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***Abstract*— This paper presents a novel observer that estimates the state of a linear system in the observer canonical form in finite time. Due to the specially selected dynamical gain, the proposed observer is easy to tune and, at the same time, it is robust to external perturbations (e.g., state disturbances and measurement noises). The tuning algorithm consists in solving a simple system of linear matrix inequalities with only two adjustable parameters. Theoretical results and a comparison with homogeneous and prescribed-time observers are illustrated by numerical simulation.**

I. INTRODUCTION

In most situations the control design requires knowledge of the entire system state. However, this requirement can hardly be met in practice since some state components cannot be measured either physically (e.g., the transient EMF of electric generators [22]) or due to the large-scale nature of the system (e.g., power grids). Fortunately, in many cases the state vector can be estimated using the available system inputs and outputs [13], [14], [16].

The problem of the state estimation (observation) becomes more sophisticated when a control goal must be achieved in finite time. Clearly, the state vector must be reconstructed in finite time as well. For example, such a time constraint can be fulfilled by shifting the observation problem to a discrete-time setting when the choice of zero eigenvalues implies a dead-beat response [9], [18]. Although such observers preserve a linear structure, their practical implementation is questionable due to their strong dependence on the exact knowledge of system matrices.

A fundamentally different approach is to design nonlinear observers [15], [17] or ones with time-varying feedback gains [10], [21]. For instance, in [15], [17] the theory of homogeneous systems is used to prove finite/fixed-time convergence as well as robustness with respect to external disturbances. However, either parameter tuning is not yet developed for high-order systems [17], or it requires solving a system of parametric linear matrix inequalities (LMIs) over a grid [15]. In contrast, the design of the so-called prescribed-time observer [10] is really simple, where all tuning parameters have clear physical meaning. Moreover, the observer reconstructs the system state in a priori chosen finite time. Nevertheless, such a method is useful only in the absence of measurement noises which, of course, always exist in practice (e.g., see Remark 20 in [6]).

Therefore, the main objective of this work is to design a finite-time observer with dynamical gains that is:

1) *easy to tune* (no parametric LMIs, minimum number of adjustable parameters having clear physical meaning);
2) *robust to external disturbances* (state perturbations and measurement noises).

In other words, we propose a robust modification of the prescribed-time observer [10] in exchange for the ability to assign the settling time a priori. We will show that this can be done by the appropriate change of the time-varying gain.

The remainder of this paper is outlined as follows. The problem formulation is given in Section II. The concepts of input-to-state stability and weighted homogeneity are recalled in Section III. The observer design and its properties are described in Section IV. Finally, a comparison with the prescribed-time [10] and homogeneous [17] observers using numerical simulation is discussed in Section VI.

Throughout this paper, the following notation is used:

- $\mathbb{R}$ is the field of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}$;
- a series of integers $1, 2, \ldots, n$ is denoted by $\overline{1,n}$;
- $\|x\| := \sqrt{x^\top x}$ is the Euclidean norm of a vector $x \in \mathbb{R}^n$;
- $L^\infty(\mathbb{R}^n)$ is the space of Lebesgue measurable essentially bounded functions $d : \mathbb{R}_+ \to \mathbb{R}^n$ with the norm defined as $\|d\|_{L^\infty} := \operatorname{ess\,sup}_{t \in \mathbb{R}_+} \|d(t)\| < +\infty$;
- $\operatorname{diag}\{\lambda_i\}_{i=1}^n$ and $\operatorname{col}\{\lambda_i\}_{i=1}^n := (\lambda_1, \lambda_2, \ldots, \lambda_n)^\top$ are the diagonal matrix and the column vector, respectively, with the elements $\lambda_i \in \mathbb{R}$, $i = \overline{1,n}$;
- $I_n \in \mathbb{R}^{n\times n}$ and $O_{n\times m} \in \mathbb{R}^{n\times m}$ are the identity and zero matrices, respectively;
- if $P = P^\top \in \mathbb{R}^{n\times n}$ then notations $P \succ 0$ ($P \prec 0$) and $P \succcurlyeq 0$ ($P \preccurlyeq 0$) mean that P is positive (negative) definite and semidefinite, respectively;
- $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimal and maximal eigenvalues of a symmetric matrix $P \in \mathbb{R}^{n\times n}$, respectively;
- $\lceil x \rfloor^\varrho := \operatorname{sign}(x)|x|^\varrho$ is the signed power of a scalar $x \in \mathbb{R}$ defined for any $\varrho > 0$.

II. PROBLEM STATEMENT

Consider a perturbed chain of integrators

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t) + b_i u(t) + d_{x,i}(t), \quad i = \overline{1, n-1}, \\ \dot{x}_n(t) &= b_n u(t) + d_{x,n}(t), \\ y(t) &= x_1(t) + d_y(t), \end{aligned} \tag{1}$$

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where $x(t) := \mathrm{col}\{x_i(t)\}_{i=1}^n \in \mathbb{R}^n$ is the state; $u(t) \in \mathbb{R}^m$ is the known control input; $y(t) \in \mathbb{R}$ is the measured output; $d_x(t) := \mathrm{col}\{d_{x,i}(t)\}_{i=1}^n \in \mathbb{R}^n$ is the state perturbation; $d_y(t) \in \mathbb{R}$ is the measurement noise; $d = (d_y, d_x)^\top \in L^\infty(\mathbb{R}^{n+1})$; $b_i \in \mathbb{R}^{1\times m}$, $i = \overline{1,n}$, are known input matrices.

The goal of this work is to design a finite-time observer characterized by two main features:

1) simple tuning procedure (no parametric LMIs);
2) robustness with respect to external disturbances d.

III. PRELIMINARIES

Consider the system of the form:

$$\dot{\xi}(t) = f(\xi(t), d(t)), \tag{2}$$

where $\xi(t) \in \mathbb{R}^p$ is the state; $d(t) \in \mathbb{R}^q$ is the external input, $d \in L^\infty(\mathbb{R}^q)$; $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$ is a continuous vector field which satisfies $f(0,0) = 0$, i.e., the origin is an equilibrium of (2). For any initial condition $\xi_0 \in \mathbb{R}^p$ and input $d \in L^\infty(\mathbb{R}^q)$ we denote the corresponding solution to (2) as $\Xi(t, \xi_0, d)$.

As it will be shown later, the robustness properties of the system (2) with respect to input d can be proven by analyzing the asymptotic stability of its autonomous ($d = 0$) counterpart:

$$\dot{\xi}(t) = f(\xi(t), 0) := f_0(\xi(t)). \tag{3}$$

Similarly to the system (2), we denote solutions to (3) with initial conditions $\xi_0 \in \mathbb{R}^p$ by $\Xi_0(t, \xi_0) = \Xi(t, \xi_0, 0)$.

A. Stability definitions and input-to-state-stability

In this subsection, we recollect the stability definitions formulated for systems in the forms (2) and (3) using the so-called *comparison functions*.

Definition 1 [12], [14]. *A continuous function $\vartheta:\mathbb{R}_+\to\mathbb{R}_+$ is said to belong to:*

- ***class** $\mathcal{K}$ if it is strictly increasing and $\vartheta(0) = 0$;*
- ***class** $\mathcal{K}_\infty$ if $\vartheta \in \mathcal{K}$ and $\lim_{s\to+\infty} \vartheta(s) = +\infty$;*
- ***generalized class** $\mathcal{K}$ ($\mathcal{GK}$) if for some $s_0 \geq 0$ it is strictly increasing on $[s_0, +\infty)$ and $\vartheta(s) = 0$ for all $s \in [0, s_0]$.*

Definition 2 [12], [14]. *A continuous function $\eta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to:*

- ***class** $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_+$, the function $\eta(\cdot, t)$ belongs to class $\mathcal{K}$, and, for each fixed $s \in \mathbb{R}_+$, the function $\eta(s, \cdot)$ is decreasing and $\eta(s, t) \to 0$ as $t \to +\infty$;*
- ***generalized class** $\mathcal{KL}$ ($\mathcal{GKL}$) if, for each fixed $t \in \mathbb{R}_+$, the function $\eta(\cdot, t)$ belongs to class $\mathcal{GK}$, and, for each fixed $s \in \mathbb{R}_+$, the function $\eta(s, \cdot)$ is decreasing and $\eta(s, t) \to 0$ as $t \to T(s)$ for some finite $T(s) \geq 0$.*

We begin with the autonomous systems (3). Let Ω be an open neighborhood of the origin in $\mathbb{R}^p$.

Definition 3 [2], [14], [20]. *The origin of the system (3) is said to be:*

- ***asymptotically stable** if there exists a function $\eta \in \mathcal{KL}$ such that $\|\Xi_0(t, \xi_0)\| \leq \eta(\|\xi_0\|, t)$ for all $t \geq 0$ and any $\xi_0 \in \Omega$;*
- ***exponentially stable** if it is asymptotically stable and $\eta(\|\xi_0\|, t) = \vartheta(\|\xi_0\|)e^{-\delta t}$ for some function $\vartheta \in \mathcal{K}$ and the convergence rate $\delta > 0$;*
- ***finite-time stable** if it is asymptotically stable and $\eta \in \mathcal{GKL}$;*
- ***prescribed-time stable** if it is finite-time stable and $\eta(\|\xi_0\|, t) = \max\{\vartheta(\|\xi_0\|)\frac{T-t}{T}, 0\}$ for some function $\vartheta \in \mathcal{K}$ and the settling time $T < +\infty$.*

*If $\Omega = \mathbb{R}^p$, then the corresponding properties are called **global**.*

To characterize the robustness of the system (2) with respect to external inputs d (e.g., state perturbations and measurement noises), we will employ the concept of input-to-state stability.

Definition 4 [7]. *The system (2) is said to be:*

- ***input-to-state stable** (ISS) if there exist functions $\eta \in \mathcal{KL}$ and $\vartheta \in \mathcal{K}$ such that for any $\xi_0 \in \mathbb{R}^p$ and $d \in L^\infty(\mathbb{R}^q)$ the following estimate holds for all $t \geq 0$:*

$$\|\Xi(t, \xi_0, d)\| \leq \eta(\|\xi_0\|, t) + \vartheta(\|d\|_{L^\infty});$$

- ***integral input-to-state stable** (iISS) if there exist functions $\eta \in \mathcal{KL}$ and $\vartheta_1, \vartheta_2 \in \mathcal{K}_\infty$ such that for any $\xi_0 \in \mathbb{R}^p$ and $d \in L^\infty(\mathbb{R}^q)$ the following estimate holds for all $t \geq 0$:*

$$\vartheta_1(\|\Xi(t, x_0, d)\|) \leq \eta(\|\xi_0\|, t) + \textstyle\int_0^t \vartheta_2(\|d(s)\|)ds.$$

Note that the definitions of finite-time ISS [12] and prescribed-time ISS [20] systems can be formulated analogously by replacing $\eta \in \mathcal{KL}$ in Definition 4 with $\eta \in \mathcal{GKL}$ and $\max\{\vartheta(\|\xi_0\|)\frac{T-t}{T}, 0\}$, respectively.

B. Weighted homogeneity

Following the works [19] and [23], for strictly positive numbers r_i, $i = \overline{1,p}$, called *homogeneous weights*, we define

- the *vector of weights* $r := \mathrm{col}\{r_i\}_{i=1}^p$;
- the *dilation matrix* $D_r(\lambda) := \mathrm{diag}\{\lambda^{r_i}\}_{i=1}^p$ for any $\lambda > 0$.

Definition 5 [19]. *A vector field $f_0 : \mathbb{R}^p \to \mathbb{R}^p$ is said to be r-homogeneous of degree $\varrho \in \mathbb{R}$, where $\varrho \geq -\min_{i=\overline{1,p}} r_i$, if for all $\xi \in \mathbb{R}^p$ and $\lambda > 0$ we have*

$$f_0(D_r(\lambda)\xi) = \lambda^\varrho D_r(\lambda) f_0(\xi). \tag{4}$$

Note that the system (3) is r-homogeneous of degree $\varrho \in \mathbb{R}$ if so is its vector field f_0. Hence, in the sequel, both definitions will be used interchangeably.

The main advantage of homogeneous systems is that their finite-time convergence can be proven comparatively easily.

Theorem 1 [4]. *If the system (3) is r-homogeneous of degree $\varrho < 0$ and asymptotically stable at the origin, then it is globally finite-time stable.*

Similar to linear systems, global asymptotic stability of the system (3) implies input-to-state stability of the corresponding system (2).

Theorem 2 [1], [11]. *Let the vector field $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$ be (r, r_d)-homogeneous of degree $\varrho \in \mathbb{R}$, where $r_d :=$*

$\mathrm{col}\{r_{d,j}\}_{j=1}^{q} \in \mathbb{R}_{+}^{q}$ *and* $\varrho \geq -\min_{i=\overline{1,p}} r_i$*, i.e., if for all* $\xi \in \mathbb{R}^p$*,* $d \in \mathbb{R}^q$ *and* $\lambda > 0$ *we have*

$$f(D_r(\lambda)\xi, D_{r_d}(\lambda)d) = \lambda^{\varrho} D_r(\lambda) f(\xi, d). \tag{5}$$

If the system (3) *is globally asymptotically stable and* $\varrho < 0$*, then the system* (2) *is:*

- *finite-time ISS if* $r_{d,\min} > 0$*;*
- *finite-time iISS if* $r_{d,\min} = 0$*,*

where $r_{d,\min} := \min_{j=\overline{1,q}} r_{d,j}$*.*

IV. OBSERVER DESIGN

Let us introduce the following state observer:

$$\begin{aligned}
&\dot{w}(t) = y(t) - \hat{x}_1(t) + z^{-\bar{r}_0}(t) L_0 w(t),\\
&\dot{\hat{x}}_i(t) = \hat{x}_{i+1}(t) + b_i u(t) - z^{-\bar{r}_i}(t) L_i w(t),\ i = \overline{1, n-1},\\
&\dot{\hat{x}}_n(t) = b_n u(t) - z^{-\bar{r}_n}(t) L_n w(t),
\end{aligned} \tag{6}$$

where $w(t) \in \mathbb{R}$ is the filtered injection input $y(t) - \hat{x}_1(t)$; $\hat{x}(t) = \mathrm{col}\{\hat{x}_i(t)\}_{i=1}^{n} \in \mathbb{R}^n$ is the state estimate; $z(t) \in \mathbb{R}_+^*$ is a dynamical gain; $\bar{r}_i := (i+1)\mu$ with $\mu \in (0, \frac{1}{n+1}]$ and $L_i < 0$, $i = \overline{0, n}$, are tuning parameters.

In this work, we will define the time-varying gain $z(t)$ as:

$$z(t) = \max\left\{v(t), \gamma_2 |w(t)|\right\}, \tag{7}$$

where $v(t) \in \mathbb{R}$ is the output of the following low-pass filter:

$$\begin{aligned}
\dot{v}(t) = \gamma_2 \big| y(t) - \hat{x}_1(t) + z^{-\bar{r}_0}(t) L_0 w(t) \big| \\
- \gamma_1 \big\lceil v(t) - \gamma_2 |w(t)| \big\rfloor^{1-\mu},
\end{aligned} \tag{8}$$

with tuning parameters $\gamma_1 > 0$ and $\gamma_2 > 0$.

Remark 1. *Note that the proposed observer* (6)–(8) *with* $v(0) = v_0 > 0$ *generalizes some well-known results up to the additional low-pass filter* $w(t)$*.*

For example, if $\gamma_2 = 0$*, then* $z(t) = v(t) = (\max\{0, v_0^{\mu} - \mu\gamma_1 t\})^{1/\mu}$*. Thus, the system* (6)–(8) *corresponds to the prescribed-time observer [10].*

If additionally $\gamma_1 = 0$*, then* $z(t) \equiv v_0$*. Hence, the system* (6)–(8) *reduces to the Luenberger observer [16] for* $v_0 = 1$ *or the high-gain observer [14] for small* v_0*.*

In this section, we will investigate the performance of the proposed observer (6)–(8) in the absence of external disturbances d. To this end, we need to rewrite the closed-loop system (1), (6)–(8) with respect to variables $e = (w, x - \hat{x})^\top$ and v:

$$\begin{aligned}
&\dot{e}(t) = \big(A + D_{\bar{r}}^{-1}(\max\{v(t), \gamma_2 |Ce(t)|\}) LC\big) e(t),\\
&\dot{v}(t) = \gamma_2 \big| g_0(e(t), v(t)) \big| - \gamma_1 \big\lceil v(t) - \gamma_2 |Ce(t)| \big\rfloor^{1-\mu},
\end{aligned} \tag{9}$$

where $g_0(e, v) := C(A + D_{\bar{r}}^{-1}(\max\{v, \gamma_2|Ce|\})LC)e$, $D_{\bar{r}}(\lambda) := \mathrm{diag}\{\lambda^{\bar{r}_i}\}_{i=0}^{n}$, $L := \mathrm{col}\{L_i\}_{i=0}^{n}$; matrices A and C are of the form

$$A = \begin{bmatrix} O_{n\times 1} & I_n \\ 0 & O_{1\times n} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & O_{1\times n} \end{bmatrix}.$$

Clearly, the system (9) is in the form (3) with the state $\xi = (e, v)^\top$. Denoting the right-hand side of (9) as $f_0(\xi)$ and selecting $r = (\mathrm{col}\{r_i\}_{i=1}^{n+1}, r_1)^\top$, where $r_i := 1 - (i-1)\mu > 0$, $i = \overline{1, n+1}$, it is easy to check that the equality (4) holds for all $\xi \in \mathbb{R}^{n+2}$ and $\lambda > 0$ with $\varrho = -\mu$. Hence, the system (9) is r–homogeneous of degree $-\mu < 0$, for which Theorem 1 can be applied. Before doing this, we have to show the asymptotic stability of the system (9) first.

Theorem 3. *Let for some* $\mu \in (0, \frac{1}{n+1})$ *and* $\gamma_1 > 0$ *the pair* $(P, Y) \in \mathbb{R}^{(n+1)\times(n+1)} \times \mathbb{R}^{(n+1)\times 1}$ *be a solution of the following LMIs:*

$$\begin{aligned}
PA + YC + A^\top P + C^\top Y^\top + \gamma_1(PH + HP) \preccurlyeq -P,\\
P \succ 0, \quad PH + HP \succcurlyeq 0,
\end{aligned} \tag{10}$$

where $H := \mathrm{diag}\{1 - (i-1)\mu\}_{i=1}^{n+1}$*. Then the system* (9) *with* $L = P^{-1}Y$ *is globally asymptotically stable.*

Proof. First, note that the solutions of the system (9) are well-defined for all $t \geq 0$ since it is r-homogeneous of degree $-\mu < 0$ (see Lemma 3.6 in [5]). Moreover, for any initial condition $\xi_0 = \xi(0) \in \mathbb{R}^n$, there exists a finite moment of time $T_0(\xi_0)$ such that $z(t) = v(t)$, i.e., $\gamma_2|w(t)| < v(t)$, for all $t \geq T_0(\xi_0)$. Indeed, denoting $\zeta = \gamma_2|w| - v$, one gets

$$\dot{\zeta}(t) = -\gamma_1 \lceil \zeta(t) \rfloor^{1-\mu} - \gamma_2 |g_0(t)| (1 - \mathrm{sign}(w(t) g_0(t)))$$

with $\zeta(0) = \gamma_2|w(0)| - v(0)$. Obviously, $\zeta(t)$ is negative for all $t \geq (\max\{0, \zeta(0)\})^{\mu}/(\gamma_1\mu) = T_0(\xi_0)$.

Therefore, for all $t \geq T_0(\xi_0)$ the system (9) reduces to

$$\begin{aligned}
&\dot{e}(t) = \big(A + D_{\bar{r}}^{-1}(v(t)) LC\big) e(t),\\
&\dot{v}(t) = \gamma_2 \big|\tilde{g}_0(e(t), v(t))\big| - \gamma_1 \big\lceil v(t) - \gamma_2 |Ce(t)| \big\rfloor^{1-\mu},
\end{aligned} \tag{11}$$

where $\tilde{g}_0(e, v) := C(A + D_{\bar{r}}^{-1}(v)LC)e$. Since for homogeneous systems attractivity implies asymptotic stability [3], it suffices to prove that solutions of the system (11) with initial conditions $(e(T_0(\xi_0)), v(T_0(\xi_0)))^\top \in \mathbb{R}^{n+1} \times \mathbb{R}_+$ satisfy the following limit property:

$$\lim_{t\to+\infty} \|e(t)\| = 0, \quad \lim_{t\to+\infty} |v(t)| = 0. \tag{12}$$

To simplify the choice of the observer parameters (μ, L, γ_1 and γ_2) under which the limit property (12) holds, we will introduce an auxiliary system that is equivalent to (11) up to some coordinate and time transformations, and for which the tuning process is trivial.

I. Auxiliary system

Consider the system in the form:

$$\begin{aligned}
&\tfrac{d}{d\tau}\varepsilon(\tau) = \big(A + LC - \big(h(\varepsilon(\tau)) - \gamma_1\big) H\big) \varepsilon(\tau),\\
&\tfrac{d}{d\tau}\nu(\tau) = \big(h(\varepsilon(\tau)) - \gamma_1\big) \nu(\tau),
\end{aligned} \tag{13}$$

where $\varepsilon(\tau) \in \mathbb{R}^{n+1}$ and $\nu(\tau) \in \mathbb{R}$ are the state variables defined in new time $\tau \in \mathbb{R}_+$; $h(\varepsilon) := \gamma_2|CA\varepsilon + L_0 C\varepsilon| + \gamma_1(1 + \lceil \gamma_2 |C\varepsilon| - 1 \rfloor^{1-\mu})$.

Let us show that the system (13) is globally exponentially stable. Indeed, for a Lyapunov function candidate $V(\varepsilon) := \varepsilon^\top P \varepsilon$, where a positive definite matrix P satisfies LMIs (10), the time derivative $\frac{d}{d\tau}V(\varepsilon)$ along (13) is given by

$$\tfrac{d}{d\tau}V(\varepsilon) = 2\varepsilon^\top P(A + LC + \gamma_1 H)\varepsilon - 2h(\varepsilon)\varepsilon^\top P H \varepsilon.$$

Taking into account that $h(\varepsilon) \geq 0$ for all $\varepsilon \in \mathbb{R}^{n+1}$, it follows from LMIs (10) that $\frac{d}{d\tau}V(\varepsilon) \leq -\varepsilon^\top P \varepsilon = -V(\varepsilon)$. Therefore, the ε-subsystem is globally exponentially stable, i.e., $\|\varepsilon(\tau)\| \leq \alpha \|\varepsilon(0)\| \exp(-\tau)$ for all $\tau \geq 0$ and any $\varepsilon(0) \in \mathbb{R}^n$, where $\alpha := \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}$.

On the other hand, the solutions of the ν-subsystem are given as follows: $\nu(\tau) = \nu(0)\exp(-\gamma_1\tau + \int_0^\tau h(\varepsilon(\sigma))d\sigma)$. Since $1 + \lceil \gamma_2|C\varepsilon| - 1\rfloor^{1-\mu} \leq 2(\gamma_2|C\varepsilon|)^{1-\mu}$ for any $\mu \in [0,1]$ (see Lemma 6 in the Appendix) and $|CA\varepsilon + L_0C\varepsilon| \leq |CA\varepsilon| + |L_0||C\varepsilon| \leq (1+|L_0|)\|\varepsilon\|$, we get for all $\tau \geq 0$

$$\begin{aligned}&\textstyle\int_0^\tau h(\varepsilon(\sigma))d\sigma \leq \int_0^{+\infty} h(\varepsilon(\sigma))d\sigma\\ &\leq \textstyle\int_0^{+\infty}\left(\gamma_2(1+|L_0|)\|\varepsilon(\sigma)\| + 2\gamma_1(\gamma_2\|\varepsilon(\sigma)\|)^{1-\mu}\right)d\sigma\\ &\leq \alpha\gamma_2(1+|L_0|)\|\varepsilon(0)\| + \tfrac{2\gamma_1}{1-\mu}(\alpha\gamma_2\|\varepsilon(0)\|)^{1-\mu} := \beta(\|\varepsilon(0)\|).\end{aligned}$$

Then $|\nu(\tau)| \leq |\nu(0)|\exp(\beta(\|\varepsilon(0)\|))\exp(-\gamma_1\tau)$ and, as a result, the system (13) is globally exponentially stable.

II. Coordinate transformation

Note that $\nu(\tau)$ remains strictly positive for any $\tau \geq 0$ provided $\nu(0) > 0$. To avoid any singularities in the sequel, we will consider only solutions $\Phi(\tau, \varphi_0) = (\varepsilon(\tau), \nu(\tau))^\top$ of the system (13) with initial conditions $\varphi_0 = (\varepsilon(0), \nu(0))^\top \in \mathbb{R}^{n+1} \times \mathbb{R}_+^*$. Having said that, we will apply the coordinate transformation $\chi : \mathbb{R}^{n+1} \times \mathbb{R}_+^* \to \mathbb{R}^{n+1} \times \mathbb{R}_+^*$ defined for any $\tau \geq 0$ as follows:

$$\chi(\Phi(\tau, \varphi_0)) := \begin{pmatrix} D_r(\nu(\tau))\varepsilon(\tau) \\ \nu(\tau)\end{pmatrix} = \begin{pmatrix}\epsilon(\tau) \\ \nu(\tau)\end{pmatrix}. \tag{14}$$

Since $(h(\varepsilon) - \gamma_1)\nu = \nu^\mu(\gamma_2|\tilde{g}_0(\epsilon, \nu)| - \gamma_1\lceil \nu - \gamma_2|C\epsilon|\rfloor^{1-\mu})$, $D_r(\nu)A = \nu^\mu AD_r(\nu)$, $D_r(\nu)LC = \nu^\mu D_{\tilde{r}}^{-1}(\nu)LCD_r(\nu)$ and $D_r(\nu)H = HD_r(\nu)$, one gets from (14) that variables $\epsilon(\tau)$ and $\nu(\tau)$ satisfy the following differential equations:

$$\begin{aligned}\tfrac{d}{d\tau}\epsilon(\tau) &= \nu^\mu(\tau)\big(A + D_{\tilde{r}}^{-1}(\nu(\tau))LC\big)\epsilon(\tau),\\ \tfrac{d}{d\tau}\nu(\tau) &= \nu^\mu(\tau)\gamma_2\big|\tilde{g}_0(\epsilon(\tau), \nu(\tau))\big|\\ &\quad - \nu^\mu(\tau)\gamma_1\lceil \nu(\tau) - \gamma_2|C\epsilon(\tau)|\rfloor^{1-\mu}\end{aligned} \tag{15}$$

with initial conditions $(\epsilon(0), \nu(0))^\top = \chi(\varphi_0)$. Note that the origin of the ϵ-subsystem is globally attractive. Indeed, assuming $\varepsilon = \mathrm{col}\{\varepsilon_i\}_{i=1}^n$, it follows from (14) that $\|\epsilon(\tau)\| = \sqrt{\sum_{i=1}^{n+1}\nu^{2r_i}(\tau)\varepsilon_i^2(\tau)}$, where $r_1 > 0$ and $r_{n+1} > 0$. On the other hand, global exponential stability of the system (13) implies that $|\nu(\tau)| \to 0$ and $|\varepsilon_i(\tau)| \to 0$, $i = \overline{1, n+1}$, as $\tau \to +\infty$. Therefore, $\|\epsilon(\tau)\| \to 0$ as $\tau \to +\infty$.

III. Time transformation

Finally, we apply the time transformation $\psi : \mathbb{R}_+ \to [T_0(\xi_0), T(\xi_0))$, where $T(\xi_0) \in \mathbb{R}_+ \cup \{+\infty\}$ is the settling time of the system (11), given by the following integral relation:

$$\psi(\tau) := T_0(\xi_0) + \textstyle\int_0^\tau \nu^\mu(\sigma)d\sigma. \tag{16}$$

Taking into account that $\frac{d}{d\tau}\psi(\tau) = \nu^\mu(\tau) > 0$ for all $\tau \geq 0$, one can easily check that systems (11) and (15) are one-to-one corresponding under the time transformation $t = \psi(\tau)$, i.e., $e(t) = \epsilon(\tau) \in \mathbb{R}^{n+1}$ and $v(t) = \nu(\tau) \in \mathbb{R}_+^*$. Clearly, the time transformation does not affect the attractivity property, as a result, $\|e(t)\| \to 0$ and $|v(t)| \to 0$ as $t \to T(\xi_0)$ which implies (12). □

Compared to [15], the main advantage of the proposed theorem is simple conditions to check: there are no parametric LMIs, but rather a simple system of LMIs that always has a solution. This simplification has been achieved due to the choice of the time-varying gain $z(t)$ in the form (7) instead of $z(t) = |w(t)|$ or $z(t) = y(t) - \hat{x}_1(t)$. Moreover, the additional filter (8) allows one to reduce the effect of high-frequency components of the measurement noise d_y on the steady-state error.

Once asymptotic stability is proven, we can proceed to finite-time convergence.

Proposition 4. *Let the conditions of Theorem 3 hold. Then the system* (9) *with $L = P^{-1}Y$ is globally finite-time stable. Furthermore, if initial conditions of the observer* (6)–(8) *are chosen such that $v(0) > \gamma_2|Ce(0)|$, then the settling time $T(e(0), v(0))$ of the system* (9) *satisfies the following:*

- *for $\gamma_2 > 0$*

$$T(e(0), v(0)) \leq \frac{v^\mu(0)}{\mu\gamma_1}\exp\left(\mu\beta(\|D_r^{-1}(v(0))e(0)\|)\right), \tag{17}$$

where $\beta(\|\varepsilon\|) = \alpha\gamma_2(1+|L_0|)\|\varepsilon\| + \frac{2\gamma_1}{1-\mu}(\alpha\gamma_2\|\varepsilon\|)^{1-\mu}$ with $\alpha = \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}$;

- *for $\gamma_2 = 0$*

$$T(e(0), v(0)) = \frac{v^\mu(0)}{\mu\gamma_1}. \tag{18}$$

Proof. Since the conditions of Theorem 3 hold, then the system (9) is globally asymptotically stable. As a result, the global finite-time stability of the system (9) immediately follows from Theorem 1.

On the other hand, repeating the proof of Theorem 3, one can see that the settling time $T(\xi_0) := \lim_{\tau\to+\infty}\psi(\tau)$ can be found from (16) as $T(\xi_0) = T_0(\xi_0) + \int_0^{+\infty}\nu^\mu(s)ds$, where $T_0(\xi_0) = 0$ due to the choice of initial conditions. Taking into account that $\nu(\tau) \leq \nu(0)\exp(\beta(\|\varepsilon(0)\|))\exp(-\gamma_1\tau)$ for $\gamma_2 > 0$ and $\nu(\tau) = \nu(0)\exp(-\gamma_1\tau)$ for $\gamma_2 = 0$, we get the expressions (17) and (18), respectively. □

Despite that Proposition 4 states the global finite-time stability of the system (9), it is more important that it also provides quantitative estimates on the settling time T for particular initial conditions. Note that $v(0)$ and $Ce(0) = w(0)$ are selected by a designer, then one might, for example, set $v(0) = 1$ and $w(0) = 0$. So, it follows from (17) and (18) that the state reconstruction can be accelerated by increasing parameters μ and γ_1. On the other hand, since $\beta(\|\varepsilon\|) \to 0$ as $\gamma_2 \to 0$, then the estimate (17) coincides with (18) in the limit case $\gamma_2 = 0$ and, moreover, becomes exact. Although the latter observation might suggest setting $\gamma_2 = 0$ in the first place [10], we will show later using numerical simulation that such a choice makes the observer very sensitive to measurement noises d_y.

V. Robustness analysis

Now we will consider the disturbed case ($d \neq 0$). Similarly to the disturbance-free case, let us rewrite the closed-loop system (1), (6)–(8) with respect to variables e and v:

$$\begin{aligned}\dot{e}(t) &= \big(A + D_{\tilde{r}}^{-1}(\max\{v(t), \gamma_2|Ce(t)|\})LC\big)e(t) + d(t),\\ \dot{v}(t) &= \gamma_2\big|g(e(t), v(t), d(t))\big| - \gamma_1\lceil v(t) - \gamma_2|Ce(t)|\rfloor^{1-\mu},\end{aligned} \tag{19}$$

where $g(e, v, d) := C(A + D_{\tilde{r}}^{-1}(\max\{v, \gamma_2|Ce|\})LC)e + Cd$.

Clearly, the system (19) is in the form (2) with the state $\xi = (e, v)^\top$. Denoting the right-hand side of (19) as $f(\xi, d)$

and selecting $r = (\text{col}\{r_i\}_{i=1}^{n+1}, r_1)^\top$ and $r_d = \text{col}\{r_{d,i}\}_{i=1}^{n+1}$, where $r_i := 1-(i-1)\mu > 0$ and $r_{d,i} := 1-i\mu$, $i = \overline{1, n+1}$, it is easy to check that the equality (5) holds for all $\xi \in \mathbb{R}^{n+2}$, $d \in \mathbb{R}^{n+1}$ and $\lambda > 0$ with $\varrho = -\mu$. Hence, the system (19) is (r, r_d)–homogeneous of degree $-\mu < 0$, for which Theorem 2 can be applied.

Proposition 5. *Let the conditions of Theorem 3 hold. Then the system* (19) *with* $L = P^{-1}Y$ *is:*

- *finite-time ISS for* $\mu \in (0, \frac{1}{n+1})$*;*
- *finite-time iISS for* $\mu = \frac{1}{n+1}$.

Proof. Since the conditions of Theorem 3 hold, then the system (9) is globally asymptotically stable. Taking into account that $r_{d,i} \geq r_{d,n+1} = 1-(n+1)\mu$, the (integral) input-to-state stability of the system (19) immediately follows from Theorem 2. □

VI. Example

Consider the system (1) with $n = 3$, $m = 1$, $b_1 = b_2 = 0$, $b_3 = 1$ and let $u(t) = Kx(t) + \cos(10t)$, where feedback gains $K = -[6,\ 11,\ 6]$ are selected such that matrix $A+BK$ is Hurwitz. Setting $\mu = 0.2$ and $\gamma_1 = 1$, one gets feedback gains $L = -[6.98,\ 28.60,\ 44.46,\ 23.04]^\top$ from LMIs (10).

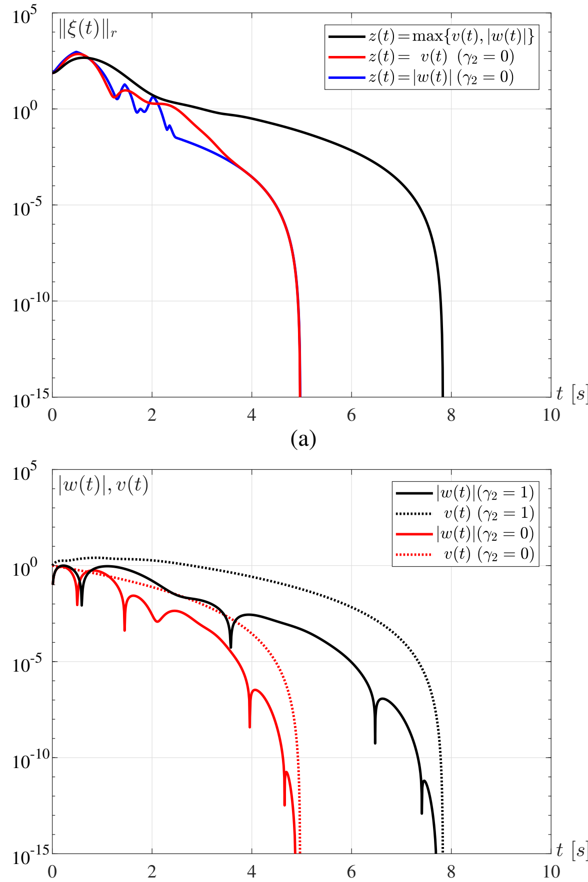
The numerical simulation of dynamical systems (1), (6) and (8) has been done in MATLAB by using the explicit Euler method with a state-dependent step [8]. The basic discretization step, the simulation time and the maximum number of iterations have been taken as $\Delta t_0 = 10^{-2}$, $t_{\max} = 10$ and $N_{\max} = 10^6$, respectively. Following the work [19], we define the r-homogeneous norm of the state ξ as follows:

$$\|\xi\|_r := |v| + \textstyle\sum_{i=1}^{n+1} |e_i|^{1/r_i}.$$

The initial conditions for systems (1), (6) and (8) have been chosen as $x(0) = [10,\ 0,\ -5]^\top$, $(w(0), \hat{x}(0))^\top = 0$ and $v(0) = 1$, respectively.

Let us compare the proposed finite-time observer (6)–(8), when $\gamma_2 = 1$ and $z(t) = \max\{v(t), |w(t)|\}$, with its prescribed-time counterpart [10], i.e., $\gamma_2 = 0$ and $z(t) = v(t)$, and another finite-time observer [17], where $z(t) = |w(t)|$. It is worth mentioning that there is no additional filter (8) in [17]. However, we will artificially introduce it with $\gamma_2 = 0$ to make a fair comparison with other observers. Obviously, such a modification does not affect either finite-time stability or robustness. Other parameters (γ_1, L_i, $\bar{r}_i$, $i = \overline{0, n}$) are the same for all the observers.

First, we consider the disturbance-free case ($d = 0$) when all the observers provide the exact state estimate in finite time. As it can be seen in Figure 1a, the r-homogeneous norm $\|\xi(t)\|_r$ of the system state converges to the origin not later than in 8 seconds. Moreover, for the prescribed-time observer (*red line*), the settling time found from the simulation matches exactly its theoretical estimate $T|_{\gamma_2=0} = v(0)/(\mu\gamma_1) = 5$ given in (18). On the other hand, it follows from Figure 1b that for the proposed observer (*black line*) the time-varying gain $v(t)$ is an upper bound for $|w(t)|$ for all $t \geq 0$ provided $v(0) > |w(0)|$.



(b)

Fig. 1: Time evolution of $\|\xi(t)\|_r$, $|w(t)|$ and $v(t)$ in the absence of external disturbances d

Now we will demonstrate the robustness property of the closed-loop system (1), (6)–(8) with respect to external disturbances d if γ_2 is chosen strictly positive. To this end, assume that $d_x(t) = 0.1 \cdot [\sin(2t),\ \omega_x(t),\ 1+\cos t]^\top$ and $d_y = 0.1\omega_y(t)$, where $\omega_x(t)$ and $\omega_y(t)$ are uniformly distributed random numbers in the interval $(-1, 1)$. One can see in Figure 2a that the proposed finite-time observer (*black line*) is indeed robust while its prescribed-time counterpart (*red line*) is not. The different behavior might be explained by taking a closer look at Figure 2b. For the case $\gamma_2 = 0$, the time-varying gain $v(t)$ decreases to zero in finite time regardless of $|w(t)|$. As a result, the closed-loop system exhibits the finite-time escape. In contrast, for $\gamma_2 = 1$, the product $v^{-\bar{r}_i}(t)w(t) = v^{1-\bar{r}_i}(t)w(t)/v(t)$, $i = \overline{0, n}$, remains bounded for all $t \geq 0$. Moreover, note that for both finite-time observers (*black and blue lines*) the magnitude of the steady-state error is of the same order. However, the proposed observer additionally filters out high-frequency components of the measurement noise d_y.

VII. Conclusion

In this paper, we have presented a novel observer that estimates the state of a linear system in the observer canonical form in finite time. Due to the specially chosen dynamical gain, the finite-time convergence of the proposed observer

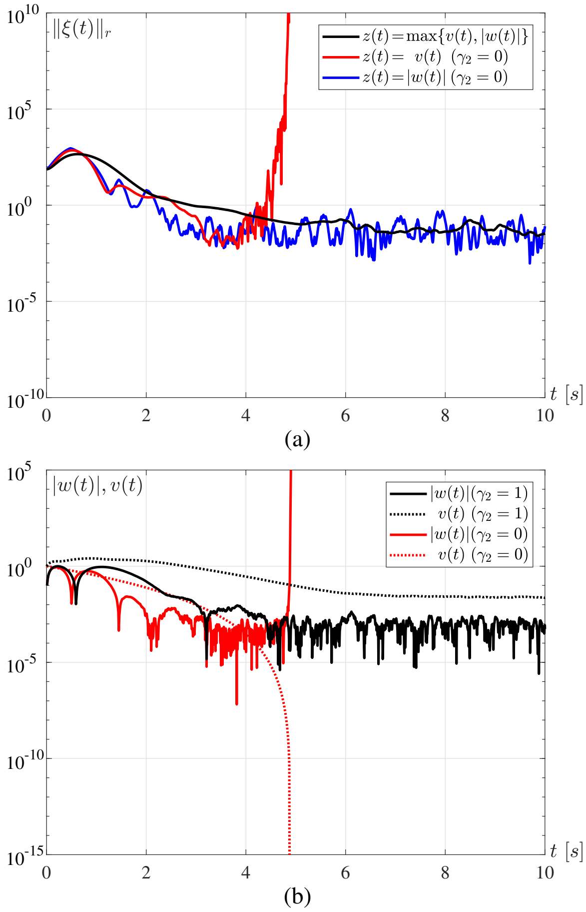


Fig. 2: Time evolution of $\|\xi(t)\|_r$, $|w(t)|$ and $v(t)$ in the presence of external disturbances d

follows from the asymptotic stability of the auxiliary system for which the tuning process is trivial. Moreover, for particular initial conditions, quantitative estimates (the exact value if $\gamma_2 = 0$) on the settling time have been given. The simulation results have confirmed that, differently from the prescribed-time observer [10], the proposed one is indeed robust with respect to external disturbances. A more detailed study on the choice of the observer parameters, especially γ_2, and their effect on the steady-state error in the presence of external disturbances is left for future research.

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Appendix

Lemma 6. *For any $\varrho \in [0,1]$ the following estimate holds for all $c \geq 0$:*

$$1 + \lceil c-1 \rfloor^{\varrho} \leq 2c^{\varrho}.$$

Proof. While the proof for $\varrho = 0$ and $\varrho = 1$ is straightforward, let us focus on the case $\varrho \in (0,1)$. If $c \in [0,1]$, then $1 + \lceil c-1 \rfloor^{\varrho} = 1 - |c-1|^{\varrho} \leq 1 - |c-1| = c \leq c^{\varrho}$. On the other hand, if $c > 1$, then $1 + \lceil c-1 \rfloor^{\varrho} = 1 + |c-1|^{\varrho} < 1 + c^{\varrho} < 2c^{\varrho}$. $\square$