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# Annular short-time stability of generalized Persidskii systems

Wenjie Mei<sup>1</sup>, Denis Efimov<sup>1</sup>, Rosane Ushirobira<sup>1</sup>

**Abstract**—This work investigates a class of generalized Persidskii systems. We study the conditions of annular short-time stability for the considered kind of systems with an essentially bounded input, which can be checked through linear matrix inequalities. Boundedness conditions on a finite interval of time, with a nonzero initial state, are obtained for this dynamics class. A neuron model example illustrates the proposed conditions.

## I. INTRODUCTION

Stability analysis of dynamical systems is a complex problem, especially for nonlinear dynamics and in the presence of external inputs [1], [2], [3]. One of the most general frameworks in this domain is the Lyapunov and the asymptotic stability concepts [4]. However, in some scenarios, it is not necessary to analyze the system behavior for all positive times, and one is interested in the compartment of trajectories on a bounded interval of time. The short-time stability concept was first introduced in the 1950s [5], [6] to address the problem that the solutions of a system stay bounded, starting from a bounded set of initial conditions, during a finite interval of time, which is of practical interest for engineers.

In this paper, we call the stability concept mentioned above, short-time stability [6] rather than finite-time stability (also frequently used to denote the same property [5]) since another stability notion, called similarly, deals with a finite time of convergence of the system trajectories to an invariant mode (*e.g.*, an equilibrium or a desired set) combined with the conventional Lyapunov stability for all positive times. At the same time, all analyses for short-time stability are performed on a bounded interval of time only.

Various studies on short-time stability and stabilization of linear time-varying systems were proposed [7], [8], [9], as well as on robust short-time stability of linear systems [10]. Further short-time stability investigations in distinct directions can be found in the scenarios of stochastic systems [11], [12], or annular (stochastic) short-time stability (ASTS) (in [13], [14], it is called annular (stochastic) finite-time stability), to mention a few examples. In this study, we are focused on investigating robust ASTS. Roughly speaking, a system is said to be ASTS if given a bounded initial domain separated from the origin, the system state stays at another bounded region during a specified time interval. The concept of ASTS was proposed to address safety problems, where

there are requirements for the state to do not exceed given lower and upper bounds, *e.g.*, a water level in a tank should not go out of prescribed thresholds [13].

The considered class of nonlinear dynamics in this paper is called the generalized Persidskii systems, which have been extensively studied in the context of neural networks [15], biological models [16], and power systems [17], and whose original form was introduced in [18], [19], [20]. Recently the conditions of input-to-state stability, input-to-output stability, and convergence, as well as the synthesis of a state observer have been established in [21], [16], [22], [23] for generalized Persidskii models. Note that most existing approaches to synthesizing Lyapunov functions for stability analysis in nonlinear dynamics involve various canonical forms of the studied differential equations, such as Lur'e systems [24], homogeneous models [25], Persidskii systems [18], and Lipschitz dynamics, and the presence of nonlinearities leads to that the established stability conditions can be rather complicated. However, for generalized Persidskii systems, the properties of the Lyapunov function proposed in [21] and its time derivative can be analyzed using linear matrix inequalities (LMIs), which is a rare case for nonlinear systems taking into account a possible highly nonlinear nature of these models.

To motivate our investigations of ASTS for generalized Persidskii systems, we would like to recall the applications of recurrent neural networks (they can be brought to the form of generalized Persidskii dynamics) to classify temporal sequences defined on a finite interval of time. In such a scenario, the output of a trained network has to approach desired levels (usually a small invariant region is allowed) in a short time, indicating a class of the input signal. Our main research direction in this work is to extend the previous results on ASTS and short-time boundedness with nonzero initial state (STBNZ), and obtain sufficient related conditions for a family of generalized Persidskii systems.

The organization of this paper is as follows. Section II presents the preliminaries and the definitions of utilized stability properties. The class of considered systems is defined in Section III. Conditions of ASTS and STBNZ for generalized Persidskii systems are given in Section IV. An example is considered for verifying the efficiency of the proposed results in Section V.

## NOTATION

- $\mathbb{N}$  and  $\mathbb{R}$  denote the sets of natural and real numbers, respectively,  $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$ ;  $\mathbb{R}^{m \times n}$  denote the vector space of  $m \times n$  real matrices. The symbol  $\|\cdot\|$  refers to the Euclidean norm on  $\mathbb{R}^n$  (and the induced

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matrix norm  $\|A\|$  for a matrix  $A \in \mathbb{R}^{m \times n}$ . The set of  $n \times n$  diagonal matrices (with nonnegative diagonal elements) is denoted by  $\mathbb{D}^n$  ( $\mathbb{D}_+^n$ ). Let  $O_{m \times n}$  refer to the  $m \times n$  zero matrix.

- For  $p, n \in \mathbb{N}$  with  $p \leq n$ , the notation  $\overline{p, n}$  is used to represent the set  $\{p, \dots, n\}$ .
- $B(\delta, x) = \{y \in \mathbb{R}^n : \|x - y\| \leq \delta\}$  stands for the closed ball of radius  $\delta > 0$  centered at a point  $x \in \mathbb{R}^n$ ;  $\text{cl}(E)$  denotes the closure of  $E \subset \mathbb{R}^n$ .
- For a Lebesgue measurable function  $u: \mathbb{R} \rightarrow \mathbb{R}^m$ , define the norm  $\|u\|_{(t_1, t_2)} = \text{ess sup}_{t \in (t_1, t_2)} \|u(t)\|$  for  $(t_1, t_2) \subseteq \mathbb{R}$ . We denote by  $\mathcal{L}_{(t_1, t_2)}^m$  (or  $\mathcal{L}_\infty^m$ ) the Banach space of functions  $u$  with  $\|u\|_{(t_1, t_2)} < +\infty$  (or  $\|u\|_\infty := \|u\|_{\mathbb{R}} < +\infty$ ).
- A continuous function  $\sigma: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{K}$  if it is strictly increasing and  $\sigma(0) = 0$ ; it belongs to class  $\mathcal{K}_\infty$  if it is also unbounded. A continuous function  $\beta: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{KL}$  if  $\beta(\cdot, r) \in \mathcal{K}$  and  $\beta(r, \cdot)$  is a decreasing function going to zero for any fixed  $r > 0$ .
- For a continuously differentiable function  $V: \mathbb{R}^n \rightarrow \mathbb{R}$ , denote by  $\nabla V(v)f(v)$  the derivative of  $V$  along the vector field  $f$  evaluated at point  $v \in \mathbb{R}^n$ .

## II. PRELIMINARIES

Consider the differential equation

$$\dot{x}(t) = F(x(t), u(t)), \quad t \in \Delta = [0, T] \subset \mathbb{R}_+, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector;  $u(t) \in \mathbb{R}^m$  is the external input,  $u \in \mathcal{L}_\infty^m$ ;  $\Delta$  is the time interval of interest with  $0 < T < +\infty$ . Moreover,  $F: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a continuous function. For an initial state  $x(0) = x_0 \in \mathbb{R}^n$  and  $u \in \mathcal{L}_\infty^m$ , we denote the corresponding solution of the system (1) by  $x(t, x_0, u)$  for the values of  $t \in \mathbb{R}_+$  the solution exists. In the sequel we assume that in (1) such a solution is uniquely defined for any  $x_0 \in \mathbb{R}^n$  and  $u \in \mathcal{L}_\infty^m$  for all  $t \in \Delta$ .

*Definition 1* ([26], [13]): The system (1) is said to be:

- 1) *short-time bounded* (STB) with respect to  $(\Delta, \gamma, \delta)$ , for given  $\gamma \geq 0$  and  $\delta > 0$ , if

$$x_0 = 0, \|u\|_\Delta \leq \gamma \Rightarrow \|x(t, x_0, u)\| \leq \delta, \quad \forall t \in \Delta.$$

- 2) *short-time bounded with nonzero initial state* (STBNZ) with respect to  $(\Delta, \varepsilon, \gamma, \delta)$ , for given  $\varepsilon \geq 0$ ,  $\gamma \geq 0$  and  $\delta > 0$ , if

$$\|x_0\| \leq \varepsilon, \|u\|_\Delta \leq \gamma \Rightarrow \|x(t, x_0, u)\| \leq \delta, \quad \forall t \in \Delta.$$

- 3) *annular short-time stable* (ASTS) with respect to  $(\Delta, \varepsilon_1, \varepsilon_2, \gamma, \delta_1, \delta_2)$ , for given  $(\varepsilon_1, \varepsilon_2) \subseteq (\delta_1, \delta_2) \subset \mathbb{R}_+$  and  $\gamma \geq 0$ , if

$$\begin{aligned} & x_0 \in \text{cl}(B(\varepsilon_2, 0) \setminus B(\varepsilon_1, 0)), \|u\|_\Delta \leq \gamma \\ \Rightarrow & x(t, x_0, u) \in \text{cl}(B(\delta_2, 0) \setminus B(\delta_1, 0)), \quad \forall t \in \Delta. \end{aligned}$$

In the rest of the paper, to lighten the notation, the time-dependency of variables might remain implicitly understood, for instance we will write  $x$  for  $x(t)$ .

## III. PROBLEM STATEMENT

Consider the following system in a generalized Persidskii form [18]:

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^M A_j F_j(x(t)) + u(t), \quad t \in \Delta = [0, T], \quad (2)$$

where  $x = [x_1 \dots x_n]^\top \in \mathbb{R}^n$  is the state,  $x(0) = x_0$ ;  $[0, T]$  is the interval of interest for some  $0 < T < +\infty$ ; the exogenous input  $u = [u_1 \dots u_n]^\top \in \mathcal{L}_\infty^n$ ;  $A_k \in \mathbb{R}^{n \times n}$  ( $k \in \overline{0, M}$ ) are constant matrices, and  $F_j$  are continuous functions,

$$F_j(x) = [f_j^1(x_1) \dots f_j^n(x_n)]^\top \in \mathbb{R}^n, \quad \forall j \in \overline{1, M}.$$

So, the nonlinearity  $F_j$  has a special structure: each component  $f_j^i$  of  $F_j$  depends only on the coordinate  $x_i$ ,  $i \in \overline{1, n}$ .

Sector (or passivity) restrictions on  $F_j$ ,  $j \in \overline{1, M}$ , are imposed in this work:

*Assumption 1:* Assume that for any  $i \in \overline{1, n}$  and  $j \in \overline{1, M}$ ,

$$v f_j^i(v) > 0, \quad \forall v \in \mathbb{R} \setminus \{0\}.$$

Under Assumption 1, with a reordering of nonlinearities and their decomposition, there exists an index  $\overline{\omega} \in \overline{0, M}$  such that for all  $i \in \overline{1, n}$ ,  $a \in \overline{1, \overline{\omega}}$

$$\lim_{v \rightarrow \pm\infty} f_a^i(v) = \pm\infty,$$

and that there exists  $\mu \in \overline{\overline{\omega}, M}$  such that for all  $i \in \overline{1, n}$ ,  $b \in \overline{1, \mu}$

$$\lim_{v \rightarrow \pm\infty} \int_0^v f_b^i(\tau) d\tau = +\infty.$$

In this case,  $\overline{\omega} = 0$  implies that all nonlinearities are bounded (at least for positive or negative argument).

In this paper, in order to conduct the stability analysis in Section IV, we also require a mild hypothesis of upper and lower bounds on the integrals of the nonlinearities, formulated in terms of products of the state and the nonlinearities themselves (note that due to Assumption 1, all these products are nonnegative):

*Assumption 2:* Assume that for any  $i \in \overline{1, n}$  and  $j, j' \in \overline{1, M}$ ,  $z \in \overline{j+1, M}$ , there exist  $\kappa_{0,j}^i$ ,  $\kappa_{1,jj'}^i$ ,  $\kappa_{2,jj'}^i$ ,  $\kappa_{3,jj'z}^i$ ,  $\eta_{0,j}^i$ ,  $\eta_{1,jj'}^i$ ,  $\eta_{2,jj'}^i$ ,  $\eta_{3,jj'z}^i \geq 0$  such that

$$\begin{aligned} \kappa_{0,j}^i x_i^2 + \sum_{j'=1}^M f_{j'}^i(x_i) \left( \kappa_{1,jj'}^i f_{j'}^i(x_i) + 2\kappa_{2,jj'}^i x_i + 2 \sum_{z=j+1}^M \kappa_{3,jj'z}^i f_z^i(x_i) \right) \\ \leq 2 \int_0^{x_i} f_j^i(s) ds \leq \\ \eta_{0,j}^i x_i^2 + \sum_{j'=1}^M f_{j'}^i(x_i) \left( \eta_{1,jj'}^i f_{j'}^i(x_i) + 2\eta_{2,jj'}^i x_i + 2 \sum_{z=j+1}^M \eta_{3,jj'z}^i f_z^i(x_i) \right) \end{aligned}$$

for all  $x \in \mathbb{R}^n$ .

Assumptions 1 and 2 are satisfied by many nonlinear functions: for example, for polynomial nonlinearities it is

sufficient to select  $\eta_{2,jj'}^i \neq 0$  (see [27] for illustration). In the sequel, we denote the diagonal matrices:

$$\begin{aligned}\kappa_{0,j} &= \text{diag}(\kappa_{0,j}^1, \dots, \kappa_{0,j}^n), \quad \kappa_{1,jj'} = \text{diag}(\kappa_{1,jj'}^1, \dots, \kappa_{1,jj'}^n), \\ \kappa_{2,jj'} &= \text{diag}(\kappa_{2,jj'}^1, \dots, \kappa_{2,jj'}^n), \quad \kappa_{3,jj'z} = \text{diag}(\kappa_{3,jj'z}^1, \dots, \kappa_{3,jj'z}^n), \\ \eta_{0,j} &= \text{diag}(\eta_{0,j}^1, \dots, \eta_{0,j}^n), \quad \eta_{1,jj'} = \text{diag}(\eta_{1,jj'}^1, \dots, \eta_{1,jj'}^n), \\ \eta_{2,jj'} &= \text{diag}(\eta_{2,jj'}^1, \dots, \eta_{2,jj'}^n), \quad \eta_{3,jj'z} = \text{diag}(\eta_{3,jj'z}^1, \dots, \eta_{3,jj'z}^n).\end{aligned}$$

#### IV. SHORT-TIME STABILITY CONDITIONS

In this section, ASTS and STBNZ sufficient conditions for the generalized Persidskii system (2) in the presence of an essentially bounded input are formulated.

Following [28], [16], [22], [23], the stability analysis of (2) can be performed using a Lyapunov function

$$V(x) = x^\top P x + 2 \sum_{j \in \overline{1,M}} \sum_{i \in \overline{1,n}} \Lambda_j^i \int_0^{x_i} f_j^i(v) dv,$$

where  $0 \leq P = P^\top \in \mathbb{R}^{n \times n}$  and  $\Lambda_j = \text{diag}(\Lambda_j^1, \dots, \Lambda_j^n) \in \mathbb{D}_+^n$  are tuning matrices. Under Assumption 1, they can always be selected in a way ensuring positive definiteness of  $V$ , then there exist  $\mathcal{K}_\infty$  functions  $\alpha_1^{P, \Lambda_1, \dots, \Lambda_M}, \alpha_2^{P, \Lambda_1, \dots, \Lambda_M}$  such that

$$\alpha_1^{P, \Lambda_1, \dots, \Lambda_M}(\|x\|) \leq V(x) \leq \alpha_2^{P, \Lambda_1, \dots, \Lambda_M}(\|x\|)$$

for all  $x \in \mathbb{R}^n$  (for instance, the upper bound can be always taken as  $\alpha_2^{P, \Lambda_1, \dots, \Lambda_M}(\tau) = \lambda_{\max}(P)\tau^2 + 2nM \max_{i \in \overline{1,n}, j \in \overline{1,M}} \{\Lambda_j^i \int_0^\tau f_j^i(v) dv\}$ ). These functions  $\alpha_1^{P, \Lambda_1, \dots, \Lambda_M}, \alpha_2^{P, \Lambda_1, \dots, \Lambda_M}$  will be used later in the proofs.

The following theorem is the main result of this paper.

*Theorem 1:* Let assumptions 1 and 2 be satisfied and  $T > 0$ ;  $\gamma_0 \geq 0$ ;  $(\varepsilon_1, \varepsilon_2) \subseteq (\delta_1, \delta_2) \subset \mathbb{R}_+$  be given. If there exist

$$0 \leq \bar{P} = \bar{P}^\top, \underline{P} = \underline{P}^\top \in \mathbb{R}^{n \times n};$$

$$\left\{ \bar{\Lambda}_j = \text{diag}(\bar{\Lambda}_j^1, \dots, \bar{\Lambda}_j^n), \underline{\Lambda}_j = \text{diag}(\underline{\Lambda}_j^1, \dots, \underline{\Lambda}_j^n) \right\}_{j=1}^M \subset \mathbb{D}_+^n;$$

$$\{\bar{\Xi}^s, \underline{\Xi}^s\}_{s=0}^M, \{\bar{\Upsilon}_{s,r}, \underline{\Upsilon}_{s,r}\}_{0 \leq s < r \leq M} \subset \mathbb{D}^n; \bar{\gamma}, \underline{\gamma} > 0$$

and

$$\beta_1, \beta_2, \bar{\rho}, \underline{\rho} \in \mathbb{R}$$

such that

$$\underline{P} + \underline{\rho} \sum_{j=1}^M \underline{\Lambda}_j > 0, \quad (3)$$

$$\underline{Q} = \underline{Q}^\top = (\underline{Q}_{a,b})_{a,b=1}^{M+2} \geq 0, \quad (4)$$

$$\begin{cases} \underline{\gamma} \leq \frac{e^{\beta_1 T} \underline{\alpha}_1(\varepsilon_1) - \alpha_2(\delta_1)}{\frac{\gamma_0}{\beta_1} [e^{\beta_1 T} - 1]}, & \text{if } \beta_1 < 0 \text{ or } \beta_1 > 0, \underline{\gamma} > \frac{\beta_1}{\gamma_0} \underline{\alpha}_1(\varepsilon_1), \\ \underline{\gamma} \leq \frac{\underline{\alpha}_1(\varepsilon_1) - \alpha_2(\delta_1)}{T \gamma_0}, & \text{if } \beta_1 = 0, \\ \underline{\alpha}_2(\delta_1) \leq \underline{\alpha}_1(\varepsilon_1), & \text{if } \beta_1 > 0, \underline{\gamma} \leq \frac{\beta_1}{\gamma_0} \underline{\alpha}_1(\varepsilon_1). \end{cases} \quad (5)$$

$$\begin{cases} -\underline{\Xi}^0 \geq \beta_1 \left( \underline{P} + \sum_{j=1}^M \underline{\Lambda}_j \eta_{0,j} \right), \\ -\underline{\Xi}^j \geq \beta_1 \underline{\Lambda}_j \sum_{j'=1}^M \eta_{1,jj'}, & \text{if } \beta_1 \geq 0, \\ -\underline{\Upsilon}_{0,j} \geq \beta_1 \underline{\Lambda}_j \sum_{j'=1}^M \eta_{2,jj'}, \\ -\underline{\Upsilon}_{j,z} \geq \beta_1 \underline{\Lambda}_j \sum_{j'=1}^M \eta_{3,jj'z}, \\ -\beta_1 \left( \underline{P} + \sum_{j=1}^M \underline{\Lambda}_j \kappa_{0,j} \right) \geq \underline{\Xi}^0, \\ -\beta_1 \underline{\Lambda}_j \sum_{j'=1}^M \kappa_{1,jj'} \geq \underline{\Xi}^j, & \text{if } \beta_1 < 0, \\ -\beta_1 \underline{\Lambda}_j \sum_{j'=1}^M \kappa_{2,jj'} \geq \underline{\Upsilon}_{0,j}, \\ -\beta_1 \underline{\Lambda}_j \sum_{j'=1}^M \kappa_{3,jj'z} \geq \underline{\Upsilon}_{j,z}. \end{cases} \quad (6)$$

$$\bar{P} + \bar{\rho} \sum_{j=1}^M \bar{\Lambda}_j > 0, \quad (7)$$

$$\bar{Q} = \bar{Q}^\top = (\bar{Q}_{a,b})_{a,b=1}^{M+2} \leq 0, \quad (8)$$

$$\begin{cases} \bar{\gamma} \leq \frac{\bar{\alpha}_1(\delta_2) - e^{\beta_2 T} \bar{\alpha}_2(\varepsilon_2)}{\frac{\gamma_0}{\beta_2} [e^{\beta_2 T} - 1]}, & \text{if } \beta_2 > 0 \text{ or } \beta_2 < 0, \bar{\gamma} > -\frac{\beta_2}{\gamma_0} \bar{\alpha}_2(\varepsilon_2) \\ \bar{\gamma} \leq \frac{\bar{\alpha}_1(\delta_2) - \bar{\alpha}_2(\varepsilon_2)}{T \gamma_0}, & \text{if } \beta_2 = 0, \\ \bar{\alpha}_2(\varepsilon_2) \leq \bar{\alpha}_1(\delta_2), & \text{if } \beta_2 < 0, \bar{\gamma} \leq -\frac{\beta_2}{\gamma_0} \bar{\alpha}_2(\varepsilon_2). \end{cases} \quad (9)$$

$$\begin{cases} \bar{\Xi}^0 \geq -\beta_2 \left( \bar{P} + \sum_{j=1}^M \bar{\Lambda}_j \eta_{0,j} \right), \\ \bar{\Xi}^j \geq -\beta_2 \bar{\Lambda}_j \sum_{j'=1}^M \eta_{1,jj'}, & \text{if } \beta_2 < 0, \\ \bar{\Upsilon}_{0,j} \geq -\beta_2 \bar{\Lambda}_j \sum_{j'=1}^M \eta_{2,jj'}, \\ \bar{\Upsilon}_{j,z} \geq -\beta_2 \bar{\Lambda}_j \sum_{j'=1}^M \eta_{3,jj'z}, \\ \beta_2 \left( \bar{P} + \sum_{j=1}^M \bar{\Lambda}_j \kappa_{0,j} \right) \geq -\bar{\Xi}^0, \\ \beta_2 \bar{\Lambda}_j \sum_{j'=1}^M \kappa_{1,jj'} \geq -\bar{\Xi}^j, & \text{if } \beta_2 \geq 0, \\ \beta_2 \bar{\Lambda}_j \sum_{j'=1}^M \kappa_{2,jj'} \geq -\bar{\Upsilon}_{0,j}, \\ \beta_2 \bar{\Lambda}_j \sum_{j'=1}^M \kappa_{3,jj'z} \geq -\bar{\Upsilon}_{j,z}. \end{cases} \quad (10)$$

where

$$\begin{aligned}\alpha_1(s) &= \alpha_1^{P, \Lambda_1, \dots, \Lambda_M}(s), \quad \alpha_2(s) = \alpha_2^{P, \Lambda_1, \dots, \Lambda_M}(s), \\ \bar{\alpha}_1(s) &= \bar{\alpha}_1^{\bar{P}, \bar{\Lambda}_1, \dots, \bar{\Lambda}_M}(s), \quad \bar{\alpha}_2(s) = \bar{\alpha}_2^{\bar{P}, \bar{\Lambda}_1, \dots, \bar{\Lambda}_M}(s).\end{aligned}$$

$$\begin{aligned}\bar{Q}_{1,1} &= A_0^\top \bar{P} + \bar{P} A_0 + \bar{\Xi}^0; \quad \bar{Q}_{j+1,j+1} = A_j^\top \bar{\Lambda}_j + \bar{\Lambda}_j A_j + \bar{\Xi}^j, \\ \bar{Q}_{1,j+1} &= \bar{P} A_j + A_0^\top \bar{\Lambda}_j + \bar{\Upsilon}_{0,j}, \quad \bar{Q}_{1,M+2} = \bar{P}, \\ \bar{Q}_{s+1,z'+1} &= A_s^\top \bar{\Lambda}_{z'} + \bar{\Lambda}_s A_{z'} + \bar{\Upsilon}_{s,z'}, \\ \bar{Q}_{j+1,M+2} &= \bar{\Lambda}_j, \quad \bar{Q}_{M+2,M+2} = -\bar{\gamma} I_n.\end{aligned}$$

$$\begin{aligned}\underline{Q}_{1,1} &= A_0^\top \underline{P} + \underline{P} A_0 + \underline{\Xi}^0; \quad \underline{Q}_{j+1,j+1} = A_j^\top \underline{\Lambda}_j + \underline{\Lambda}_j A_j + \underline{\Xi}^j, \\ \underline{Q}_{1,j+1} &= \underline{P} A_j + A_0^\top \underline{\Lambda}_j + \underline{\Upsilon}_{0,j}, \quad \underline{Q}_{1,M+2} = \underline{P}, \\ \underline{Q}_{s+1,z'+1} &= A_s^\top \underline{\Lambda}_{z'} + \underline{\Lambda}_s A_{z'} + \underline{\Upsilon}_{s,z'}, \\ \underline{Q}_{j+1,M+2} &= \underline{\Lambda}_j, \quad \underline{Q}_{M+2,M+2} = \underline{\gamma} I_n.\end{aligned}$$

$$j, j' \in \overline{1, M}; \quad z \in \overline{j+1, M}; \quad s \in \overline{1, M-1}; \quad z' \in \overline{s+1, M}.$$

Then the system (2) is ASTS with respect to  $([0, T], \varepsilon_1, \varepsilon_2, \gamma_0, \delta_1, \delta_2)$ .

*Proof:* The proof contains two steps for estimating the upper and the lower bounds of  $\|x(t)\|$  over the time interval  $[0, T]$ , and both steps follow a regular methodology in the short-time stability analyses.

1) Consider a candidate Lyapunov function

$$\bar{V}(x) = x^\top \bar{P} x + 2 \sum_{j=1}^M \sum_{i=1}^n \bar{\Lambda}_j^i \int_0^{x_i} f_j^i(\tau) d\tau.$$

By Finsler's Lemma, the condition (7) is equivalent to positive definiteness of  $\bar{V}(x)$  (under these conditions the matrix  $\bar{P}$  is positive definite on the subspace where the integral terms are zero). Thus, under the restriction (7), it follows that there exist some  $\mathcal{K}_\infty$  functions  $\bar{\alpha}_1, \bar{\alpha}_2$  such that

$$\bar{\alpha}_1(\|x\|) \leq \bar{V}(x) \leq \bar{\alpha}_2(\|x\|), \quad \forall x \in \mathbb{R}^n$$

due to Assumption 1.

Next, consider the time derivative of  $\bar{V}$ , for which an upper estimate holds true under the restriction (8):

$$\begin{aligned}\dot{\bar{V}} &= \nabla \bar{V}(x) \dot{x} \\ &= \dot{x}^\top \bar{P} x + x^\top \bar{P} \dot{x} + 2 \sum_{j=1}^M \sum_{i=1}^n \bar{\Lambda}_j^i f_j^i(x_i) \dot{x}_i \\ &= \begin{bmatrix} x \\ F_1(x) \\ \vdots \\ F_M(x) \\ u \end{bmatrix}^\top \bar{Q} \begin{bmatrix} x \\ F_1(x) \\ \vdots \\ F_M(x) \\ u \end{bmatrix} - x^\top \bar{\Xi}^0 x \\ &\quad - \sum_{j=1}^M F_j(x)^\top \bar{\Xi}^j F_j(x) - 2 \sum_{j=1}^M x^\top \bar{\Upsilon}_{0,j} F_j(x) \\ &\quad - 2 \sum_{s=1}^{M-1} \sum_{z=s+1}^M F_s(x)^\top \bar{\Upsilon}_{s,z} F_z(x) + \bar{\gamma} u^\top u \\ &\leq -x^\top \bar{\Xi}^0 x - 2 \sum_{s=1}^{M-1} \sum_{z=s+1}^M F_s(x)^\top \bar{\Upsilon}_{s,z} F_z(x) \\ &\quad - \sum_{j=1}^M F_j(x)^\top \bar{\Xi}^j F_j(x) - 2 \sum_{j=1}^M x^\top \bar{\Upsilon}_{0,j} F_j(x) + \bar{\gamma} u^\top u.\end{aligned}$$

Now we want to show that there exists  $\beta_2 \in \mathbb{R}$  such that

$$\begin{aligned}\beta_2 \bar{V}(x) &\geq -x^\top \bar{\Xi}^0 x - \sum_{j=1}^M F_j(x)^\top \bar{\Xi}^j F_j(x) \\ &\quad - 2 \sum_{j=1}^M x^\top \bar{\Upsilon}_{0,j} F_j(x) - 2 \sum_{s=1}^{M-1} \sum_{z=s+1}^M F_s(x)^\top \bar{\Upsilon}_{s,z} F_z(x),\end{aligned}$$

which is true under Assumption 2 and the conditions (7), (10). Therefore, we have

$$\dot{\bar{V}} \leq \beta_2 \bar{V} + \bar{\gamma} u^\top u,$$

so that

$$\begin{aligned}\bar{V}(x(t)) &\leq e^{\beta_2 t} \bar{V}(x_0) + \int_0^t e^{\beta_2(t-s)} \bar{\gamma} u(s)^\top u(s) ds \\ &\leq e^{\beta_2 t} \bar{\alpha}_2(\|x_0\|) + \frac{\bar{\gamma} \gamma_0^2}{\beta_2} (e^{\beta_2 t} - 1) \\ &\leq e^{\beta_2 t} \bar{\alpha}_2(\varepsilon_2) + \frac{\bar{\gamma} \gamma_0^2}{\beta_2} (e^{\beta_2 t} - 1).\end{aligned}$$

It further holds that

$$\begin{aligned}\sup_{t \in [0, T]} \bar{V}(x(t)) &\leq \sup_{t \in [0, T]} \{a \eta(t) + b(\eta(t) - 1)\} \\ &= \begin{cases} a \eta(T) + b(\eta(T) - 1), & \text{if } \beta_2 > 0, \\ a + \bar{\gamma} \gamma_0^2 T, & \text{if } \beta_2 = 0, \\ a \eta(0) + b(\eta(0) - 1), & \text{if } \beta_2 < 0, a + b \geq 0, \\ a \eta(T) + b(\eta(T) - 1), & \text{if } \beta_2 < 0, a + b < 0 \end{cases}\end{aligned}$$

with the settings of

$$\eta(t) = e^{\beta_2 t}, \quad b = \frac{\bar{\gamma} \gamma_0^2}{\beta_2}, \quad a = \bar{\alpha}_2(\varepsilon_2).$$

Therefore, under the conditions (9) we obtain

$$\bar{V}(x(t)) \leq e^{\beta_2 t} \bar{\alpha}_2(\varepsilon_2) + \frac{\bar{\gamma} \gamma_0^2}{\beta_2} (e^{\beta_2 t} - 1) \leq \bar{\alpha}_1(\delta_2), \quad \forall t \in [0, T],$$

by which we see that

$$\bar{\alpha}_1(\|x(t)\|) \leq \bar{V}(x(t)) \leq \bar{\alpha}_1(\delta_2) \quad \Rightarrow \quad \|x(t)\| \leq \delta_2, \quad \forall t \in [0, T],$$

and the norm of the state  $x$  is upper bounded on the interval of time  $[0, T]$ .

2) Now let us evaluate the behavior of  $\|x(t)\|$  from below on  $t \in [0, T]$ , for which another candidate Lyapunov function is considered:

$$\underline{V}(x) = x^\top \underline{P} x + 2 \sum_{j=1}^M \sum_{i=1}^n \underline{\Lambda}_j^i \int_0^{x_i} f_j^i(\tau) d\tau.$$

In a similar way as in the previous development, it follows that there exist some functions  $\underline{\alpha}_1, \underline{\alpha}_2 \in \mathcal{K}_\infty$  such that

$$\underline{\alpha}_1(\|x\|) \leq \underline{V}(x) \leq \underline{\alpha}_2(\|x\|), \quad \forall x \in \mathbb{R}^n$$

due to the condition (3), Finsler's Lemma and Assumption 1.

Calculating the time derivative of  $\underline{V}$  along trajectories of (2), under (4) we have

$$\begin{aligned}\dot{\underline{V}} &\geq -x^\top \underline{\Xi}^0 x - 2 \sum_{s=1}^{M-1} \sum_{z=s+1}^M F_s(H_s x)^\top H_s \underline{\Upsilon}_{s,z} H_z^\top F_z(H_z x) \\ &\quad - \sum_{j=1}^M F_j(H_j x)^\top \underline{\Xi}^j F_j(H_j x) - 2 \sum_{j=1}^M x^\top H_j^\top \underline{\Upsilon}_{0,j} F_j(H_j x) \\ &\quad - \underline{\gamma} u^\top u.\end{aligned}$$

Applying similar methods, there exists  $\beta_1 \in \mathbb{R}$  such that

$$\begin{aligned} \beta_1 \underline{V}(x) &\leq -x^\top \underline{\Xi}^0 x - \sum_{j=1}^M F_j(H_j x)^\top \underline{\Xi}^j F_j(H_j x) \\ &- 2 \sum_{j=1}^M x^\top H_j^\top \underline{\Upsilon}_{0,j} F_j(H_j x) - 2 \sum_{s=1}^{M-1} \sum_{z=s+1}^M F_s(H_s x)^\top H_s \underline{\Upsilon}_{s,z} H_z^\top F_z(H_z x) \end{aligned}$$

due to the condition (6) and Assumption 2. Then, it can be shown that under the conditions (5), the property

$$e^{\beta_1 t} \underline{\alpha}_1(\varepsilon_1) - \frac{\gamma_0^2}{\beta_1} (e^{\beta_1 t} - 1) \geq \underline{\alpha}_2(\delta_1), \quad \forall t \in [0, T]$$

holds, which induces

$$\underline{\alpha}_2(\delta_1) \leq \underline{V}(x(t)) \leq \underline{\alpha}_2(\|x(t)\|) \Rightarrow \|x(t)\| \geq \delta_1, \quad \forall t \in [0, T].$$

This completes the proof.  $\blacksquare$

*Remark 1:* Since the conditions admit  $\beta_1, \beta_2$  to take positive or negative values, and  $\gamma_0$  to be not smaller than  $\|u\|_{[0,T]}$  (for practical verification, one can select them heuristically), the used Lyapunov functions  $\underline{V}$  and  $\bar{V}$  serve for establishing stable or unstable bounding compartments, which is not restrictive and the given LMIs can always be fulfilled.

For the formulation of the STBNZ conditions for the system (2), we set  $\varepsilon_1 = \delta_1 = 0$  in Theorem 1 and obtain the following corollary:

*Corollary 1:* Assume that all conditions of Theorem 1 are satisfied under the substitutions  $\varepsilon_1 \rightarrow 0, \delta_1 \rightarrow 0$  and the eliminations of (3), (4), (5), (6). Then the system (2) is STBNZ with respect to  $([0, T], \varepsilon_2, \gamma_0, \delta_2)$ .

*Proof:* Note that the conditions (3), (4), (5), (6) under the substitutions in Corollary 1 should be omitted due to  $\|x\| \geq 0$  for  $x \in \mathbb{R}^n$ .  $\blacksquare$

The obtained conditions of ASTS are rather sophisticated, but they are based on verification of LMIs, which is a rare case considering the remarkable nonlinearity of the system (2), and that the lower and upper estimates on the behavior of such a system are derived.

Theorem 1 and Corollary 1 can be used in several safety and fault detection applications, where the state of a process, having a model in the form of (2), has to belong to a prescribed domain during a bounded interval of time. Next, we will demonstrate the non-conservativeness of the derived conditions and the bounds for a neural model example.

## V. EXAMPLE

FitzHugh-Nagumo (FN) models [29] are frequently used to investigate a characteristic excursion observed in neural dynamics in the presence of an external stimulus exceeding a certain threshold. It is often the case that there are required bounds (both lower and upper ones) for the membrane voltage, which have to be tracked during transients for the recovery variable (see [30] for more details) during a finite operation time. Thus, the study of ASTS conditions for FN models has practical meaning.

The FN dynamics can be presented as follows:

$$\begin{aligned} \dot{v}(t) &= v(t) - \frac{v(t)^3}{3} - w(t) + I, \\ \tau \dot{w}(t) &= v(t) + \lambda - bw(t), \end{aligned} \quad (11)$$

where  $v(t) \in \mathbb{R}$  is the membrane potential;  $w(t) \in \mathbb{R}$  is a recovery variable;  $\tau > 0, \lambda, b$  are constants satisfying

$$1 - \frac{2b}{3} < \lambda < 1, \quad 0 < b < 1, \quad b < \tau^2;$$

$I \in \mathbb{R}$  is the exogenous stimulus current.

The model (11) can be rewritten as:

$$\dot{H}(t) = R_0 H(t) + R_1 \sigma(H(t)) + \omega(t),$$

which is clearly in the form of generalized Persidskii systems, where

$$\begin{aligned} H(t) &= \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}, \quad R_0 = \begin{bmatrix} 1 & -1 \\ \frac{1}{\tau} & -\frac{b}{\tau} \end{bmatrix}, \\ R_1 &= \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \sigma(H(t)) = \begin{bmatrix} \frac{v(t)^3}{3} \\ \frac{w(t)^3}{3} \end{bmatrix}, \quad \omega(t) = \begin{bmatrix} I \\ \frac{\lambda}{\tau} \end{bmatrix}. \end{aligned}$$

For illustration, let

$$\tau = 12, \quad b = 0.8, \quad I = 0.04, \quad \lambda = 0.672.$$

Assumption 1 is fulfilled due to the form of the function  $\sigma$ . Also, Assumption 2 is satisfied for

$$\begin{aligned} \kappa_{0,1}^1 &= \kappa_{0,1}^2 = 0, \quad \kappa_{1,11}^1 = \kappa_{1,11}^2 = 0, \\ \kappa_{2,11}^1 &= \kappa_{2,11}^2 = \frac{1}{4}; \\ \eta_{0,1}^1 &= \eta_{0,1}^2 = 0, \quad \eta_{1,11}^1 = \eta_{1,11}^2 = 0, \\ \eta_{2,11}^1 &= \eta_{2,11}^2 = \frac{1}{4}, \end{aligned}$$

since  $\sigma$  is composed by polynomial functions.

The LMIs proposed in Theorem 1 are verified by

$$\begin{aligned} T &= 10^{-3}, \quad \|\omega\|_{[0, 10^{-3}]} \leq \gamma_0 = 0.05, \quad \varepsilon_1 = 0.5, \quad \varepsilon_2 = 0.7, \\ \delta_1 &= 0.3, \quad \delta_2 = 2.5, \quad \bar{P} = 10^{-4} \times \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \\ \bar{\Lambda}_1 &= \begin{bmatrix} 5.7316 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{\Xi}_0 = \begin{bmatrix} -4.2666 & 0 \\ 0 & -3.9275 \end{bmatrix}, \\ \bar{\Xi}_1 &= O_{2 \times 2}, \quad \bar{\Upsilon}_{0,1} = \begin{bmatrix} -5.2609 & 0 \\ 0 & 0 \end{bmatrix}, \\ \beta_2 &= 10, \quad \bar{\gamma} = 7.16 \times 10^5; \\ \underline{P} &= 10^{-3} \times \begin{bmatrix} 0.8912 & 0 \\ 0 & 0.8929 \end{bmatrix}, \quad \underline{\Delta}_1 = \underline{\Xi}_1 = \underline{\Upsilon}_{0,1} = O_{2 \times 2}, \\ \underline{\Xi}_0 &= \begin{bmatrix} 0.0891 & 0 \\ 0 & 0.0869 \end{bmatrix}, \quad \beta_1 = -100, \quad \underline{\gamma} = 11.1582, \end{aligned}$$

and the trajectories with different initial states are presented in Fig. 1. From this plot, we see that the system trajectories initiated into the dashed annulus  $\text{cl}(B(0.7, 0) \setminus B(0.5, 0))$  remain within the solid one  $\text{cl}(B(2.5, 0) \setminus B(0.3, 0))$ . It is also worth highlighting that the given estimates are rather tight since some trajectories closely approach the borders.

## VI. CONCLUSION

This paper proposed sufficient conditions for the robust annular short-time stability (ASTS) and short-time boundedness with nonzero initial state (STBNZ) for generalized Persidskii systems. The formulated conditions were obtained in the form of linear algebraic and matrix inequalities.

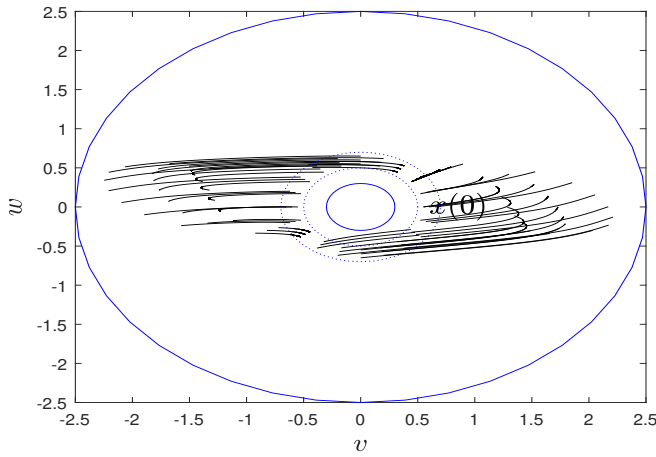


Fig. 1. The phase space of (11)

Therefore, they can be constructively verified. A simulation scenario with a neuron model was presented to examine the proposed results. The future research directions include the output short-time analysis for the considered systems, the study of more efficient numerical tools for solving the proposed conditions, and the investigation of other practical applications, *e.g.*, recurrent neural networks.

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