generating condition discussed in Section 1.4 because  $\mathcal{F}_0$  discards all Lie brackets like  $[X^1, [X^0, X^1]]$  or  $[X^1, X^2]$  involving control vector fields more than once, compare (13), (14) and Definition 8. In fact, full rank of  $\mathcal{F}_0$  amounts to controllability of the linear approximation; see, e.g., the discussion in [5, Section 3.2] around Equation (3.35) for the case where  $U$  is a neighborhood of the origin, or (33) and Lemma 16 in the present paper for the case where  $U$  is not. Studying controllability of systems that are bracket generating but do not satisfy our condition on  $\mathcal{F}_0$ , while the control is constrained to a set  $U$  that is not a neighborhood of the origin, is outside the scope of this paper. Example 25 displays some systems in this category for which we prove controllability by ad-hoc methods, whose generalisation could be a topic of further investigation.

*Example 24* (illustration of the paper's results). Consider the following system, of the form (6), with  $F^1, F^2$  that depend on  $\varphi$  only,  $I = (I_1, I_2, I_3) \in O = M = \mathbb{R}^3$ ,  $\varphi \in \mathbb{S}^1$  and control  $u = (u_1, u_2)$ .

$$(41) \quad \begin{pmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{I}_3 \end{pmatrix} = L(\theta, \varphi, u) := \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} u_1 \\ 0 \\ u_2 \end{pmatrix},$$

$$\dot{\varphi} = 1,$$

$$u \in U = \{(u_1, u_2) \in \mathbb{R}^2, u_1^2 + u_2^2 \leq 1, |u_2| \leq u_1 \tan \alpha\},$$

where  $\theta, \alpha$  are fixed parameters,  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ ,  $0 < \alpha < \frac{\pi}{2}$ ,  $0 < |\alpha| + |\theta| < \frac{\pi}{2}$ ; the set  $U$  is depicted in Figure 2. It is also of the form (1) on  $\mathcal{M} = \mathbb{R}^3 \times \mathbb{S}^1$ .

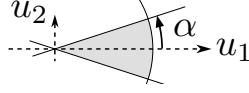


FIG. 2. Control set  $U \subset \mathbb{R}^2$  for System (41).

The definition (19) of  $\check{E}^U(I) \subset T_I M = \mathbb{R}^3$  yields  $\check{E}^U(I) = \cup_{\varphi \in \mathbb{S}^1} L(\theta, \varphi, U)$  for the present system, where  $L(\theta, \varphi, U)$  is (in spherical coordinates language) a planar piece in the meridian half plane of longitude  $\varphi$ , isometric to  $U$  (see Figure 2), rotated around the origin to bring its axis to latitude  $\theta$ ; taking the union over  $\varphi$  yields the volume pictured in Figure 3, qualitatively different depending whether  $|\theta|$  is larger or smaller than  $\alpha$ . Analytically, this volume is  $\{v \in \mathbb{R}^3, \|v\| \leq 1 \text{ and } \sqrt{v_1^2 + v_2^2} \tan(\theta - \alpha) \leq v_3 \leq \sqrt{v_1^2 + v_2^2} \tan(\theta + \alpha)\}$ . If  $|\theta| < \alpha$ ,  $\check{E}^U(I)$  has the effect shown in Figure 3-(a), its conic

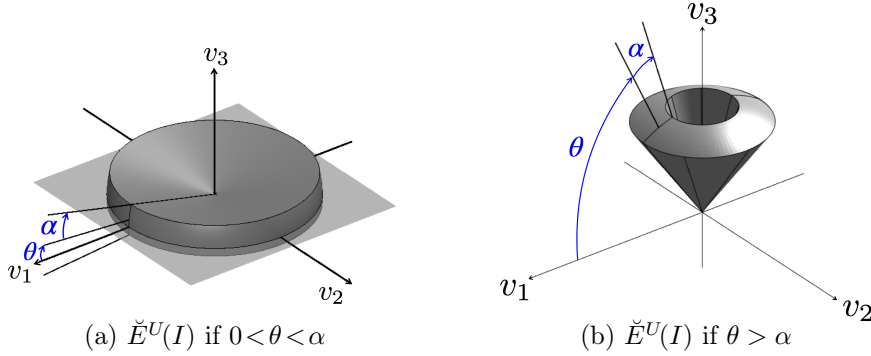


FIG. 3. Set  $\check{E}^U(I)$  for System (41).

hull is the whole  $T_I M$ , Condition (22) holds, Theorem 11 yields global controllability and Theorem 17 local controllability over one period. If  $\theta > \alpha$ ,  $\check{E}^U(I)$  has the effect shown in Figure 3-(b), with a symmetric picture contained in the lower half space if  $\theta < -\alpha$ ; in both cases, its dual cone is a cone of revolution of half angle  $|\theta| - \alpha$ , with nonempty interior; as a conclusion, Proposition 19 yields non-controllability if  $|\theta| > \alpha$ . If  $|\theta| = \alpha$ , one sees by inspection that the system is not controllable ( $\dot{I}_3$  has a fixed sign), but no result from the paper applies; indeed, cone  $\check{E}^U(I)$  is a closed half space, so that Theorem 11 (or 17) does not apply, but neither does Proposition 19 because the dual cone to cone  $\check{E}^U(I)$  is only a half line.

From the point of view of the remarks on Lie brackets made just before this example, it is clear that  $\mathcal{F}_0(I, \varphi)$  spans  $T_I M$  at all points (use  $F^1, F^2, \partial F^1/\partial \varphi, \partial F^2/\partial \varphi$ ), for any value of the parameters: we clearly see that full rank of  $\mathcal{F}_0$  is not, here, sufficient for controllability, that occurs for some values of the parameters only.

Let us now enrich this academic example by making  $\theta$  the fourth component  $I_4$  of the state,  $\alpha \in [0, \pi/4]$  being still a constant parameter:

$$(43) \quad \begin{pmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{I}_3 \end{pmatrix} = L(I_4, \varphi, (u_1, u_2)), \quad \dot{I}_4 = u_3 \cos^2(2I_4), \quad \dot{\varphi} = 1, \quad u \in U' = U \times [-1, 1],$$

where the control is now  $(u_1, u_2, u_3)$  and the state  $(I_1, I_2, I_3, I_4, \varphi) \in \mathcal{M} = M \times \mathbb{S}^1$ ,  $M = \mathbb{R}^3 \times (-\frac{\pi}{4}, \frac{\pi}{4})$ . The factor  $\cos^2(2I_4)$  in  $\dot{I}_4$  prevents solutions from leaving  $\mathcal{M}$

in finite time with admissible controls. If  $\alpha \geq \pi/4$ , Condition (22) holds everywhere and conclusions are the same as for System (41) in the case  $|\theta| < \alpha$ . If  $\alpha < \pi/4$ , Condition (22) holds in the region  $R_1 = \{(I_1, I_2, I_3, I_4, \varphi), |I_4| < \alpha\}$  and fails in  $R_0 = \{(I_1, I_2, I_3, I_4, \varphi), I_4 = \pm\alpha\}$  and  $R_2 = \{(I_1, I_2, I_3, I_4, \varphi), |I_4| > \alpha\}$ . [Theorem 11](#) does not apply globally, [Theorem 17](#) yields local controllability in one period in region  $R_1$ , [Proposition 19](#) tells us that the system is not locally controllable around its periodic solutions in region  $R_2$ , and we see by inspection that this local controllability does not hold either in  $R_0$ . Interestingly, this system is also globally controllable if  $\alpha < \pi/4$  although local controllability does not hold in half of the space and [Theorem 11](#) does not predict controllability. Indeed, starting from some  $(I_1^0, I_2^0, I_3^0, I_4^0, \varphi^0)$  to reach  $(I_1^f, I_2^f, I_3^f, I_4^f, \varphi^f)$ , one may always use the control  $u_3$  to reach the region  $R_1$  (for instance go to  $(I_1^0, I_2^0, I_3^0, 0, \varphi^0)$ ), then use the controls  $u_1, u_2$  to reach  $I_1^f, I_2^f, I_3^f$  (possible as in (41) with  $I_4 = 0$ ) with constant  $I_4$  and then use again  $u_3$  to steer  $I_4$  from 0 to  $I_4^f$ . This displays a case where global controllability holds without local controllability holding everywhere, hence illustrating a point in [Remark 7](#).

*Example 25* (going beyond the paper's results). Consider the following system of the form (6) with scalar control ( $m = 1$ ,  $\mathcal{M} = M \times \mathbb{S}^1$ ,  $M = \mathbb{R}^3$ ,  $d = 4$ ):

$$(44) \quad \begin{pmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{I}_3 \end{pmatrix} = u \left( \cos \varphi \begin{pmatrix} 1 \\ 0 \\ -I_2/2 \end{pmatrix} + \sin \varphi \begin{pmatrix} 0 \\ 1 \\ I_1/2 \end{pmatrix} \right), \\ \dot{\varphi} = 1, \quad u \in U \subset \mathbb{R}.$$

The set  $U$  will be either  $[-1, 1]$  or  $[0, 1]$ . The system also reads  $\dot{x} = X^0(x) + uX^1(x)$ , with  $x = (I_1, I_2, I_3, \varphi)$  and the vector fields  $X^0$  and  $X^1$  defined by:

$$(45) \quad X^0 = \frac{\partial}{\partial \varphi}, \quad X^1 = \cos \varphi \left( \frac{\partial}{\partial I_1} - \frac{1}{2} I_2 \frac{\partial}{\partial I_3} \right) + \sin \varphi \left( \frac{\partial}{\partial I_2} + \frac{1}{2} I_1 \frac{\partial}{\partial I_3} \right).$$

With reference to the second paragraph of this [Section 3](#), the vector fields  $X^0$ ,  $X^1$ ,  $[X^0, X^1]$  and  $[X^1, [X^0, X^1]]$  are linearly independent, hence the bracket generating condition is satisfied (this implies global controllability from [Theorem 9](#) if  $U = [-1, 1]$ ), while the rank of  $\mathcal{F}_0$  is only 2: it is spanned by  $X^1$  and  $[X^0, X^1]$  and vanishes on the differential forms  $\frac{1}{2} I_2 dI_1 - \frac{1}{2} I_1 dI_2 + dI_3$  and  $d\varphi$ . We are therefore in the case where our sufficient conditions cannot hold. Indeed, a simple computation from (18), (16) and (19) yields,  $E^U(I, \varphi) = \check{E}^U(I) \times \{0\}$  with

$$(46) \quad \check{E}^U(I) = \{(v_1, v_2, v_3) \in T_I M = \mathbb{R}^3, \frac{1}{2} I_2 v_1 - \frac{1}{2} I_1 v_2 + v_3 = 0 \text{ and } v_1^2 + v_2^2 \leq 1\},$$

whether  $U$  is  $[-1, 1]$  or  $[0, 1]$  (in the former case, it is “covered twice”), so that cone  $\check{E}^U(I)$  is a proper vector subspace of  $T_I M$  and none of the sufficient conditions in [Theorems 11](#), [17](#) or [13](#) hold true, while the polar cone of  $\check{E}^U(I)$  is a line in  $T_I^* M$ , hence has an empty interior, and one may not either apply [Proposition 18](#) or [Proposition 19](#) to infer non-controllability. No result from the present paper applies here.

Let us however investigate local controllability in the spirit of the conclusions of [Theorem 17](#), by ad-hoc techniques outside the scope of the results of the paper. We are going to prove the following three properties for System (44) (recall that the state  $x$  is  $(I, \varphi)$ , with  $I \in \mathbb{R}^3$ ,  $\pi(I, \varphi) = I$ , and the period  $T(x)$  is  $T(I) = 2\pi$ ):

- (a)  $\pi(\mathcal{A}_{T(I)}^{[-\varepsilon, \varepsilon]}(I, \varphi))$  is a neighborhood of  $I$ ,
- (b)  $\pi(\mathcal{A}_{T(I)}^{[0, \varepsilon]}(I, \varphi))$  is not a neighborhood of  $I$ ,

(c)  $\pi(\mathcal{A}_{2T(I)}^{[0,\varepsilon]}(I, \varphi))$  is a neighborhood of  $I$ ,  
for any  $\varepsilon$ ,  $0 < \varepsilon \leq 1$ . Point (a) is in addition of the global controllability mentioned above when  $U = [-1, 1]$ , while points (b) and (c) concerning the case  $U = [0, 1]$ , more in the spirit of the paper, display an interesting behavior where local controllability is obtained over two periods but not over one. Before establishing them, note that the general solution of (44) starting from  $(I_1^0, I_2^0, I_3^0, \varphi^0)$  at time zero is given by  $\varphi(t) = \varphi^0 + t$  and, with complex notations for  $(I_1, I_2)$ ,

$$(47) \quad \begin{aligned} I_1(t) + i I_2(t) &= I_1^0 + i I_2^0 + \int_0^t u(s) e^{i(\varphi^0 + s)} ds, \\ I_3(t) &= I_3^0 + \frac{1}{2} \iint_{0 \leq r \leq s \leq t} u(s) u(r) \sin(s-r) dr ds \\ &\quad + \frac{1}{2} \Im \left( (I_1^0 - i I_2^0) (I_1(t) - I_1^0 + i (I_2(t) - I_2^0)) \right), \end{aligned}$$

For intuition behind equations, recall that  $\dot{I}_3 = \frac{1}{2}(I_1 \dot{I}_2 - I_2 \dot{I}_1)$  infers that the variation  $I_3(t_2) - I_3(t_1)$  is the area swept from the origin by the plane curve  $t \mapsto (I_1(t), I_2(t))$  on the interval  $[t_1, t_2]$  and  $\varphi$  is (if  $u$  is positive) the polar angle of the velocity vector on that curve. It turns out that the area swept from the origin by a closed curve generated by a non-negative control over a time interval of length at most  $2\pi$  cannot be negative:

$$(48) \quad \left( u(\cdot) \geq 0 \text{ and } I_1(2\pi) = I_1(0) \text{ and } I_2(2\pi) = I_2(0) \right) \Rightarrow I_3(2\pi) \geq I_3(0).$$

To prove this, first decompose the integral in the expression for  $I_3(2\pi)$  in (47):

$$(49) \quad \iint_{0 \leq r \leq s \leq 2\pi} = \iint_{0 \leq r \leq s \leq \pi} + \iint_{\pi \leq r \leq s \leq 2\pi} + \iint_{(s,r) \in [\pi, 2\pi] \times [0, \pi]}.$$

From (47),  $(I_1(2\pi), I_2(2\pi)) = (I_1^0, I_2^0)$  implies  $\int_0^{2\pi} u(s) \sin(s-r) ds = 0$  for any  $r$ , hence  $\iint_{(s,r) \in [0, 2\pi] \times [0, \pi]} u(s) u(r) \sin(s-r) dr ds = 0$ , implying that the last term in (49) is equal to  $-\iint_{(s,r) \in [0, \pi] \times [0, \pi]} \sin(s-r) u(s) u(r) ds dr$ , which is zero by symmetry around the axis  $r = s$ ; hence the integral giving  $I_3(2\pi)$  in (47) reduces to the region  $\{0 \leq r \leq s \leq \pi\} \cup \{\pi \leq r \leq s \leq 2\pi\}$ , where  $u(s) u(r) \sin(s-r)$  is non negative; this proves (48). Point (b) follows.

Let us now turn to Point (c). Consider the solution starting from an arbitrary  $(I_1^0, I_2^0, I_3^0)$  with the piecewise constant control depicted in Figure 4 (left) for  $t$  in  $[0, \frac{10}{3}\pi]$ , continued by zero on  $[\frac{10}{3}\pi, 4\pi]$ , with constants  $\varepsilon_1, \varepsilon_2, \varepsilon_3$  chosen such that

$$(50) \quad \varepsilon_1 = \sqrt{\frac{(2\pi+3)\sqrt{3}-5\pi}{2\pi-3\sqrt{3}}} \varepsilon_2, \quad \varepsilon_3 = (\sqrt{3}-1)\varepsilon_2, \quad 0 < \varepsilon_1 < \varepsilon_3 < \varepsilon_2 < \varepsilon.$$

The second relation amounts to  $(I_1(\frac{10\pi}{3}), I_2(\frac{10\pi}{3})) = (I_1^0, I_2^0)$ , and the first one to  $I_3(\frac{10\pi}{3}) = I_3^0$  (use formulas (47))<sup>5</sup>. On the time interval  $[0, \frac{\pi}{3}]$ , the control is constant equal to  $\varepsilon_1$ , that belongs to the topological interior of  $[0, \varepsilon]$ ; the linear approximation on that interval hence reads  $\dot{I} = A(t)\delta I + B(t)\delta u$ , with no constraint on  $\delta u$  and

$$A(t) = \varepsilon_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\sin \varphi(t) & \cos \varphi(t) & 0 \end{pmatrix} \text{ and } B(t) = \varepsilon_1 \begin{pmatrix} \cos \varphi(t) \\ \sin \varphi(t) \\ I_1(t) \sin \varphi(t) - I_2(t) \cos \varphi(t) \end{pmatrix}.$$

<sup>5</sup>Once the 2<sup>nd</sup> condition ensures that the projection on the  $(I_1, I_2)$  plane is a closed curve like the one in Figure 4 resembling a goldfish,  $I_3(\frac{10\pi}{3}) = I_3^0$  means that the area of the fish's "body", run counter-clockwise, equals the area of the "tail", run clockwise. See the two lines after (47).

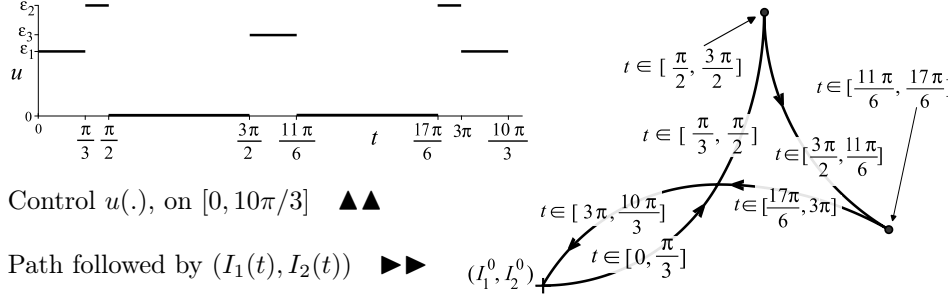


FIG. 4. Left: the control used for Point (a) in Example 25. The constants  $\varepsilon_1, \varepsilon_2, \varepsilon_3$  are tuned according to (50), where  $\varepsilon_2$  may be taken arbitrarily small. Right: the closed curve  $t \mapsto (I_1(t), I_2(t))$ ,  $t \in [0, \frac{10\pi}{3}]$ . We drew it from (47) with  $u(\cdot)$  given above,  $\varepsilon_2 = 1$  and  $\varphi(0) = 0$ . Since the curve in the  $(I_1, I_2)$ -plane scales according to  $\varepsilon_2$ , and the choice of  $\varphi(0)$  rotates the curve, no axis or units appear on the figure; changing  $(I_1^0, I_2^0)$  translates the curve and changes the way the  $I_3$  component of the solution scales with  $\varepsilon_2$ .

The vectors  $B(t)$ ,  $(\frac{d}{dt} - A(t))B(t)$ ,  $(\frac{d}{dt} - A(t))^2 B(t)$  are always a basis of  $\mathbb{R}^3$ , as seen in the following formulas, where the  $t$  argument is omitted in the right-hand sides ( $\varphi$  should be  $\varphi(t)$ , or  $\varphi^0 + t$ ):

$$(\frac{d}{dt} - A(t))B(t) = \varepsilon_1^2 \begin{pmatrix} -\sin \varphi \\ \cos \varphi \\ I_1 \cos \varphi + I_2 \sin \varphi \end{pmatrix}, \quad (\frac{d}{dt} - A(t))^2 B(t) = -\varepsilon_1^2 B(t) + \begin{pmatrix} 0 \\ 0 \\ 2\varepsilon_1^3 \end{pmatrix}.$$

Controllability of the linear approximation follows, according to [22] (see also our remarks after (56)). This implies that  $\pi(\mathcal{A}_{\pi/3}^{[0, \varepsilon]}(I_1^0, I_2^0, I_3^0, \varphi^0))$  is a neighborhood of  $(I_1(\frac{\pi}{3}), I_2(\frac{\pi}{3}), I_3(\frac{\pi}{3}))$  according to the classical linear test for controllability (implicit function theorem applied to the end-point mapping) because the control set  $U = [0, \varepsilon]$  is a neighborhood of the reference control around which linearization is performed. This implies Point (c) because  $\mathcal{A}_{2T(I)}^{[0, \varepsilon]}(I^0, \varphi^0)$  contains the image of  $\mathcal{A}_{\pi/3}^{[0, \varepsilon]}(I^0, \varphi^0)$  by the time-varying flow corresponding to the reference control from Figure 4 on the time-interval  $[\pi/3, 4\pi]$ .

Point (a) is easier than Point (c), negative controls being allowed: skip the two intervals or length  $\pi$  on which  $u \equiv 0$  in Figure 4, left, and replace  $\varepsilon_3$  with  $-\varepsilon_3$ ; the obtained control on  $[0, 4\pi/3]$  yields the same curve on Figure 4, right, without the stops at each cusp, but with the same linearized controllability.

## Appendix A. Controllability of time-varying linear systems with constrained control.

Consider a time-varying linear control system, with constraints on the control:

$$(51) \quad \dot{z} = A(t)z + B(t)v, \quad v \in V,$$

where the state  $z$  belongs to  $\mathbb{R}^d$ ,  $V$  is a compact subset of  $\mathbb{R}^m$ , and  $t \mapsto A(t)$  and  $t \mapsto B(t)$  are smooth maps  $\mathbb{R} \rightarrow \mathbb{R}^{d \times d}$  and  $\mathbb{R} \rightarrow \mathbb{R}^{d \times m}$  respectively. This appendix is devoted to giving conditions for

$$(52) \quad \mathcal{A}_{t_0, t_f}^V(z_0) = \{z(t_f), \text{ with } t \mapsto (z(t), v(t)) \text{ solution of (51), } z(t_0) = z_0\}$$

to be a neighborhood of the origin. It is known that  $\mathcal{A}_{t_0, t_f}^V(z_0)$  is convex, at least if  $V$  is compact, whether  $V$  is convex or not, see, e.g., [16, Chapter 2, Theorem 1A (appendix), p.164].

It is customary to define the transition matrix (of  $A(\cdot)$ ) as the map  $(t_1, t_2) \mapsto \phi(t_2, t_1) \in \mathbb{R}^{d \times d}$  such that, for any  $t_0 \in \mathbb{R}$  and  $z_0 \in \mathbb{R}^d$ ,  $t \mapsto z(t) = \phi(t, t_0)z_0$  is the solution of  $\dot{z} = A(t)z$ ,  $z(t_0) = z_0$ ; it satisfies

$$(53) \quad \frac{\partial \phi}{\partial t_2}(t_2, t_1) = A(t_2) \phi(t_2, t_1), \quad \frac{\partial \phi}{\partial t_1}(t_2, t_1) = -\phi(t_2, t_1) A(t_1), \quad \phi(t, t) = I.$$

The change of variables  $\zeta = \phi(t_0, t)z$  yields, for any  $t_0$ , another form of System (51):

$$(54) \quad \begin{aligned} \dot{\zeta}(t) &= \phi(t_0, t)B(t)v(t), \\ z(t) &= \phi(t, t_0)\zeta(t), \end{aligned}$$

and the general formula for solutions of (51) follows through variation of constants:

$$(55) \quad z(t) = \phi(t, t_0)z(t_0) + \int_{t_0}^t \phi(t, \tau)B(\tau)v(\tau)d\tau.$$

If the convex hull of  $V$  is a neighborhood of the origin (in particular if  $V = \mathbb{R}^m$ ), one has the following characterization, after [13, Section 9.2], [26], [22]:  $A_{t_0, t}^V(0)$  is a neighborhood of the origin if and only if, for  $p \in (\mathbb{R}^d)^*$  (a line vector),

$$(56) \quad \left( p \phi(t_0, s)B(s) = 0 \text{ for all } s \text{ in } [t_0, t_f] \right) \Rightarrow p = 0.$$

This property is stated in [13] as “the rows of  $s \mapsto \phi(t_0, s)B(s)$  are linearly independent on the interval  $[t_0, t_f]$ ”, and is equivalent to

$$(57) \quad \text{Span} \{ \phi(t_0, s)B(s)v, s \in [t_0, t_f], v \in V \} = \mathbb{R}^d.$$

The case where the convex hull of  $V$  is *not* a neighborhood of the origin, the one of interest here, is treated in [21], and sketched in [16, Section 2.2]. Let us state the precise results needed in Section 2.2 (it is difficult to refer to [21], that characterizes  $\cup_{t \in (-\infty, t_f]} A_{t, t_f}^V(0)$  rather than  $A_{t_0, t_f}^V(0)$  for a fixed  $t_0$ ).

**PROPOSITION 26.** *Assume that  $V$  is a compact subset of  $\mathbb{R}^m$  containing zero, and that  $t_0, t_f$  are two real numbers. The accessible set  $A_{t_0, t_f}^V(0)$  is a neighborhood of the origin if and only if*

$$(58) \quad \text{cone} \{ \phi(t_0, s)B(s)v, s \in [t_0, t_f], v \in V \} = \mathbb{R}^d.$$

We also need the following characterization of when some projection of the accessible set is a neighborhood of the origin in the projection of the state space (controllability with respect to a projection, see Definition 5, part II).

**PROPOSITION 27.** *Assume that  $V$  is a compact subset of  $\mathbb{R}^m$  containing zero, and that  $t_0, t_f$  are two real numbers. Let  $n$  be a positive integer,  $n \leq d$ , and  $P$  a surjective linear map  $\mathbb{R}^d \rightarrow \mathbb{R}^n$ . The set  $P A_{t_0, t_f}^V(0)$  (image of  $A_{t_0, t_f}^V(0)$  by  $P$ ) is a neighborhood of the origin in  $\mathbb{R}^n$  if and only if*

$$(59) \quad \text{cone} \{ P \phi(t_f, s)B(s)v, s \in [t_0, t_f], v \in V \} = \mathbb{R}^n.$$

Note that, by the separating hyperplane theorem, (58) and (59) are equivalent to

$$(60) \quad (p \phi(t_0, s)B(s)v \geq 0 \text{ for all } v \in V, s \in [t_0, t_f]) \Rightarrow p = 0$$

$$(61) \quad \text{and } (p P \phi(t_f, s)B(s)v \geq 0 \text{ for all } v \in V, s \in [t_0, t_f]) \Rightarrow p = 0$$



respectively,  $p \in (\mathbb{R}^d)^*$ ,  $p \in (\mathbb{R}^n)^*$  in (60), (61). Conditions are given under this form in the literature, rather than (58) or (59).

Proposition 26 is a particular case of Proposition 27: take  $M = \mathcal{M}$ ,  $n = d$ , and  $P = I$  (identity map), and factor out the linear isomorphism  $\phi(t_0, t_f)$  in (59) to “replace  $t_f$  with  $t_0$ ”. When  $n < d$ , the controllability property characterized in Proposition 27 is strictly weaker than the one in Proposition 26. Let us end this appendix with a short proof of Proposition 27.

*Proof of Proposition 27.* Let us prove necessity first. If (59) does not hold, there exists a nonzero  $p$  such that  $pP\phi(t_f, s)B(s)v \geq 0$  for any  $v \in V$  and  $s \in [t_0, t_f]$ , and formula (55), with  $z_0 = 0$ , obviously implies  $pPz(t_f) \geq 0$  for any solution, implying that  $PA_{t_0, t_f}^V(0)$  is contained in the half-space  $\{y \in \mathbb{R}^n, py \geq 0\}$ , hence is not a neighborhood of 0.

To prove the converse, assume that  $PA_{t_0, t_f}^V(0)$  is not a neighborhood of 0 in  $\mathbb{R}^n$ . Since  $PA_{t_0, t_f}^V(0)$  is convex (this was recalled at the beginning of this appendix, see [16, Chapter 2, Theorem 1A (appendix), p. 164]), it is contained in the half space  $\{y \in \mathbb{R}^n, py \geq 0\}$ , for some nonzero  $p \in (\mathbb{R}^n)^* \setminus \{0\}$ , i.e. any solution  $(z(\cdot), v(\cdot))$  of (51) with  $z(t_0) = 0$  and  $v(\cdot)$  valued in  $V$  satisfies  $pPz(t_f) \geq 0$ . It is not too difficult to see that, for the same  $p$ , one must then have  $pP\phi(t_f, s)B(s)v \geq 0$  for all  $v$  in  $V$  and all  $s$  in  $[t_0, t_f]$ ; this defeats (61), hence (59).  $\square$

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