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Hamiltonian formulation of the stochastic surface wave problem

Evgueni Dinvai, Etienne Mémin

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Abstract

We devise a stochastic Hamiltonian formulation of the water wave problem. This stochastic representation is built within the framework of the modelling under location uncertainty. Starting from restriction to the free surface of the general stochastic fluid motion equations, we show how one can naturally deduce Hamiltonian structure under a small noise assumption. Moreover, as in the classical water wave theory, the non-local Dirichlet-Neumann operator appears explicitly in the energy functional. This, in particular, allows us, in the same way as in deterministic setting, to conduct systematic approximations of the Dirichlet-Neumann operator and to infer different simplified wave models including noise in a natural way.

1 Introduction

In physical oceanography there is a certain interest to describe the dynamical properties of the solutions of water wave equations, as for example the Korteweg-de Vries (KdV) equation [17], in the presence of random perturbations, depending on the space and time variables. As a matter of fact deterministic modelling of waves requires to rely on strong approximations or simplifications to describe the complex motion of ocean surface waves. Non linear interactions, and important wave physics phenomenon such as white capping, wave breaking, wind stress and bottom drag need to be simplified. The resulting models, though giving rise to accurate predictive numerical codes, do not fully account for the reality and incorporate many sources of uncertainty that are difficult to handle and quantify.

There are in particular two possible motivations that could asymptotically lead to useful stochastic models. The first one is related to the case in which the pressure field or the wind forcing are non homogeneous, and modelled by a stationary space-time process with small correlation length compared to the wavelength of the surface waves. The second motivation comes from the consideration of random bottom topography. For example, the case when the bottom is modelled by a stationary ergodic process with small correlation length compared to the surface waves has been studied in [16]. However, up to our knowledge, there is no rigorous derivation of such asymptotic models starting from the full water wave problem.

The objective of this development is to propose an equivalent of the classical water wave problem formulation in a stochastic setting. The idea will be to proceed in a way that stays as close as possible to the deterministic context. In order to do so, we will cope with the same flow regime pertaining to the derivation of the classical deterministic wave solutions, together with a decomposition of the Lagrangian velocity in terms of a smooth in time velocity component and a random uncorrelated uncertainty component.

It is important to stress that here we do not impose randomness as an adhoc perturbation of the classical deterministic linear waves or of the dispersion relation. Instead, stochasticity is imposed right from the start in the flow constitutive equations by assuming a decomposition of the fluid particles displacement as

$$d\mathbf{X}_t = \mathbf{u}(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{B}. \quad (1.1)$$

Splitting velocity in this way we imply that there is an uncertainty of the white noise type in location of fluid particles. Such consideration is based on the work by the second author [38], where this was used to deduce a stochastic analogue of the Reynolds transport theorem. The divergence-free random field involved in the Lagrangian formulation (1.1) is defined over the fluid domain Ω , through the deterministic kernel function $\check{\sigma}(\cdot, \cdot, t)$ of the diffusion operator $\boldsymbol{\sigma}(\cdot, t)$ as

$$\forall \mathbf{x} \in \Omega, \quad (\boldsymbol{\sigma}[\mathbf{f}](\mathbf{x}, t))^i \triangleq \sum_j \int_{\Omega} \check{\sigma}^{ij}(\mathbf{x}, \mathbf{y}, t) f^j(\mathbf{y}, t) d\mathbf{y}, \quad i, j = 1, \dots, d,$$

where either $d = 2$ for the two dimensional fluid domain Ω or $d = 3$ for the three dimensional fluid domain Ω . Normally, we will use the indices x, z in the first situation and the indices x, y, z in the second case. The

kernel is assumed to be in $L^2(\Omega \times \Omega)$, which leads to an Hilbert-Schmidt integral operator. The covariance of the random turbulent component is as a consequence well defined and reads as

$$Q_{ij}(\mathbf{x}, \mathbf{x}', t, t') = \mathbb{E}((\boldsymbol{\sigma}[\mathbf{d}\mathbf{B}_t](\mathbf{x}, t))_i (\boldsymbol{\sigma}[\mathbf{d}\mathbf{B}_t](\mathbf{x}', t'))_j) = c_{ij}(\mathbf{x}, \mathbf{x}', t) \delta(t - t') dt,$$

and the diagonal of the covariance tensor defined as $a_{ij}(\mathbf{x}, t) = c_{ij}(\mathbf{x}, \mathbf{x}, t)$, corresponds to the quadratic variation terms associated to the noise; it has the dimension of a diffusion (m^2/s) and plays the role of a *generalized* matrix-valued eddy viscosity. In the following it is referred to as the variance tensor. For simplicity reasons, we will work in a 2D domain $\Omega(x, z)$, with x being the horizontal direction and z the vertical axis, and $\mathbf{x} = (x, z)^T$. However, the extension to 3D with a transverse direction of homogeneity is almost immediate and will be shortly presented in Section 7. Below, we pass in review the different hypothesis used to derive the linear Airy waves.

Let us point out that the uncertainty in the fluid particle displacement is governed by a white noise process in time, whereas in space the flow is assumed smooth. There are two reasons for proceeding in this way. Firstly, admitting randomness both in space and time, makes the problem much harder, so it does not seem possible to arrive at a reasonable formulation. Secondly, in the particular case when the noise is due to a random bottom, it turns out that the rough bottom variations are naturally regularised by inherent smoothing properties of the Dirichlet-Neumann operator, as was shown in [14]. Mathematically, it can be explained by appearance of $\text{sech}(hD)$ in the expansion of the Dirichlet-Neumann operator in front of the bottom topography. Physically, it means that the surface does not feel sharp variations of the bottom. Thus it seems natural to restrict ourselves to the Lagrangian decomposition (1.1) containing a Q -Wiener process \mathbf{B} . Therefore, passing to the continuous Eulerian limit, the free surface elevation $\eta(x, t)$ will be a stochastic process with respect to time, yet remaining smooth in space. This is demonstrated numerically below.

Let us note that another approach to noise modelling in fluids was proposed in [30]. This approach ensues from a variational formulation, whereas modelling under location uncertainty (LU) corresponds to a stochastic Newtonian formulation. Both techniques lead to different conservation properties. Namely, circulation conservation is imposed in the former, while energy conservation is directly associated to the latter. However, as we wish here to stick closely to a water wave formulation deduced directly from the fluid flow equations, LU is an easier setting to work with. This framework will indeed enable us to obtain direct stochastic representation of almost all the classical approximations of the surface waves, starting with the simplest one: Airy waves.

The LU framework has recently been shown to perform very well for oceanic quasi-geostrophic flow models [2, 3, 39], rotating shallow water system [7] and large eddies simulation [8, 9, 32]. It provided in particular much better results than classical deterministic models at coarse resolution in terms of variabilities, extreme events, long-terms statistics and data assimilation issues [9, 24]. Interestingly, a LU version of the reduced order Lorenz-63 model, derived in the very same way as the original model [37], has been shown numerically to allow a faster exploration of the strange attractor region than classical viscous models and to lead to more accurate statistics than ad hoc stochastic models built with multiplicative forcings [10]. This latter study has shown also good convergence behaviour of the stochastic system for vanishing noise. This has been theoretically confirmed recently in [18]. It was demonstrated that LU Navier-Stokes models have martingale solutions in 3D and a unique strong solution - in the probabilistic sense - in 2D. In the 3D case, in the limit of vanishing noise, it has been demonstrated there exists a subsequence converging in law toward a weak solution of the deterministic Navier-Stokes equations and that in 2D the whole sequence converges toward the unique solution. As such these results warrant the use of the LU setting as a consistent large-scale stochastic representation of flow dynamics. The questions of wave solutions and surface waves representations in LU have not been yet studied. As motivated earlier stochastic extensions of surface waves representation are important in the objective of somewhat alleviating the approximations performed in those models, but also and more importantly, to provide coarse representations of waves in ocean models at coarse scales. As a matter of fact, surface waves or internal waves are very badly represented in large-scale ocean models. This is obviously very detrimental to climate or oceanic circulation simulations as waves are key drivers in the energy redistribution and in the interaction between ocean and atmosphere. The forms that should take surface waves representations in this stochastic setting is the main objective followed here. This purpose can be summed up through the following questions. What becomes of the classical water waves models in the LU setting? Do we still remain within a Hamiltonian formulation? What would be then the form of the noise considered as well as of the associated solutions? All these questions will be partly answered here for different types of models with a gradual complexity.

2 Fluid motion under location uncertainty

A two dimensional water wave problem with the gravity g and the undisturbed water depth h is under consideration. The fluid domain is the layer $\Omega_\eta = \{(x, z) \in \mathbb{R}^2 \mid -h < z < \eta(x, t)\}$ extending to infinity in the positive and negative horizontal x -direction. Here $\eta(x, t)$ represents the elevation of the free surface at the point x and

time moment t . It is a random variable though following a common convention we omit the dependence on probability variable. The sea bottom is assumed to be flat and rigid. It is represented by the lower boundary $z = -h$, so that the unperturbed fluid at rest corresponds to the domain $\Omega_0 = \mathbb{R} \times (-h, 0)$. The flow is assumed to be incompressible.

As was shown in [38] from the stochastic Lagrangian velocity decomposition (1.1) one can deduce hydrodynamical equations. For an incompressible flow the mass conservation reads

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot (\boldsymbol{\sigma} d\mathbf{B}) = 0.$$

As classically assumed, the viscous forces, the surface tension, wind induced stress and pressure are neglected as well as the Coriolis correction. The gravity force is consequently dominating. Physically, we set ourselves hence in context with waves longer than few centimetres and shorter than few kilometres. The LU momentum equations are given by [2, 7, 38]

$$d_t \mathbf{u} + (\mathbf{u}^* \cdot \nabla) \mathbf{u} dt + (\boldsymbol{\sigma} d\mathbf{B} \cdot \nabla) \mathbf{u} - \frac{1}{2} \nabla \cdot ((\mathbf{a} \cdot \nabla) \mathbf{u}) dt = \mathbf{g} dt - \sqrt{\frac{h}{\rho^2 g}} \nabla dp, \quad (2.1a)$$

$$\mathbf{u}^* = \mathbf{u} - \frac{1}{2} \nabla \cdot \mathbf{a}, \quad (2.1b)$$

where $\mathbf{g} = (0, -g)$ stands for the gravity acceleration, directed downward along z -axis, and dp denotes the pressure. The variance tensor \mathbf{a} is defined by $a_{ij} dt = \langle \boldsymbol{\sigma} d\mathbf{B}_i, \boldsymbol{\sigma} d\mathbf{B}_j \rangle$, where the brackets $\langle f, g \rangle$ denote the quadratic covariation term of any two stochastic processes f and g . Below we show how it can be calculated for particular models of the noise vector $\boldsymbol{\sigma} d\mathbf{B}$. It can be noticed that these equations are very much alike the deterministic momentum equation. In the same way as classical large eddies formulation they include a diffusion term (last left-hand side term in the first equation) depending on the variance tensor. It plays the same role as the subgrid tensor with a matrix eddy diffusivity provided by the variance tensor. The second term is an effective advection involving the Ito-Stokes drift $-1/2 \nabla \cdot \mathbf{a}$, which can be interpreted as a generalization of the Stokes drift velocity component associated to waves orbital velocity. The third term represents the advection of the large-scale velocity component by the small-scale random component. The energy brought by this random term can be shown to be exactly compensated by the energy loss by the diffusion term [40]. Interested reader may refer to [2, 3, 38, 39, 40] for further explanations and analysis in several flow configurations.

The momentum equations 2.1a are complemented by boundary conditions. At the bottom $z = -h$ we consider a slip condition for the slow velocity component

$$u_z = 0 \quad \text{at } z = -h. \quad (2.2)$$

At the free surface $z = \eta(x, t)$ we suppose the pressure to be constant, which implies $dp = 0$ and so simplifies the momentum conservation (2.1a) at the upper boundary. This constitutes the so called dynamical boundary condition.

The stochastic transport of the surface elevation is balanced by the vertical velocity as

$$u_z dt + \boldsymbol{\sigma} d\mathbf{B}_z = \mathbb{D}_t \eta,$$

with $\mathbb{D}_t q$ denoting the transport operator introduced in [40] by the equality

$$\mathbb{D}_t q \triangleq d_t q + \nabla \cdot (q \mathbf{u}^*) dt + \boldsymbol{\sigma} d\mathbf{B} \cdot \nabla q - \frac{1}{2} \nabla \cdot (\mathbf{a} \nabla q) dt, \quad (2.3)$$

and which corresponds to a stochastic expression of the material derivative for a transported scalar. This leads to the so called kinematical boundary condition

$$u_z dt + \boldsymbol{\sigma} d\mathbf{B}_z = d_t \eta + \nabla \cdot (\eta \mathbf{u}^*) dt + \boldsymbol{\sigma} d\mathbf{B} \cdot \nabla \eta - \frac{1}{2} \nabla \cdot (\mathbf{a} \nabla \eta) dt \quad (2.4)$$

at the free surface.

As classically done in the deterministic setting, the large-scale flow is then assumed to be potential

$$\mathbf{u}(\mathbf{x}, t) = \nabla \varphi(x, z, t) \quad \text{with} \quad \Delta \varphi = \partial_x^2 \varphi + \partial_z^2 \varphi = 0. \quad (2.5)$$

The noise is divergence free and can be written in terms of a potential function as well, by introducing an operator φ^σ in a way that $\varphi^\sigma(\mathbf{x}, t) d\mathbf{B}$ is a scalar function and the relation

$$\boldsymbol{\sigma}(\mathbf{x}, t) d\mathbf{B} = \nabla^\perp \varphi^\sigma(\mathbf{x}, t) d\mathbf{B}$$

holds true. Here $\nabla^\perp = (-\partial_z, \partial_x)^T$ represents the orthogonal gradient operator in 2D, with the curl operator defined as $\nabla^\perp \cdot \mathbf{u}$. In terms of kernel representation, the noise reads

$$\forall \mathbf{x} \in \Omega, \int_{\Omega} \check{\sigma}^{ij}(\mathbf{x}, \mathbf{y}, t) d\mathbf{B}_t(\mathbf{y}) = \int_{\Omega} \nabla^\perp \check{\varphi}^T(\mathbf{x}, \mathbf{y}, t) d\mathbf{B}_t(\mathbf{y}).$$

Let us note that in this expression the potential kernel is vectorial. The large-scale flow component is analytic (i.e. divergence free and curl free) while the noise is not necessarily irrotational. It concentrates all the vorticity of the complete flow. Compared to the deterministic case, in which the whole flow is potential, this stochastic representation extends thus immediately the deterministic setting in which eddies are not at all taken into account.

After neglecting the pressure fluctuations at the surface $z = \eta(x, t)$, the dynamical boundary condition (2.1a) takes the form

$$d_t \nabla \varphi - \mathbf{g} dt + \frac{1}{2} (\nabla |\nabla \varphi|^2 - ((\nabla \cdot \mathbf{a}) \cdot \nabla) \nabla \varphi - \nabla \cdot ((\mathbf{a} \cdot \nabla) \nabla \varphi)) dt + \left(\nabla^\perp \varphi^\sigma d\mathbf{B} \cdot \nabla \right) \nabla \varphi = 0. \quad (2.6)$$

Note that in general situation one cannot get the Bernoulli integral from this expression, which makes an analysis below more demanding. In order to reduce the problem to the surface we need to rewrite this equation in terms of derivatives of φ . For this we expand derivatives as follows

$$\begin{aligned} \frac{1}{2} ((\nabla \cdot \mathbf{a}) \cdot \nabla) f + \frac{1}{2} \nabla \cdot ((\mathbf{a} \cdot \nabla) f) &= \sum_{i,j} \left[(\partial_i a_{ij}) \partial_j f + \frac{1}{2} a_{ij} \partial_i \partial_j f \right] \\ &= (\partial_x a_{xx}) \partial_x f + (\partial_z a_{xz}) \partial_x f + (\partial_x a_{xz}) \partial_z f + (\partial_z a_{zz}) \partial_z f + \frac{1}{2} a_{xx} \partial_x^2 f + \frac{1}{2} a_{zz} \partial_z^2 f + a_{xz} \partial_x \partial_z f, \end{aligned}$$

where f stands for any smooth function, for example, one can take $f = \nabla \varphi$.

3 Reduction to surface

Let us define the value of the potential at the surface, the so-called potential trace

$$\Phi(x, t) = \varphi(x, \eta(x, t), t). \quad (3.1)$$

Combining the divergence free condition (2.5) with the bottom condition (2.2) we have

$$\begin{cases} \Delta \varphi = 0 & \text{in } \Omega_\eta, \\ \partial_z \varphi = 0 & \text{at } z = -h, \\ \varphi = \Phi & \text{at } z = \eta. \end{cases} \quad (3.2)$$

One can associate with this elliptic problem the Dirichlet-Neumann operator $G(\eta)$ which assigns to Φ the normal derivative of φ , that is,

$$G(\eta) \Phi = \partial_z \varphi - (\partial_x \varphi) \partial_x \eta \text{ at } z = \eta. \quad (3.3)$$

Note that the kinematical condition (2.4) takes the form $\partial_t \eta = G(\eta) \Phi$ in the deterministic case. In other words introduction of the Dirichlet-Neumann operator allows us to reduce the problem to the surface. In the stochastic case as one can see below, this reduction incorporates additional terms. Let us also point out that this operator was thoroughly studied in literature, see [34] and references therein. The crucial property in our case concerns the fact that $G(\eta)$ can be approximated via a Taylor series, see Appendix B.2. Introducing this operator, the kinematical boundary condition (2.4) can be rewritten as

$$d\eta = \left(G(\eta) \Phi + (\partial_x a_{xx} + \partial_z a_{zx}) \partial_x \eta + \frac{1}{2} a_{xx} \partial_x^2 \eta \right) dt + \boldsymbol{\sigma} d\mathbf{B}_z - \boldsymbol{\sigma} d\mathbf{B}_x \partial_x \eta. \quad (3.4)$$

Let us now consider the reduction of the dynamical boundary condition (2.6) to the surface. For this purpose we need to express derivatives of φ in terms of derivatives of η , Φ . The gradient $\nabla \varphi$ is found from the definitions of Φ and G . It reads

$$\nabla \varphi = \begin{pmatrix} 1 & \partial_x \eta \\ -\partial_x \eta & 1 \end{pmatrix}^{-1} \begin{pmatrix} \partial_x \Phi \\ G\Phi \end{pmatrix} = \frac{1}{1 + (\partial_x \eta)^2} \begin{pmatrix} \partial_x \Phi - \partial_x \eta G\Phi \\ G\Phi + \partial_x \eta \partial_x \Phi \end{pmatrix}. \quad (3.5)$$

Differentiating twice Expression (3.1) and once Expression (3.3), one arrives to the system

$$\begin{cases} \partial_x^2 \varphi + \partial_z^2 \varphi = 0, \\ \partial_x^2 \varphi + (\partial_x \eta)^2 \partial_z^2 \varphi + 2\partial_x \eta \partial_x \partial_z \varphi = \partial_x^2 \Phi - \partial_z \varphi \partial_x^2 \eta, \\ -\partial_x \eta \partial_x^2 \varphi + \partial_x \eta \partial_z^2 \varphi + (1 - (\partial_x \eta)^2) \partial_x \partial_z \varphi = \partial_x(G\Phi) + \partial_x \varphi \partial_x^2 \eta, \end{cases}$$

that needs to be resolved with respect to the second derivatives of potential φ . After a direct calculation and using (3.5) we obtain

$$\begin{pmatrix} \partial_x^2 \varphi \\ \partial_z^2 \varphi \\ \partial_x \partial_z \varphi \end{pmatrix} = \frac{1}{(1 + (\partial_x \eta)^2)^2} \begin{pmatrix} 1 - (\partial_x \eta)^2 & -2\partial_x \eta \\ (\partial_x \eta)^2 - 1 & 2\partial_x \eta \\ 2\partial_x \eta & 1 - (\partial_x \eta)^2 \end{pmatrix} \begin{pmatrix} \partial_x^2 \Phi \\ \partial_x(G\Phi) \end{pmatrix} \\ + \frac{\partial_x^2 \eta}{(1 + (\partial_x \eta)^2)^3} \begin{pmatrix} (\partial_x \eta)^3 - 3\partial_x \eta & 3(\partial_x \eta)^2 - 1 \\ 3\partial_x \eta - (\partial_x \eta)^3 & 1 - 3(\partial_x \eta)^2 \\ 1 - 3(\partial_x \eta)^2 & (\partial_x \eta)^3 - 3\partial_x \eta \end{pmatrix} \begin{pmatrix} \partial_x \Phi \\ G\Phi \end{pmatrix} \quad (3.6)$$

Finally, the third derivatives of potential φ can be found by solving the system

$$\begin{cases} \partial_x^3 \varphi + \partial_x \partial_z^2 \varphi = 0, \\ \partial_x^2 \partial_z \varphi + \partial_z^3 \varphi = 0, \\ \partial_x^3 \varphi + 3\partial_x \eta \partial_x^2 \partial_z \varphi + 3(\partial_x \eta)^2 \partial_x \partial_z^2 \varphi + (\partial_x \eta)^3 \partial_z^3 \varphi = \partial_x^3 \Phi + 3\partial_x^2 \varphi \partial_x \eta \partial_x^2 \eta - 3\partial_x \partial_z \varphi \partial_x^2 \eta - \partial_z \varphi \partial_x^3 \eta, \\ -\partial_x \eta \partial_x^3 \varphi + (1 - 2(\partial_x \eta)^2) \partial_x^2 \partial_z \varphi + (2\partial_x \eta - (\partial_x \eta)^3) \partial_x \partial_z^2 \varphi + (\partial_x \eta)^2 \partial_z^3 \varphi \\ = \partial_x^2(G\Phi) + 3\partial_x^2 \varphi \partial_x^2 \eta + 3\partial_x \partial_z \varphi \partial_x \eta \partial_x^2 \eta + \partial_x \varphi \partial_x^3 \eta. \end{cases}$$

Resolving this system and using (3.5), (3.6) one obtains

$$\begin{pmatrix} \partial_x^3 \varphi \\ \partial_x^2 \partial_z \varphi \\ \partial_x \partial_z^2 \varphi \\ \partial_z^3 \varphi \end{pmatrix} = \frac{1}{(1 + (\partial_x \eta)^2)^3} \begin{pmatrix} 1 - 3(\partial_x \eta)^2 & (\partial_x \eta)^3 - 3\partial_x \eta \\ 3\partial_x \eta - (\partial_x \eta)^3 & 1 - 3(\partial_x \eta)^2 \\ 3(\partial_x \eta)^2 - 1 & 3\partial_x \eta - (\partial_x \eta)^3 \\ (\partial_x \eta)^3 - 3\partial_x \eta & 3(\partial_x \eta)^2 - 1 \end{pmatrix} \begin{pmatrix} \partial_x^3 \Phi \\ \partial_x^2(G\Phi) \end{pmatrix} \\ + \frac{1}{(1 + (\partial_x \eta)^2)^4} \begin{pmatrix} 4(\partial_x \eta)^3 - 4\partial_x \eta & 6(\partial_x \eta)^2 - (\partial_x \eta)^4 - 1 \\ (\partial_x \eta)^4 + 1 - 6(\partial_x \eta)^2 & 4(\partial_x \eta)^3 - 4\partial_x \eta \\ 4\partial_x \eta - 4(\partial_x \eta)^3 & (\partial_x \eta)^4 + 1 - 6(\partial_x \eta)^2 \\ 6(\partial_x \eta)^2 - (\partial_x \eta)^4 - 1 & 4\partial_x \eta - 4(\partial_x \eta)^3 \end{pmatrix} \left[3\partial_x^2 \eta \begin{pmatrix} \partial_x^2 \Phi \\ \partial_x(G\Phi) \end{pmatrix} + \partial_x^3 \eta \begin{pmatrix} \partial_x \Phi \\ G\Phi \end{pmatrix} \right] \\ + \frac{3(\partial_x^2 \eta)^2}{(1 + (\partial_x \eta)^2)^5} \begin{pmatrix} 10(\partial_x \eta)^2 - 5(\partial_x \eta)^4 - 1 & (\partial_x \eta)^5 - 10(\partial_x \eta)^3 + 5\partial_x \eta \\ 10(\partial_x \eta)^3 - (\partial_x \eta)^5 - 5\partial_x \eta & 10(\partial_x \eta)^2 - 5(\partial_x \eta)^4 - 1 \\ 5(\partial_x \eta)^4 + 1 - 10(\partial_x \eta)^2 & 10(\partial_x \eta)^3 - (\partial_x \eta)^5 - 5\partial_x \eta \\ (\partial_x \eta)^5 - 10(\partial_x \eta)^3 + 5\partial_x \eta & 5(\partial_x \eta)^4 + 1 - 10(\partial_x \eta)^2 \end{pmatrix} \begin{pmatrix} \partial_x \Phi \\ G\Phi \end{pmatrix} \quad (3.7)$$

Now differentiation of the fluid surface potential Φ with respect to t results in

$$d\Phi = d(t \mapsto \varphi(x, \eta(x, t), t)) = d_t \varphi + (\partial_z \varphi) d\eta + d\langle \partial_z \varphi, \eta \rangle + \frac{1}{2} \partial_z^2 \varphi d\langle \eta, \eta \rangle,$$

where the brackets $\langle f, g \rangle$ denote the quadratic covariation term of any two stochastic processes f and g .

To be able to use the boundary condition (2.6) one needs to differentiate this expression once more with respect to the horizontal variable x , so that

$$d\partial_x \Phi = d_t \partial_x \varphi + d_t \partial_z \varphi \partial_x \eta + \partial_z \varphi \partial_x d\eta + (\partial_x \partial_z \varphi + \partial_z^2 \varphi \partial_x \eta) d\eta + \partial_x \left(d\langle \partial_z \varphi, \eta \rangle + \frac{1}{2} \partial_z^2 \varphi d\langle \eta, \eta \rangle \right). \quad (3.8)$$

Using Equation (3.4) one obtains the expression of the quadratic variation

$$d\langle \eta, \eta \rangle = \langle \sigma d\mathbf{B}_z, \sigma d\mathbf{B}_z \rangle - 2\langle \sigma d\mathbf{B}_z, \sigma d\mathbf{B}_x \partial_x \eta \rangle + \langle \sigma d\mathbf{B}_x \partial_x \eta, \sigma d\mathbf{B}_x \partial_x \eta \rangle = (a_{zz} - 2a_{xz} \partial_x \eta + a_{xx} (\partial_x \eta)^2) dt.$$

From Equations (2.6) and (3.4) we deduce

$$d\langle \partial_z \varphi, \eta \rangle = \langle \sigma d\mathbf{B}_x \partial_x \partial_z \varphi, \sigma d\mathbf{B}_x \partial_x \eta \rangle - \langle \sigma d\mathbf{B}_x \partial_x \partial_z \varphi, \sigma d\mathbf{B}_z \rangle + \langle \sigma d\mathbf{B}_z \partial_z^2 \varphi, \sigma d\mathbf{B}_x \partial_x \eta \rangle - \langle \sigma d\mathbf{B}_z \partial_z^2 \varphi, \sigma d\mathbf{B}_z \rangle \\ = (a_{xx} \partial_x \partial_z \varphi \partial_x \eta - a_{xz} \partial_x \partial_z \varphi + a_{xz} \partial_z^2 \varphi \partial_x \eta - a_{zz} \partial_z^2 \varphi) dt.$$

Thus

$$\begin{aligned}
& \partial_x \left(d\langle \partial_z \varphi, \eta \rangle + \frac{1}{2} \partial_z^2 \varphi d\langle \eta, \eta \rangle \right) \\
&= \left(a_{xx} \partial_x \partial_z \varphi \partial_x^2 \eta + a_{xx} \partial_z^2 \varphi \partial_x \eta \partial_x^2 \eta + \partial_x a_{xx} \partial_x \partial_z \varphi \partial_x \eta - \partial_x a_{xz} \partial_x \partial_z \varphi - \frac{1}{2} \partial_x a_{zz} \partial_z^2 \varphi + \frac{1}{2} \partial_x a_{xx} \partial_z^2 \varphi (\partial_x \eta)^2 \right. \\
&\quad + \partial_z a_{xx} \partial_x \partial_z \varphi (\partial_x \eta)^2 - \partial_z a_{xz} \partial_x \partial_z \varphi \partial_x \eta - \frac{1}{2} \partial_z a_{zz} \partial_z^2 \varphi \partial_x \eta + \frac{1}{2} \partial_z a_{xx} \partial_z^2 \varphi (\partial_x \eta)^3 + a_{xx} \partial_x^2 \partial_z \varphi \partial_x \eta \\
&\quad \left. - a_{xz} \partial_x^2 \partial_z \varphi - \frac{1}{2} a_{zz} \partial_x \partial_z^2 \varphi + \frac{3}{2} a_{xx} \partial_x \partial_z^2 \varphi (\partial_x \eta)^2 - a_{xz} \partial_x \partial_z^2 \varphi \partial_x \eta - \frac{1}{2} a_{zz} \partial_z^3 \varphi \partial_x \eta + \frac{1}{2} a_{xx} \partial_z^3 \varphi (\partial_x \eta)^3 \right) dt.
\end{aligned}$$

It completes Formula (3.8). The final equation is obtained after substitution into this formula of expressions (2.6), (3.4) and (3.5), which gives us the acceleration potential of the inviscid fluid on the surface

$$\begin{aligned}
d\partial_x \Phi &= \partial_x \left(-g\eta - \frac{1}{2} (\partial_x \Phi)^2 + \frac{(G\Phi + \partial_x \Phi \partial_x \eta)^2}{2(1 + (\partial_x \eta)^2)} \right) dt - \sigma d\mathbf{B}_x \partial_x^2 \Phi \\
&\quad + \partial_z \varphi (\partial_x \sigma d\mathbf{B}_z - \partial_x \sigma d\mathbf{B}_x \partial_x \eta + \partial_z \sigma d\mathbf{B}_z \partial_x \eta - \partial_z \sigma d\mathbf{B}_x (\partial_x \eta)^2) \\
&\quad + \left(\frac{3}{2} a_{xx} \partial_x \partial_z \varphi \partial_x^2 \eta + \frac{3}{2} a_{xx} \partial_z^2 \varphi \partial_x \eta \partial_x^2 \eta + 3\partial_x a_{xx} \partial_x \partial_z \varphi \partial_x \eta - \frac{1}{2} \partial_x a_{zz} \partial_z^2 \varphi + \frac{3}{2} \partial_x a_{xx} \partial_z^2 \varphi (\partial_x \eta)^2 \right. \\
&\quad + \partial_z a_{xx} \partial_x \partial_z \varphi (\partial_x \eta)^2 + \frac{1}{2} \partial_z a_{zz} \partial_z^2 \varphi \partial_x \eta + \frac{1}{2} \partial_z a_{xx} \partial_z^2 \varphi (\partial_x \eta)^3 + \frac{3}{2} a_{xx} \partial_x^2 \partial_z \varphi \partial_x \eta \\
&\quad + \frac{3}{2} a_{xx} \partial_x \partial_z^2 \varphi (\partial_x \eta)^2 + \frac{1}{2} a_{xx} \partial_z^3 \varphi (\partial_x \eta)^3 + \partial_z a_{xz} \partial_z^2 \varphi (\partial_x \eta)^2 + \partial_x a_{xx} \partial_x^2 \varphi + \partial_z a_{xz} \partial_x^2 \varphi + \partial_z a_{zz} \partial_x \partial_z \varphi \\
&\quad + \frac{1}{2} a_{xx} \partial_x^3 \varphi + \partial_z a_{xz} \partial_x \partial_z \varphi \partial_x \eta + \partial_x a_{xz} \partial_z^2 \varphi \partial_x \eta + \partial_z \varphi \left(\frac{3}{2} \partial_x a_{xx} \partial_x^2 \eta + \partial_z a_{xz} \partial_x^2 \eta + \frac{1}{2} a_{xx} \partial_x^3 \eta \right. \\
&\quad \left. + \partial_x^2 a_{xx} \partial_x \eta + \partial_x \partial_z a_{xz} \partial_x \eta + \partial_x \partial_z a_{xx} (\partial_x \eta)^2 + \partial_z^2 a_{xz} (\partial_x \eta)^2 + \frac{1}{2} \partial_z a_{xx} \partial_x \eta \partial_x^2 \eta \right) \right) dt. \quad (3.9)
\end{aligned}$$

By means of the Dirichlet-Neumann operator $G(\eta)$, we thus transformed the initial two dimensional problem to the one dimensional problem (3.4), (3.9). Here the derivatives of potential φ are defined by formulas (3.5), (3.6), (3.7). The noise $\sigma d\mathbf{B}$ and so the variance \mathbf{a} are modelled separately, and their expressions are assumed to be known. So far all calculations are formally exact and no approximation has been performed. In the next sections, we proceed to several simplifications of this model. The first one will focus on the constitution of Hamiltonian stochastic solutions whereas the following will consider classical weakly non linear approximation and their stochastic counterparts.

4 Hamiltonian representation under small noise assumption

As was shown by Zakharov [42], the deterministic water wave problem enjoys the Hamiltonian structure

$$\partial_t \eta = \delta \mathcal{H} / \delta \Phi, \quad \partial_t \Phi = -\delta \mathcal{H} / \delta \eta$$

with the total energy

$$\mathcal{H} = \frac{1}{2} \int_{\mathbb{R}} (g\eta^2 + \Phi G(\eta)\Phi) dx \quad (4.1)$$

which is a conserved quantity for the deterministic problem.

After introducing a new velocity variable $u = \partial_x \Phi$, one may notice (see Appendix A.1) that System (3.4), (3.9) can shortly be written down as

$$d \begin{pmatrix} \eta \\ u \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}}_J \begin{pmatrix} \delta \mathcal{H} / \delta \eta \\ \delta \mathcal{H} / \delta u \end{pmatrix} dt + \begin{pmatrix} d\eta^\sigma \\ du^\sigma \end{pmatrix}.$$

The aim of the current section is to approximate the noise $d\eta^\sigma, du^\sigma$, keeping its linear part unchanged, and in a way such that the energy $\mathcal{H}(\eta, u)$ is conserved. More precisely, we will show that up to the noise linearization, System (3.4), (3.9) can be written in Stratonovich form as

$$d \begin{pmatrix} \eta \\ u \end{pmatrix} = J \nabla \mathcal{H} dt + \sum_i J_i \nabla \mathcal{H} \circ dW_i, \quad (4.2)$$

with anti-symmetric operators J_i that will be precised below. The notation $f \circ dW$ denotes Stratonovich stochastic integral. Here $\{W_i\}$ is a sequence of independent scalar Wiener processes. We recall that in general a cylindrical or a Q -Wiener process is infinite dimensional, since it is defined by the diffusion operator σ . One of the most used ways to define such noise is to use an infinite sequence $\{W_i\}$, see details in [15]. In practice one may typically need up to a hundred of them, as for example in [35], where a quasi-geostrophic model is considered. As one shall see below, accounting all the simplifications regarded here, we will arrive to an essentially one dimensional noise, which means that it is enough to have only one scalar Brownian motion. However, we do not assume it a priori, and so we stick to the general case of the infinite sequence $\{W_i\}$. This also potentially may lead to different generalisations, see Conclusion for more discussion. In order to compare Expression (4.2) with (3.4), (3.9) we need to represent it in the Itô form. Upon using the classical relation between Itô and Stratonovich integrals [33], we obtain

$$J_i \nabla \mathcal{H} \circ dW_i = J_i \nabla \mathcal{H} dW_i + \frac{1}{2} \langle J_i d\nabla \mathcal{H}, dW_i \rangle = J_i \nabla \mathcal{H} dW_i + \frac{1}{2} J_i \nabla \mathcal{H}' J_i \nabla \mathcal{H} dt, \quad (4.3)$$

where we presume that each J_i is time independent. Here $\nabla \mathcal{H}'$ is the Jacobi matrix given in (B.2).

Thus

$$d \begin{pmatrix} \eta \\ u \end{pmatrix} = J \nabla \mathcal{H} dt + \sum_i J_i \nabla \mathcal{H} dW_i + \frac{1}{2} \sum_i J_i \nabla \mathcal{H}' J_i \nabla \mathcal{H} dt,$$

which can be compared with System (3.4), (3.9) to choose the best fit operators J_i . Indeed,

$$d\eta = J^{1\bullet} \nabla \mathcal{H} dt + \sigma d\mathbf{B}_z - \sigma d\mathbf{B}_x \partial_x \eta + \left((\partial_x a_{xx} + \partial_z a_{xz}) \partial_x \eta + \frac{1}{2} a_{xx} \partial_x^2 \eta \right) dt,$$

where $J_i^{1\bullet}$ denotes first row of J . According to the divergence free assumption the noise vector at the surface is modelled as

$$\sigma d\mathbf{B} = \sum_i \nabla^\perp \varphi_i(x, \eta(x, t)) dW_i.$$

Note that the noise part does not depend on the velocity variable u , which means that these equations can be compared only approximately, since both coordinates of the gradient $\nabla \mathcal{H}$ involve velocity. Linearizing the gradient in the stochastic part as

$$\nabla \mathcal{H} \approx \begin{pmatrix} g\eta \\ hK^2 u \end{pmatrix},$$

where we have used $G(\eta) \approx G(0) = -hK^2 \partial_x^2$, defined from Fourier transform through $\mathcal{F}(Kf)(\xi) = \sqrt{\frac{\tanh h\xi}{h\xi}} \widehat{f}(\xi)$, see Appendix B.2, one gets an expression for the noise in the Hamiltonian formulation that can be easily compared with the noise coming from the location uncertainty principle. We want to identify J_i so that

$$\begin{aligned} \sum_i (gJ_i^{11}\eta + hJ_i^{12}K^2u) dW_i &= \sigma d\mathbf{B}_z - \sigma d\mathbf{B}_x \partial_x \eta \\ &= \sum_i (\partial_x \varphi_i(x, \eta(x, t)) + \partial_z \varphi_i(x, \eta(x, t)) \partial_x \eta) dW_i \approx \sum_i (\partial_x \varphi_i(x, 0) + \partial_x (\partial_z \varphi_i(x, 0)\eta)) dW_i. \end{aligned}$$

Immediately, $J_i^{12} = 0$ and so $J_i^{21} = -J_i^{12*} = 0$. On the other hand to respect both $J_i^{11*} = -J_i^{11}$ and $gJ_i^{11}\eta = \partial_x \varphi_i(x, 0) + \partial_x (\partial_z \varphi_i(x, 0)\eta)$, we have to admit

$$\partial_x \varphi_i(x, 0) = 0,$$

$$\partial_x \partial_z \varphi_i(x, 0) = 0, \quad (\gamma_i := \partial_z \varphi_i(x, 0)),$$

which results in $J_i^{11} = \gamma_i \partial_x / g$. Now let us check that this conclusion is in line with the Itô correction term. Indeed,

$$a_{xx} dt = \langle \sigma d\mathbf{B}_x, \sigma d\mathbf{B}_x \rangle = \sum_i (\partial_z \varphi_i)^2 dt \approx \sum_i \gamma_i^2 dt,$$

and similarly,

$$\partial_x a_{xx} = 2 \sum_i \partial_z \varphi_i \partial_x \partial_z \varphi_i \approx 0,$$

$$\partial_z a_{xz} = - \sum_i \partial_z (\partial_x \varphi_i \partial_z \varphi_i) = - \sum_i (\partial_x \varphi_i \partial_z^2 \varphi_i + \partial_x \partial_z \varphi_i \partial_z \varphi_i) \approx 0.$$

Hence

$$\begin{aligned} \left((\partial_x a_{xx} + \partial_z a_{xz}) \partial_x \eta + \frac{1}{2} a_{xx} \partial_x^2 \eta \right) dt &\approx \frac{1}{2} \sum_i \gamma_i^2 \partial_x^2 \eta dt \\ &= \frac{1}{2} \sum_i J_i^{1\bullet} \begin{pmatrix} g & 0 \\ 0 & hK^2 \end{pmatrix} J_i \begin{pmatrix} g\eta \\ hK^2 u \end{pmatrix} dt \approx \frac{1}{2} \sum_i J_i^{1\bullet} \nabla \mathcal{H}' J_i \nabla \mathcal{H} dt, \end{aligned}$$

where

$$J_i = \begin{pmatrix} \gamma_i g^{-1} \partial_x & 0 \\ 0 & J_i^{22} \end{pmatrix}$$

and $J_i^{1\bullet}$ denotes its first row. It is left to find J_i^{22} in a similar way, namely, we want to get

$$\begin{aligned} \sum_i h J_i^{22} K^2 u dW_i &= -\sigma d\mathbf{B}_x \partial_x^2 \Phi + \partial_z \varphi (\partial_x \sigma d\mathbf{B}_z - \partial_x \sigma d\mathbf{B}_x \partial_x \eta + \partial_z \sigma d\mathbf{B}_z \partial_x \eta - \partial_z \sigma d\mathbf{B}_x (\partial_x \eta)^2) \\ &\approx \sum_i (\partial_z \varphi_i(x, \eta(x, t)) \partial_x^2 \Phi + \partial_x^2 \varphi_i(x, \eta(x, t)) G \Phi) dW_i \approx \sum_i (\partial_z \varphi_i(x, 0) \partial_x^2 \Phi + \partial_x^2 \varphi_i(x, 0) G \Phi) dW_i \\ &= \sum_i \gamma_i \partial_x u dW_i. \end{aligned}$$

Hence $J_i^{22} = \gamma_i h^{-1} K^{-2} \partial_x$, and one can easily check that this conclusion is in line with the Itô correction term as above. Finally, in variables η, u the structure map has the form

$$J_i^{(\eta, u)} = J_i = \gamma_i \begin{pmatrix} g^{-1} \partial_x & 0 \\ 0 & h^{-1} K^{-2} \partial_x \end{pmatrix}.$$

4.1 Canonical representation

One may try to return to initial canonical variables η, Φ . According to the change of variable explained in Appendix A.1, one can get

$$J^{(\eta, \Phi)} = \begin{pmatrix} 1 & 0 \\ 0 & \partial_x^{-1} \end{pmatrix} J^{(\eta, u)} \begin{pmatrix} 1 & 0 \\ 0 & -\partial_x^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$J_j^{(\eta, \Phi)} = \begin{pmatrix} 1 & 0 \\ 0 & \partial_x^{-1} \end{pmatrix} J_j^{(\eta, u)} \begin{pmatrix} 1 & 0 \\ 0 & -\partial_x^{-1} \end{pmatrix} = \gamma_j \begin{pmatrix} g^{-1} \partial_x & 0 \\ 0 & -h^{-1} K^{-2} \partial_x^{-1} \end{pmatrix}.$$

Note that

$$J_j^{(\eta, \Phi)} \nabla_{(\eta, \Phi)} \mathcal{H} = \begin{pmatrix} \gamma_j g^{-1} \partial_x \delta \mathcal{H} / \delta \eta & \\ 0 & \gamma_j h^{-1} K^{-2} (\mathcal{H}(\eta) \partial_x \Phi) \end{pmatrix}$$

is well defined regardless of the precise definition of ∂_x^{-1} . We finally obtain the following canonical stochastic representation of water waves

$$d \begin{pmatrix} \eta \\ \Phi \end{pmatrix} = J^{(\eta, \Phi)} \nabla_{(\eta, \Phi)} \mathcal{H} dt + \sum_j J_j^{(\eta, \Phi)} \nabla_{(\eta, \Phi)} \mathcal{H} \circ dW_j.$$

The second equation of this system, containing $d\Phi$, constitutes a stochastic extension of the Bernoulli surface wave equation. Under a small noise assumption, this model remains in a form that is fairly close to the original one.

4.2 On noise modelling

We assumed at the beginning that both the coarse and stochastic parts of the fluid velocity are divergence-free. For this it is enough to suppose incompressibility and that

$$\nabla \cdot (\nabla \cdot \mathbf{a}) = 0,$$

where

$$a_{jk} = \sum_i \left(\nabla^\perp \varphi_i \right)_j \left(\nabla^\perp \varphi_i \right)_k.$$

Then

$$\sum_i ((\partial_x \partial_z \varphi_i)^2 - \partial_x^2 \varphi_i \partial_z^2 \varphi_i) = 0$$

It is also natural to assume the non-penetration condition at the bottom for the stochastic component of velocity. Thus eventually we have a family parameterised by index i of problems

$$\begin{aligned} (\partial_x \partial_z \varphi_i)^2 &= \partial_x^2 \varphi_i \partial_z^2 \varphi_i, \\ \partial_x \varphi_i(x, 0) &= 0, \\ \partial_z \varphi_i(x, 0) &= \gamma_i, \\ \partial_x \varphi_i(x, -h) &= 0. \end{aligned}$$

A possible simple solution is

$$\varphi_i(x, z) = \Psi_i(z) \text{ with } \Psi_i'(0) = \gamma_i,$$

for example.

5 Weakly nonlinear approximations

In this section we proceed to approximations of the water waves formulation in a similar way as it is done in the deterministic setting. The objective is to provide stochastic representations of classical water wave representations. Airy waves, Whitham-Boussinesq, Boussinesq, Benjamin-Bona-Mahony, fully dispersive unidirectional, and Whitham model waves will be systematically passed in review.

For all these models, the simplification of both kinematical and dynamical boundary conditions is performed through a scale analysis and a small slope assumption of the waves.

5.1 Airy stochastic waves

We start by the simplest wave model, in which both boundary conditions are fully linearized. Let us first introduce a new velocity-type variable $v = K^2 u = K^2 \partial_x \Phi$ that will be useful in the following. The linear wave model can then be obtained by taking the Hamiltonian simply to be

$$\mathcal{H} = \mathcal{H}_0(\eta, v) = \frac{1}{2} \int_{\mathbb{R}} \left(g\eta^2 + h(K^{-1}v)^2 \right) dx,$$

and so the system takes the form

$$d \begin{pmatrix} \eta \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \sum_i \gamma_i^2 \partial_x^2 & -h\partial_x \\ -gK^2 \partial_x & \frac{1}{2} \sum_i \gamma_i^2 \partial_x^2 \end{pmatrix} \begin{pmatrix} \eta \\ v \end{pmatrix} dt + \sum_i \gamma_i \partial_x \begin{pmatrix} \eta \\ v \end{pmatrix} dW_i.$$

Note that the noise matrices are unitary up to the multiplier $\gamma_i \partial_x$, and so all matrices in this equation commute with each other. This system can hence easily be solved exactly. The fundamental solution has the form

$$\mathcal{S}(t, t_0) = S(t - t_0) S_\sigma(t, t_0),$$

where

$$S(t - t_0) = \exp \left(\begin{pmatrix} 0 & -h\partial_x \\ -gK^2 \partial_x & 0 \end{pmatrix} (t - t_0) \right) = \begin{pmatrix} \cos(U(t - t_0)) & -ihD \frac{\sin(U(t - t_0))}{U} \\ -igDK^2 \frac{\sin(U(t - t_0))}{U} & \cos(U(t - t_0)) \end{pmatrix}$$

with $U = \sqrt{gG_0}$, the Fourier multiplier $D = -i\partial_x$ and

$$S_\sigma(t, t_0) = \exp \sum_i \begin{pmatrix} \gamma_i \partial_x & 0 \\ 0 & \gamma_i \partial_x \end{pmatrix} (W_i(t) - W_i(t_0)).$$

In a diagonal form it reads

$$\mathcal{S}(t, t_0) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ K & -K \end{pmatrix} \begin{pmatrix} e^{-i(t-t_0)U \operatorname{sgn} D + \sum_j i\gamma_j D(W_j(t) - W_j(t_0))} & 0 \\ 0 & e^{i(t-t_0)U \operatorname{sgn} D + \sum_j i\gamma_j D(W_j(t) - W_j(t_0))} \end{pmatrix} \begin{pmatrix} 1 & K^{-1} \\ 1 & -K^{-1} \end{pmatrix}.$$

Clearly, that for any times t, t_0 operator $\mathcal{S}(t, t_0)$ is unitary in the Sobolev space $X^s = H^s \times H^{s+1/2}$ equipped with the norm

$$\|(\eta, v)\|_{X^s}^2 = \|\eta\|_{H^s}^2 + \|K^{-1}v\|_{H^s}^2 \quad (5.1)$$

and we get a solution that is similar to the standard one.

Note that if $\eta_d(x, t)$ stands for the deterministic wave with the initial wave given at the time moment t_0 , then the stochastic wave with the same initial data has the form

$$\eta(x, t) = e^{\sum_j i\gamma_j D(W_j(t) - W_j(t_0))} \eta_d(x, t) = \eta_d \left(x + \sum_j \gamma_j (W_j(t) - W_j(t_0)), t \right). \quad (5.2)$$

In other words, stochastic linear waves are Airy waves shifted randomly in space.

5.2 Linear noise models

The previous characterization of stochastic waves extends indeed to any linear noise models.

Consider models of the form

$$d \begin{pmatrix} \eta \\ v \end{pmatrix} = J \nabla \mathcal{H} dt + \sum_j J_j \nabla \mathcal{H}_0 \circ dW_j,$$

with anti-symmetric operators J_j as above. Here \mathcal{H} can stand either for the full total energy (4.1) or for an approximation of it, such as (5.4) that will be exhibited later on, for example.

It turns out that for most long wave approximations, \mathcal{H} is a conserved quantity, and this system reduces to the corresponding deterministic one.

In order to show energy conservation let us denote $u = (\eta, v)^T$. Then

$$\begin{aligned} \mathcal{H}(u(t)) - \mathcal{H}(u(0)) &= \int_0^t \langle \nabla \mathcal{H}(u(t')), J \nabla \mathcal{H}(u(t')) \rangle_{L^2 \times L^2} dt' \\ &+ \sum_j \int_0^t \langle \nabla \mathcal{H}(u(t')), J_j \nabla \mathcal{H}_0(u(t')) \rangle_{L^2 \times L^2} \circ dW_j(t') = \sum_j \gamma_j \int_0^t \int \left(\frac{\delta \mathcal{H}}{\delta \eta} \partial_x \eta + \frac{\delta \mathcal{H}}{\delta v} \partial_x v \right) dx \circ dW_j = 0 \end{aligned}$$

provided

$$\mathcal{H} = \int H(\psi(D)u(x)) dx,$$

for example, as in (5.4). This property remains valid for any approximation of $G(\eta)$ in (4.1) via Taylor expansion, and so \mathcal{H} given in (4.1) is a conserved quantity for the full Euler system with linear noise.

Notating its non-linear part $F = J \nabla (\mathcal{H} - \mathcal{H}_0)$ we can rewrite it in the form

$$\begin{pmatrix} \eta \\ v \end{pmatrix} (t) = \mathcal{S}(t, t_0) \left(\begin{pmatrix} \eta \\ v \end{pmatrix} (t_0) + \int_{t_0}^t \mathcal{S}^{-1}(s, t_0) F(\eta(s), v(s)) ds \right),$$

where \mathcal{S} is defined above. Note that for any real number α we have

$$(e^{i\alpha D} \eta) e^{i\alpha D} v = e^{i\alpha D} (\eta v)$$

and that $e^{i\alpha D}$ commute with any Fourier multiplier. Thus the stochastic system with linear noise has still a solution of the form (5.2).

5.3 Whitham-Boussinesq model

Here we regard a simplified model that was derived in the deterministic case from the Hamiltonian long wave approximation [22]. We will essentially repeat the arguments of Section 4. The main difference comes from the view of the Hamiltonian \mathcal{H} that now will have an explicit expression. Note that in (4.1) the dependence on η is implicit, since there is no exact explicit expression for the Dirichlet-Neumann operator $G(\eta)$ standing in the definition of \mathcal{H} in (4.1). This, of course, simplifies and clarifies the derivation presented above. Moreover, it could serve as an alternative derivation to the one given in Section 4, since the main idea there was the fully dispersive linearisation of the noise given in two systems: (3.4), (3.9) and (4.2). The model currently

under consideration is fully dispersive, and so up to a change of variables we will get the same anti-symmetric operators J_i as we obtained in Section 4. In variables η and $v = K^2 u = K^2 \partial_x \Phi$ it reads

$$\begin{cases} d\eta = -h\partial_x v dt - K^2 \partial_x(\eta v) dt + d\eta^\sigma, \\ dv = -gK^2 \partial_x \eta dt - K^2 \partial_x (v^2/2) dt + dv^\sigma, \end{cases} \quad (5.3)$$

where

$$K = \sqrt{\frac{\tanh hD}{hD}}$$

with $D = -i\partial_x$ being a Fourier multiplier. The problem is to model the noise $d\eta^\sigma, dv^\sigma$ in a way that the energy

$$\mathcal{H} = \frac{1}{2} \int_{\mathbb{R}} \left(g\eta^2 + h(K^{-1}v)^2 + \eta v^2 \right) dx \quad (5.4)$$

remains conserved along time for any solution. This quantity serves as a Hamiltonian for the corresponding deterministic system, which means

$$d \begin{pmatrix} \eta \\ v \end{pmatrix} = \begin{pmatrix} 0 & -K^2 \partial_x \\ -K^2 \partial_x & 0 \end{pmatrix} \begin{pmatrix} \delta \mathcal{H} / \delta \eta \\ \delta \mathcal{H} / \delta v \end{pmatrix} dt + \begin{pmatrix} d\eta^\sigma \\ dv^\sigma \end{pmatrix}.$$

As in the full water wave problem we approximate the noise in such a way that

$$d \begin{pmatrix} \eta \\ v \end{pmatrix} = J \nabla \mathcal{H} dt + \sum_i J_i \nabla \mathcal{H} \circ dW_i,$$

with

$$J = \begin{pmatrix} 0 & -K^2 \partial_x \\ -K^2 \partial_x & 0 \end{pmatrix} \quad \text{and} \quad J_i = \begin{pmatrix} J_i^{11} & J_i^{12} \\ J_i^{21} & J_i^{22} \end{pmatrix}.$$

Note that $J_i^{jk*} = -J_i^{kj}$ for any i, j, k . We need to rewrite it in the Itô form in order to compare with System (3.4), (3.9). One can easily see that

$$\nabla \mathcal{H} = \begin{pmatrix} \delta \mathcal{H} / \delta \eta \\ \delta \mathcal{H} / \delta v \end{pmatrix} = \begin{pmatrix} g\eta + v^2/2 \\ hK^{-2}v + \eta v \end{pmatrix}$$

and so

$$d \nabla \mathcal{H} = \begin{pmatrix} g & v \\ v & hK^{-2} + \eta \end{pmatrix} \begin{pmatrix} d\eta \\ dv \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \langle dv, dv \rangle \\ \langle d\eta, dv \rangle \end{pmatrix} = \begin{pmatrix} g & v \\ v & hK^{-2} + \eta \end{pmatrix} \sum_i J_i \nabla \mathcal{H} \circ dW_i + \dots,$$

where the rest terms are of bounded variation, so they go away when one calculates the quadratic covariation while passing from Stratonovich to Itô integration, cf. (4.3). Thus

$$d \begin{pmatrix} \eta \\ v \end{pmatrix} = J \nabla \mathcal{H} dt + \sum_i J_i \nabla \mathcal{H} dW_i + \frac{1}{2} \sum_i J_i \begin{pmatrix} g & v \\ v & hK^{-2} + \eta \end{pmatrix} J_i \nabla \mathcal{H} dt,$$

which can be compared with System (3.4), (3.9) to choose the best fit operators J_i . Indeed,

$$d\eta = J^1 \bullet \nabla \mathcal{H} dt + \sigma d\mathbf{B}_z - \sigma d\mathbf{B}_x \partial_x \eta + \left((\partial_x a_{xx} + \partial_z a_{xz}) \partial_x \eta + \frac{1}{2} a_{xx} \partial_x^2 \eta \right) dt,$$

where the noise vector

$$\sigma d\mathbf{B} = \sum_i \nabla^\perp \varphi_i(x, \eta(x, t)) dW_i.$$

Note that the noise part does not depend on the velocity variable v , which means that these equations can be compared only approximately, since both coordinates of the gradient $\nabla \mathcal{H}$ contain velocity. Linearising the gradient in the stochastic part as

$$\nabla \mathcal{H} \approx \begin{pmatrix} g\eta \\ hK^{-2}v \end{pmatrix}$$

we want to obtain

$$\begin{aligned} \sum_i (gJ_i^{11} \eta + hJ_i^{12} K^{-2} v) dW_i &= \sigma d\mathbf{B}_z - \sigma d\mathbf{B}_x \partial_x \eta \\ &= \sum_i (\partial_x \varphi_i(x, \eta(x, t)) + \partial_z \varphi_i(x, \eta(x, t)) \partial_x \eta) dW_i \approx \sum_i (\partial_x \varphi_i(x, 0) + \partial_x (\partial_z \varphi_i(x, 0) \eta)) dW_i. \end{aligned}$$

Immediately, $J_i^{12} = 0$ and so $J_i^{21} = -J_i^{12*} = 0$. On the other hand to respect both $J_i^{11*} = -J_i^{11}$ and $gJ_i^{11}\eta = \partial_x\varphi_i(x,0) + \partial_x(\partial_z\varphi_i(x,0)\eta)$, we have to admit

$$\partial_x\varphi_i(x,0) = 0$$

$$\partial_x\partial_z\varphi_i(x,0) = 0, \quad (\gamma_i := \partial_z\varphi_i(x,0))$$

which results in $J_i^{11} = \gamma_i\partial_x/g$. Now let us check that this conclusion is in line with the Itô correction. Indeed,

$$a_{xx}dt = \langle \sigma d\mathbf{B}_x, \sigma d\mathbf{B}_x \rangle = \sum_i (\partial_z\varphi_i)^2 dt \approx \sum_i \gamma_i^2 dt,$$

and similarly,

$$\partial_x a_{xx} = 2 \sum_i \partial_z\varphi_i \partial_x \partial_z\varphi_i \approx 0,$$

$$\partial_z a_{xz} = - \sum_i \partial_z (\partial_x\varphi_i \partial_z\varphi_i) = - \sum_i (\partial_x\varphi_i \partial_z^2\varphi_i + \partial_x\partial_z\varphi_i \partial_z\varphi_i) \approx 0.$$

Hence

$$\left((\partial_x a_{xx} + \partial_z a_{xz}) \partial_x \eta + \frac{1}{2} a_{xx} \partial_x^2 \eta \right) dt \approx \frac{1}{2} \sum_i \gamma_i^2 \partial_x^2 \eta dt \approx \frac{1}{2} \sum_i J_i^1 \begin{pmatrix} g & v \\ v & hK^{-2} + \eta \end{pmatrix} J_i \nabla \mathcal{H} dt,$$

where

$$J_i = \begin{pmatrix} \gamma_i g^{-1} \partial_x & 0 \\ 0 & J_i^{22} \end{pmatrix}$$

and J_i^1 is its first row. It is left to find J_i^{22} in a similar way, namely, we want to get

$$\begin{aligned} \sum_i hK^{-2} J_i^{22} K^{-2} v dW_i &= -\sigma d\mathbf{B}_x \partial_x^2 \Phi + \partial_z \varphi (\partial_x \sigma d\mathbf{B}_z - \partial_x \sigma d\mathbf{B}_x \partial_x \eta + \partial_z \sigma d\mathbf{B}_z \partial_x \eta - \partial_z \sigma d\mathbf{B}_x (\partial_x \eta)^2) \\ &\approx \sum_i (\partial_z \varphi_i(x, \eta(x, t)) \partial_x^2 \Phi + \partial_x^2 \varphi_i(x, \eta(x, t)) G \Phi) dW_i \approx \sum_i (\partial_z \varphi_i(x, 0) \partial_x^2 \Phi + \partial_x^2 \varphi_i(x, 0) G \Phi) dW_i \\ &= \sum_i \gamma_i \partial_x^2 \Phi dW_i. \end{aligned}$$

Hence $J_i^{22} = \gamma_i K^2 \partial_x / h$, and one can easily check that this conclusion is in line with the Itô correction term as above. As a result

$$J_i = \gamma_i \begin{pmatrix} g^{-1} \partial_x & 0 \\ 0 & h^{-1} K^2 \partial_x \end{pmatrix},$$

and so we obtain finally the following stochastic Whitham-Boussinesq system

$$d \begin{pmatrix} \eta \\ v \end{pmatrix} = -K^2 \partial_x \begin{pmatrix} hK^{-2}v + \eta v \\ g\eta + v^2/2 \end{pmatrix} dt + \sum_j \gamma_j \partial_x \begin{pmatrix} \eta + g^{-1}v^2/2 \\ v + h^{-1}K^2(\eta v) \end{pmatrix} \circ dW_j. \quad (5.5)$$

Some numerical solutions of this system will be provided in section 6 for different numerical schemes. An exponential scheme will in particular allow us to numerically highlight the energy conservation of this stochastic model.

5.4 Boussinesq model

In the deterministic water wave theory the following four parameter family of equations

$$\begin{cases} (1 - b\partial_x^2)\partial_t\eta + h(1 + a\partial_x^2)\partial_x w + \partial_x(\eta w) = 0, \\ (1 - d\partial_x^2)\partial_t w + g(1 + c\partial_x^2)\partial_x \eta + w\partial_x w = 0 \end{cases} \quad (5.6)$$

is of a particular interest. It was derived in [5]. Its Cauchy problem was studied in [6]. This model exhibits solitary wave solutions, as was shown in [11, 12, 21, 23]. Here η is the surface elevation as usual, whereas w is a velocity with physical meaning depending on a particular choice of the real coefficients a, b, c, d . In order for System (5.6) to be Hamiltonian with the total energy coinciding approximately with the total energy of the full water wave problem, one needs to impose that $b = d$. Moreover, in order to be a valid ocean model in the Boussinesq regime, it is required to set $c = 0$ and $a + b + c + d = h^2/3$ as well. A naive assignment $w = u$,

$a = h^2/3$ and $b = c = d = 0$ gives a system consistent with the deterministic full water wave problem, however, it reveals ill posed [6].

Consequently, in order to restrict ourselves to consideration of (5.6) when it is a good Hamiltonian well posed approximation in the Boussinesq regime of the full water wave problem, we impose $a \leq 0$, $b = d \geq 0$, $c = 0$, with their sum fixed as previously. This turns Equations (5.6) into a one parameter family of systems.

We introduce a new velocity variable through the expression

$$w = K_b^{-1}u = (1 - b\partial_x^2)^{-1}u, \quad (5.7)$$

and conduct the long wave approximation $\mathcal{H} \approx \mathcal{H}(\eta, w)$ with the new energy

$$\mathcal{H}(\eta, w) = \frac{1}{2} \int_{\mathbb{R}} (g\eta^2 + hwK_a w + \eta w^2) dx, \quad (5.8)$$

where we impose

$$a = \frac{h^2}{3} - 2b \leq 0 \quad (5.9)$$

and set

$$K_a = 1 - |a|\partial_x^2. \quad (5.10)$$

One can easily calculate the gradient

$$\nabla \mathcal{H} = \begin{pmatrix} \delta \mathcal{H} / \delta \eta \\ \delta \mathcal{H} / \delta w \end{pmatrix} = \begin{pmatrix} g\eta + w^2/2 \\ hK_a w + \eta w \end{pmatrix},$$

and repeating the arguments from Section 5.3 one deduces that

$$J = \begin{pmatrix} 0 & -\partial_x K_b^{-1} \\ -\partial_x K_b^{-1} & 0 \end{pmatrix}, \quad J_j = \begin{pmatrix} g^{-1} & 0 \\ 0 & h^{-1}K_a^{-1} \end{pmatrix} \gamma_j \partial_x.$$

Finally, we arrive to the following one parameter family of Stochastic Boussinesq equations

$$d \begin{pmatrix} \eta \\ w \end{pmatrix} = -\partial_x K_b^{-1} \begin{pmatrix} hK_a w + \eta w \\ g\eta + w^2/2 \end{pmatrix} dt + \sum_j \gamma_j \partial_x \begin{pmatrix} \eta + g^{-1}w^2/2 \\ w + h^{-1}K_a^{-1}(\eta w) \end{pmatrix} \circ dW_j, \quad (5.11)$$

which is a stochastic extension of (5.6) with Relation (5.9).

5.5 Benjamin-Bona-Mahony model

In order to derive a unidirectional model in the Boussinesq regime, one may notice that the transformation

$$r = \frac{1}{2} \left(\eta + \sqrt{hK_a/gw} \right), \quad l = \frac{1}{2} \left(\eta - \sqrt{hK_a/gw} \right) \quad (5.12)$$

diagonalises the linear deterministic part of System (5.11). Physically, these new variables approximately represent right- and left-moving waves, respectively. According to the rule explained in Appendix A.1 we have

$$J = J^{(r,l)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2} \sqrt{\frac{h}{g} K_a K_b^{-1}} \partial_x, \quad J_j = J_j^{(r,l)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\gamma_j}{2g} \partial_x.$$

Inserting $\eta = \eta(r, l)$ and $w = w(r, l)$ from (5.12) into (5.8) one obtains the Hamiltonian $\mathcal{H}(r, l)$ that under the long wave approximation simplifies to the form

$$\mathcal{H}(r, l) = g \int_{\mathbb{R}} \left(r^2 + l^2 + \frac{1}{2h} (r+l)(r-l)^2 \right) dx. \quad (5.13)$$

Now neglecting the coupling between the right- and left-moving waves, one can admit that $\mathcal{H}(r, l) \approx \mathcal{H}(r) + \mathcal{H}(l)$, where

$$\mathcal{H}(r) = g \int_{\mathbb{R}} \left(r^2 + \frac{1}{2h} r^3 \right) dx, \quad (5.14)$$

which is justified, for instance, if the waves are moving essentially in one direction. This approximation can be also used in case of a very short interaction between the waves moving in the opposite directions, for example, in the problem of collision of two solitons. The Gateaux derivative of $\mathcal{H}(r)$ is

$$\frac{\delta \mathcal{H}}{\delta r} = 2g \left(r + \frac{3}{4h} r^2 \right).$$

Thus we obtain the following stochastic Benjamin-Bona-Mahony (BBM) equation

$$dr = -\sqrt{ghK_aK_b^{-1}}\partial_x \left(r + \frac{3}{4h}r^2 \right) dt + \sum_j \gamma_j \partial_x \left(r + \frac{3}{4h}r^2 \right) \circ dW_j, \quad (5.15)$$

where K_a, K_b are defined by (5.10) and parameters a, b are related by (5.9). The deterministic BBM model corresponding to $\gamma_j \equiv 0$ and $a = 0$ firstly appeared in [4]. It describes right-moving surface waves in the Boussinesq regime. Moreover, $\mathcal{H}(r)$ defined by (5.14) coincides with the total energy (4.1) with the same order of error. The classical deterministic BBM equation is known to conserve the H^1 -norm, namely, the integral $\int rK_b r dx$. This invariant plays an important role in its mathematical analysis. For the stochastic model this norm is unfortunately not anymore conserved. Indeed, the noise affects dramatically this invariant. As a matter of fact, for $a = 0$ and $b > 0$ we have

$$\begin{aligned} d \int rK_b r dx &= -2\sqrt{gh} \int r \partial_x \left(r + \frac{3}{4h}r^2 \right) dx dt + 2 \sum_j \gamma_j \int r (1 - b\partial_x^2) \partial_x \left(r + \frac{3}{4h}r^2 \right) dx \circ dW_j \\ &= \frac{3b}{2h} \sum_j \gamma_j \int (\partial_x r)^3 dx \circ dW_j, \end{aligned}$$

which is not zero in general. This Stratonovich integral is in addition of non zero expectation.

5.6 Modified Benjamin-Bona-Mahony model

The shortcoming of the previous stochastic BBM model (5.15) motivates us to propose some modifications of this model, yet staying at the same level of accuracy. To that end, we introduce the following functional

$$\mathcal{Q}(r) = g \int_{\mathbb{R}} \left(rK_a^{-1/2}K_b r + \frac{1}{2h}r^3 \right) dx, \quad (5.16)$$

which coincides with the energy (5.14) in the shallow water regime ($K_a \approx K_b \approx 1$). It has the variational derivative

$$\frac{\delta \mathcal{Q}}{\delta r} = 2g \left(K_a^{-1/2}K_b r + \frac{3}{4h}r^2 \right).$$

We propose the following model

$$dr = -\frac{1}{2}\sqrt{\frac{h}{g}}K_aK_b^{-2}\partial_x \frac{\delta \mathcal{Q}}{\delta r} dt + \frac{1}{2g} \sum_j \gamma_j \sqrt{K_aK_b^{-1}}\partial_x \frac{\delta \mathcal{Q}}{\delta r} \circ dW_j,$$

that respects the conservation of \mathcal{Q} . Indeed, since the differential of \mathcal{Q} with respect to the variable r is defined on test functions via the L^2 -inner product as $d\mathcal{Q}(r)(\psi) = (\delta \mathcal{Q}/\delta r, \psi)$, then taking into account that Stratonovich differentiation satisfies the usual chain rule, one obtains

$$d(\mathcal{Q}(r(t))) = -\frac{1}{2}\sqrt{\frac{h}{g}} \left(\frac{\delta \mathcal{Q}}{\delta r}, K_aK_b^{-2}\partial_x \frac{\delta \mathcal{Q}}{\delta r} \right) dt + \frac{1}{2g} \sum_j \gamma_j \left(\frac{\delta \mathcal{Q}}{\delta r}, \sqrt{K_aK_b^{-1}}\partial_x \frac{\delta \mathcal{Q}}{\delta r} \right) \circ dW_j = 0.$$

More explicitly the modified BBM model reads

$$dr = -\sqrt{gh}\partial_x \left(\sqrt{K_aK_b^{-1}}r + \frac{3}{4h}K_aK_b^{-2}r^2 \right) dt + \sum_j \gamma_j \partial_x \left(r + \frac{3}{4h}\sqrt{K_aK_b^{-1}}r^2 \right) \circ dW_j. \quad (5.17)$$

Note that both (5.17) and (5.15) are of the same order of accuracy. However, Energy (5.16) constitutes a poorer approximation of (4.1) than (5.14). This flaw may be genuinely considered in view of the modelling of energy exchanges between coarse and fine scales. In other words, the energy accuracy is relaxed here and replaced by a modified conserved total energy. We believe that the conservation of the functional \mathcal{Q} will be useful in the analysis of Equation (5.17). Well posedness of the corresponding Cauchy problem is in particular studied in a subsequent paper [20].

5.7 Fully dispersive unidirectional model

Similarly to what was done in Section 5.5, we introduce here a fully dispersive unidirectional model. We start with the description given in Section 5.3 and split again the waves under consideration in terms of right- and left-moving waves. The final equation of the Whitham type has the form

$$dr = -\sqrt{gh}K\partial_x \left(r + \frac{3}{4h}r^2 \right) dt + \sum_j \gamma_j \partial_x \left(r + \frac{3}{4h}r^2 \right) \circ dW_j, \quad (5.18)$$

and it enjoys the conservation of Functional (5.14). In the deterministic framework this model appeared in [22], however to our knowledge it was not studied further in later works.

5.8 Whitham model

Introducing the energy functional

$$\mathcal{Q}(r) = g \int_{\mathbb{R}} \left(rKr + \frac{1}{2h}r^3 \right) dx, \quad (5.19)$$

which again coincides with the energy (5.14) in the shallow water regime, which has the variational derivative

$$\frac{\delta \mathcal{Q}}{\delta r} = 2g \left(Kr + \frac{3}{4h}r^2 \right),$$

we consider an equation of the following structure

$$dr = -\frac{1}{2} \sqrt{\frac{h}{g}} \partial_x \frac{\delta \mathcal{Q}}{\delta r} dt + \frac{1}{2g} \sum_j \gamma_j K^{-1} \partial_x \frac{\delta \mathcal{Q}}{\delta r} \circ dW_j,$$

that obviously conserves \mathcal{Q} . Explicitly, the stochastic Whitham equation has the form

$$dr = -\sqrt{gh} \partial_x \left(Kr + \frac{3}{4h}r^2 \right) dt + \sum_j \gamma_j \partial_x \left(r + \frac{3}{4h}K^{-1}r^2 \right) \circ dW_j. \quad (5.20)$$

Its deterministic analogue has been paid to a lot of attention recently. Local well-posedness and solitary wave existence were proved in [25] and [26], respectively. The latter was significantly improved in [41]. Cusped waves were studied in [27] and [28]. Wave braking was proved in [31].

6 Numerical experiments

Here we provide with some numerical results obtained with different numerical schemes for the conservative equations of the form (A.1), that fits all the weakly nonlinear models given above in Section 5. We work below with exponential integrators, since they exhibit in general good stability results. More precisely, we will assess and compare the explicit Euler scheme for the mild equation (A.3), as well as the explicit Euler and Milstein for the Duhamel equation (A.5). In all these examples the spatial discretization is performed in the Fourier domain. We evaluated in particular these three schemes on the model described in Section 5.3. Our numerical experiments suggest that the Duhamel form (A.5) deserves a special attention, since it provides a fast and accurate treatment of the stochastic water wave equations. It is in line with the findings of [29].

For all the schemes the noise was simulated as follows, $\Delta t = t_{n+1} - t_n$, $W_j^{n+1} - W_j^n = \sqrt{\Delta t} Z_n^j$, $n = 0, 1, 2, \dots$ and $\{Z_n^j\}_{n=0}^{\infty}$ are sequences of independent $N(0, 1)$ -distributed random variables. Note that the quadratic variation $\frac{1}{2} \sum_i \gamma_i^2$ has the dimension of a viscosity in m^2/s . Let us introduce a non-dimensional noise parameter ϵ such that

$$\frac{1}{2} \sum_i \gamma_i^2 = \sqrt{gh^3} \epsilon,$$

and enabling us to quantify the noise level magnitude.

In the next sections we present thoroughly the three discrete temporal schemes explored in these experiments.

6.1 Euler discretisation of mild form

The explicit Euler time discretisation applied to the mild form (A.3) of Equation (A.1) has the form

$$u(t) \approx e^{\tilde{A}(t-t_0)} \left(u(t_0) + F(u(t_0))(t-t_0) + \sum_j (B_j u(t_0) + g_j(u(t_0))) (W_j(t) - W_j(t_0)) \right),$$

provided $t_0 \leq t$ are close. Note that for all considered above models $B_j = \gamma_j \partial_x$ and $g_j(u) = \gamma_j \mathbf{g}(u)$, where the later stays for the noise nonlinearity, compare the general equation (A.1) with particular models (5.5), (5.11), (5.15), (5.17), (5.18), (5.20). For example, for the stochastic BBM equation (5.15) we have $\mathbf{g}(u) = 3\partial_x(u^2)/(4h)$. Moreover, $\partial_x \mathbf{g}(u) = \mathbf{g}'(u) \partial_x u$, since $\mathbf{g}(u)$ is a composition of polynomials and Fourier multipliers. Hence the Itô corrected nonlinearity $F(u)$ can be slightly simplified as

$$F(u) = f(u) + \frac{1}{2} \sum_j \gamma_j^2 (2\partial_x \mathbf{g}(u) + \mathbf{g}'(u) \mathbf{g}(u)).$$

Finally, our mild Euler exponential integrator reads

$$u(t_{n+1}) \approx u_{n+1} = e^{\tilde{A}\Delta t} \left(u_n + F(u_n)\Delta t + (\partial_x u_n + \mathbf{g}(u_n)) \sum_j \gamma_j Z_n^j \sqrt{\Delta t} \right). \quad (6.1)$$

6.2 Euler discretisation of Duhamel form

The explicit Euler time discretisation applied to the mild form (A.5) of Equation (A.1) has the form

$$u(t) \approx \mathcal{S}(t, t_0) \left(u(t_0) + \tilde{f}(u(t_0))(t-t_0) + \sum_j g_j(u(t_0))(W_j(t) - W_j(t_0)) \right),$$

provided $t_0 \leq t$ are close. As above the Duhamel nonlinearity $\tilde{f}(u)$ can be slightly simplified as

$$\tilde{f}(u) = f(u) + \frac{1}{2} \sum_j \gamma_j^2 \mathbf{g}'(u) \mathbf{g}(u).$$

Finally, our Duhamel-Euler exponential integrator reads

$$u(t_{n+1}) \approx u_{n+1} = \mathcal{S}(t_{n+1}, t_n) \left(u_n + \tilde{f}(u_n)\Delta t + \mathbf{g}(u_n) \sum_j \gamma_j Z_n^j \sqrt{\Delta t} \right), \quad (6.2)$$

where the operator matrix $\mathcal{S}(t_{n+1}, t_n)$ is defined by Formula (A.4).

6.3 Milstein discretisation of Duhamel form

In order to obtain the Milstein type discretisation of (A.5), we need to expand time dependence as follows. Firstly, note that

$$\mathcal{S}(t_0, s) - 1 = \int_{t_0}^s d\mathcal{S}(t_0, r) = - \sum_k B_k \int_{t_0}^s dW_k(r) + \mathcal{O}(s-t_0),$$

and so

$$\mathcal{S}^{-1}(s, t_0) = \mathcal{S}(t_0, s) = 1 - \sum_k B_k \int_{t_0}^s dW_k(r) + \mathcal{O}(s-t_0).$$

In particular, we have that

$$\int_{t_0}^t \mathcal{S}^{-1}(s, t_0) \tilde{f}(u(s)) ds = \tilde{f}(u(t_0))(t-t_0) + \mathcal{O}\left((t-t_0)^{3/2}\right).$$

Secondly, note that

$$g_j(u(s)) = g_j(u(t_0)) + g'_j(u(t_0)) \sum_k (B_k u(t_0) + g_k(u(t_0))) \int_{t_0}^s dW_k(r) + \mathcal{O}(s-t_0),$$

where we have approximated the difference $u(s) - u(t_0)$ with the help of equation (A.2). Thus one obtains

$$\begin{aligned} \mathcal{S}^{-1}(s, t_0)g_j(u(s)) &= g_j(u(t_0)) \\ &+ \sum_k [g'_j(u(t_0))B_k u(t_0) + g'_j(u(t_0))g_k(u(t_0)) - B_k g_j(u(t_0))] \int_{t_0}^s dW_k(r) + \mathcal{O}(s - t_0) \end{aligned}$$

which after integration leads to

$$\begin{aligned} \sum_j \int_{t_0}^t \mathcal{S}^{-1}(s, t_0)g_j(u(s))dW_j(s) &= \sum_j g_j(u(t_0)) \int_{t_0}^t dW_j(s) \\ &+ \sum_{j,k} [g'_j(u(t_0))B_k u(t_0) + g'_j(u(t_0))g_k(u(t_0)) - B_k g_j(u(t_0))] \int_{t_0}^t \int_{t_0}^s dW_k(r)dW_j(s) + \mathcal{O}\left((t - t_0)^{3/2}\right). \end{aligned}$$

As in the previous two treatments the final expressions can be simplified taking into account that all the models under consideration admit $B_j = \gamma_j \partial_x$ and $g_j(u) = \gamma_j \mathfrak{g}(u)$. Moreover, it turns out that one does not need to sample the corresponding Lévy areas, since in our framework the expression in the square brackets [...] is symmetric with respect to j, k . Indeed, as thoroughly explained for example in [36], this symmetry obviously leads to

$$\begin{aligned} \sum_{j,k} [\dots] \int_{t_0}^t \int_{t_0}^s dW_k(r)dW_j(s) &= \frac{1}{2} \sum_{j,k} [\dots] \left(\int_{t_0}^t \int_{t_0}^s dW_k(r)dW_j(s) + \int_{t_0}^t \int_{t_0}^s dW_j(r)dW_k(s) \right) \\ &= \frac{1}{2} \sum_{j,k} [\dots] ((W_j(t) - W_j(t_0))(W_k(t) - W_k(t_0)) - \delta_{jk}(t - t_0)). \end{aligned}$$

One can in addition notice that the last sum $\frac{1}{2} \sum_{j,k} [\dots] \delta_{jk}$ coincides exactly with the difference between $\tilde{f}(u(t_0))$ and $f(u(t_0))$. Thus

$$\begin{aligned} u(t) \approx \mathcal{S}(t, t_0) &\left(u(t_0) + f(u(t_0))(t - t_0) + \sum_j g_j(u(t_0))(W_j(t) - W_j(t_0)) \right. \\ &\left. + \frac{1}{2} \sum_{j,k} [g'_j(u(t_0))B_k u(t_0) + g'_j(u(t_0))g_k(u(t_0)) - B_k g_j(u(t_0))] (W_j(t) - W_j(t_0))(W_k(t) - W_k(t_0)) \right), \end{aligned}$$

provided $t_0 \leq t$ are close. Finally, our Duhamel-Milstein exponential integrator reads

$$u(t_{n+1}) \approx u_{n+1} = \mathcal{S}(t_{n+1}, t_n) \left(u_n + f(u_n)\Delta t + \mathfrak{g}(u_n) \sum_j \gamma_j Z_n^j \sqrt{\Delta t} + \frac{1}{2} \mathfrak{g}'(u_n) \mathfrak{g}(u_n) \left(\sum_j \gamma_j Z_n^j \right)^2 \Delta t \right), \quad (6.3)$$

where the operator matrix $\mathcal{S}(t_{n+1}, t_n)$ is defined by Formula (A.4).

6.4 Simulations

We test all the numerical schemes given above on the system introduced in Section 5.3. Here

$$\begin{aligned} u &= \begin{pmatrix} \eta \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -h\partial_x \\ -gK^2\partial_x & 0 \end{pmatrix}, \quad B_j = \gamma_j \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix}, \\ f(u) &= - \begin{pmatrix} K^2\partial_x(\eta v) \\ K^2\partial_x(v^2/2) \end{pmatrix}, \quad g_j(u) = \gamma_j \mathfrak{g}(u) = \gamma_j \begin{pmatrix} g^{-1}v\partial_x v \\ h^{-1}K^2\partial_x(\eta v) \end{pmatrix}. \end{aligned}$$

In order to find nonlinear mappings in Schemes (6.1), (6.2), (6.3) we calculate the derivative

$$\mathfrak{g}'(u) = \begin{pmatrix} 0 & g^{-1}\partial_x(v \cdot) \\ h^{-1}K^2\partial_x(v \cdot) & h^{-1}K^2\partial_x(\eta \cdot) \end{pmatrix},$$

and so

$$\mathfrak{g}'(u)\mathfrak{g}(u) = \begin{pmatrix} (gh)^{-1}\partial_x(vK^2\partial_x(\eta v)) \\ (gh)^{-1}K^2\partial_x(v^2\partial_x v) + h^{-2}K^2\partial_x(\eta K^2\partial_x(\eta v)) \end{pmatrix}.$$

Hence

$$F(u) = - \left(\frac{K^2 \partial_x(\eta v)}{K^2 \partial_x(v^2/2)} \right) + \frac{1}{2} \sum_i \gamma_i^2 \left(\frac{1}{g} \partial_x^2 v^2 + \frac{1}{gh} \partial_x(v K^2 \partial_x(\eta v)) \right. \\ \left. \frac{2}{h} K^2 \partial_x^2(\eta v) + \frac{1}{3gh} K^2 \partial_x^2 v^3 + \frac{1}{h^2} K^2 \partial_x(\eta K^2 \partial_x(\eta v)) \right),$$

$$\tilde{f}(u) = - \left(\frac{K^2 \partial_x(\eta v)}{K^2 \partial_x(v^2/2)} \right) + \frac{1}{2} \sum_i \gamma_i^2 \left(\frac{1}{3gh} K^2 \partial_x^2 v^3 + \frac{1}{h^2} K^2 \partial_x(\eta K^2 \partial_x(\eta v)) \right).$$

Finally, after substituting these identities in Schemes (6.1), (6.2), (6.3) we are ready to simulate evolution of waves, described by System (5.5).

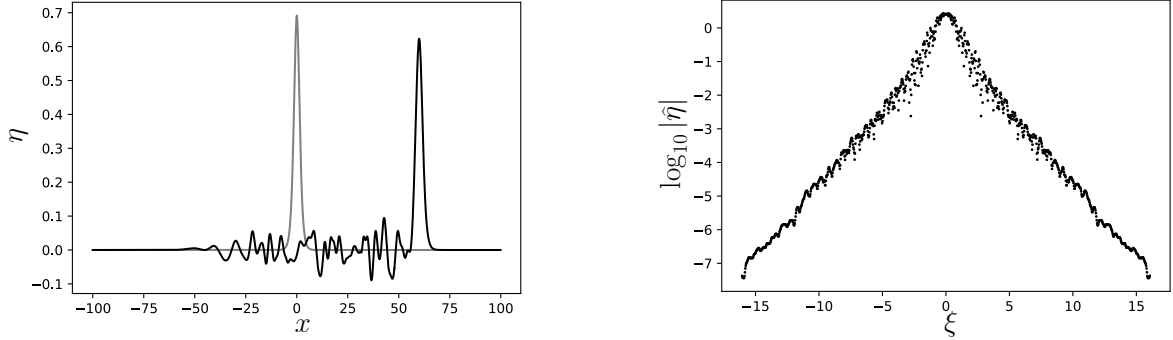


Figure 1: Surface wave evolution on the left with its Fourier transform on the right. The corresponding initial wave is marked by the gray color.

We take $g = h = 1$, set the noise level $\epsilon = 0.1$. The time step is $\Delta t = 0.0005$ that corresponds to $\sqrt{\Delta t} \approx 0.022$. The spatial discretisation is done by a Fourier series with $N = 1024$ modes. The corresponding grid of the computational domain $[-100, 100]$ is uniform. As an initial data $u(0)$ we take a solitary wave associated with the deterministic model; the corresponding algorithm can be found in [19]. In Figure 1 one can see how a solitary wave, initially localised around $x = 0$, evolves by the time moment $t = 50$. In order to assess precision of these calculations we evaluate the energy \mathcal{H} given by (5.4), where the spatial integral is calculated by the trapezoidal rule. Due to energy conservation one anticipates to get a horizontal straight curve. However, since each scheme produces a stochastic process that is not the exact solution, we can see noisy fluctuations of the total energy in Figure 2.

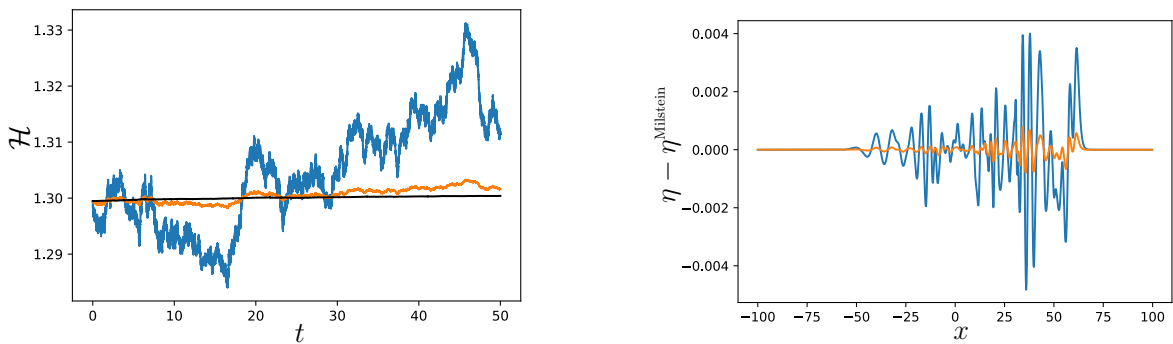


Figure 2: The energy on the left with the difference between different numerical schemes on the right. Blue corresponds to (6.1), orange to (6.2) and black to (6.3). The Milstein scheme (6.3) is taken as a reference on the second picture.

As can be observed from this result, the solitary waves propagate together with noisy wavy structures of much smaller height. On the left part of figure 2 the energy can be seen to be numerically well preserved for the Milstein discretisation of the Duhamel form in comparison to the two other schemes. In particular, the Euler scheme associated to the mild solution reveals the most unstable in terms of energy conservation. At each time moment the surface elevation stays smooth, which can be seen from its Fourier transform depicted on Figure 1 for the time moment $t = 50$. Different schemes give very close results. And so in Figure 1 we have

chosen to demonstrate the Milstein scheme solution that is presumably more precise, which is supported by Figure 2, where on the right the difference of solution of schemes (6.1) and (6.2) with (6.3) is shown at the same time moment $t = 50$. Figure 2 also demonstrates that Scheme (6.2) is more accurate than Scheme (6.1), which suggests that it is preferable to use the Duhamel integral form (A.5).

7 Hamiltonian water wave formulation for three dimensional flows

In this section we focus now on the extension of the development of Section 4 to a three dimensional fluid layer. We show how a similar strategy can be applied in the 3D case. It leads to the two dimensional surface wave problem

$$d \begin{pmatrix} \eta \\ u_1 \\ u_2 \end{pmatrix} = J \nabla \mathcal{H} dt + \sum_j J_j \nabla \mathcal{H} \circ dW_j,$$

where $\eta(x, y, t)$ is the surface elevation and $u_1 = \partial_x \Phi$, $u_2 = \partial_y \Phi$ are derivatives of the surface velocity potential $\Phi(x, y, t)$. The energy \mathcal{H} has exactly the same form as above, given by (4.1) with the Dirichlet-Neumann operator

$$G(\eta)\Phi = \partial_z \varphi - (\partial_x \varphi) \partial_x \eta - (\partial_y \varphi) \partial_y \eta \text{ at } z = \eta$$

associated now with the three dimensional elliptic problem. The latter is given in (3.2) now with the Laplacian $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$. The structure map $J = -J^*$ has the form

$$J = \begin{pmatrix} 0 & -\partial_x & -\partial_y \\ -\partial_x & 0 & 0 \\ -\partial_y & 0 & 0 \end{pmatrix}.$$

The anti-symmetric operators $J_j = (J_j^{ik})$, standing in the Stratonovich noise, are derived below from the small noise assumption and in a similar way as for the 1D waves. One can repeat the arguments of Sections 3, 4. However, to simplify the exposition we will conduct surface reduction and linearisation simultaneously, in order to avoid long expressions analogous to the ones detailed in Section 3.

The kinematical boundary condition gives us the first equation

$$d\eta = G\Phi dt + \sigma d\mathbf{B}_z - \sigma d\mathbf{B}_x \partial_x \eta - \sigma d\mathbf{B}_y \partial_y \eta + \left(\sum_{l \in \{x, y, z\}, m \in \{x, y\}} \partial_l a_{lm} \partial_m \eta + \frac{1}{2} \sum_{l, m \in \{x, y\}} a_{lm} \partial_l \partial_m \eta \right) dt,$$

where the divergence-free noise vector at the surface is modelled via the vector noise potentials $\psi^j(x, y, z)$ as

$$\sigma d\mathbf{B} = \sum_j \nabla \times \psi^j(x, y, \eta(x, y, t)) dW_j.$$

As above we use approximation

$$\nabla \times \psi^j(x, y, \eta(x, y, t)) = \nabla \times \psi^j(x, y, 0) + \nabla \times \partial_z \psi^j(x, y, 0) \eta(x, y, t).$$

Calculating $\nabla \mathcal{H}$ and linearising it, one arrives to

$$\frac{\delta \mathcal{H}}{\delta \eta} = g\eta, \quad \frac{\delta \mathcal{H}}{\delta u_1} = hK^2 u_1, \quad \frac{\delta \mathcal{H}}{\delta u_2} = hK^2 u_2,$$

where $K = \sqrt{\tanh(h|D|)/|h|D|}$ with $D = (-i\partial_x, -i\partial_y)$. Thus

$$\begin{aligned} gJ_j^{11}\eta + hJ_j^{12}K^2 u_1 + hJ_j^{13}K^2 u_2 &= \partial_x \psi_y^j(x, y, 0) - \partial_y \psi_x^j(x, y, 0) \\ &+ \partial_x (\partial_z \psi_y^j(x, y, 0)\eta) - \partial_y (\partial_z \psi_x^j(x, y, 0)\eta) - \partial_y \psi_z^j(x, y, 0) \partial_x \eta + \partial_x \psi_z^j(x, y, 0) \partial_y \eta. \end{aligned}$$

Clearly, $J_j^{12} = J_j^{13} = 0$ and so $J_j^{21} = J_j^{31} = 0$. This Hamiltonian structure implies that the noise is multiplicative which results in the expression

$$\partial_x \psi_y^j(x, y, 0) = \partial_y \psi_x^j(x, y, 0). \quad (7.1)$$

The first diagonal element is defined by the following expression

$$gJ_j^{11}\eta = \partial_x (\partial_z \psi_y^j(x, y, 0)\eta) - \partial_y (\partial_z \psi_x^j(x, y, 0)\eta) - \partial_y \psi_z^j(x, y, 0) \partial_x \eta + \partial_x \psi_z^j(x, y, 0) \partial_y \eta. \quad (7.2)$$

Note that anti-symmetry of J_j^{11} implies the following restrictions on the noise potential

$$\partial_x \partial_z \psi_y^j(x, y, 0) = \partial_y \partial_z \psi_x^j(x, y, 0). \quad (7.3)$$

Linearising derivatives as $\partial_z \varphi = G\Phi$, $\partial_x \varphi = \partial_x \Phi$, $\partial_y \varphi = \partial_y \Phi$, $\partial_x^2 \varphi = \partial_x^2 \Phi$, $\partial_y^2 \varphi = \partial_y^2 \Phi$, $\partial_x \partial_y \varphi = \partial_x \partial_y \Phi$ and continuing to neglect nonlinear terms one can calculate

$$d\partial_x \Phi = d\partial_x \varphi + d\partial_z \varphi \partial_x \eta + \partial_x d\langle \partial_z \varphi, \eta \rangle.$$

Hence one obtains

$$d\partial_x \Phi = -g\partial_x \eta dt - \sigma d\mathbf{B}_x \partial_x^2 \Phi - \sigma d\mathbf{B}_y \partial_x \partial_y \Phi + \text{noise diffusion}$$

and similarly

$$d\partial_y \Phi = -g\partial_y \eta dt - \sigma d\mathbf{B}_x \partial_x \partial_y \Phi - \sigma d\mathbf{B}_y \partial_y^2 \Phi + \text{noise diffusion}$$

that gives rise to

$$\begin{cases} hJ_j^{22} K^2 u_1 + hJ_j^{23} K^2 u_2 = \gamma_x^j \partial_x u_1 + \gamma_y^j \partial_x \partial_y \Phi, \\ hJ_j^{32} K^2 u_1 + hJ_j^{33} K^2 u_2 = \gamma_x^j \partial_x \partial_y \Phi + \gamma_y^j \partial_x u_2, \end{cases}$$

where we have introduced the following functions

$$\begin{cases} \gamma_x^j(x, y) = \partial_z \psi_y^j(x, y, 0) - \partial_y \psi_z^j(x, y, 0), \\ \gamma_y^j(x, y) = \partial_x \psi_z^j(x, y, 0) - \partial_z \psi_x^j(x, y, 0). \end{cases}$$

Now presenting $\partial_x \partial_y \Phi$ as $\partial_y u_1$ in the first equation and as $\partial_x u_2$ in the second one, we obtain that $J_j^{23} = J_j^{32} = 0$ and

$$hJ_j^{22} K^2 = hJ_j^{33} K^2 = \gamma_x^j \partial_x + \gamma_y^j \partial_y.$$

Due to the noise restriction given in (7.3) the operator $\gamma_x^j \partial_x + \gamma_y^j \partial_y$ turns out to be anti-symmetric as one can easily check. This, however, leads to the fact that operators J_j^{22} , J_j^{33} can be anti-symmetric if and only if the differential operator K^2 commutes with functions γ_x^j , γ_y^j . Hence γ_x^j , γ_y^j are constants. Finally, we conclude that all operators J_j are diagonal with

$$J_j^{11} = g^{-1} \gamma^j \cdot \nabla, \quad J_j^{22} = J_j^{33} = h^{-1} K^{-2} \gamma^j \cdot \nabla.$$

8 Conclusion

In this study, we explored stochastic representations of classical wave formulations within the setting of the modelling under location uncertainty. These models are derived in a way that remains close to the deterministic derivation. In particular, we paid attention to stochastic formulations preserving the Hamiltonian structure of the deterministic models. This strong constraint leads to consider only homogeneous noise, which do not depend on space. As a matter of fact, as one can notice it turns out that all the antisymmetric operators J_i appearing in the Hamiltonian formulation (4.2) differ from each other by scalars γ_j . One of the possible extensions is to consider instead of scalars γ_j Fourier multipliers with symbols emphasising particular frequencies, in the form of characteristic or δ -function. In other words, this consists in considering an homogeneous random field defined from a finite linear combination of Fourier harmonics with particular wave numbers. In future works we would like to study further wave solutions of the shallow water system in order to revisit classical theories of geostrophic adjustment as well as interactions between surface waves and wind forcing. The purpose pursued would be to reinterpret classical models enriched with a noise component. Another future work of interest will concern the development of stochastic representations of nonlinear Shallow Water theories for coastal waves[1].

A Stochastic equations

A.1 Change of variables

A stochastic Hamiltonian evolutionary system in variable X is a system of partial differential equations of the form

$$dX = J^X \nabla_X \mathcal{H} dt + \sum_j J_j^X \nabla_X \mathcal{H} \circ dW_j,$$

where structure maps J^X, J_j^X do not depend on X . They may be different from each other and may have different views in different variables X . Here we investigate how J^X, J_j^X change under a transformation $Y = Y(X)$. Firstly, one can notice that

$$\nabla_X \mathcal{H} = \frac{\partial Y^*}{\partial X} \nabla_Y \mathcal{H}.$$

Indeed, for any test function ψ the inner product

$$(\nabla_X \mathcal{H}, \psi) = d\mathcal{H}(X)(\psi) = d\mathcal{H}(Y(X)) \frac{\partial Y}{\partial X}(\psi) = \left(\nabla_Y \mathcal{H}, \frac{\partial Y}{\partial X} \psi \right) = \left(\frac{\partial Y^*}{\partial X} \nabla_Y \mathcal{H}, \psi \right).$$

Differentiating the transformation $Y(X)$ one obtains

$$dY = \frac{\partial Y}{\partial X} \circ dX = \frac{\partial Y}{\partial X} J^X \nabla_X \mathcal{H} dt + \frac{\partial Y}{\partial X} \sum_j J_j^X \nabla_X \mathcal{H} \circ dW_j = J^Y \nabla_Y \mathcal{H} dt + \sum_j J_j^Y \nabla_Y \mathcal{H} \circ dW_j,$$

where the new maps have the forms

$$J^Y = \frac{\partial Y}{\partial X} J^X \frac{\partial Y^*}{\partial X}, \quad J_j^Y = \frac{\partial Y}{\partial X} J_j^X \frac{\partial Y^*}{\partial X}.$$

A.2 Duhamel formula

Here we provide with two alternative integral formulations of stochastic partial differential equations, taking advantage that the models under consideration admit exact solutions after linearization. This eventually allows us to use exponential integrators for numerical simulations [29, 36], which reveals usually to be much more stable than other explicit schemes. The models under consideration can be generally written down in the following Stratonovich form

$$du = (Au + f(u))dt + \sum_j (B_j u + g_j(u)) \circ dW_j, \quad (\text{A.1})$$

where A and all B_j are linear operators, whereas f and all g_j are nonlinear. The stochastic process u can stand here for any quantity of the fluid flow, depending on the concrete model. For any continuous martingale the Stratonovich integral

$$dX = h \circ dW,$$

is well defined [33] and can be turned into the corresponding Itô integral as

$$h \circ dW = hdW + \frac{1}{2}d\langle h, W \rangle.$$

In our case, after using the bilinearity property of the quadratic variation, this leads to

$$\begin{aligned} (B_j u + g_j(u)) \circ dW_j &= (B_j u + g_j(u))dW_j + \frac{1}{2}\langle d(B_j u + g_j(u)), dW_j \rangle = \dots \\ &+ \frac{1}{2} (B_j + g'_j(u)) \langle du, dW_j \rangle = (B_j u + g_j(u))dW_j + \frac{1}{2} (B_j + g'_j(u)) (B_j u + g_j(u))dt. \end{aligned}$$

Thus the final Itô form reads

$$du = \left(\tilde{A}u + F(u) \right) dt + \sum_j (B_j u + g_j(u))dW_j, \quad (\text{A.2})$$

where

$$\tilde{A} = A + \frac{1}{2} \sum_j B_j^2$$

and

$$F(u) = f(u) + \frac{1}{2} \sum_j (B_j g_j(u) + g'_j(u)B_j u + g'_j(u)g_j(u)).$$

In the mild integral form this equation reads

$$u(t) = e^{\tilde{A}(t-t_0)} u(t_0) + \int_{t_0}^t e^{\tilde{A}(t-s)} F(u(s)) ds + \sum_j \int_{t_0}^t e^{\tilde{A}(t-s)} (B_j u(s) + g_j(u(s))) dW_j(s), \quad (\text{A.3})$$

where $0 \leq t_0 \leq t$. So far we did not use any specific information about the linear operators A and B_j .

Now let us suppose that all operators A , B_j commute between each other. Then the corresponding linear stochastic equation

$$dv = \tilde{A}vdt + \sum_j B_j v dW_j$$

can be solved analytically as

$$v(t) = \mathcal{S}(t, t_0)v(t_0),$$

where the fundamental matrix

$$\mathcal{S}(t, t_0) = \exp \left[A(t - t_0) + \sum_j B_j (W_j(t) - W_j(t_0)) \right]. \quad (\text{A.4})$$

Indeed, introducing the notation Y for $\mathcal{S} = \exp Y$, we deduce that

$$d\mathcal{S} = \exp Y dY + \frac{1}{2} \exp Y \langle dY, dY \rangle = \mathcal{S} \left(A dt + \sum_j B_j dW_j + \frac{1}{2} \sum_j B_j^2 dt \right).$$

Clearly, there exists the inverse stochastic matrix operator $\mathcal{S}^{-1}(t, t_0) = \mathcal{S}(t_0, t)$. Now, let us show that the nonlinear equation (A.2) can be written in the following Duhamel form

$$u(t) = \mathcal{S}(t, t_0) \left(u(t_0) + \int_{t_0}^t \mathcal{S}^{-1}(s, t_0) \tilde{f}(u(s)) ds + \sum_j \int_{t_0}^t \mathcal{S}^{-1}(s, t_0) g_j(u(s)) dW_j(s) \right), \quad (\text{A.5})$$

where

$$\tilde{f}(u) = F(u) - \sum_j B_j g_j(u) = f(u) + \frac{1}{2} \sum_j (g_j'(u) B_j u + g_j'(u) g_j(u) - B_j g_j(u)).$$

Indeed, differentiating the right part of the Duhamel representation we obtain

$$d(\mathcal{S}(t, t_0)u(t_0)) = \tilde{A}\mathcal{S}(t, t_0)u(t_0)dt + \sum_j B_j \mathcal{S}(t, t_0)u(t_0)dW_j,$$

$$d \left(\mathcal{S}(t, t_0) \int_{t_0}^t \mathcal{S}^{-1}(s, t_0) \tilde{f}(u(s)) ds \right) = \left(\tilde{A}dt + \sum_j B_j dW_j \right) \mathcal{S}(t, t_0) \int_{t_0}^t \mathcal{S}^{-1}(s, t_0) \tilde{f}(u(s)) ds + \tilde{f}(u(t))dt,$$

$$\begin{aligned} & d \left(\mathcal{S}(t, t_0) \sum_j \int_{t_0}^t \mathcal{S}^{-1}(s, t_0) g_j(u(s)) dW_j(s) \right) \\ &= \left(\tilde{A}dt + \sum_k B_k dW_k \right) \mathcal{S}(t, t_0) \sum_j \int_{t_0}^t \mathcal{S}^{-1}(s, t_0) g_j(u(s)) dW_j(s) + \sum_j g_j(u(s)) dW_j(s) \\ &\quad + \left\langle \left(\tilde{A}dt + \sum_k B_k dW_k \right) \mathcal{S}(t, t_0), \sum_j \mathcal{S}^{-1}(t, t_0) g_j(u(t)) dW_j(t) \right\rangle, \end{aligned}$$

where the last cross variation reduces to the sum $\sum_j B_j g_j(u(t))dt$ that is exactly the difference between $F(u)dt$ and $\tilde{f}(u)dt$. Thus summing up we arrive to the original Itô differential equation (A.2).

B Dirichlet-Neumann operator

B.1 Properties of Dirichlet-Neumann operator

The operator $G(\eta)$ is non-negative and self-adjoint on a dense subspace of $L^2(\mathbb{R})$. There is an operator $\mathcal{X}(\eta)$ such that

$$G(\eta) = D\mathcal{X}(\eta)D,$$

where D is the Fourier multiplier $D = -i\partial_x$. The operator $\mathcal{K}(\eta)$ is non-negative and self-adjoint on a dense subspace of $L^2(\mathbb{R})$. More precisely,

$$\left\| (1 + |D|)^{-1/2} u \right\|_{L^2}^2 \lesssim \int_{\mathbb{R}} u \mathcal{K}(\eta) u dx \lesssim \left\| (1 + |D|)^{-1/2} u \right\|_{L^2}^2$$

for any $u \in H^{-1/2}(\mathbb{R})$ [34, Proposition 3.12]. The Dirichlet-Neumann operator is analytic [34, Theorem 3.21] and its first-order derivative has the form

$$dG(\eta)(\theta)\psi = -G(\eta) \left(\theta \frac{G(\eta)\psi + \partial_x \eta \partial_x \psi}{1 + (\partial_x \eta)^2} \right) - \partial_x \left(\theta \left(\partial_x \psi - \frac{G(\eta)\psi + \partial_x \eta \partial_x \psi}{1 + (\partial_x \eta)^2} \partial_x \eta \right) \right). \quad (\text{B.1})$$

Let us calculate $\nabla \mathcal{H}$ in variables η, Φ , for example, using this formula. The Gâteaux derivative $\delta \mathcal{H} / \delta \eta$ is defined by means of an arbitrary smooth real-valued square integrable function θ from the variation

$$\int \frac{\delta \mathcal{H}}{\delta \eta} \theta dx = \left. \frac{d}{d\tau} \mathcal{H}(\eta + \tau\theta, \Phi) \right|_{\tau=0} = \int \left(g\eta\theta + \frac{1}{2} \Phi dG(\eta)(\theta)\Phi \right) dx.$$

Substituting (B.1) and using symmetric properties of $G(\eta)$ together with integration by parts one deduces

$$\int \frac{\delta \mathcal{H}}{\delta \eta} \theta dx = \int \left(g\eta + \frac{1}{2} (\partial_x \Phi)^2 - \frac{(G(\eta)\Phi + \partial_x \eta \partial_x \Phi)^2}{2(1 + (\partial_x \eta)^2)} \right) \theta dx,$$

which defines $\delta \mathcal{H} / \delta \eta$ since θ is arbitrary. Calculation of $\delta \mathcal{H} / \delta \Phi$ is trivial. Clearly, $\mathcal{K}(\eta)$ is differentiable and

$$d\mathcal{K}(\eta)(\theta)\psi = D^{-1} dG(\eta)(\theta) D^{-1} \psi.$$

$$\nabla \mathcal{H}'(\eta, u) = \begin{pmatrix} d_\eta \frac{\delta \mathcal{H}}{\delta \eta}(\eta, u) & d_u \frac{\delta \mathcal{H}}{\delta \eta}(\eta, u) \\ d_\eta \frac{\delta \mathcal{H}}{\delta u}(\eta, u) & d_u \frac{\delta \mathcal{H}}{\delta u}(\eta, u) \end{pmatrix} \quad (\text{B.2})$$

where

$$d_\eta \frac{\delta \mathcal{H}}{\delta \eta}(\eta, u)(\theta) = g\theta - \frac{u \partial_x \eta - \partial_x(\mathcal{K}(\eta)u)}{(1 + (\partial_x \eta)^2)^2} (u \partial_x \theta + \partial_x(\mathcal{K}(\eta)u) \partial_x \eta \partial_x \theta - (1 + (\partial_x \eta)^2) \partial_x(d\mathcal{K}(\eta)(\theta)u))$$

$$d_u \frac{\delta \mathcal{H}}{\delta \eta}(\eta, u)(w) = uw - \frac{u \partial_x \eta - \partial_x(\mathcal{K}(\eta)u)}{1 + (\partial_x \eta)^2} (w \partial_x \eta - \partial_x(\mathcal{K}(\eta)w))$$

$$d_\eta \frac{\delta \mathcal{H}}{\delta u}(\eta, u)(\theta) = d\mathcal{K}(\eta)(\theta)u$$

$$d_u \frac{\delta \mathcal{H}}{\delta u}(\eta, u)(w) = \mathcal{K}(\eta)w$$

B.2 Approximation of Dirichlet-Neumann operator

It is well known that $G(\eta) = \sum G_j(\eta)$, where each $G_j(\eta)$ is homogeneous of order $j = 0, 1, \dots$ with respect to η . Let us find the first two approximations: G_0 independent of η and G_1 linear in η . Operator G is defined by the elliptic problem imposed on φ that in Fourier space has the form

$$\begin{cases} \partial_z^2 \widehat{\varphi} - \xi^2 \widehat{\varphi} = 0, \\ \partial_z \widehat{\varphi}(\xi, -h) = 0, \end{cases}$$

where without loss of generality we can omit the dependence on time. Hence

$$\widehat{\varphi}(\xi, z) = A(\xi) \cosh \xi(z + h),$$

and so

$$\varphi(x, z) = \frac{1}{2\pi} \int A(\xi) \cosh \xi(z + h) e^{i\xi x} d\xi, \quad (\text{B.3})$$

where A is defined from the boundary condition $\Phi(x) = \varphi(x, \eta(x))$, i. e. from the integral equation

$$\Phi(x) = \frac{1}{2\pi} \int A(\xi) \cosh \xi(\eta(x) + h) e^{i\xi x} d\xi. \quad (\text{B.4})$$

Now we can calculate $G_0 = G(0)$. Indeed, for any given function Φ we can find the potential φ having trace Φ at the flat surface $\eta \equiv 0$ exploiting (B.3), (B.4). From (B.4) one deduces that $A(\xi) = \widehat{\Phi}(\xi)/\cosh(h\xi)$ for the flat surface, and so applying (3.3), (B.3) one obtains

$$G_0\Phi(x) = \frac{1}{2\pi} \int A(\xi)\xi \sinh(h\xi)e^{i\xi x} d\xi = \frac{1}{2\pi} \int \widehat{\Phi}(\xi)\xi \tanh(h\xi)e^{i\xi x} d\xi = D \tanh(hD)\Phi(x),$$

with $D = -i\partial_x$. In order to find $G_1(\eta)$ we expand hyperbolic functions in (3.3) with φ defined by (B.3) as follows

$$G(\eta)\Phi(x) = \frac{1}{2\pi} \int A(\xi)\xi \sinh(h\xi)e^{i\xi x} d\xi + \frac{1}{2\pi} \int A(\xi)\xi^2 \cosh(h\xi)e^{i\xi x} d\xi \eta(x) - \frac{1}{2\pi} \int A(\xi)i\xi \cosh(h\xi)e^{i\xi x} d\xi \partial_x \eta(x).$$

Now expanding the hyperbolic cosine in (B.4) we evaluate

$$\begin{aligned} G_0\Phi(x) &= G_0 \frac{1}{2\pi} \int A(\xi)(\cosh(h\xi) + \xi \sinh(h\xi)\eta(x))e^{i\xi x} d\xi \\ &= \frac{1}{2\pi} \int A(\xi)\xi \sinh(h\xi)e^{i\xi x} d\xi + G_0 \left(\eta(x) \frac{1}{2\pi} \int A(\xi)\xi \sinh(h\xi)e^{i\xi x} d\xi \right). \end{aligned}$$

Thus continuing neglecting quadratic terms in η we obtain

$$\begin{aligned} G_1(\eta)\Phi(x) &= G(\eta)\Phi(x) - G_0\Phi(x) = \frac{1}{2\pi} \int A(\xi)\xi^2 \cosh(h\xi)e^{i\xi x} d\xi \eta(x) \\ &\quad - \frac{1}{2\pi} \int A(\xi)i\xi \cosh(h\xi)e^{i\xi x} d\xi \partial_x \eta(x) - G_0 \left(\eta(x) \frac{1}{2\pi} \int A(\xi)\xi \sinh(h\xi)e^{i\xi x} d\xi \right) \end{aligned}$$

that is linear in η provided $A(\xi) = \widehat{\Phi}(\xi)/\cosh(h\xi)$, which leads finally to

$$G_1(\eta)\Phi = -\eta\partial_x^2\Phi - (\partial_x\eta)\partial_x\Phi - G_0(\eta G_0\Phi) = -\partial_x(\eta\partial_x\Phi) - G_0(\eta G_0\Phi).$$

A recursion formula for the next approximations $G_j(\eta)$ can be found in [13].

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