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## WAVENUMBER-EXPLICIT STABILITY AND CONVERGENCE ANALYSIS OF $hp$ FINITE ELEMENT DISCRETIZATIONS OF HELMHOLTZ PROBLEMS IN PIECEWISE SMOOTH MEDIA

M. BERNKOPF, T. CHAUMONT-FRELET, AND J.M. MELENK

ABSTRACT. We present a wavenumber-explicit convergence analysis of the  $hp$  Finite Element Method applied to a class of heterogeneous Helmholtz problems with piecewise analytic coefficients at large wavenumber  $k$ . Our analysis covers the heterogeneous Helmholtz equation with Robin, exact Dirichlet-to-Neumann, and second order absorbing boundary conditions, as well as perfectly matched layers.

### 1. INTRODUCTION

Time-harmonic wave propagation problems play a major role in a wide range of physical and industrial applications. For the numerical treatment of such problems in heterogeneous media, Galerkin discretization methods of finite element type constitute one of the most efficient approaches currently available. Still, in the high-wavenumber regime where the computational domain spans many wavelengths, the problem becomes highly indefinite, which leads to substantially increased computational costs due to increased dispersion errors. As a result, significant research efforts have focused in the past decades on understanding these dispersion errors and the question of how to design finite element methods (FEM) able to cope with them.

In the present work, as in [Ber21], we focus on the scalar Helmholtz equation, which is arguably the simplest, yet interesting, PDE model to analyze phenomena appearing in the high-wavenumber regime. Specifically, given a domain  $\Omega$  with analytic boundary  $\Gamma := \partial\Omega$  and  $f : \Omega \rightarrow \mathbb{C}$  our model problem is to find  $u : \Omega \rightarrow \mathbb{C}$  such that

$$\begin{cases} -k^2\mu u - \nabla \cdot (A\nabla u) & = f & \text{in } \Omega, \\ A\nabla u \cdot n - T_k u & = 0 & \text{on } \Gamma, \end{cases}$$

where  $\mu, A$  are given piecewise analytic coefficients (with analytic interfaces where  $\mu, A$  may jump), and  $T_k$  is an operator acting on functions on the boundary. For instance,  $T_k$  could be the Dirichlet-to-Neumann (DtN) map of the exterior Helmholtz problem in  $\mathbb{R}^d \setminus \bar{\Omega}$  or  $T_k = ik$  in the simpler case of impedance boundary conditions. The “homogeneous” Helmholtz equation where  $\mu = 1$  and  $A = I$

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is already largely covered in the literature, and the present work focuses on the “heterogeneous” case where these coefficients are allowed to vary on  $\Omega$ .

A first milestone towards a better understanding of the behavior of finite element methods was achieved in the 90s in the seminal works [IB95, IB97]. There, the authors focus on homogeneous one-dimensional media, and present error estimates that are explicit in terms of the wavenumber  $k$ , the mesh size  $h$ , and the polynomial degree  $p$ . They also coined the term “pollution effect” for the dispersion errors observed in the numerics, which expresses the observation that in the high-wavenumber regime, quasi-optimality of the finite element solution is lost, unless the number of degrees of freedom per wavelength is increased as  $k$  is increased. Similar estimates were obtained for two and three-dimensional homogeneous media in the same period, but limited to the lowest-order case where  $p = 1$  [Mel95].

The pollution effect has been further analyzed in the 2000s, in particular using dispersion analysis [Ain04]. While dispersion analysis is extremely insightful from a practical viewpoint, it is limited to homogeneous media discretized with translation-invariant grids, and right-hand sides as well as boundary conditions are not taken into account.

The direct extension of the first-order result of [Mel95] to higher-order elements is non-trivial. Indeed, the error analysis follows a duality technique commonly known as “Schatz argument” [Sch74], where the finite element error is abstractly used as a right-hand side of an adjoint problem. One is then led to study the regularity of the solution to the adjoint problem with an  $L^2$  right-hand side, even if the actual right-hand side to the physical problem is more regular. This is an issue when analyzing high-order methods, since their high convergence rate do not apply to solutions with low regularity. The situation was unlocked in the 2010s thanks to the key concept of regularity splitting [MS10, MS11]. The idea is to identify two distinct contributions in the solution, namely, to write  $u = u_F + u_A$ , where  $u_F$  only possesses finite regularity, but behaves well in the high-wavenumber regime, whereas  $u_A$  is oscillatory but analytic.

The splitting proposed in [MS10, MS11] is based on low- and high-pass filters defined via the Fourier transformation. For homogeneous media with analytic boundaries, it permits to establish the quasi-optimality of the finite element solution under the assumption that

$$(1.1) \quad \frac{kh}{p} + k^\theta \left( \frac{kh}{\sigma p} \right)^p \leq c,$$

for some problem-dependent  $\theta$  and  $\sigma$  and a generic constant  $c$  that does not depend on  $k$ ,  $h$ , or  $p$ . We can also rewrite (1.1) as

$$(1.2) \quad \frac{kh}{p} \leq c_1 \quad \text{and} \quad p \geq c_2 \log k,$$

where  $c_1$  and  $c_2$  are again independent of  $k$ ,  $h$ , and  $p$ . Informally, (1.2) suggests that the FEM is stable if the number of degrees of freedom per wavelength is kept constant ( $kh/p \leq c_1$ ) and the polynomial degree is mildly increased in the high-wavenumber regime ( $p \geq c_2 \log k$ ). It also means that  $hp$ -FEMs are pollution free, provided the polynomial degree  $p$  is increased logarithmically with the wavenumber. Together with the dispersion analysis [Ain04], the stability condition (1.2) strongly encourages the use of high-order methods to solve high-wavenumber problems to reduce the pollution effect. In fact, the performance of high-order methods is also

clearly observed numerically, and therefore, they are now widely used in industry [BPG16, TZNHD17].

The idea of regularity splitting is actually not limited to finite element methods, and has been used repeatedly in the literature. Important examples include the sharp treatment of corner singularities [CFN18] and unfitted meshes [CF16], as well as other discretization techniques such as discontinuous Galerkin [MPS13], continuous interior penalty [DW15, ZW13] and multiscale [CFV20] methods. Besides, the technique is not only useful to derive *a priori* error estimates, but is also important in the context of *a posteriori* error estimation [CFEV21, DS13]. We finally mention that the idea can also be extended to more complicated wave propagation problems [MS21, CFV22].

Regularity splittings are thus an important ingredient of wavenumber-explicit analysis of FEMs. Yet, the technique proposed in [MS10, MS11] heavily relies on the Green’s functions of the Helmholtz operator, and, as a result, it appears to be limited to homogeneous media. So far, two alternative procedures to build a regularity splitting in heterogeneous media have been proposed. On the one hand, an iterative argument based on elliptic regularity is proposed in [CFN19]. This technique can handle general wave operators in piecewise smooth media, but unfortunately, it only provides (1.1) with a constant  $c$  depending on  $p$ , which prevents the analysis of  $hp$ -FEMs, and does not lead to (1.2). On the other hand, novel ideas based on semi-classical analysis have been recently introduced in [LSW22, GLSW23, GLSW22]. While this last splitting provides (1.1) and (1.2) with  $p$ -robust constants, it requires *globally* smooth parameters. In particular, the important case of piecewise constant parameters, modelling a medium consisting of different materials, is not covered. An advantage of [LSW22, GLSW23, GLSW22] over the present splitting, however, is that only finite (although global) regularity of the coefficients is required, as compared to (only piecewise) analyticity here. The two approaches are thus complementary.

It should be stressed that both approaches, like in fact all  $k$ -explicit analyses of numerical methods for Helmholtz problems, rely on  $k$ -explicit bounds  $C_{\text{sol},k}^-$  for the solution operator of the continuous problem. If  $C_{\text{sol},k}^-$  grows only polynomially in  $k$ , then exponential approximability of high order methods can be exploited, as is done in both approaches. We refer to a more detailed discussion of  $C_{\text{sol},k}^-$  in Remark 5.3. In this work, we modify the initial Fourier filtering approach of [MS10, MS11] and adapt it to be able to develop a regularity splitting of solutions of wave propagation problems in piecewise smooth media and subsequently apply a duality argument (“Schatz argument”) to Galerkin discretizations. We cover several classes of scalar Helmholtz problems in heterogeneous media with different types of boundary conditions, providing quasi-optimality under the resolution conditions (1.1) and (1.2) with  $p$ -robust constants. The approach taken in the present work is quite general and thus not limited to the scalar Helmholtz problems under consideration.

The remainder of this work is organized as follows. Section 2 presents the settings and key notations for our work. We devise our regularity splitting in Section 3 and subsequently apply it to Galerkin discretizations in Section 4. The analysis of Sections 3 and 4 is performed in an abstract setting under general assumptions. In Section 5, we verify that these abstract assumptions indeed hold true for several classes of relevant scalar Helmholtz problems, namely, the heterogeneous

Helmholtz equation with a) the classical impedance boundary conditions and perfectly matched layers on a circular/spherical domain; b) exact boundary conditions expressed in terms of the Dirichlet-to-Neumann operator; c) second order absorbing boundary conditions. We also refer to [Ber21] where in addition to the present examples the case of the elasticity equation on circular domains with exact Dirichlet-to-Neumann boundary conditions is analyzed with the present techniques. Appendix A presents technical results about Dirichlet-to-Neumann operators which are of independent interest.

## 2. ASSUMPTIONS AND PROBLEM SPECIFIC NOTATION

**2.1. Wavenumber.** We use the letter  $k$  for the wavenumber. Since we are especially interested in the high-wavenumber regime, we assume for the sake of simplicity that  $k$  is bounded away from zero, i.e.,  $k \geq k_0 > 0$  for some  $k_0 > 0$  that is fixed throughout the present work.

**2.2. Generic constants.** Throughout this work,  $C$  will denote a generic constant that is independent of the wavenumber  $k$  and other critical parameters such as the mesh size  $h$  and the polynomial degree  $p$ . The constant may, however, depend on the operators appearing in assumptions **(WP)** and **(AP)** introduced in Sections 2.6 and 2.7 below. It may also depend on the constant  $C_{\text{affine}}$  and  $C_{\text{metric}}$  introduced in Assumption 4.4 that describes the “shape regularity” of the elements of the mesh. If  $a, b, \dots$  are other quantities appearing in the analysis, we employ the notation  $C_{a,b,\dots}$  for a constant that is additionally allowed to depend on  $a, b, \dots$ . Finally, we will employ explicit names together with a subscript  $k$  for constants allowed to depend on the wavenumber.

**2.3. Geometry.** We consider a Lipschitz domain  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{2, 3\}$ , with boundary  $\Gamma := \partial\Omega$ . We set  $\Omega^+ := \mathbb{R}^d \setminus \bar{\Omega}$ . We introduce the interior trace  $\gamma_0^{\text{int}} u := (u|_{\Omega})|_{\Gamma}$  and the exterior trace  $\gamma_0^{\text{ext}} u := (u|_{\Omega^+})|_{\Gamma}$ . We assume that  $\Omega$  is partitioned into a set  $\mathcal{P}$  of non-overlapping Lipschitz subdomains such that  $\cup_{P \in \mathcal{P}} \bar{P} = \bar{\Omega}$ . The internal interface  $(\cup_{P \in \mathcal{P}} \partial P) \setminus \Gamma$  will be denoted  $\Gamma_{\text{interf}}$ . While the abstract results in Sections 3 and 4 do not require much regularity of  $\Gamma$  and  $\Gamma_{\text{interf}}$ , we flag at this point already that in the applications discussed in Section 5 the assumptions on the geometry are such that  $\Gamma$  and the pieces  $\partial P$ ,  $P \in \mathcal{P}$ , of  $\Gamma_{\text{interf}}$  are analytic (Assumption 5.2). Notice that this assumption prevents the presence of corners and edges, and therefore intersection points in particular.

**2.4. Function spaces.** The notations  $L^2(\Omega)$  and  $L^2(\Gamma)$  stand for the usual Lebesgue spaces of square-integrable complex-valued functions defined on  $\Omega$  and  $\Gamma$ , respectively. The notations  $\|\cdot\|_{0,\Omega}$  and  $\|\cdot\|_{0,\Gamma}$  stand for the usual norms of these spaces. We also write  $(\cdot, \cdot)$  and  $\langle \cdot, \cdot \rangle$  for the inner product of  $L^2(\Omega)$  and  $L^2(\Gamma)$  and (with a standard abuse of notation) also for the usual (anti-)duality pairings. For  $s, t \geq 0$ ,  $H^s(\Omega)$  and  $H^t(\Gamma)$  are (possibly) fractional Sobolev spaces of functions defined on  $\Omega$  and  $\Gamma$ .  $\tilde{H}^{-s}(\Omega)$  and  $H^{-t}(\Gamma)$  are then the (topological) antidual of  $H^s(\Omega)$  and  $H^t(\Gamma)$ . For all  $s$ , we denote by  $\|\cdot\|_{s,\Omega}$  the norm of  $H^s(\Omega)$ , and if  $s > 0$ , by  $|\cdot|_{s,\Omega}$  its semi-norm. We also use similar notations for  $H^s(\Gamma)$ . For  $0 \leq t \leq 1$ , we introduce the “energy” space  $H^{1,t}(\Omega, \Gamma) := \{u \in H^1(\Omega) : u \in H^t(\Gamma)\}$  with  $k$ -dependent norm

$$(2.1) \quad \|u\|_{1,t,k}^2 := k^2 \|u\|_{0,\Omega}^2 + k^{1-2t} |u|_{t,\Gamma}^2 + |u|_{1,\Omega}^2.$$

We introduce this space with  $t = 1$  to handle second-order absorbing boundary conditions that involve surface differential operators. One easily sees that for  $t \leq 1/2$  the space  $H^{1,t}(\Omega, \Gamma)$  coincides with  $H^1(\Omega)$ , but we still employ the notation  $H^{1,t}(\Omega, \Gamma)$  to unify our presentation.

For  $r \geq 0$ , we will also use the broken Sobolev spaces

$$H^r(\mathcal{P}) := \{v \in L^2(\Omega) \mid v|_P \in H^r(P) \forall P \in \mathcal{P}\}$$

with their norm and and semi-norm given by

$$\|v\|_{s,\mathcal{P}}^2 := \sum_{P \in \mathcal{P}} \|v\|_{s,P}^2, \quad |v|_{s,\mathcal{P}}^2 := \sum_{P \in \mathcal{P}} |v|_{s,P}^2.$$

In addition to Sobolev spaces, we will employ spaces of (piecewise) analytic functions. For open sets  $\omega \subset \mathbb{R}^d$  and  $M, \gamma \geq 0$ , we set

$$(2.2) \quad \mathfrak{A}(M, \gamma, \omega) := \{v \in L^2(\omega) \mid \|\nabla^n v\|_{0,\omega} \leq M\gamma^n \max\{n, k\}^n \quad \forall n \in \mathbb{N}_0\},$$

where  $\mathbb{N}_0 := \{1, 2, \dots\}$  and  $|\nabla^n v(x)|^2 = \sum_{|\alpha|=n} \frac{n!}{\alpha!} |D^\alpha v(x)|^2$ . The analyticity class

$$(2.3) \quad \mathfrak{A}(M, \gamma, \mathcal{P}) := \{v \in L^2(\Omega) \mid v|_P \in \mathfrak{A}(M, \gamma, P) \quad \forall P \in \mathcal{P}\}$$

is a class of piecewise analytic functions on  $\Omega$ . We will also employ analytic functions defined on the boundary  $\Gamma$ . For the sake of simplicity, analyticity classes on  $\Gamma$  are defined as traces of analytic functions. That is, for an open neighborhood  $T$  of  $\Gamma$  and  $\gamma > 0$ , we set

$$\mathfrak{A}(M, \gamma, T, \Gamma) := \{\gamma_0^{int} V \mid V \in \mathfrak{A}(M, \gamma, T \cap \Omega)\}.$$

By a slight abuse of notation, for a function  $g \in L^2(\Gamma)$  we will write  $g \in \mathfrak{A}(M, \gamma, T, \Gamma)$  to indicate the existence of a function  $G \in \mathfrak{A}(M, \gamma, T \cap \Omega, \Gamma)$  with  $\gamma_0^{int} G = g$ .

**Remark 2.1.** (i) Functions  $G \in \mathfrak{A}(M, \gamma, T \cap \Omega)$  are analytic on  $\overline{T \cap \Omega}$  and therefore have a trace on  $\Gamma$ . Indeed, for  $T \cap \Omega$  Lipschitz we have by the Sobolev embedding theorem for  $t \in \mathbb{N}$  with  $t > d/2$  that  $H^t(T \cap \Omega) \subset C(\overline{T \cap \Omega})$ . Hence,  $\|\nabla^n G\|_{L^\infty(T \cap \Omega)} \leq C\gamma^{n+t} \max\{n+2, k\}^{n+t}$  for all  $n \in \mathbb{N}_0$  for suitable  $C$ . In turn, this implies that the Taylor series of  $G$  about each point  $x_0 \in T \cap \Omega$  converges with radius of convergence  $1/(e\gamma)$ .

(ii) The choice of interior traces in  $\mathfrak{A}(M, \gamma, T, \Gamma)$  is arbitrary. One could equivalently define them as traces of functions analytic in subsets of  $\Omega^+$ : It is shown in [Mel12, Lemma B.5] that if  $G \in \mathfrak{A}(M, \gamma, T \cap \Omega)$ , then there exist a neighborhood  $T'$  of  $\Gamma$  (depending only on  $\gamma$ ), constants  $M', \gamma'$ , and a function  $G' \in \mathfrak{A}(M', \gamma', T' \cap \Omega^+)$  with  $\gamma_0^{int} G = \gamma_0^{ext} G'$ .

(iii) Let  $T$  be an open neighborhood of  $\Gamma$  and  $G \in \mathfrak{A}(M, \gamma, T \setminus \Gamma)$ . By (i), the traces  $\gamma_0^{int} G$  and  $\gamma_0^{ext} G$  exist so that the jump  $\llbracket G \rrbracket := \gamma_0^{ext} G - \gamma_0^{int} G$  exists. In view of the arguments given in (ii), we have  $\llbracket G \rrbracket \in \mathfrak{A}(M', \gamma', T', \Gamma)$  for some  $\gamma', T'$  depending only on  $\gamma$  and  $T$  and some  $M'$  depending additionally on  $M$ .  $\blacksquare$

We note that if  $\gamma \leq \gamma'$  and  $M \leq M'$ , then

$$\mathfrak{A}(M, \gamma, \mathcal{P}) \subset \mathfrak{A}(M', \gamma', \mathcal{P}), \quad \mathfrak{A}(M, \gamma, T, \Gamma) \subset \mathfrak{A}(M', \gamma', T, \Gamma).$$

We also notice that the trace inequality

$$(2.4) \quad k^{1/2} \|v\|_{s,\Gamma} + \|v\|_{s+1/2,\Gamma} \leq C_s (k \|v\|_{s,\mathcal{P}} + \|v\|_{s+1,\mathcal{P}})$$

holds true for every  $s \geq 0$ . The analyticity classes are invariant under analytic changes of variables and multiplication by analytic functions:

**Lemma 2.2** ([MS21, Lemma 2.6]). *Let  $\omega_1, \omega_2 \subset \mathbb{R}^d$  be bounded open and  $g : \omega_1 \rightarrow \omega_2$  be a bijection and analytic on the closed set  $\bar{\omega}_1$ . Let  $f_1$  be analytic on the closed set  $\bar{\omega}_2$  and  $f_2 \in \mathfrak{A}(M_f, \gamma_f, \omega_2)$ . Then there are constants  $C_1, \gamma_1 > 0$  depending only on  $f_1, g, \omega_1, \omega_2$ , and  $\gamma_f$  such that  $f_1(f_2 \circ g) \in \mathfrak{A}(C_1 M_f, \gamma_1, \omega_1)$ .*

**2.5. Informal explanation of the proof of the main result and the assumptions.** We perform a general analysis under abstract assumptions given in Sections 2.6 and 2.7 below. Before rigorously presenting the proof and stating the requirements, let us rather informally present the key arguments and motivate the main assumptions.

We consider a time-harmonic wave propagation problems of the form

$$(2.5) \quad \begin{cases} -T_{k,\Omega}^- u - \nabla \cdot (A \nabla u) & = f & \text{in } \Omega, \\ \partial_n u + T_{k,\Gamma}^- u & = g & \text{on } \Gamma, \end{cases}$$

where  $A : \Omega \rightarrow \mathbb{R}^{d \times d}$  is a piecewise analytic coefficient. The prototypical example is the heterogeneous Helmholtz equation with impedance boundary conditions with

$$T_{k,\Omega}^- u = k^2 \mu u, \quad T_{k,\Gamma}^- u = iku,$$

where  $\mu : \Omega \rightarrow \mathbb{R}$  is a piecewise analytic coefficient, and the superscript  $-$  reminds us that, in general, there is negative zero-order term in the PDE. In what follows, we will take  $g = 0$  for simplicity.

Our goal is, given  $f \in L^2(\Omega)$ , to split the solution  $u$  as  $u = u_F + u_A$  where  $u_F \in H^2(\mathcal{P})$  with  $|u_F|_{2,\mathcal{P}} \leq C \|f\|_{0,\Omega}$  and  $u_A$  is (piecewise) analytic, but with a norm growing (in a controlled way) with  $k$ . An intuitive (but slightly inexact) explanation of how this splitting is obtained is as follows.

**Step 1.** We introduce the following stable decomposition of the right-hand side:

$$f = f_F + f_A,$$

where  $f_A$  is (piecewise) analytic, and  $f_F \in L^2(\Omega)$ . This splitting is obtained through filter operators defined using cutoffs in the Fourier domain. These are introduced in Section 3.1 below. Crucially,  $f_A$  contains the ‘‘low frequencies’’ of  $f$  so that it is analytic, and  $f_F$  consists of the remaining ‘‘high frequencies’’.

**Step 2.** The analytic part  $u_A$  of the splitting is essentially obtained by solving (2.5) with  $f$  replaced by  $f_A$ . Assuming that (2.5) possesses appropriate regularity shifts,  $u_A$  will indeed be (piecewise) analytic, but include oscillations with frequency  $O(k)$  that have to be expected for Helmholtz problems.

**Step 3.** The subtle part of the analysis is to show that the  $H^2(\mathcal{P})$  norm of  $u_F := u - u_A$  does not grow with  $k$ . The reason why this can happen is that  $u_F$  solves (2.5) with a right-hand side  $f_F$  that does not contain the natural frequency  $k$  of the equation (it is made up of the higher-frequency content of  $f$ ). To rigorously quantify this in our analysis we consider an auxiliary coercive problem that acts as a parametrix for high-frequency data. Specifically, we show that, in an appropriate



sense,  $u_F \sim \tilde{u}_F$ , where  $\tilde{u}_F$  solves the auxiliary problem

$$(2.6) \quad \begin{cases} -T_{k,\Omega}^+ \tilde{u}_F - \nabla \cdot (A \nabla \tilde{u}_F) = f_F & \text{in } \Omega, \\ \partial_n \tilde{u}_F + T_{k,\Gamma}^+ \tilde{u}_F = 0 & \text{on } \Gamma. \end{cases}$$

The operators  $T_{k,\Omega}^+$  and  $T_{k,\Gamma}^+$  appear only in the analysis, to which they are tuned, but not in the numerical realization. For the Helmholtz equation with Robin boundary conditions, we can select  $T_{k,\Omega}^+ = -T_{k,\Omega}^- = -k^2\mu$  and  $T_{k,\Gamma}^+ = 0$ . The superscript  $+$  reminds us that, in general, the PDE in (2.6) will contain a positive zero-order term. Since  $f_F \in L^2(\Omega)$ , we cannot expect more than  $H^2(\mathcal{P})$  regularity for  $\tilde{u}_F$ . However, since (2.6) is a coercive problem, we have by Lax-Milgram and elliptic regularity  $\|\tilde{u}_F\|_{2,\mathcal{P}} \leq C\|f_F\|_{0,\Omega} \leq C\|f\|_{0,\Omega}$  which is the desired scaling. A nice way of summarizing this last step is that for high-frequency data, we are allowed to flip the “bad sign” in the Helmholtz problem.

We now summarize the key properties that makes such a proof work for the case  $T_{k,\Omega}^- = -T_{k,\Omega}^+ = k^2\mu$ ,  $T_{k,\Gamma}^- = ik$  and  $T_{k,\Gamma}^+ = 0$ . We then expand on how these properties may be generalized, leading to our main assumptions below.

We formulate a first set of assumptions on the wave propagation problem itself. These are given in **(WP)** in Section 2.6 below. Consider first the case  $T_{k,\Omega}^- = -k^2\mu$  and  $T_{k,\Gamma}^- = ik$ , for which the following properties are essential: (1) The sesquilinear form associated with the problem is uniformly continuous w.r.t. the  $\|\cdot\|_{1,t,k}$  norm (with  $t = 1/2$  in this case). (2) For  $f \in L^2(\Omega)$ , (2.5) admits a unique solution  $u \in H^1(\Omega)$ . In general,  $\|u\|_{1,t,k}$  is not bounded by  $C\|f\|_{0,\Omega}$  with a  $C > 0$  independent of in  $k$ , but we will require that the growth be merely polynomial in  $k$ . (3) If the right-hand side  $f$  in (2.5) is (piecewise) analytic, then so is the solution  $u$ .

These are exactly the assumptions given in **(WP1)**, **(WP2)**, and **(WP3)**, up to one possible generalization. Namely, in **(WP1)**, we allow for the continuity constant of the sesquilinear form to grow (polynomially) in  $k$ . In exchange, we require in **(WP4)** that the operators  $T_{k,\Omega}^-$  and  $T_{k,\Omega}^+$  may be split into operators that are uniformly bounded in  $k$  and smoothening operators that maps into sets of analytic functions. Roughly speaking, we allow polynomial growth of the continuity constant, if this additional growth is compensated by analytic regularity. Such considerations are crucial when dealing with DtN operators on general surfaces, since the DtN operator does not have to be uniformly bounded in  $k$ . We also allow for more general second-order terms than  $-\nabla(A\nabla\cdot)$  in **(WP5)** as long as the corresponding sesquilinear form  $a$  is uniformly continuous in  $k$ .

We then need a set of assumptions on the auxiliary problem (2.6). First, as seen from the above discussion, it is key that (2.6) is coercive and that it possess a  $k$ -uniform  $L^2(\Omega)$  to  $H^2(\mathcal{P})$  regularity shift. These requirements are stated in **(AP1)** and **(AP2)**. The final assumption is that the wave propagation problem (2.5) and the auxiliary problem (2.6) are suitably linked to one another, in such a way that  $u_F \sim \tilde{u}_F$ . These assumptions are expressed by considering the differences  $T_{k,\Omega}^\Delta := T_{k,\Omega}^+ - T_{k,\Omega}^-$  and  $T_{k,\Gamma}^\Delta := T_{k,\Gamma}^+ - T_{k,\Gamma}^-$ . Notice that in the case of the impedance boundary condition considered above, we have  $T_{k,\Omega}^\Delta = 2k^2\mu$  and  $T_{k,\Gamma}^\Delta = ik$ . The crucial point that enables our analysis is then that  $T_{k,\Omega}^\Delta$  and  $T_{k,\Gamma}^\Delta$  are compact operator (considering functions in  $H^1(\Omega)$  and  $H^{1/2}(\Gamma)$ ) with norm suitably controlled



in  $k$ . Specifically, the powers of  $k$  in the operator norms are compensated by the smoothening degree of the operators. **(AP3)** essentially states this requirement, while allowing for a generalization similar to **(WP4)**. More precisely, we allow the difference operator  $T_{k,\Omega}^\Delta$  and  $T_{k,\Gamma}^\Delta$  to have norms polynomially bounded in  $k$ , if this is compensated by smoothening properties, i.e., if the parts of the operators responsible for the growth in  $k$  map into sets of analytic functions. Again, such a generalization is required to handle, e.g., DtN operators.

We finally mention that in the discussion above, we have considered regularity shifts from  $L^2(\Omega)$  to  $H^2(\mathcal{P})$ . However, our assumptions are slightly more general, as we also consider shifts between  $H^r(\mathcal{P})$  and  $H^{r+2}(\mathcal{P})$ . The goal of this last generalization is to derive regularity splittings where the regular part lies in  $H^{r+2}(\mathcal{P})$  with  $r > 0$ , provided that  $f \in H^r(\mathcal{P})$ .

**2.6. Time-harmonic wave propagation problem.** Given  $f \in L^2(\Omega)$  and  $g \in L^2(\Gamma)$ , our model problem consists in finding  $u \in H^{1,t}(\Omega, \Gamma)$  such that

$$(2.7) \quad b_k^-(u, v) = (f, v) + \langle g, v \rangle \quad \forall v \in H^{1,t}(\Omega, \Gamma)$$

with

$$(2.8) \quad b_k^-(u, v) := a(u, v) - (T_{k,\Omega}^- u, v) - \langle T_{k,\Gamma}^- u, v \rangle,$$

where  $a(\cdot, \cdot)$  is a sesquilinear form on  $H^{1,t}(\Omega, \Gamma)$  and the operators  $T_{k,\Omega}^-$  and  $T_{k,\Gamma}^-$  are differential operators acting on “volume” and “surface” functions. Before providing a detailed, formal description of these operators, we first explain why  $b_k^-$  is decomposed in the form given.

The sesquilinear form  $a(\cdot, \cdot)$  is meant to represent the “elliptic part” of the wave operator. A typical example is the “grad-grad” form in the scalar Helmholtz case, but one may imagine other situation such as the elasticity operator. The operator  $T_{k,\Omega}^-$  then accounts for the “frequency part” of the operator, i.e., the time-derivative transformed into  $ik$  in the frequency domain. The most basic situation is then  $T_{k,\Omega}^- u := k^2 u$ , but we allow for more generality. Finally,  $T_{k,\Gamma}^-$  is an operator on  $\Gamma$ ; relevant examples are the DtN map or some approximation of it.

We now summarize our assumptions **(WP)** on the wave propagation problem.

**(WP1)** The sesquilinear form  $b_k^- : H^{1,t}(\Omega, \Gamma) \times H^{1,t}(\Omega, \Gamma) \rightarrow \mathbb{C}$  is continuous, i.e., there exists a constant  $C_{\text{cont},k}^- \geq 0$ , possibly depending on  $k$ , such that

$$|b_k^-(u, v)| \leq C_{\text{cont},k}^- \|u\|_{1,t,k} \|v\|_{1,t,k} \quad \forall u, v \in H^{1,t}(\Omega, \Gamma).$$

**(WP2)** Problem (2.7) is well posed, i.e., for every  $f \in L^2(\Omega)$  and  $g \in L^2(\Gamma)$  it admits a unique weak solution  $S_k^-(f, g) := u \in H^{1,t}(\Omega, \Gamma)$ , and the stability estimate

$$(2.9) \quad \|S_k^-(f, g)\|_{1,t,k} \leq C_{\text{sol},k}^- (\|f\|_{0,\Omega} + k^{1/2} \|g\|_{0,\Gamma})$$

holds true with a constant  $C_{\text{sol},k}^-$  that may depend on  $k$ , but is independent of  $f$  and  $g$  (See also Remark 2.4).

Assumptions **(WP1)** and **(WP2)** simply state that the problem is well-posed and that the direct and inverse operators are bounded with continuity constants  $C_{\text{cont},k}^-$  and  $C_{\text{sol},k}^-$ . For our quasi-optimality result we also need the solution to be (piecewise) analytic when the right-hand sides are (piecewise) analytic.

**(WP3)** For each tubular neighborhood  $T$  of  $\Gamma$  there exists a constant  $\gamma_0$  and there exists a non-decreasing function  $\vartheta : [\gamma_0, +\infty) \rightarrow \mathbb{R}$ , both independent of  $k$ , and there exists a possibly  $k$ -dependent constant  $C_{\text{ana},k}^-$  such that for all  $\gamma \geq \gamma_0$  the following holds: for all piecewise analytic data  $f \in \mathfrak{A}(M_f, \gamma, \mathcal{P})$  and all boundary data  $g \in \mathfrak{A}(M_g, \gamma, T, \Gamma)$  the solution  $S_k^-(f, g)$  to Problem (2.7) satisfies  $S_k^-(f, g) \in \mathfrak{A}(M_u, \vartheta(\gamma), \mathcal{P})$ , i.e., it is again piecewise analytic, and

$$(2.10) \quad M_u \leq C_{\text{ana},k}^- C_{\text{sol},k}^- k^{-1} (M_f + kM_g).$$

**Remark 2.3.** **(WP3)** permits the data  $f, g$  to depend on  $k$ . In particular, the constants  $M_f, M_g$  may depend on  $k$  but **(WP3)** requires the constant  $M_u$  to be bounded as given in (2.10).  $\blacksquare$

For many model problems, the continuity constant  $C_{\text{cont},k}^-$  of  $b_k^-$  does not depend on  $k$ , and the three above assumptions are sufficient to proceed with our analysis. However,  $C_{\text{cont},k}^-$  may depend on  $k$ , for example, when considering DtN operators on general surfaces  $\Gamma$ , or when dealing with Maxwell's equations [MS21]. As a result, we include an additional assumption to treat these cases.

**(WP4)** The linear operators  $T_{k,\Omega}^- : H^{1,t}(\Omega, \Gamma) \rightarrow H^{1,t}(\Omega, \Gamma)'$  and  $T_{k,\Gamma}^- : H^t(\Gamma) \rightarrow H^{-t}(\Gamma)$  admit splittings into linear operators

$$(2.11) \quad T_{k,\Omega}^- = R_{k,\Omega}^- + A_\Omega^-, \quad T_{k,\Gamma}^- = R_{k,\Gamma}^- + A_\Gamma^-$$

such that the “ $R$  part” is uniformly bounded in  $k$ , while the “ $A$  part” maps into analytic functions: We assume

$$(2.12) \quad |(R_{k,\Omega}^- u, v)| + |\langle R_{k,\Gamma}^- u, v \rangle| \leq C \|u\|_{1,t,k} \|v\|_{1,t,k} \quad \forall u, v \in H^{1,t}(\Omega, \Gamma),$$

and we assume the existence of a constant  $C_{A,k}^-$  (possibly depending on  $k$ ) and a constant  $\gamma_A^-$  and a tubular neighborhood  $T$  (both independent of  $k$ ) such that for all  $u \in H^{1,t}(\Omega, \Gamma)$  we have  $A_\Omega^- u \in \mathfrak{A}(M_u, \gamma_A^-, \mathcal{P})$  and  $A_\Gamma^-(u|_\Gamma) \in \mathfrak{A}(M_v, \gamma_A^-, T, \Gamma)$  with

$$M_u \leq C_{A,k}^- \|u\|_{1,t,k}, \quad kM_v \leq C_{A,k}^- \|u\|_{1,t,k}.$$

**(WP5)** We assume that  $a(\cdot, \cdot)$  is uniformly-in- $k$  continuous, i.e.,

$$(2.13) \quad |a(u, v)| \leq C \|u\|_{1,t,k} \|v\|_{1,t,k} \quad \forall u, v \in H^{1,t}(\Omega, \Gamma).$$

**2.7. Auxiliary positive problem.** Our analysis hinges on an auxiliary problem that is meant to be a “positive” version of the time-harmonic problem. We thus introduce the sesquilinear form

$$(2.14) \quad b_k^+(u, v) := a(u, v) - (T_{k,\Omega}^+ u, v) - \langle T_{k,\Gamma}^+ u, v \rangle \quad \forall u, v \in H^{1,t}(\Omega, \Gamma),$$

where  $T_{k,\Omega}^+ : H^{1,t}(\Omega) \rightarrow (H^{1,t}(\Omega))'$  and  $T_{k,\Gamma}^+ : H^t(\Omega) \rightarrow H^{-t}(\Omega)$  are problem specific (continuous linear) operators chosen to conduct the analysis. In the simplest case when  $T_{k,\Omega}^- = +k^2$  and  $T_{k,\Gamma}^- = -ik$ , we can simply select  $T_{k,\Omega}^+ = -k^2$  and  $T_{k,\Gamma}^+ = 0$ . However, allowing for more generality enables us to treat a much wider class of problems.

The assumptions **(AP)** on the auxiliary problem are listed as follows:

**(AP1)**  $b_k^+$  is coercive: there exist  $C_{\text{coer}}^+ > 0$  (independent of  $k$ ) and  $\sigma \in \mathbb{C}$  with  $|\sigma| = 1$  such that

$$(2.15) \quad \text{Re}(\sigma b_k^+(u, u)) \geq C_{\text{coer}}^+ \|u\|_{1,t,k}^2 \quad \forall u \in H^{1,t}(\Omega, \Gamma).$$

Thus, for  $f \in \tilde{H}^{-1}(\Omega)$  and  $g \in H^{-t}(\Gamma)$ , we may define  $S_k^+(f, g)$  as the unique element of  $H^{1,t}(\Omega, \Gamma)$  such that

$$(2.16) \quad b_k^+(S_k^+(f, g), v) = (f, v) + \langle g, v \rangle \quad \forall v \in H^{1,t}(\Omega, \Gamma).$$

**(AP2)** There exists an integer  $s_{\max} \geq 0$  and for each  $0 \leq s \leq s_{\max}$  there exists a  $C_{\text{shift},s}^+ > 0$  independent of  $k$  such that if  $f \in H^s(\mathcal{P})$  and  $g \in H^{s+1/2}(\Gamma)$  we have  $S_k^+(f, g) \in H^{s+2}(\mathcal{P})$  with

$$(2.17) \quad \|S_k^+(f, g)\|_{s+2, \mathcal{P}} \leq C_{\text{shift},s}^+ \left[ k^s \|f\|_{0, \Omega} + \|f\|_{s, \mathcal{P}} + k^{s+1/2} \|g\|_{0, \Gamma} + \|g\|_{s+1/2, \Gamma} \right].$$

We further need the auxiliary problem to have similar Sobolev regularity properties as the original wave propagation problem. This is measured by the sesquilinear form

$$b_k^\Delta(u, v) := b_k^-(u, v) - b_k^+(u, v) = (T_{k,\Omega}^\Delta u, v) + \langle T_{k,\Gamma}^\Delta u, v \rangle \quad \forall u, v \in H^{1,t}(\Omega, \Gamma),$$

where  $T_{k,\Omega}^\Delta = T_{k,\Omega}^+ - T_{k,\Omega}^-$  and  $T_{k,\Gamma}^\Delta = T_{k,\Gamma}^+ - T_{k,\Gamma}^-$ . Similar to the discussion on assumption **(WP4)** it is enough for many applications to assume that  $b_k^\Delta$  is bounded uniformly in  $k$ . To be fully general, however, we need a last splitting assumption.

**(AP3)** We have the splittings

$$(2.18) \quad T_{k,\Omega}^\Delta = R_{k,\Omega}^\Delta + A_{k,\Omega}^\Delta, \quad T_{k,\Gamma}^\Delta = R_{k,\Gamma}^\Delta + A_{k,\Gamma}^\Delta.$$

The linear operator  $R_{k,\Omega}^\Delta$  is of order at most one and satisfies for  $0 \leq s \leq s_{\max}$

$$(2.19) \quad \|R_{k,\Omega}^\Delta u\|_{s, \mathcal{P}} \leq C_s (k^2 \|u\|_{s, \mathcal{P}} + k \|u\|_{s+1, \mathcal{P}}) \quad \forall u \in H^{s+1}(\mathcal{P}),$$

whereas the linear operator  $R_{k,\Gamma}^\Delta$  is of order zero and satisfies

$$(2.20) \quad \|R_{k,\Gamma}^\Delta v\|_{s+1/2, \Gamma} \leq C \left( k^{3/2} \|v\|_{s, \Gamma} + k \|v\|_{s+1/2, \Gamma} \right) \quad \forall v \in H^{s+1/2}(\Gamma).$$

Finally, there exists a constant  $C_{A,k}^\Delta$ , possibly depending on  $k$ , and there exist a constant  $\gamma_A^\Delta$  and a tubular neighborhood  $T$  of  $\Gamma$  (both independent of  $k$ ) such that for all  $u \in H^{1,t}(\Omega, \Gamma)$  we have  $A_{k,\Omega}^\Delta u \in \mathfrak{A}(M_u, \gamma_A^\Delta, \mathcal{P})$  and  $A_{k,\Gamma}^\Delta(u|_\Gamma) \in \mathfrak{A}(M_v, \gamma_A^\Delta, T, \Gamma)$  with

$$(2.21) \quad M_u \leq C_{A,k}^\Delta \|u\|_{1,t,k}, \quad kM_v \leq C_{A,k}^\Delta \|u\|_{1,t,k}.$$

**Remark 2.4** (On the stability assumption **(WP2)**). The stability estimate (2.9) does not make minimal regularity assumptions on the data  $f, g$ . However, the stability constant  $C_{\text{sol},k}^-$  (2.9) is closely related to the inf-sup constant

$$(2.22) \quad \gamma_{\text{inf-sup}} := \inf_{u \in H^{1,t}(\Omega, \Gamma)} \sup_{v \in H^{1,t}(\Omega, \Gamma)} \frac{|b_k^-(u, v)|}{\|u\|_{1,t,k} \|v\|_{1,t,k}}$$

if **(AP1)** and **(AP3)** hold. Let  $B_k^- : H^{1,t}(\Omega, \Gamma) \rightarrow H^{1,t}(\Omega, \Gamma)'$  be the operator induced by  $b_k^-$ . We claim that if  $B_k^-$  is an isomorphism, in which case  $1/\gamma_{\text{inf-sup}} = \|(B_k^-)^{-1}\|_{H^{1,t}(\Omega, \Gamma) \leftarrow H^{1,t}(\Omega, \Gamma)'}$ , then (2.9) holds with  $C_{\text{sol},k}^- = O(1/(k\gamma_{\text{inf-sup}}))$ . To see this, note that for the solution  $u = S_k^-(f, g)$  of (2.7) there is  $v \in H^{1,t}(\Omega, \Gamma)$  with  $\|v\|_{1,t,k} = 1$  such that

$$\gamma_{\text{inf-sup}} \|u\|_{1,t,k} \leq |b_k^-(u, v)| = |(f, v) + \langle g, v \rangle| \lesssim (k^{-1} \|f\|_{0, \Omega} + k^{-1/2} \|g\|_{0, \Gamma}) \|v\|_{1,t,k}$$

so that (2.9) holds with  $C_{\text{sol},k}^- = O(1/(k\gamma_{\text{inf-sup}}))$ . Conversely, if (2.9) holds, then the inf-sup constant satisfies (2.23) below as we now show using the arguments used,

e.g., in [Mel95, Prop. 8.2.7] or [EM12, Thm. 2.5]. The assumption **(AP3)** implies that  $B_k^-$  is a compact perturbation of a coercive operator. Since (2.9) implies uniqueness, the Fredholm alternative provides that  $B_k^-$  is an isomorphism. Since the norm of  $(B_k^-)^{-1}$  equals the norm of  $((B_k^-)')^{-1}$ , we compute the latter. To see this, let  $u \in H^{1,t}(\Omega, \Gamma)$  be arbitrary and  $z \in H^{1,t}(\Omega, \Gamma)$  solve, for all  $w \in H^{1,t}(\Omega, \Gamma)$ ,

$$\begin{aligned} b_k^-(z, w) &= b_k^-(u, w) - b_k^+(u, w) \\ &= (T_{k,\Omega}^\Delta u, w) + \langle T_{k,\Gamma}^\Delta u, w \rangle = (R_{k,\Omega}^\Delta u + A_{k,\Omega}^\Delta u, w) + \langle R_{k,\Gamma}^\Delta u + A_{k,\Gamma}^\Delta u, w \rangle. \end{aligned}$$

Noting

$$\begin{aligned} |(T_{k,\Omega}^\Delta u, w)| &\lesssim (k + C_{A,k}^\Delta) \|u\|_{1,t,k} \|w\|_{0,\Omega}, \\ |\langle T_{k,\Gamma}^\Delta u, w \rangle| &\lesssim \|T_{k,\Gamma}^\Delta u\|_{0,\Gamma} \|w\|_{0,\Gamma} \lesssim \left( \|R_{k,\Gamma}^\Delta u\|_{1/2,\Gamma} + k^{-1/2} C_{A,k}^\Delta \|u\|_{1,t,k} \right) \|w\|_{0,\Gamma} \\ &\lesssim \left( k^{3/2} \|u\|_{0,\Gamma} + k \|u\|_{1/2,\Gamma} + C_{A,k}^\Delta k^{-1/2} \|u\|_{1,t,k} \right) \|w\|_{0,\Gamma} \\ &\lesssim (k + C_{A,k}^\Delta k^{-1/2}) \|u\|_{1,t,k} \|w\|_{0,\Gamma}. \end{aligned}$$

By (2.9) and these two bounds, we get

$$\|z\|_{1,t,k} \lesssim (k^{3/2} + C_{A,k}^\Delta) C_{\text{sol},k}^- \|u\|_{1,t,k}.$$

We conclude

$$\sup_{v \neq 0} \frac{|b_k^-(v, u)|}{\|u\|_{1,t,k} \|v\|_{1,t,k}} \stackrel{v=u-z}{\geq} \frac{|b_k^+(u, u)|}{\|u\|_{1,t,k} \|u-z\|_{1,t,k}} \stackrel{\text{(AP1)}}{\gtrsim} \frac{\|u\|_{1,t,k}^2}{\|u\|_{1,t,k} (\|u\|_{1,t,k} + \|z\|_{1,t,k})}$$

and arrive at

$$(2.23) \quad \gamma_{\text{infsup}} \gtrsim \frac{1}{C_{\text{sol},k}^- (k^{3/2} + C_{A,k}^\Delta)}.$$

■

### 3. ABSTRACT CONTRACTION ARGUMENT

**3.1. Filtering operators.** Following [MS10, MS11], a key ingredient of our splittings are filtering operators defined using cut-offs in the Fourier domain. We need to filter both surface and volume functions, leading to two pairs of filters.

Before providing proper mathematical definitions, we summarize the ideas behind the construction of the filtering operators. To this end, consider a function  $f \in L^2(\mathbb{R}^d)$  together with its Fourier transform  $\widehat{f}$ . The decomposition

$$\widehat{f} = \chi_\eta \widehat{f} + (1 - \chi_\eta) \widehat{f},$$

where  $\chi_\eta$  is the characteristic function of the ball  $B_\eta(0)$ , induces the splitting

$$f = L^\eta f + H^\eta f,$$

where  $L^\eta$  and  $H^\eta$  are the inverse Fourier transforms of  $\chi_\eta \widehat{f}$  and  $(1 - \chi_\eta) \widehat{f}$ , respectively. By construction the Fourier transform of  $L^\eta f$  is supported in  $\overline{B(0, \eta)}$  so that we call  $L^\eta$  a low-pass filter, whereas  $H^\eta$  is referred to as a high-pass filter. Crucially,  $L^\eta f$  is analytic, which is key in the construction of the analytic part of the solution splitting. Next, we extend the above construction to handle functions defined on the boundary  $\Gamma$ , and deal with piecewise analyticity instead of global analyticity.

On the boundary  $\Gamma$ , we can employ the surface filters introduced in [MS11, Lemmas 4.2, 4.3]. Proposition 3.1 sums up their essential properties.

**Proposition 3.1** (Surface filters). *For each  $\eta > 1$ , there exist two operators  $L_{k,\Gamma}^\eta, H_{k,\Gamma}^\eta: L^2(\Gamma) \rightarrow L^2(\Gamma)$  such that  $H_{k,\Gamma}^\eta + L_{k,\Gamma}^\eta = \text{id}$ . For  $0 \leq s' \leq s$  the operator  $H_{k,\Gamma}^\eta$  satisfies*

$$(3.1) \quad \|H_{k,\Gamma}^\eta g\|_{H^{s'}(\Gamma)} \leq C_{s,s'}(\eta k)^{s'-s} \|g\|_{H^s(\Gamma)} \quad \forall g \in H^s(\Gamma).$$

*There exists a tubular neighborhood  $T$  of  $\Gamma$  that depends solely on the analyticity properties of  $\Gamma$  and there exists a constant  $\eta' \geq \eta$  depending only on  $\eta$  and  $\Gamma$  such that  $L_{k,\Gamma}^\eta g \in \mathfrak{A}(M, \eta', T, \Gamma)$  with*

$$(3.2) \quad M \leq C_s \|g\|_{-1/2, \Gamma}.$$

*Proof.* The operators  $H_{k,\Gamma}^\eta$  and  $L_{k,\Gamma}^\eta$  are defined in [MS11, Lemmas 4.2, 4.3] and the properties of  $H_{k,\Gamma}^\eta$  are shown there. For the properties of  $L_{k,\Gamma}^\eta$ , we note the following: Let  $T$  be a tubular neighborhood on which the analytic continuation  $\mathbf{n}^*$  of the normal vector  $\mathbf{n}$  exists and let, as in construction in [MS11],  $G \in H^{s+3/2}(\Omega)$  be such that  $\partial_n G = g$  on  $\Gamma$  and  $\|G\|_{s'+3/2, \Omega} \lesssim \|g\|_{s', \Gamma}$  for  $-1/2 \leq s' \leq s$ . Then  $L_{k,\Gamma}^\eta g$  is defined as the restriction to  $\Gamma$  of  $\mathbf{n}^* \cdot \nabla L_{k,\Omega}^\eta G$ . By construction,  $L_{k,\Omega}^\eta G$  is analytic on  $\mathbb{R}^d$  with

$$\|\nabla^n L_{k,\Omega}^\eta G\|_{0, \mathbb{R}^d} \leq C(\eta k)^{n-1} \|G\|_{1, \Omega} \quad \forall n \in \mathbb{N}.$$

We conclude from Lemma 2.2 that  $\mathbf{n}^* \cdot \nabla L_{k,\Omega}^\eta G \in \mathfrak{A}(C \|G\|_{1, \Omega}, \eta', T, \Gamma)$  for some  $C$ ,  $\eta'$  depending only on  $\eta$  and  $\Gamma$ . Without loss generality,  $\eta' \geq \eta$ . We conclude with the estimate  $\|G\|_{1, \Omega} \lesssim \|g\|_{-1/2, \Gamma}$ .  $\square$

**Proposition 3.2** (Volume filters). *For each  $\eta > 1$ , there exist two operators  $L_{k,\Omega}^\eta, H_{k,\Omega}^\eta: L^2(\Omega) \rightarrow L^2(\Omega)$  such that  $H_{k,\Omega}^\eta + L_{k,\Omega}^\eta = \text{id}$ . For  $0 \leq s' \leq s$  and  $0 \leq \varepsilon < 1/2$ , the operator  $H_{k,\Omega}^\eta$  satisfies*

$$(3.3) \quad \|H_{k,\Omega}^\eta f\|_{s', \mathcal{P}} \leq C_{s,s'}(\eta k)^{s'-s} \|f\|_{s, \mathcal{P}} \quad \forall f \in H^s(\mathcal{P}),$$

$$(3.4) \quad \|H_{k,\Omega}^\eta f\|_{\tilde{H}^{-\varepsilon}(\Omega)} \leq C_\varepsilon(\eta k)^{-\varepsilon} \|f\|_{0, \Omega} \quad \forall f \in L^2(\Omega)$$

*for constants  $C_{s,s'}$ ,  $C_\varepsilon$  independent of  $f$ ,  $k$ , and  $\eta$ . Finally, we have  $L_{k,\Omega}^\eta f \in \mathfrak{A}(M, \eta, \mathcal{P})$  with*

$$(3.5) \quad M \leq C_\Omega \|f\|_{0, \Omega}.$$

*Proof.* For all  $P \in \mathcal{P}$ , we set

$$\left( H_{k,\Omega}^\eta f \right) \Big|_P = (H_{\eta k}(E_P f)) \Big|_P \quad \left( L_{k,\Omega}^\eta f \right) \Big|_P = (L_{\eta k}(E_P f)) \Big|_P,$$

where  $E_P: H^s(P) \rightarrow H^s(\mathbb{R}^d)$  is the Stein extension operator, [Ste70, Chap. VI], and  $H_{\eta k}$  and  $L_{\eta k}$  are the ‘‘full space’’ high and low pass filters introduced in [MS11]. Then, the proof of (3.3) and (3.5) can be found in [MS11, Lemmas 4.2, 4.3].

Regarding (3.4), note that for  $0 \leq \varepsilon < 1/2$  the space of compactly supported smooth functions  $\bigcup_{P \in \mathcal{P}} C_0^\infty(P)$  is dense in  $H^\varepsilon(\Omega)$ . It is easy to check that for all  $0 \leq s \leq 1$

$$(3.6) \quad \|H_{\eta k} f\|_{-s, \mathbb{R}^d} \leq C_s(\eta k)^{-s} \|f\|_{0, \mathbb{R}^d} \quad \forall f \in L^2(\mathbb{R}^d),$$

[MS11, Section 4.1.1]. For  $v \in C_0^\infty(P)$  let  $\tilde{v}$  denote the trivial extension by zero on all of  $\mathbb{R}^d$  of  $v$ . These observations give

$$\begin{aligned} |(H_{\eta k}^\Omega f, v)_\Omega| &= \left| \sum_{P \in \mathcal{P}} (H_{\eta k} E_P(f), v)_{L^2(P)} \right| = \left| \sum_{P \in \mathcal{P}} (H_{\eta k} E_P(f), \tilde{v}_P)_{L^2(\mathbb{R}^d)} \right| \\ &\leq \sum_{P \in \mathcal{P}} \|H_{\eta k} E_P(f)\|_{-\varepsilon, \mathbb{R}^d} \|\tilde{v}_P\|_{\varepsilon, \mathbb{R}^d}. \end{aligned}$$

Additionally, we have

$$\begin{aligned} \|H_{\eta k} E_P(f)\|_{-\varepsilon, \mathbb{R}^d} &\stackrel{(3.6)}{\leq} C_\varepsilon (\eta k)^{-\varepsilon} \|E_P(f)\|_{0, \mathbb{R}^d} \leq C_\varepsilon (\eta k)^{-\varepsilon} \|f\|_{0, P}, \\ \|\tilde{v}\|_{\varepsilon, \mathbb{R}^d} &\stackrel{\varepsilon < 1/2}{\lesssim} C_\varepsilon \|v\|_{\varepsilon, P}, \end{aligned}$$

which leads to

$$\left| (H_{k, \Omega}^\eta f, v)_{L^2(\Omega)} \right| \leq C_\varepsilon (\eta k)^{-\varepsilon} \sum_{P \in \mathcal{P}} \|f\|_{0, P} \|v\|_{\varepsilon, P} \leq C_\varepsilon (\eta k)^{-\varepsilon} \|f\|_{0, \Omega} \|v\|_{\varepsilon, \mathcal{P}},$$

and the result follows since  $\|v\|_{\varepsilon, \mathcal{P}} \leq \|v\|_{\varepsilon, \Omega}$ .  $\square$

**Remark 3.3** (Other constructions of high and low pass filters). The presented high and low pass filters are by no means the only possible constructions. It is, e.g., possible to construct the low-frequency filter on bounded domains by truncating the expansion in eigenfunctions of the Laplacian endowed with Neumann boundary conditions. See also [Mel12, Sec. 6.1] for similar considerations.  $\blacksquare$

**3.2. A contraction argument.** We employ the low and high pass filters to formulate a preliminary decomposition result. For a given right-hand side, the decomposition consists of splitting of the desired form plus a remainder that is the solution to the time-harmonic problem with new right-hand side that is strictly smaller in norm than the original one. We thus call the preliminary result a ‘‘contraction’’ argument. This contraction argument is key in our analysis, and established in Lemma 3.4 below.

Due to the duality between  $H^\varepsilon(\Omega)$  and  $\tilde{H}^{-\varepsilon}(\Omega)$ , we have for  $f \in L^2(\Omega)$  and  $g \in L^2(\Gamma)$  with a constant  $C_\varepsilon$  depending only on  $\varepsilon \in [0, 1]$

$$|(f, v) + (g, v)| \leq C_\varepsilon \left( k^{\varepsilon-1} \|f\|_{\tilde{H}^{-\varepsilon}(\Omega)} + k^{-1/2} \|g\|_{0, \Gamma} \right) \|v\|_{1, k, \Omega} \quad \forall v \in H^1(\Omega).$$

Since  $b_k^+$  is coercive by (AP1), this leads to

$$(3.7) \quad \|S_k^+(f, g)\|_{1, t, k} \leq C_\varepsilon \left( k^{\varepsilon-1} \|f\|_{\tilde{H}^{-\varepsilon}(\Omega)} + k^{-1/2} \|g\|_{0, \Gamma} \right).$$

**Lemma 3.4** (Unified contraction argument). *Let (WP) and (AP) hold. Let  $q \in (0, 1)$  and  $s \in [0, s_{\max}]$  be given. Then, there exists a parameter  $\eta > 1$  depending on  $q$  and  $s$  such that for all  $f \in H^s(\mathcal{P})$  and  $g \in H^{s+1/2}(\Gamma)$  we can write*

$$S_k^-(f, g) = u_F + u_A + S_k^-(\tilde{f}, \tilde{g}),$$

where

$$u_F := S_k^+(H_{k, \Omega}^\eta f, H_{k, \Gamma}^\eta g) \quad u_A := S_k^-(L_{k, \Omega}^\eta f, L_{k, \Gamma}^\eta g) + S_k^-(A_{k, \Omega}^\Delta u_F, A_{k, \Gamma}^\Delta u_F),$$

and the pair  $(\tilde{f}, \tilde{g}) \in H^s(\mathcal{P}) \times H^{s+1/2}(\Gamma)$  satisfies

$$(3.8) \quad \|\tilde{f}\|_{s, \mathcal{P}} + \|\tilde{g}\|_{s+1/2, \Gamma} \leq q(\|f\|_{s, \mathcal{P}} + \|g\|_{s+1/2, \Gamma}).$$

In addition, we have  $u_F \in H^{s+2}(\mathcal{P})$  with

$$(3.9) \quad \|u_F\|_{s+2, \mathcal{P}} \leq C_s (\|f\|_{s, \mathcal{P}} + \|g\|_{s+1/2, \Gamma}),$$

and  $u_A \in \mathfrak{A}(M, \eta', \mathcal{P})$  for some  $\eta' > 1$  depending only on  $\eta$  and  $\Gamma$  with

$$(3.10) \quad M \leq CC_{\text{ana}, k}^- C_{\text{sol}, k}^- (1 + k^{-2} C_{A, k}^\Delta) (\|f\|_{0, \Omega} + \|g\|_{1/2, \Gamma}).$$

*Proof.* Let  $\eta > 1$  to be fixed later on. For  $f \in H^s(\Omega)$  and  $g \in H^{s+1/2}(\Gamma)$ , we set  $u := S_k^-(f, g)$ . We introduce

$$u_F := S_k^+(H_{k, \Omega}^\eta f, H_{k, \Gamma}^\eta g), \quad u_{A, I} := S_k^-(L_{k, \Omega}^\eta f, L_{k, \Gamma}^\eta g)$$

so that  $u = u_F + u_{A, I} + r_I$  with

$$r_I := -S_k^-(T_{k, \Omega}^\Delta u_F, T_{k, \Gamma}^\Delta u_F) = -S_k^-(R_{k, \Omega}^\Delta u_F, R_{k, \Gamma}^\Delta u_F) - S_k^-(A_{k, \Omega}^\Delta u_F, A_{k, \Gamma}^\Delta u_F),$$

where we employed **(AP3)**. We then introduce

$$u_{A, II} := -S_k^-(A_{k, \Omega}^\Delta u_F, A_{k, \Gamma}^\Delta u_F), \quad \tilde{f} := -R_{k, \Omega}^\Delta u_F, \quad \tilde{g} := -R_{k, \Gamma}^\Delta u_F,$$

leading to the splitting  $u = u_F + u_A + S_k^-(\tilde{f}, \tilde{g})$  with

$$u_A := u_{A, I} + u_{A, II} = S_k^-(L_{k, \Omega}^\eta f, L_{k, \Gamma}^\eta g) - S_k^-(A_{k, \Omega}^\Delta u_F, A_{k, \Gamma}^\Delta u_F).$$

Before establishing the contraction estimate (3.8), we prove the regularity estimate (3.9). Using (2.17) from **(AP2)** as well as the mapping properties (3.1) and (3.3) of the high pass filters, we have

$$\begin{aligned} & \|u_F\|_{s+2, \mathcal{P}} \\ & \stackrel{\text{(AP2)}}{\leq} C_s \left( \|H_{k, \Omega}^\eta f\|_{s, \mathcal{P}} + k^s \|H_{k, \Omega}^\eta f\|_{0, \Omega} + \|H_{k, \Gamma}^\eta g\|_{s+1/2, \Gamma} + k^{s+1/2} \|H_{k, \Gamma}^\eta g\|_{0, \Gamma} \right) \\ & \stackrel{(3.1), (3.3)}{\leq} C_s \left( \|f\|_{s, \mathcal{P}} + k^s (k\eta)^{-s} \|f\|_{s, \mathcal{P}} + \|g\|_{s+1/2, \Gamma} + k^{s+1/2} (k\eta)^{-s-1/2} \|g\|_{s+1/2, \Gamma} \right) \\ & \leq C_s (\|f\|_{s, \mathcal{P}} + \|g\|_{s+1/2, \Gamma}), \end{aligned}$$

which proves (3.9).

We next establish (3.8). Recalling the definition of  $\tilde{f}$  and  $\tilde{g}$ , estimates (2.19) and (2.20) combined with trace inequality (2.4) show that

$$(3.11) \quad \|\tilde{f}\|_{s, \mathcal{P}} + \|\tilde{g}\|_{s+1/2, \Gamma} \leq C (k^2 \|u_F\|_{s, \mathcal{P}} + k \|u_F\|_{s+1, \mathcal{P}}).$$

The case  $s = 0$  is treated separately. Fixing  $\varepsilon = 1/4$ , we have from (3.7) and the properties (3.1) and (3.4) of the high-pass filters

$$\begin{aligned} k \|u_F\|_{1, t, k} & \stackrel{(3.7)}{\leq} C_\varepsilon k \left( k^{\varepsilon-1} \|H_{k, \Omega}^\eta f\|_{\tilde{H}^{-\varepsilon}(\Omega)} + k^{-1/2} \|H_{k, \Gamma}^\eta g\|_{0, \Gamma} \right) \\ & \stackrel{(3.1), (3.4)}{\leq} C_\varepsilon k \left( k^{\varepsilon-1} (k\eta)^{-\varepsilon} \|f\|_{0, \Omega} + k^{-1/2} (k\eta)^{-1/2} \|g\|_{1/2, \Gamma} \right) \\ (3.12) \quad & \leq C \eta^{-\varepsilon} (\|f\|_{0, \Omega} + \|g\|_{1/2, \Gamma}), \end{aligned}$$

where, since  $\varepsilon = 1/4$  is fixed, we absorbed the constant  $C_\varepsilon$  in a constant  $C$  independent of  $k$  and  $\varepsilon$ ; we also used that  $\eta^{-1/2} \leq \eta^{-\varepsilon}$  since  $\eta > 1$ . We conclude from (3.11) and (3.12)

$$(3.13) \quad \|\tilde{f}\|_{0, \Omega} + \|\tilde{g}\|_{1/2, \Gamma} \leq C \eta^{-\varepsilon} (\|f\|_{0, \Omega} + \|g\|_{1/2, \Gamma}).$$



We turn to the case  $0 < s \leq s_{\max}$ . With  $\varepsilon = 0$  in (3.7) we have together with the mapping properties (3.1) and (3.3) of the high-pass filters that

$$\begin{aligned} k^2 \|u_{\text{F}}\|_{0,\Omega} &\leq Ck \left( k^{-1} \|H_{k,\Omega}^\eta f\|_{0,\Omega} + k^{-1/2} \|H_{k,\Gamma}^\eta g\|_{0,\Gamma} \right) \\ &\leq C_s (k\eta)^{-s} \|f\|_{s,\mathcal{P}} + k^{1/2} (k\eta)^{-s-1/2} \|g\|_{s+1/2,\Gamma}. \end{aligned}$$

Hence, we find

$$(3.14) \quad k^{s+2} \|u_{\text{F}}\|_{0,\Omega} \leq C_s \eta^{-s} (\|f\|_{s,\mathcal{P}} + \|g\|_{s+1/2,\Gamma}).$$

We note the multiplicative interpolation estimate for  $0 \leq r \leq s+2$ :

$$(3.15) \quad \|u_{\text{F}}\|_{r,\mathcal{P}} \leq C_r \|u_{\text{F}}\|_{0,\Omega}^{\frac{s+2-r}{s+2}} \|u_{\text{F}}\|_{s+2,\mathcal{P}}^{\frac{r}{s+2}}.$$

Inserting (3.14) and (3.9) into (3.15) we find

$$(3.16) \quad \|u_{\text{F}}\|_{r,\mathcal{P}} \leq C_r k^{r-s-2} \eta^{-s \frac{s+2-r}{s+2}} (\|f\|_{s,\mathcal{P}} + \|g\|_{s+1/2,\Gamma}).$$

Using (3.16) with  $r = s$  and with  $r = s+1$  in (3.11) we find

$$(3.17) \quad \begin{aligned} \|\tilde{f}\|_{s,\Omega} + \|\tilde{g}\|_{s+1/2,\Gamma} &\leq C_s (k^2 \|u_{\text{F}}\|_{s,\mathcal{P}} + k \|u_{\text{F}}\|_{s+1,\mathcal{P}}) \\ &\leq C_s \left( \eta^{-s \frac{2}{s+2}} + \eta^{-s \frac{1}{s+2}} \right) (\|f\|_{s,\mathcal{P}} + \|g\|_{s+1/2,\Gamma}). \end{aligned}$$

At this point, since the constants appearing in (3.13) and (3.17) are independent of  $\eta$ ,  $f$ ,  $g$ , we can select  $\eta$  large enough ( $\eta$  may depend on  $s$ ) to obtain (3.8) with  $q < 1$ . For the sake of simplicity, we additionally select  $\eta$  larger than the constants  $\gamma_0$  of (WP4) and  $\gamma_{\text{A}}^\Delta$  of (AP3).

We next establish (3.10). For the first component  $u_{\text{A,I}}$ , properties (3.2) and (3.5) of the low pass filters show that

$$L_{k,\Omega}^\eta f \in \mathfrak{A}(M_f, \eta, \mathcal{P}), \quad L_{k,\Gamma}^\eta g \in \mathfrak{A}(M_g, \eta', T, \Gamma)$$

with

$$M_f \leq C \|f\|_{0,\Omega}, \quad M_g \leq C \|g\|_{-1/2,\Gamma}$$

and a tubular neighborhood  $T$  of  $\Gamma$  and a constant  $\eta' \geq \eta$  that depend solely on  $\eta$  and  $\Gamma$ . From (WP3) and  $\eta' \geq \eta \geq \gamma_0$ , we get  $u_{\text{A,I}} \in \mathfrak{A}(M_{\text{I}}, \vartheta(\eta'), \mathcal{P})$  with

$$M_{\text{I}} \leq C_{\text{ana},k}^- C_{\text{sol},k}^- k^{-1} (M_f + kM_g) \leq CC_{\text{ana},k}^- C_{\text{sol},k}^- k^{-1} (\|f\|_{0,\Omega} + k\|g\|_{-1/2,\Gamma}).$$

We turn to estimating  $u_{\text{A,II}}$ . From (3.12) we get  $\|u_{\text{F}}\|_{1,t,k} \lesssim k^{-1} (\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma})$ . Using (AP3), we arrive at

$$f_{\text{II}} := A_{k,\Omega}^\Delta u_{\text{F}} \in \mathfrak{A}(M_{f,\text{II}}, \gamma_{\text{A}}^\Delta, \mathcal{P}), \quad g_{\text{II}} := A_{k,\Gamma}^\Delta u_{\text{F}} \in \mathfrak{A}(M_{g,\text{II}}, \gamma_{\text{A}}^\Delta, T, \mathcal{P})$$

with

$$M_{f,\text{II}} + kM_{g,\text{II}} \leq C_{\text{A},k}^\Delta k^{-1} (\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma}).$$

Since we chose  $\eta$  larger than  $\gamma_0$  and  $\gamma_{\text{A}}^\Delta$ , we also have

$$f_{\text{II}} := A_{k,\Omega}^\Delta u_{\text{F}} \in \mathfrak{A}(M_{f,\text{II}}, \eta', \mathcal{P}), \quad g_{\text{II}} := A_{k,\Gamma}^\Delta u_{\text{F}} \in \mathfrak{A}(M_{g,\text{II}}, \eta', T, \Gamma),$$

and it follows from (WP3) that  $u_{\text{A,II}} \in \mathfrak{A}(M_{\text{II}}, \vartheta(\eta'), \mathcal{P})$  with

$$M_{\text{II}} \leq CC_{\text{ana},k}^- C_{\text{sol},k}^- k^{-1} (M_{f,\text{II}} + kM_{g,\text{II}}) \leq CC_{\text{ana},k}^- C_{\text{sol},k}^- k^{-2} C_{\text{A},k}^\Delta (\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma}).$$

Estimate (3.10) now follows since

$$u_{\text{A}} := u_{\text{A,I}} + u_{\text{A,II}} \in \mathfrak{A}(M_{\text{I}} + M_{\text{II}}, \vartheta(\eta'), \mathcal{P}).$$

Relabelling the parameter  $\vartheta(\eta')$  as  $\eta'$  and estimating generously  $\|g\|_{-1/2,\Gamma} \lesssim \|g\|_{1/2,\Gamma}$  gives the result.  $\square$

**3.3. Regularity splitting of time-harmonic solutions.** The contraction result of Lemma 3.4 can be iterated to yield a splitting of the solution of Helmholtz problems into a part  $u_F$  with finite regularity and an analytic part  $u_A$ :

**Theorem 3.5** (Abstract regularity splitting). *Assume **(WP)** and **(AP)**. For all  $0 \leq s \leq s_{\max}$ , and for each  $f \in H^s(\mathcal{P})$  and  $g \in H^{s+1/2}(\Gamma)$  there is a splitting*

$$S_k^-(f, g) = u_F + u_A$$

with  $u_F \in H^{s+2}(\mathcal{P})$  satisfying

$$(3.18) \quad \|u_F\|_{s+2,\mathcal{P}} \leq C_s (\|f\|_{s,\mathcal{P}} + \|g\|_{s+1/2,\Gamma}),$$

and  $u_A \in \mathfrak{A}(M, \gamma_s^*, \mathcal{P})$  with

$$(3.19) \quad M \leq C_{\text{ana},k}^- C_{\text{sol},k}^- (1 + k^{-2} C_{A,k}^\Delta) (\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma}),$$

and a constant  $\gamma_s^*$  depending solely on  $s$  and the parameters appearing in **(WP)**, **(AP)**.

*Proof.* The proof iterates the contraction argument in Lemma 3.4. To fix ideas, we select  $q := 1/2$  and denote by  $\eta_s, \eta'_s$  the parameters  $\eta, \eta'$  given by Lemma 3.4. Letting  $f^{(0)} := f$  and  $g^{(0)} := g$ , the splitting

$$S_k^-(f^{(0)}, g^{(0)}) = u_F^{(0)} + u_A^{(0)} + S_k^-(f^{(1)}, g^{(1)}),$$

holds true with contracted data

$$\|f^{(1)}\|_{s,\mathcal{P}} + \|g^{(1)}\|_{s+1/2,\Gamma} \leq q (\|f^{(0)}\|_{s,\mathcal{P}} + \|g^{(0)}\|_{s+1/2,\Gamma});$$

the regular part  $u_F^{(0)} \in H^{s+2}(\mathcal{P})$  satisfies

$$\|u_F^{(0)}\|_{s+2,\mathcal{P}} \leq C_s (\|f^{(0)}\|_{s,\mathcal{P}} + \|g^{(0)}\|_{s+1/2,\Gamma})$$

and the analytic part  $u_A^{(0)} \in \mathfrak{A}(M^{(0)}, \eta'_s, \mathcal{P})$  with

$$M^{(0)} \leq C C_{\text{sol},k}^- (1 + k^{-2} C_{A,k}^\Delta) (\|f^{(0)}\|_{0,\Omega} + \|g^{(0)}\|_{1/2,\Gamma}).$$

We can then repeat this argument to split  $S_k^-(f^{(1)}, g^{(1)})$ , and so on, resulting in an inductive definition of  $(f^{(i)}, g^{(i)}) \in H^s(\mathcal{P}) \times H^{s+1/2}(\Gamma)$ ,  $u_F^{(i)} \in H^{s+2}(\mathcal{P})$  and  $u_A^{(i)} \in \mathfrak{A}(M^{(i)}, \eta'_s, \mathcal{P})$  with

$$\begin{aligned} \|f^{(i)}\|_{s,\mathcal{P}} + \|g^{(i)}\|_{s+1/2,\Gamma} &\leq q^i (\|f^{(0)}\|_{s,\mathcal{P}} + \|g^{(0)}\|_{s+1/2,\Gamma}), \\ \|u_F^{(i)}\|_{s+2,\mathcal{P}} &\leq C_s q^i (\|f^{(0)}\|_{s,\mathcal{P}} + \|g^{(0)}\|_{s+1/2,\Gamma}) \\ M^{(i)} &\leq q^i C C_{\text{sol},k}^- (1 + k^{-2} C_{A,k}^\Delta) (\|f^{(0)}\|_{0,\Omega} + \|g^{(0)}\|_{1/2,\Gamma}). \end{aligned}$$

It follows that we have

$$u = \underbrace{\sum_{i \in \mathbb{N}_0} u_F^{(i)}}_{=: u_F} + \underbrace{\sum_{i \in \mathbb{N}_0} u_A^{(i)}}_{=: u_A},$$

where the two series converge due to the geometric decrease of the factors  $q^i$ . Then, introducing  $\gamma_s^* := \eta'_s$ , estimates (3.18) and (3.19) simply follow since  $q = 1/2$  and  $\sum_{q \in \mathbb{N}_0} q^i = 2$ .  $\square$

## 4. STABILITY AND CONVERGENCE OF ABSTRACT GALERKIN DISCRETIZATIONS

In this section, we utilize the splitting introduced in Theorem 3.5 to analyze Galerkin discretizations. We first establish quasi-optimality and error estimates for abstract discrete spaces. These results are then applied to the particular case of the  $hp$  finite element method, where explicit stability conditions are derived.

The proofs of this section hinge on duality techniques, and require properties of the adjoint wave propagation problem. For the sake of simplicity, we assume here that the considered wave propagation and auxiliary problems are symmetric in the sense that

$$(4.1) \quad b_k^-(v, u) = b_k^-(\bar{u}, \bar{v}), \quad b_k^+(v, u) = b_k^+(\bar{u}, \bar{v}) \quad \forall u, v \in H^{1,t}(\Omega, \Gamma).$$

We mention that (4.1) is not mandatory; however, without (4.1) one needs to assume **(WP)** and **(AP)** for the adjoint problems rather than the primal problem, which we avoid here to simplify the presentation.

**4.1. Abstract Galerkin discretizations.** We start by considering a generic finite-dimensional subspace  $V_h \subset H^{1,t}(\Omega, \Gamma)$ . The corresponding Galerkin approximation  $u_h \in V_h$  to the exact solution  $u \in H^{1,t}(\Omega, \Gamma)$  of (2.7) is given by

$$(4.2) \quad b_k^-(u_h, v_h) = (f, v_h) + \langle g, v_h \rangle \quad \forall v_h \in V_h.$$

The convergence analysis of the Galerkin method is based on a duality argument and thus hinges on the ability of  $V_h$  to approximate solution to the (adjoint) Helmholtz problem. Following previous works on the subject [CFN19, LSW22, MS10, Sau06, Sch74], the key approximation properties of  $V_h$  are encoded in real numbers  $\eta$ , called approximation factors. Specifically, for  $m \geq 1$ ,  $\gamma > 0$ , and the (fixed) tubular neighborhood  $T$  of  $\Gamma$  given as the intersection of the two tubular neighborhoods given by **(WP4)** and **(AP3)**, these adjoint approximation factors  $\eta^{(m)}$ ,  $\eta_\gamma^{(\text{exp})}$  are defined as follows:

$$(4.3a) \quad \eta^{(m)} := \sup_{\substack{(f,g) \in H^{m-1}(\mathcal{D}) \times H^{m-1/2}(\Gamma) \\ \|f\|_{m-1,\mathcal{D}} + \|g\|_{m-1/2,\Gamma} = 1}} \inf_{v_h \in V_h} \|S_k^-(f, g) - v_h\|_{1,t,k},$$

$$(4.3b) \quad \eta_\gamma^{(\text{exp})} := \sup_{\substack{(f,g) \in \mathfrak{A}(M_f, \gamma, \mathcal{D}) \times \mathfrak{A}(M_g, \gamma, T, \Gamma) \\ M_f + kM_g = 1/2}} \inf_{v_h \in V_h} \|S_k^-(f, g) - v_h\|_{1,t,k},$$

To ease the presentation, we introduce the projection

$$\Pi_h v := \arg \min_{v_h \in V_h} \|v - v_h\|_{1,t,k}.$$

**Lemma 4.1.** *Let  $u \in H^{1,t}(\Omega, \Gamma)$  and assume that  $u_h \in V_h$  satisfies*

$$(4.4) \quad b_k^-(u - u_h, v_h) = 0 \quad \forall v_h \in V_h.$$

*Then, the estimate*

$$(4.5) \quad |b_k^-(u - u_h, v)| \leq C \left( 1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A)}^{(\text{exp})} \right) \|u - u_h\|_{1,t,k} \|v\|_{1,t,k}$$

*holds true for all  $v \in H^{1,t}(\Omega, \Gamma)$ , where  $C$  is a constant independent of  $k$ ,  $u$ , and  $v$ .*

*Proof.* For the sake of brevity we write  $e_h := u - u_h$ . For an arbitrary  $v \in H^{1,t}(\Omega, \Gamma)$ , we have

$$\begin{aligned} |b_k^-(e_h, v)| &\stackrel{(4.1)}{=} |b_k^-(\bar{v}, \bar{e}_h)| = |a(\bar{v}, \bar{e}_h) - (T_{k,\Omega}^- \bar{v}, \bar{e}_h) - \langle T_{k,\Gamma}^- \bar{v}, \bar{e}_h \rangle| \\ &\stackrel{(2.11)}{\leq} |a(\bar{v}, \bar{e}_h)| + |(R_{k,\Omega}^- \bar{v}, \bar{e}_h)| + |\langle R_{k,\Gamma}^- \bar{v}, \bar{e}_h \rangle| + |(A_{k,\Omega}^- \bar{v}, \bar{e}_h) + \langle A_{k,\Gamma}^- \bar{v}, \bar{e}_h \rangle| \\ &\stackrel{(2.12), (2.13)}{\leq} C \|e_h\|_{1,t,k} \|v\|_{1,t,k} + |(A_{k,\Omega}^- \bar{v}, \bar{e}_h) + \langle A_{k,\Gamma}^- \bar{v}, \bar{e}_h \rangle|. \end{aligned}$$

The analytic part is now treated using a duality argument: Using (4.1), we have by setting  $\bar{\phi} := S_k^-(A_{k,\Omega}^- \bar{v}, A_{k,\Gamma}^- \bar{v})$

$$(A_{k,\Omega}^- \bar{v}, \bar{e}_h) + \langle A_{k,\Gamma}^- \bar{v}, \bar{e}_h \rangle \stackrel{(4.1)}{=} b_k^-(S_k^-(A_{k,\Omega}^- \bar{v}, A_{k,\Gamma}^- \bar{v}), \bar{e}_h) = b_k^-(e_h, \phi) = b_k^-(e_h, \phi - \Pi_h \phi).$$

Then, using (WP4), we have

$$\|\phi - \Pi_h \phi\|_{1,t,k} = \|\bar{\phi} - \Pi_h \bar{\phi}\|_{1,t,k} = \inf_{w_h \in V_h} \|S_k^-(A_{k,\Omega}^- \bar{v}, A_{k,\Gamma}^- \bar{v}) - w_h\|_{1,t,k} \leq C_{A,k}^- \eta_{\vartheta(\gamma_A)}^{(\text{exp})} \|v\|_{1,t,k},$$

and the conclusion follows from (WP1) since

$$|(A_{k,\Omega}^- \bar{v}, \bar{e}_h) + \langle A_{k,\Gamma}^- \bar{v}, \bar{e}_h \rangle| = |b_k^-(e_h, \phi - \Pi_h \phi)| \leq C_{\text{cont},k}^- \|e_h\|_{1,t,k} \|\phi - \Pi_h \phi\|_{1,t,k}. \quad \square$$

**Lemma 4.2.** *Assume that  $u \in H^{1,t}(\Omega, \Gamma)$  and  $u_h \in V_h$  are such that*

$$(4.6) \quad b_k^-(u - u_h, v_h) = 0 \quad \forall v_h \in V_h.$$

*Then, there exist two constants  $\rho$  and  $\mu$  independent of  $k$ ,  $u$ , and  $u_h$  such that*

$$(4.7) \quad \begin{aligned} &\left( \mu - \left( 1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A)}^{(\text{exp})} \right) \left( k\eta^{(1)} + C_{A,k}^\Delta \eta_{\vartheta(\gamma_A)}^{(\text{exp})} \right) \right) \|u - u_h\|_{1,t,k} \\ &\leq \rho \left( 1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A)}^{(\text{exp})} \right) \|u - \Pi_h u\|_{1,t,k}. \end{aligned}$$

*Proof.* We introduce the Galerkin error  $e_h := u - u_h$ . Recalling (2.15) and (4.6), we estimate with  $C_0 := 1/C_{\text{coer}}^+$

$$\begin{aligned} \frac{1}{C_0} \|e_h\|_{1,t,k}^2 &\leq \text{Re } b_k^+(e_h, \sigma e_h) \leq |b_k^+(e_h, e_h)| \leq \\ &|b_k^\Delta(e_h, e_h)| + |b_k^-(e_h, e_h)| = |b_k^\Delta(e_h, e_h)| + |b_k^-(e_h, u - \Pi_h u)|. \end{aligned}$$

With Lemma 4.1 we may then estimate

$$\|e_h\|_{1,t,k}^2 \leq C_0 |b_k^\Delta(e_h, e_h)| + C_1 \left( 1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A)}^{(\text{exp})} \right) \|e_h\|_{1,t,k} \|u - \Pi_h u\|_{1,t,k},$$

where  $C_1$  does not depend on  $k$ ,  $u$  or  $u_h$ . On the other hand, we have upon setting  $\bar{\phi}_F := S_k^-(R_{k,\Omega}^\Delta \bar{e}_h, R_{k,\Gamma}^\Delta \bar{e}_h)$  and  $\bar{\phi}_A := S_k^-(A_{k,\Omega}^\Delta \bar{e}_h, A_{k,\Gamma}^\Delta \bar{e}_h)$ :

$$\begin{aligned} b_k^\Delta(e_h, e_h) &\stackrel{(4.1)}{=} b_k^\Delta(\bar{e}_h, \bar{e}_h) \\ &\stackrel{(2.18)}{=} (R_{k,\Omega}^\Delta \bar{e}_h, \bar{e}_h) + \langle R_{k,\Gamma}^\Delta \bar{e}_h, \bar{e}_h \rangle + (A_{k,\Omega}^\Delta \bar{e}_h, \bar{e}_h) + \langle A_{k,\Gamma}^\Delta \bar{e}_h, \bar{e}_h \rangle \\ &= b_k^-(S_k^-(R_{k,\Omega}^\Delta \bar{e}_h, R_{k,\Gamma}^\Delta \bar{e}_h), \bar{e}_h) + b_k^-(S_k^-(A_{k,\Omega}^\Delta \bar{e}_h, A_{k,\Gamma}^\Delta \bar{e}_h), \bar{e}_h) \\ &= b_k^-(e_h, \phi_F) + b_k^-(e_h, \phi_A). \end{aligned}$$

Using Lemma 4.1, we can then write

$$\begin{aligned}
 |b_k^-(e_h, \phi_F)| &= |b_k^-(e_h, \phi_F - \Pi_h \phi_F)| \\
 &\stackrel{\text{L. 4.1}}{\leq} C \left(1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})}\right) \|e_h\|_{1,t,k} \|\phi_F - \Pi_h \phi_F\|_{1,t,k} \\
 &\stackrel{(4.3a)}{\leq} C \left(1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})}\right) \eta^{(1)} \|e_h\|_{1,t,k} (\|R_{k,\Omega}^\Delta \bar{e}_h\|_{0,\Omega} + \|R_{k,\Gamma}^\Delta \bar{e}_h\|_{1/2,\Omega}) \\
 &\stackrel{(\text{AP3})}{\leq} C_2 \left(1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})}\right) k \eta^{(1)} \|e_h\|_{1,t,k}^2,
 \end{aligned}$$

where  $C_2$  is independent of  $k$ ,  $u$ , and  $u_h$ . Note that in the above estimate, we only use **(AP3)** for the operators  $R_{k,\Omega}^\Delta$  and  $R_{k,\Gamma}^\Delta$ , i.e., equations (2.19) and (2.20), which does not introduce any dependence on  $k$  in the constant  $C_2$ . Similarly, we have

$$\begin{aligned}
 |b_k^-(e_h, \phi_A)| &\stackrel{\text{L. 4.1}}{\leq} C \left(1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})}\right) \|e_h\|_{1,t,k} \|\phi_A - \Pi_h \phi_A\|_{1,t,k} \\
 &\leq C \left(1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})}\right) C_{A,k}^\Delta \eta_{\vartheta(\gamma_A^\Delta)}^{(\text{exp})} \|e_h\|_{1,t,k}^2,
 \end{aligned}$$

which leads to

$$|b_k^\Delta(e_h, e_h)| \leq C_3 \left(1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})}\right) \left(k \eta^{(1)} + C_{A,k}^\Delta \eta_{\vartheta(\gamma_A^\Delta)}^{(\text{exp})}\right) \|e_h\|_{1,t,k}^2,$$

with  $C_3$  independent of  $k$ ,  $u$ , and  $u_h$ . Therefore,

$$\begin{aligned}
 (4.8) \quad &\left(1 - C_0 C_3 \left(1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})}\right) \left(k \eta^{(1)} + C_{A,k}^\Delta \eta_{\vartheta(\gamma_A^\Delta)}^{(\text{exp})}\right)\right) \|e_h\|_{1,t,k}^2 \leq \\
 &C_1 \left(1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})}\right) \|e_h\|_{1,t,k} \|u - \Pi_h u\|_{1,t,k}.
 \end{aligned}$$

Estimate (4.7) then follows by setting  $1/\mu := C_0 \max(1, C_3)$  and  $\rho := \mu C_1$  and multiplying both sides of (4.8) by  $\mu$ .  $\square$

**Theorem 4.3** (abstract quasi-optimality). *Assume **(WP)** and **(AP)**. Let  $s \in [0, s_{\max}]$ ,  $f \in H^s(\mathcal{P})$ , and  $g \in H^{s+1/2}(\Gamma)$ . Recall the constant  $\mu$  from Lemma 4.2, and assume that*

$$(4.9) \quad k \eta^{(1)} + C_{A,k}^\Delta \eta_{\vartheta(\gamma_A^\Delta)}^{(\text{exp})} \leq \frac{\mu}{4}, \quad C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})} \leq 1.$$

*Then, there exists  $C > 0$  independent of  $k$ ,  $f$ ,  $g$ , and there exists a unique  $u_h \in V_h$  such that (4.2) holds, and for  $u := S_k^-(f, g)$ , we have*

$$(4.10) \quad \|u - u_h\|_{1,t,k} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{1,t,k}.$$

*Proof.* Fix  $s \in [0, s_{\max}]$ ,  $f \in H^s(\mathcal{P})$ , and  $g \in H^{s+1/2}(\Gamma)$ . Let  $u_h$  be any element of  $V_h$  such that (4.2) holds true. We easily see that  $u := S_k^-(f, g)$  and  $u_h$  satisfy the assumptions of Lemma 4.2. Injecting (4.9) in (4.7), we have

$$\frac{\mu}{2} \|u - u_h\|_{1,t,k} \leq 2\rho \|u - \Pi_h u\|_{1,t,k} = 2\rho \inf_{v_h \in V_h} \|u - v_h\|_{1,t,k},$$

so that (4.10) holds for all  $u_h$  satisfying (4.2) with  $C := 4\rho/\mu$ .

Next, we observe that having established (4.10) for all  $u_h$  satisfying (4.2) implies, by linearity, the uniqueness of the solution to (4.10). Since (4.10) corresponds to finite-dimensional linear system, this proves the existence and uniqueness of  $u_h$ .  $\square$

**4.2. Application to  $hp$ -FEM.** We now focus on the case where the discretization subspace  $V_h$  is an  $hp$ -FEM space of piecewise (mapped) polynomials. We consider the case where the mesh  $\mathcal{T}_h$  associated with the finite element space is aligned with the partition  $\mathcal{P}$  and excludes hanging nodes. Since we are working with analytic interfaces, dedicated assumptions are required [MS10].

**Assumption 4.4** (Quasi-uniform regular fitted meshes). Let  $\widehat{K}$  be the reference simplex in spatial dimension  $d = 2, 3$ . The mesh  $\mathcal{T}_h$  is a partition of  $\Omega$  into non-overlapping elements  $K$  such that  $\cup_{K \in \mathcal{T}_h} \widehat{K} = \widehat{\Omega}$ . For each  $K \in \mathcal{T}_h$ , there exists a bijective mapping  $F_K: \widehat{K} \rightarrow K$  that can be written as  $F_K = R_K \circ A_K$ , where  $A_K$  is affine and  $R_K$  is an analytic map. Specifically, there exist constants  $C_{\text{affine}}, C_{\text{metric}}, \gamma > 0$  independent of  $K$  such that

$$\begin{aligned} \|A'_K\|_{L^\infty(\widehat{K})} &\leq C_{\text{affine}} h_K, & \|(A'_K)^{-1}\|_{L^\infty(\widehat{K})} &\leq C_{\text{affine}} h_K^{-1}, \\ \|(R'_K)^{-1}\|_{L^\infty(\widehat{K})} &\leq C_{\text{metric}}, & \|\nabla^n R_K\|_{L^\infty(\widehat{K})} &\leq C_{\text{metric}} \gamma^n n! \quad \forall n \in \mathbb{N}_0. \end{aligned}$$

Here,  $\tilde{K} = A_K(\widehat{K})$  and  $h_K > 0$  denotes the element diameter. We assume (“no hanging nodes”) that the element maps of elements sharing an edge or a face induce the same parametrization on that edge or face. (See [MS22, Sec. 8.1] for a precise formulation for the case  $d = 3$  and [Mel02, Def. 2.4.1] for the case  $d = 2$ .) Furthermore, we assume that the mesh  $\mathcal{T}_h$  resolves the interfaces of  $\mathcal{P}$ , i.e., each element  $K$  lies entirely in one subdomain  $P \in \mathcal{P}$ .  $\blacksquare$

For  $p \geq 1$ , we introduce the space  $S_p(\mathcal{T}_h)$  as the space of piecewise mapped polynomials of degree  $p$ :

$$S_p(\mathcal{T}_h) := \left\{ u \in H^1(\Omega) : u|_K \circ F_K \in \mathcal{P}_p(\widehat{K}) \text{ for all } K \in \mathcal{T}_h \right\}.$$

We note  $S^p(\mathcal{T}_h) \subset H^{1,t}(\Omega, \Gamma)$ . We will require approximation operators that approximate functions in the  $\|\cdot\|_{1,t,k}$ -norm:

**Proposition 4.5** (Interpolation errors). *Let Assumption 4.4 be valid. There is an operator  $I_h : H^2(\mathcal{P}) \cap H^1(\Omega) \rightarrow S_p(\mathcal{T}_h)$  with the following properties:*

- (i)  $I_h$  is defined elementwise, i.e.,  $(I_h v)|_K$  depends solely on  $v|_{\widehat{K}}$ .
- (ii) For  $1 \leq m \leq p$  and for all  $v \in H^{m+1}(\mathcal{P})$ , we have

$$(4.11) \quad \|v - I_h v\|_{1,t,k} \leq C_m \left( 1 + \left( \frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left( \frac{h}{p} \right)^m \|v\|_{m+1, \mathcal{P}}.$$

- (iii) Fix  $\tilde{C} > 0$  and assume  $kh/p \leq \tilde{C}$ . For all  $\gamma$ , there exist constants  $\sigma_\gamma, C_\gamma$  solely depending on  $\gamma, C_{\text{metric}}$  of Assumption 4.4, and  $\tilde{C}$  such that for all  $v \in \mathfrak{A}(M, \gamma, \mathcal{P})$ ,

$$(4.12) \quad \|v - I_h v\|_{1,t,k} \leq C_\gamma \left( 1 + \left( \frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left( \left( \frac{h}{h + \sigma_\gamma} \right)^p + k \left( \frac{kh}{\sigma_\gamma p} \right)^p \right) M.$$

- (iv) Finally, for  $v \in \mathfrak{A}(M, \gamma, \mathcal{P})$  and  $0 \leq q \leq p$ , we have

$$(4.13) \quad \|v - I_h v\|_{1,t,k} \leq C_{q,\gamma} \left( 1 + \left( \frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left( \left( \frac{h}{p} \right)^q + k \left( \frac{kh}{\sigma_\gamma p} \right)^p \right) M.$$

*Proof.* The construction is essentially that of [MS10, Thm. 5.5], where the approximation in the norm  $\|\nabla \cdot\|_{0,\mathcal{P}} + k\|\cdot\|_{0,\Omega}$  is considered instead of  $\|\cdot\|_{1,t,k}$ . We will therefore be brief. The operator  $I_h$  is defined on the reference simplex and taken to be the one of [MS22, Lemma 8.3], which in turn is actually the operator  $\widehat{\Pi}_p^{\text{grad},3d}$  (for  $d = 3$ ) or the operator  $\widehat{\Pi}_p^{\text{grad},2d}$  (for  $d = 2$ ) of [MR20].

*Proof of (ii):* The  $p$ -dependence of the approximation properties are spelled out in [MS22, Lemma 8.3] and the  $h$ -dependence follows from scaling arguments.

*Proof of (iii):* Denote by  $\widehat{I}_h$  the operator on the reference simplex, and introduce for  $K \in \mathcal{T}_h$  and  $t \in \{0, 1\}$  the norm  $\|v\|_{1,t,k,K}^2 := \|\nabla v\|_{L^2(K)}^2 + k^2\|v\|_{L^2(K)}^2 + k^{1-2t}|v|_{H^t(\partial K)}^2$ . By standard scaling arguments and the approximation properties of  $\widehat{I}_h$  (see the discussion in the proof of [MS22, Lemma 8.3]) we get

$$\|v - I_h v\|_{1,t,k,K} \lesssim h^{d/2-1} p^{-1} \left( 1 + \frac{kh}{p} + \left( \frac{kh}{p} \right)^{1/2-t} \right) |\widehat{v}|_{2,\widehat{K}},$$

where  $\widehat{u} := u \circ F_K$  is the pull-back of  $u$  to  $\widehat{K}$ . Noting that  $\widehat{I}_h$  is a projection onto  $\mathcal{P}_p(\widehat{K})$  (and thus  $I_h$  a projection onto  $S_p(\mathcal{T}_h)$ ) we arrive at

$$\|v - I_h v\|_{1,t,k,K} \lesssim h^{d/2-1} p^{-1} \left( 1 + \frac{kh}{p} + \left( \frac{kh}{p} \right)^{1/2-t} \right) \inf_{w \in \mathcal{P}_p(\widehat{K})} |\widehat{v} - w|_{2,\widehat{K}}.$$

For  $v \in \mathfrak{A}(M, \gamma, \mathcal{P})$ , we can now proceed as in the proof of [MS10, Thm. 5.5] using in particular the approximation result [MS10, Lemma C.2] for  $\inf_{w \in \mathcal{P}_p(\widehat{K})} |\widehat{v} - w|_{2,\widehat{K}}$ . This leads to the desired result for  $t \in \{0, 1\}$ . For  $t \in (0, 1)$  the boundary estimate  $|v - I_h v|_{t,\Gamma}$  then follows by the interpolation inequality.

*Proof of (iv):* If  $q \leq p$ , the function

$$h \mapsto \left( \frac{h}{h + \sigma_\gamma} \right)^{p-q}$$

is non-decreasing. Hence, since  $h \leq C$ , we have

$$\begin{aligned} \left( \frac{h}{h + \sigma_\gamma} \right)^p &= \left( \frac{h}{h + \sigma_\gamma} \right)^q \left( \frac{h}{h + \sigma_\gamma} \right)^{p-q} \\ &\leq \sigma_\gamma^{-q} h^q \left( \frac{C}{C + \sigma_\gamma} \right)^{p-q} = \sigma_\gamma^{-q} h^q \rho_\gamma^{p-q} = (\rho_\gamma \sigma_\gamma)^{-q} h^q \rho_\gamma^p, \end{aligned}$$

where  $\rho_\gamma < 1$ . It is then clear that  $\rho_\gamma^p \leq C_{\gamma,q} p^{-q}$ , and it follows that

$$\left( \frac{h}{h + \sigma_\gamma} \right)^p \leq C_{q,\gamma} \left( \frac{h}{p} \right)^q.$$

Then, estimate (4.13) follows from (4.12).  $\square$

Our key result concerning finite element discretizations is established under an additional assumption of polynomial well-posedness that we detail now:



**Assumption 4.6** (Polynomial well-posedness). There exist constants  $\alpha_a, \alpha_s, \alpha_c, C \geq 0$ , independent of  $k$  such that

$$(4.14) \quad C_{\text{ana},k}^- \leq Ck^{\alpha_a},$$

$$(4.15) \quad C_{\text{sol},k}^- \leq Ck^{\alpha_s},$$

$$(4.16) \quad C_{\text{cont},k}^- + C_{A,k}^- + k^{-2}C_{A,k}^\Delta \leq Ck^{\alpha_c}.$$

**Lemma 4.7** (Approximation factors for  $hp$  finite elements). *Assume that  $p \geq 1 + \alpha_a + 2\alpha_c + \alpha_s$ . Then, we have*

$$(4.17) \quad \eta^{(m)} \leq C_{m,\alpha_s,\alpha_c} \left( 1 + \left( \frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left( \left( \frac{h}{p} \right)^m + \frac{h}{p} \left( \frac{kh}{p} \right)^{\alpha_a + \alpha_s + \alpha_c} + k^{\alpha_a + \alpha_s + \alpha_c + 1} \left( \frac{kh}{\sigma_{\gamma_m^*} p} \right)^p \right)$$

for all  $m \geq 1$ , and

$$(4.18) \quad \eta_\gamma^{(\text{exp})} \leq C_{\gamma,\alpha_s} k^{-1} \left( 1 + \left( \frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left( \left( \frac{h}{p} \right)^{2\alpha_c + 1} \left( \frac{kh}{p} \right)^{\alpha_a + \alpha_s} + k^{\alpha_a + \alpha_s + 1} \left( \frac{kh}{\sigma_{\vartheta(\gamma)} p} \right)^p \right)$$

for all  $\gamma > 0$ .

*Proof.* Let  $1 \leq m \leq p$ , and fix  $(f, g) \in H^{m-1}(\mathcal{P}) \times H^{m-1/2}(\Gamma)$  with

$$\|f\|_{m-1,\mathcal{P}} + \|g\|_{m-1/2,\Gamma} = 1.$$

For  $u := S_k^-(f, g)$ , Theorem 3.5 provides the splitting  $u = u_F + u_A$ , where  $u_F \in H^{m+1}(\Omega)$  and  $u_A \in \mathfrak{A}(M, \gamma_m^*, \mathcal{P})$  with

$$(4.19) \quad \begin{aligned} \|u_F\|_{m+1,\mathcal{P}} &\leq C_{m-1}, \\ M &\leq CC_{\text{ana},k}^- C_{\text{sol},k}^- (1 + k^{-2}C_{A,k}^\Delta) \leq Ck^{\alpha_a + \alpha_s + \alpha_c}. \end{aligned}$$

It is convenient to abbreviate

$$(4.20) \quad F_{t,k,h,p} := \left( 1 + \left( \frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right).$$

Applying (4.11), we have

$$\|u_F - I_h u_F\|_{1,t,k} \leq C_m F_{t,k,h,p} \left( \frac{h}{p} \right)^m \|u_F\|_{m+1,\mathcal{P}} \leq C'_m F_{t,k,h,p} \left( \frac{h}{p} \right)^m.$$

On the other hand, since  $p \geq 1 + \alpha_a + \alpha_s + 2\alpha_c$  by assumption, we can use (4.13) with  $q = \alpha_a + \alpha_s + \alpha_c + 1$  and  $\gamma = \gamma_m^*$ , showing that

$$\begin{aligned} \|u_A - I_h u_A\|_{1,t,k} &\leq C_q F_{t,k,h,p} \left( \left( \frac{h}{p} \right)^{\alpha_a + \alpha_s + \alpha_c + 1} + k \left( \frac{kh}{\sigma_{\gamma_m^*} p} \right)^p \right) M \\ &\stackrel{(4.19)}{\leq} C_q k^{\alpha_a + \alpha_s + \alpha_c} F_{t,k,h,p} \left( \left( \frac{h}{p} \right)^{\alpha_a + \alpha_s + \alpha_c + 1} + k \left( \frac{kh}{\sigma_{\gamma_m^*} p} \right)^p \right) \\ &\leq C_q F_{t,k,h,p} \left( \frac{h}{p} \left( \frac{kh}{p} \right)^{\alpha_a + \alpha_s + \alpha_c} + k^{\alpha_a + \alpha_s + \alpha_c + 1} \left( \frac{kh}{\sigma_{\gamma_m^*} p} \right)^p \right). \end{aligned}$$

Then, we have

$$\begin{aligned} \|u - I_h u\|_{1,t,k} &\leq \|u_F - I_h u_F\|_{1,t,k} + \|u_A - I_h u_A\|_{1,t,k} \\ &\leq C_{m,q} F_{t,k,h,p} \left( \left( \frac{h}{p} \right)^m + \frac{h}{p} \left( \frac{kh}{p} \right)^{\alpha_a + \alpha_s + \alpha_c} + k^{\alpha_a + \alpha_s + \alpha_c + 1} \left( \frac{kh}{\sigma_{\gamma_m^*} p} \right)^p \right), \end{aligned}$$

and (4.17) follows from the definition of  $\eta^{(m)}$  in (4.3a).

Consider now  $(f, g) \in \mathfrak{A}(M_f, \gamma, \mathscr{P}) \times \mathfrak{A}(M_g, \gamma, T, \Gamma)$  with  $M_f + kM_g = 1/2$ , and set  $u := S_k^-(f, g)$ . We know from (WP3) that  $u \in \mathfrak{A}(M_u, \vartheta(\gamma), \mathscr{P})$  with

$$M_u \leq CC_{\text{ana},k}^- C_{\text{sol},k}^- k^{-1} (M_f + kM_g) = C_{\text{ana},k}^- CC_{\text{sol},k}^- k^{-1} \leq C k^{\alpha_a + \alpha_s - 1}.$$

It then follows from (4.13) with  $q = \alpha_a + \alpha_s + 2\alpha_c + 1$  and  $\gamma = \vartheta(\gamma)$  that

$$\|u - I_h u\|_{1,t,k} \leq C_{\gamma,\alpha} k^{\alpha_a + \alpha_s - 1} F_{t,k,h,p} \left( \left( \frac{h}{p} \right)^{\alpha_a + \alpha_s + 2\alpha_c + 1} + k \left( \frac{kh}{\sigma_{\vartheta(\gamma)} p} \right)^p \right),$$

and (4.18) follows from the definition of  $\eta_\gamma^{(\text{exp})}$  in (4.3b).  $\square$

Our main result is then a direct consequence of Theorem 4.3 and Lemma 4.7.

**Theorem 4.8** (discrete stability of  $hp$ -FEM). *Let (WP) and (AP) as well as Assumptions 4.4 and 4.6 hold true. Then, for all  $c_2 > 0$  there exist constants  $c_1, C$  depending solely on  $c_2$  and the constants appearing in (WP), (AP), Assumptions 4.6, 4.4 such that under the scale resolution condition*

$$(4.21) \quad \frac{kh}{p} \leq c_1 \quad \text{and} \quad p \geq 1 + 2\alpha_c + \alpha_a + \alpha_s + c_2 \log k$$

the Galerkin solution  $u_{hp} \in S_p(\mathcal{T}_h)$  of (4.2) exists, is unique, and satisfies

$$(4.22) \quad \|u - u_{hp}\|_{1,t,k} \leq C \inf_{v_{hp} \in S_p(\mathcal{T}_h)} \|u - v_{hp}\|_{1,t,k}.$$

*Proof.* We may assume that  $kh/p \leq c_1 < 1$  so that easy calculations show

$$(4.23) \quad \sup_{t \in [0,1]} F_{t,k,h,p} = F_{1,k,h,p},$$

$$(4.24) \quad F_{1,k,h,p} \left( \frac{kh}{p} \right)^{1/2} \leq 1 + \left( \frac{kh}{p} \right)^{1/2} + \left( \frac{kh}{p} \right)^{3/2} \leq 1 + c_1^{1/2} + c_1^{3/2} \leq 3.$$

Let  $c_2 > 0$  and fix  $p \geq 1 + 2\alpha_c + \alpha_a + \alpha_s + c_2 \log k$ . Then, we have in particular that  $\log k \leq (p - 1/2)/c_2$ , so that for any  $\sigma, b > 0$

$$k^b \leq \left( e^{b/c_2} \right)^{p-1/2},$$

and

$$(4.25) \quad k^b \left( \frac{kh}{\sigma p} \right)^{p-1/2} \leq \left( e^{b/c_2} \right)^{p-1/2} \left( \frac{c_1}{\sigma} \right)^{p-1/2} = \left( \frac{e^{b/c_2}}{\sigma} c_1 \right)^{p-1/2}.$$

Since  $p \geq 1 + 2\alpha_c + \alpha_a + \alpha_s$  by assumption, we can use (4.17) with  $m = 1$ . Since  $t \leq 1$ , picking  $\sigma = \sigma_{\gamma_1^*}$  and  $b = \alpha_a + \alpha_s + \alpha_c + 2$  in (4.25), we get

$$\begin{aligned} k\eta^{(1)} &\stackrel{(4.17),(4.23)}{\leq} CF_{1,k,h,p} \left( \frac{kh}{p} + \left( \frac{kh}{p} \right)^{\alpha_a + \alpha_s + \alpha_c + 1} + k^{\alpha_a + \alpha_s + \alpha_c + 2} \left( \frac{kh}{\sigma_{\gamma_1^*} p} \right)^p \right) \\ &\stackrel{(4.24)}{\leq} 3C \left( \left( \frac{kh}{p} \right)^{1/2} + \left( \frac{kh}{p} \right)^{\alpha_a + \alpha_s + \alpha_c + 1/2} + k^{\alpha_a + \alpha_s + \alpha_c + 2} \sigma_{\gamma_1^*}^{-1/2} \left( \frac{kh}{\sigma_{\gamma_1^*} p} \right)^{p-1/2} \right) \\ &\leq 3C \left( c_1^{1/2} + c_1^{\alpha_a + \alpha_s + \alpha_c + 1/2} + \sigma_{\gamma_1^*}^{-1/2} \left( \frac{e^{(\alpha_a + \alpha_s + \alpha_c + 2)/c_2}}{\sigma_{\gamma_1^*}} c_1 \right)^{p-1/2} \right). \end{aligned}$$

We then employ (4.18) with  $\gamma = \vartheta(\gamma_A^-)$ , leading to

$$\begin{aligned} &C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})} \\ &\leq C_{\alpha_s, \alpha_c} k^{2\alpha_c - 1} F_{t,k,h,p} \left( \left( \frac{h}{p} \right)^{2\alpha_c + 1} \left( \frac{kh}{p} \right)^{\alpha_a + \alpha_s} + k^{\alpha_a + \alpha_s + 1} \left( \frac{kh}{\sigma_{\vartheta(\gamma)} p} \right)^p \right) \\ &\stackrel{(4.23),(4.24)}{\leq} 3C \left( k^{-1} \left( \frac{h}{p} \right)^1 \left( \frac{kh}{p} \right)^{2\alpha_c + \alpha_a + \alpha_s - 1/2} + \sigma_{\vartheta(\gamma)}^{-1/2} k^{2\alpha_c + \alpha_a + \alpha_s} \left( \frac{kh}{\sigma_{\vartheta(\gamma)} p} \right)^{p-1/2} \right) \\ &\stackrel{(4.25)}{\leq} 3C \left( c_1^{2\alpha_c + \alpha_a + \alpha_s + 1/2} + \sigma_{\vartheta(\gamma)}^{-1/2} \left( \frac{e^{(2\alpha_c + \alpha_a + \alpha_s)/c_2}}{\sigma_{\vartheta(\gamma_A^-)}} c_1 \right)^{p-1/2} \right), \end{aligned}$$

where we have used (4.25) with  $\sigma = \sigma_{\vartheta(\gamma_A^-)}$  and  $b = 2\alpha_c + \alpha_a + \alpha_s$ . Finally, since  $k \geq k_0$ , a similar reasoning shows that

$$\begin{aligned} C_{A,k}^\Delta \eta_{\vartheta(\gamma_A^\Delta)}^{(\text{exp})} &\leq C k^{\alpha_c} \eta_{\vartheta(\gamma_A^\Delta)}^{(\text{exp})} \leq C k^{2\alpha_c} \eta_{\vartheta(\gamma_A^\Delta)}^{(\text{exp})} \\ &\leq 3C_\alpha \left( c_1^{2\alpha_c + \alpha_a + \alpha_s + 1/2} + \sigma_{\vartheta(\gamma_A^\Delta)}^{-1/2} \left( \frac{e^{(2\alpha_c + \alpha_a + \alpha_s)/c_2}}{\sigma_{\vartheta(\gamma_A^\Delta)}} c_1 \right)^{p-1/2} \right). \end{aligned}$$

At this point, we assume in addition to  $c_1 \leq 1$  that

$$(4.26) \quad \frac{e^{(\alpha_a + \alpha_s + \alpha_c + 2)/c_2}}{\sigma_{\gamma_1^*}} c_1 \leq 1, \quad \frac{e^{(2\alpha_c + \alpha_a + \alpha_s)/c_2}}{\sigma_{\vartheta(\gamma_A^-)}} c_1 \leq 1, \quad \frac{e^{(2\alpha_c + \alpha_a + \alpha_s)/c_2}}{\sigma_{\vartheta(\gamma_A^\Delta)}} c_1 \leq 1,$$

which leads to the simplified expressions (using  $p \geq 1$ )

$$k\eta^{(1)} + C_{A,k}^\Delta \eta_{\vartheta(\gamma_A^\Delta)}^{(\text{exp})} \leq C_{\alpha_s, \alpha_c} \left[ 1 + \sigma_{\gamma_1^*}^{-1/2} \sqrt{\frac{e^{(\alpha_a + \alpha_c + \alpha_s + 2)/c_2}}{\sigma_{\gamma_1^*}}} + \sigma_{\vartheta(\gamma_A^\Delta)}^{-1/2} \sqrt{\frac{e^{(2\alpha_c + \alpha_a + \alpha_s)/c_2}}{\sigma_{\vartheta(\gamma_A^\Delta)}}} \right] c_1^{1/2}$$

and

$$C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})} \leq C_{\alpha_c, \alpha_s} \left( 1 + \sigma_{\vartheta(\gamma_A^-)}^{-1/2} \sqrt{\frac{e^{(2\alpha_c + \alpha_a + \alpha_s)/c_2}}{\sigma_{\vartheta(\gamma_A^-)}}} \right) c_1^{1/2}.$$

Then, we see that requiring, in addition to (4.26), that

$$C_{\alpha_c, \alpha_s} \left( 1 + \frac{\sqrt{e^{(\alpha_a + \alpha_c + \alpha_s + 2)/c_2}}}{\sigma_{\gamma_1^*}} + \frac{\sqrt{e^{(2\alpha_c + \alpha_a + \alpha_s)/c_2}}}{\sigma_{\vartheta(\gamma_A^-)}} + \frac{\sqrt{e^{(2\alpha_c + \alpha_a + \alpha_s)/c_2}}}{\sigma_{\vartheta(\gamma_A^+)}} \right) c_1^{1/2} \\ \leq \min(\mu/4, 1),$$

then the requirements of Theorem 4.3 are met, which leads to the result.  $\square$

## 5. APPLICATION TO WAVE PROAGATION PROBLEMS

In this section, we verify that the Assumptions **(WP)**, **(AP)** of Section 2 are in fact satisfied for a variety of time-harmonic wave propagation problems. We emphasize that Assumption **(WP2)** and, in particular, the polynomial stability assumption expressed in (4.16) is assumed and has to be proved separately. We discuss cases where such a stability property holds true in Remark 5.3 below.

Throughout, we will consider the heterogeneous Helmholtz equation, and in all examples the sesquilinear form  $a(\cdot, \cdot)$  will be of the form

$$(5.1) \quad a(u, v) = (A(x)\nabla u, \nabla v).$$

The strong form of problem (2.7) is

$$(5.2) \quad L_k^- u := -k^2 n^2 u - \nabla \cdot (A(x)\nabla u) = f \quad \text{in } \Omega, \quad \partial_{n_A} u - T_{k, \Gamma}^- u = g \quad \text{on } \Gamma,$$

with the co-normal derivative  $\partial_{n_A} v := \mathbf{n} \cdot (A\nabla v)|_\Gamma$  and the outer normal vector  $\mathbf{n}$ . Hence, in all examples the volume operator is given by  $T_{k, \Omega}^- = k^2 n^2$ . The complex-valued coefficients  $A \in L^\infty(\Omega, \text{GL}(\mathbb{C}^d))$  and  $n \in L^\infty(\Omega, \mathbb{C})$  are assumed to satisfy the following:  $A$  is uniformly elliptic, i.e.,

$$(5.3a) \quad \text{Re}(\xi^H A(x)\xi) \geq a_{\min} |\xi|^2 \quad \forall \xi \in \mathbb{C}^d \quad \forall x \in \Omega.$$

Furthermore,  $A$  and  $n$  are piecewise analytic, specifically,

$$(5.3b) \quad \|\nabla^p A\|_{L^\infty(\Omega \setminus \Gamma_{\text{interf}})} \leq C_A \gamma_A^p p! \quad \forall p \in \mathbb{N}_0,$$

$$(5.3c) \quad \|\nabla^p n\|_{L^\infty(\Omega \setminus \Gamma_{\text{interf}})} \leq C_n \gamma_n^p p! \quad \forall p \in \mathbb{N}_0,$$

for fixed constants  $C_A, \gamma_A$  and  $C_n, \gamma_n$ .

**Remark 5.1.** The coefficients  $A$  and  $n$  are allowed to depend on  $k$ . We merely require that the constants in (5.3a)–(5.3c) not depend on  $k$ . Therefore, equations including volume damping, classically written in the form  $-k^2 n^2 u + ikmu - \nabla \cdot (A\nabla u)$  with some  $m$  satisfying the condition that  $\|\nabla^p m\|_{L^\infty(\Omega \setminus \Gamma_{\text{interf}})} \leq C_m \gamma_m^p p!$  for all  $p \in \mathbb{N}_0$  are included in this setting. Also certain PML-variants with  $k$ -dependent matrices  $A$  are included in this setting.  $\blacksquare$

The strong form of problem (2.16) will be

$$(5.4) \quad L_k^+ u := -\nabla \cdot (A(x)\nabla u) + k^2 u = f \quad \text{in } \Omega, \quad \partial_{n_A} u - T_{k, \Gamma}^+ u = g \quad \text{on } \Gamma$$

so that  $T_{k, \Omega}^+ = k^2$ . The problems (5.2), (5.4), which we discuss in more detail in Sections 5.1–5.3, thus differ in the boundary conditions, i.e., the operators  $T_{k, \Gamma}^-$  and  $T_{k, \Gamma}^+$ .

Throughout this Section 5, we assume:

**Assumption 5.2.** The boundary  $\Gamma$  and the interface  $\Gamma_{\text{interf}}$  are analytic.  $\blacksquare$

**Remark 5.3.** The polynomial stability assumption is known to hold true in a variety of situations. The growth of the stability constant is governed by the trajectories of the rays (solutions to the bi-characteristic equation) associated with the wave propagation problem. This is intuitive to understand: if some rays are trapped, more time will be needed for the energy to leak out of the domain, resulting in a larger stability constant. On the mathematical level, the connection between waves and rays can be made rigorous through semi-classical analysis [LP64, Mel75, Vai75], and three scenarios must be distinguished.

First of all, we say that the problem is non-trapping if all rays escape the domain in finite time, i.e., there are no trapped rays [GSW20]. In this case, it is well-known that  $C_{\text{sol},k}^-$  is bounded uniformly in  $k$ . Early works on the subject have focused on the scattering problem from a Dirichlet obstacle (see e.g. [LP64, Mel75]), but the estimate also applies to problems with heterogeneous media under radial monotonicity conditions on the coefficients (see, e.g., [MS19]).

The next situation to consider is the case of “weak” trapping [CWSGS20]. In this case, rays may be trapped, but all the trajectories of trapped rays are unstable. This is for instance the case for the scattering problem from a Dirichlet obstacle consisting of two balls or two squares. In this case, although rays may be trapped between the two parts of the obstacle, polynomial well-posedness holds true. For instance, it is known that  $C_{\text{sol},k}^- \lesssim \log k$  for the case of two balls [Ika88], and that  $C_{\text{sol},k}^- \lesssim k^2$  for the case of two squares [CWSGS20]. Similar estimates holds true when the weak trapping is generated by space-varying coefficients [CFS23].

Finally, the case of “strong” trapping arises when trapped rays have stable trajectories. This is for instance the case when scattering from a “ $C$  shaped” obstacle. It can be shown that polynomial stability does not hold true for all frequencies [MS19]. However, the set of frequencies causing the super-algebraic growth is, in a sense, small. Specifically, it is shown in [LSW21] that polynomial stability holds true even in this case with  $C_{\text{sol},k}^- \lesssim k^{(5d+1)/2}$ , up to excluding a set of frequencies of arbitrarily small measure.

The above stability bounds are stated for a Helmholtz problem with a radiation condition at infinity or, equivalently, an exact DtN operator. However, these bounds still hold true if the DtN operator is replaced by a PML, see [GLS23]. Bounds for impedance boundary conditions are also given, e.g., in [BCFG17, Mel95]. Higher-order absorbing boundary conditions are also considered in [GLS21]. ■

**5.1. Heterogeneous Helmholtz problem with Robin boundary conditions and PML.** The heterogeneous Helmholtz problem with Robin boundary conditions corresponds to the choice

$$T_{k,\Gamma}^- u := ikGu$$

in (5.2) for some function  $G$  that is analytic in a fixed tubular neighborhood  $T$  of  $\Gamma$ . In (5.4), we select  $T_{k,\Gamma}^+ := 0$ . Note that we allow for complex-valued coefficients  $A$  and  $n$ , which allows us to cover forms of PML where the path deformation into the complex plane is based on the use of polar/spherical coordinate [CM98, GLSW22]. Specifically, it can be seen following the discussion in [CFV22, Remark 2.1] and the expression of  $A$  given in [GLSW22, Eq (1.11)] that (WP3) holds true if the damping parameter is small enough (the width of the PML, however, is not constrained).

**Lemma 5.4** ((AP) for Robin b.c.). *Let Assumption 5.2 and (5.3) be valid. Let  $G$  be analytic in a tubular neighborhood  $T$  of  $\Gamma$ . For  $L_k^-, L_k^+$  given by (5.2) and (5.4) with  $T_{k,\Gamma}^- = ikG$  and  $T_{k,\Gamma}^+ = 0$  the following holds with  $t = 1/2$ :*

- (i)  $b_k^+$  is bounded uniformly in  $k$ , and **(AP1)** holds with  $\sigma = 1$ .
- (ii)  $\|S_k^+(f, g)\|_{1,t,k} \leq C [k^{-1}\|f\|_{0,\Omega} + k^{-1/2}\|g\|_{0,\Gamma}]$ , and **(AP2)** holds for any fixed  $s_{\max} \geq 0$ .
- (iii) The splittings of  $T_{k,\Omega}^- - T_{k,\Omega}^+$  and  $T_{k,\Gamma}^- - T_{k,\Gamma}^+$  of **(AP3)** are given by  $R_{k,\Omega}^\Delta u = k^2(n^2 + 1)u$ ,  $A_{k,\Omega}^\Delta = 0$  and  $R_{k,\Gamma}^\Delta u = ikGu$ ,  $A_{k,\Gamma}^\Delta = 0$  and hence satisfy **(AP3)** for any fixed  $s_{\max} \geq 0$ .

*Proof.* Proof of (i), (iii): Continuity of  $b_k^+$  follows by inspection. The coercivity **(AP1)** of  $b_k^+$  with  $\sigma = 1$  follows from (5.3a) and  $T_{k,\Gamma}^+ = 0$ .

*Proof of (ii):* The coercivity of  $b_k^+$  implies for  $u := S_k^+(f, g)$

$$\|u\|_{1,t,k}^2 \lesssim |(f, u)| + |(g, u)_\Gamma| \lesssim k^{-1}\|f\|_{0,\Omega}\|u\|_{0,\Omega} + k^{-1/2}\|g\|_{0,\Gamma}k^{1/2}\|u\|_{0,\Gamma}.$$

The trace inequality  $k\|v\|_{0,\Gamma} \lesssim \|v\|_{1,\Omega} + k\|v\|_{0,\Omega} \lesssim \|v\|_{1,t,k}$  then implies the a priori bound. The validity of **(AP2)** follows from standard elliptic regularity theory for Neumann problems since  $\Gamma$  and  $\Gamma_{\text{interf}}$  are smooth. The fact that  $A$  is  $\mathbb{C}^{d \times d}$ -valued instead of pointwise SPD is discussed in more detail in the proof of Lemma 5.5 below.

*Proof of (iii):* follows by inspection. □

**Lemma 5.5** ((WP) for Robin b.c.). *Let Assumption 5.2 and (5.3) be valid. Let  $G$  be analytic in a tubular neighborhood  $T$  of  $\Gamma$ . Assume **(WP2)**. Then:*

- (i) **(WP1)** holds with  $C_{\text{cont},k}^- = O(1)$ , and **(WP3)** holds with  $C_{\text{ana},k}^- = O(1)$  (both uniformly in  $k$ ).
- (ii) **(WP4)** holds with  $R_{k,\Omega}^- u = k^2 n^2 u$ ,  $A_{k,\Omega}^- = 0$  and  $R_{k,\Gamma}^- = ikGu$ ,  $A_{k,\Gamma}^- = 0$ . In particular, for the constants in Assumption 4.6, we have  $C_{\text{cont},k}^- = O(1)$  (uniformly in  $k$ ) and  $C_{A,k}^- = 0$  and  $C_{A,k}^\Delta = 0$ .
- (iii) **(WP5)** holds.

*Proof.* Proof of (i):  $C_{\text{cont},k}^- = O(1)$  follows by inspection. To see **(WP3)**, we first discuss the case that the matrix  $A$  is pointwise SPD as this case is essentially covered by the literature. Specifically, we observe that the function  $u = S_k^-(\tilde{f}, \tilde{g})$  with  $\tilde{f} \in \mathfrak{A}(C_{\tilde{f}}, \gamma_{\tilde{f}}, \mathcal{P})$ ,  $\tilde{g} \in \mathfrak{A}(C_{\tilde{g}}, \gamma_{\tilde{g}}, T, \Gamma)$ , satisfies, upon setting  $\varepsilon = 1/k$ ,

$$(5.5) \quad \begin{aligned} -\varepsilon^2 \nabla \cdot (A \nabla u) - n^2 u &= \varepsilon^2 \tilde{f} && \text{in } \Omega, \\ \varepsilon^2 \partial_{n_A} u &= \varepsilon(\varepsilon \tilde{g} + iGu) && \text{on } \Gamma, \end{aligned}$$

The regularity of solutions of these problems form has been addressed in [MS11, Lemma 4.13] and [Mel02, Prop. 5.4.5, Rem. 5.4.6]. The procedure is as follows: the regularity can be analyzed locally and the boundary or the interface may be flattened by (local) changes of variables so that one is reduced to the analysis on balls (interior estimates and transmission problems) or half-balls (boundary estimates). Note that that Lemma 2.2 (and [Mel02, Lemma 4.3.4]) provide that the analyticity classes are invariant under bi-analytic changes of variables. Hence, one may apply [Mel02, Prop. 5.5.1] (interior analytic regularity), [Mel02, Prop. 5.5.3] (boundary analytic regularity for Neumann problems) as well as [Mel02, Prop. 5.5.4] (interface analytic regularity) if  $\Gamma_{\text{interf}} \neq \emptyset$ , to problem (5.5). Considering as an

example the case of boundary regularity, we apply [Mel02, Prop. 5.5.3] with the following choices

$$\begin{aligned} C_A &= O(C_A), C_b = 0, C_c = O(C_{n^2}), C_f = O(\varepsilon^2 C_{\tilde{f}}), C_{G_1} = O(\varepsilon C_{\tilde{g}}), C_{G_2} = O(1), \\ \gamma_A &= O(\gamma_A), \gamma_b = 0, \gamma_c = O(\gamma_{n^2}), \gamma_f = O(\gamma_{\tilde{f}}), \quad \gamma_{G_1} = O(\gamma_{\tilde{g}}), \quad \gamma_{G_2} = O(1), \end{aligned}$$

where  $\gamma_{n^2}$  and  $C_{n^2}$  are the analyticity constants of the coefficient  $n^2$  and the  $O(\cdot)$ -notation absorbs the modification of constants due to the analytic change of variables when transforming to a half-ball. By combining the finitely many local contributions, we arrive at

$$\begin{aligned} \|\nabla^p u\|_{L^2(\Omega \setminus \Gamma_{\text{interf}})} &\lesssim C\gamma^p \max\{k, p\}^p (k^{-2}C_{\tilde{f}} + k^{-1}C_{\tilde{g}} + k^{-1}\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \\ &\lesssim C\gamma^p \max\{k, p\}^p (k^{-2}C_{\tilde{f}} + k^{-1}C_{\tilde{g}} + C_{\text{sol},k}^- k^{-1}(C_{\tilde{f}} + kC_{\tilde{g}})) \\ &\lesssim C\gamma^p \max\{k, p\}^p C_{\text{sol},k}^- k^{-1}(C_{\tilde{f}} + kC_{\tilde{g}}) \quad \forall p \geq 2, \end{aligned}$$

where we applied the stability estimate (2.9) as well as  $C_{\text{sol},k}^- \gtrsim 1$ . The stability estimate (2.9) shows that the bound is valid for  $p \in \{0, 1\}$  as well, so that  $S_k^-(\tilde{f}, \tilde{g}) \in \mathfrak{A}(CC_{\text{sol},k}^- k^{-1}(C_{\tilde{f}} + kC_{\tilde{g}}), \gamma, \mathcal{P})$ , i.e., **(WP3)** holds with  $C_{\text{ana},k}^- = O(1)$ .

For the case that the matrix  $A$  is not SPD but merely satisfies (5.3a), we note that the proof of [Mel02, Props. 5.5.1, 5.5.3, 5.5.4] relies on a (piecewise)  $H^2$ -regularity result for the principal part of the operator (cf. [Mel02, Lemmas 5.5.5, 5.5.8, 5.5.9]), which in turns hinges on the difference quotient technique of Nirenberg. We note that the condition (5.3a) provides a coercivity that makes the difference quotient technique applicable for Neumann and transmission problems so that also [Mel02, Props. 5.5.1, 5.5.3, 5.5.4] holds for complex-valued coefficients  $A$ ,  $n$  satisfying (5.3). *Proof of (ii), (iii):* By inspection.  $\square$

Under the assumption **(WP2)** we have the following quasi-optimality result for the heterogeneous Helmholtz equation with Robin boundary conditions:

**Theorem 5.6** (Robin b.c. and PML). *Let  $t = 1/2$ . Let Assumption 5.2 and (5.3) be valid. Let  $G$  be analytic on a tubular neighborhood of  $\Gamma$ . For problem (5.2) with  $T_{k,\Gamma}^- = ikG$  assume the polynomial well-posedness of the solution operator  $S_k^-$  of Assumption 4.6. In the discretization setting of Assumption 4.4, for each  $c_2 > 0$  there are constants  $c_1, C > 0$  independent of  $h, p$ , and  $k$  such that under the scale-resolution condition (4.21) the quasi-optimality result (4.22) holds.*

*Proof.* Combine Lemmas 5.4 and 5.5 with the stability Assumption 4.6.  $\square$

**5.2. Heterogeneous Helmholtz problem in the full space.** The heterogeneous Helmholtz problem with Sommerfeld radiation condition in full space  $\mathbb{R}^d$  is to find  $U \in H_{\text{loc}}^1(\mathbb{R}^d)$  such that

$$(5.6) \quad \begin{aligned} -k^2 n^2 U - \nabla \cdot (A \nabla U) &= f && \text{in } \mathbb{R}^d, \\ |\partial_r U - ikU| &= o\left(\|x\|^{\frac{1-d}{2}}\right) && \text{for } \|x\| \rightarrow \infty, \end{aligned}$$

is satisfied in the weak sense. Here,  $\partial_r$  denotes the derivative in the radial direction. We assume  $f, A$ , and  $n$  (which are defined on  $\mathbb{R}^d$ ) to be local in the sense that  $f, A - I$ , and  $n - 1$  are compactly supported.

To approximate the problem with a FEM, we introduce a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^d$  such that  $\text{supp } f \subset \bar{\Omega}$ ,  $\text{supp } (A - I) \subset \Omega$ , and  $\text{supp } (n - 1) \subset \bar{\Omega}$ . Since we



have a lot of freedom in the choice of  $\Omega$ , we will assume that it has been designed so that

$$(5.7) \quad \Gamma := \partial\Omega \text{ is analytic and } \Omega^+ = \mathbb{R}^d \setminus \bar{\Omega} \text{ is non-trapping, [BSW16, Def. 1.1].}$$

Problem (5.6) can then be reformulated on  $\Omega$  using the Dirichlet-to-Neumann operator  $\text{DtN}_k: H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ , which is given by  $g \mapsto \text{DtN}_k g := \partial_n w := \mathbf{n} \cdot \nabla w$ , and  $w \in H_{\text{loc}}^1(\mathbb{R}^d \setminus \Omega)$  is the unique weak solution to

$$\begin{aligned} -\Delta w - k^2 w &= 0 && \text{in } \Omega^+, \\ w &= g && \text{on } \Gamma, \\ |\partial_r w - ikw| &= o\left(\|x\|^{\frac{1-d}{2}}\right) && \text{for } \|x\| \rightarrow \infty. \end{aligned}$$

The heterogeneous Helmholtz problem in full space is to find  $u \in H^1(\Omega)$  such that

$$(5.8) \quad \begin{aligned} L_k^- u &:= -k^2 n^2 u - \nabla \cdot (A \nabla u) = f && \text{in } \Omega, \\ \partial_{n_A} u - T_{k,\Gamma}^- u &:= \partial_{n_A} u - \text{DtN}_k u = g && \text{on } \Gamma. \end{aligned}$$

For  $g = 0$ , the solution  $u$  of (5.8) is then the restriction to  $\Omega$  of the solution of (5.6). If  $d = 2$ , we assume that  $\text{diam } \Omega < 1$ , which ensures that the single layer operator  $V_0: H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$  appearing below is boundedly invertible.

We define the operator  $\text{DtN}_0$  as the map  $\text{DtN}_0: g \mapsto \partial_n u^{\text{ext}}$ , where  $u^{\text{ext}}$  solves

$$(5.9a) \quad -\Delta u^{\text{ext}} = 0 \quad \text{in } \Omega^+,$$

$$(5.9b) \quad u^{\text{ext}} = g \quad \text{on } \Gamma,$$

$$(5.9c) \quad u^{\text{ext}}(x) = \begin{cases} O(1/\|x\|) & \text{as } \|x\| \rightarrow \infty \text{ for } d = 3 \\ b \ln \|x\| + O(1/\|x\|) & \text{as } \|x\| \rightarrow \infty \text{ for some } b \in \mathbb{R} \text{ for } d = 2. \end{cases}$$

**Remark 5.7.** The reason for choosing the particular condition (5.9c) at  $\infty$  is that it ensures the representation formula

$$u^{\text{ext}}(x) = -(\tilde{V}_0 \partial_n u^{\text{ext}})(x) + (\tilde{K}_0 u^{\text{ext}})(x)$$

with the usual single layer  $\tilde{V}_0$  and double layer potential  $\tilde{K}_0$  for the Laplacian; see, e.g., [McL00, Thm. 8.9] or [CS85, Lem. 3.5]. This particular choice will reappear in the analysis of the difference  $\text{DtN}_k - \text{DtN}_0$  in Lemma 5.8. For our choice of condition at  $\infty$ , one has the Calderón identity

$$(5.10) \quad \begin{pmatrix} u^{\text{ext}} \\ \partial_n u^{\text{ext}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + K_0 & -V_0 \\ -D_0 & \frac{1}{2} - K'_0 \end{pmatrix} \begin{pmatrix} u^{\text{ext}} \\ \partial_n u^{\text{ext}} \end{pmatrix},$$

where  $V_0$ ,  $K_0$ ,  $K'_0$ , and  $D_0$  denote the single layer, double layer, adjoint double layer, and the hypersingular operator, respectively (see, e.g., [SS11, Ste08] for the precise definition). By definition, we have  $\partial_n u^{\text{ext}} = \text{DtN}_0 u^{\text{ext}}$ . From the Calderón identity (5.10), one easily concludes

$$(5.11) \quad (\text{DtN}_0 u^{\text{ext}}, u^{\text{ext}})_{L^2(\Gamma)} = -(D_0 u^{\text{ext}}, u^{\text{ext}})_{L^2(\Gamma)} - (V_0 \partial_n u^{\text{ext}}, \partial_n u^{\text{ext}})_{L^2(\Gamma)} \leq 0,$$

where we employed that  $D_0$  is positive semidefinite and  $V_0$  is positive definite, [Ste08, Thm. 6.23, Cor. 6.25].  $\blacksquare$

For the auxiliary problem (5.4), we select  $T_{k,\Gamma}^+ = \text{DtN}_0$ .

In order to show that Problem (5.8) fits into the framework of Section 4 we need to analyze the operators  $\text{DtN}_k$  and  $\text{DtN}_0$ :

**Lemma 5.8.** *Let  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{2, 3\}$ , be a bounded Lipschitz domains with smooth boundary  $\Gamma$ . Then the following holds:*

- (i)  $-\langle \text{DtN}_0 u, u \rangle \geq 0$  for all  $u \in H^{1/2}(\Gamma)$ .
- (ii) The operator  $\text{DtN}_0$  is a bounded linear operator  $H^{s+1/2}(\Gamma) \rightarrow H^{s-1/2}(\Gamma)$  for every  $s \geq 0$ .
- (iii) Assume (5.7). Let the ball  $B_R(0)$  satisfy  $\bar{\Omega} \subset B_R(0)$ , set  $\Omega_R := B_R(0) \setminus \Gamma$ . Let  $s \geq 0$  be given. Then

$$\text{DtN}_k - \text{DtN}_0 = kB + \llbracket \partial_n \tilde{A} \rrbracket$$

where the linear operators  $B: H^s(\Gamma) \rightarrow H^s(\Gamma)$  and  $\tilde{A}: H^s(\Gamma) \rightarrow C^\infty(\Omega_R)$  satisfy for all  $u \in H^s(\Gamma)$

$$\|Bu\|_{s,\Gamma} \lesssim \|u\|_{s,\Gamma}, \quad \tilde{A}u \in \mathfrak{A}(Ck^\beta \|u\|_{s,\Gamma}, \gamma, \Omega_R)$$

with  $\beta = 7/2 + d/2$ , and constants  $C, \gamma \geq 0$  independent of  $k$ . Here, the operator  $\llbracket \partial_n \tilde{A} \rrbracket$  realizes the jump of the normal derivative of  $\tilde{A}v$  and is given by  $\llbracket \partial_n \tilde{A} \rrbracket v := \mathbf{n} \cdot \nabla(\tilde{A}v)|_{\Omega^+} - \mathbf{n} \cdot \nabla(\tilde{A}v)|_{\Omega}$  (see (A.9)).

- (iv) Let  $\Gamma = \partial B_1(0) \subset \mathbb{R}^3$  be the unit ball in dimension  $d = 3$ . Then  $\text{DtN}_k - \text{DtN}_0: H^s(\Gamma) \rightarrow H^s(\Gamma)$  satisfies, for every  $s \geq 0$ ,

$$\|\text{DtN}_k u - \text{DtN}_0 u\|_{s,\Gamma} \lesssim k \|u\|_{s,\Gamma} \quad \forall u \in H^s(\Gamma).$$

*Proof.* Proof of (i):  $-\langle \text{DtN}_0 u, u \rangle \geq 0$  is asserted in (5.11).

*Proof of (ii):* Follows from regularity properties of elliptic boundary value problems. Alternatively, one could appeal to the representation of the (exterior)  $\text{DtN}_0$  as  $\text{DtN}_0 = V_0^{-1}(-1/2 + K_0)$  and the mapping properties of  $V_0$  and  $K_0$  as given in, e.g., [SS11, Thm. 3.2.2].

*Proof of (iii):* See Appendix A.

*Proof of (iv):* For  $\Gamma = \partial B_1(0) \subset \mathbb{R}^3$ , the operator  $\text{DtN}_k$  has an explicit series representation in terms of spherical harmonics. That is, denoting by  $Y_\ell^m$  the standard spherical harmonics, a function  $u \in L^2(\Gamma)$  can be expanded as

$$u = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_\ell^m Y_\ell^m.$$

One has the norm equivalence  $\|u\|_{H^s(\Gamma)}^2 \sim \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} |u_\ell^m|^2 (\ell+1)^{2s}$  for every  $s \geq 0$ , see, e.g., [Néd01, Sec. 2.5.1]. The operators  $\text{DtN}_0$  and  $\text{DtN}_k$  can be written as

$$\text{DtN}_0 u = - \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell+1) u_\ell^m Y_\ell^m, \quad \text{DtN}_k u = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} z_\ell(k) u_\ell^m Y_\ell^m,$$

with explicit estimates for the symbol  $z_\ell(k)$  given in [Néd01, Sec. 2.6.3]. The formulas for  $\text{DtN}_0$  and  $\text{DtN}_k$  immediately give

$$(5.12) \quad \text{DtN}_k u - \text{DtN}_0 u = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (z_\ell(k) + \ell + 1) u_\ell^m Y_\ell^m,$$

From [DI01, Lem. 3.2, eqn. (3.28)], where the operator  $\text{DtN}_k$  has the opposite sign, we have

$$(5.13) \quad \ell + 1 - k \leq -\text{Re } z_\ell(k) \leq \ell + 1 + k.$$

Consequently, we immediately have

$$(5.14) \quad |\text{Re } z_\ell(k) + \ell + 1| \leq k.$$

From [Néd01, Thm. 2.6.1, eqn. (2.6.24)] we have

$$(5.15) \quad 0 \leq \text{Im } z_\ell(k) \leq k.$$

Combining (5.14) and (5.15) yields

$$(5.16) \quad |z_\ell(k) + \ell + 1| \leq 2k.$$

For  $u \in H^s(\Gamma)$  and with the previous estimate we have

$$\begin{aligned} \|\text{DtN}_k u - \text{DtN}_0 u\|_{s,\Gamma}^2 &\sim \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell + 1)^{2s} |z_\ell(k) + \ell + 1|^2 |u_\ell^m|^2 \\ &\leq (2k)^2 \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell + 1)^{2s} |u_\ell^m|^2 Y_\ell^m \sim (2k)^2 \|u\|_{s,\Gamma}^2, \end{aligned}$$

which yields the result.  $\square$

**Lemma 5.9** ((AP) for full space problem). *Let Assumption 5.2 and (5.3), (5.7) be valid. For  $L_k^-, L_k^+$  given by (5.2), (5.4) with  $T_{k,\Gamma}^- = \text{DtN}_k$  and  $T_{k,\Gamma}^+ = \text{DtN}_0$  the following holds with  $t = 1/2$ :*

- (i)  $b_k^+$  is bounded uniformly in  $k$ , and (AP1) holds with  $\sigma = 1$ .
- (ii)  $\|S_k^+(f, g)\|_{1,t,k} \leq C [k^{-1}\|f\|_{0,\Omega} + k^{-1/2}\|g\|_{0,\Gamma}]$ , and (AP2) holds for any fixed  $s_{\max} \geq 0$ .
- (iii) The splitting of  $T_{k,\Omega}^- - T_{k,\Omega}^+$  in (AP3) is given by  $R_{k,\Omega}^\Delta u = k^2(1 + n^2)u$ ,  $A_{k,\Omega}^\Delta = 0$ . The splitting of  $T_{k,\Gamma}^- - T_{k,\Gamma}^+$  in (AP3) is given by  $R_{k,\Gamma}^\Delta u = kB$  and  $A_{k,\Gamma}^\Delta = \llbracket \partial_n \tilde{A} \rrbracket$  with the operators  $B, \llbracket \partial_n \tilde{A} \rrbracket$  given by Lemma 5.8(iii). (See also Remark 2.1(iii) for the interpretation of  $\llbracket \partial_n \tilde{A} \rrbracket$  as a mapping into analyticity classes of the form  $\mathfrak{A}(M, \gamma, T, \Gamma)$ .)
- (iv) For the case  $\Gamma = \partial B_1(0) \subset \mathbb{R}^3$ , the statement (iii) holds true with the operator  $A_{k,\Gamma}^\Delta = 0$  and the operator  $R_{k,\Gamma}^\Delta = B = \text{DtN}_k - \text{DtN}_0$  of order zero bounded uniformly in  $k$  as described in Lemma 5.8(iv).

*Proof.* *Proof of (i):* Continuity of  $b_k^+$  follows by inspection. The coercivity (AP1) of  $b_k^+$  with  $\sigma = 1$  follows from (5.3a) and the positivity of  $-T_{k,\Gamma}^+ = -\text{DtN}_0$  by (5.11).

*Proof of (ii):* Set  $w := S_k^+(f, g)$ . The a priori bound

$$(5.17) \quad \|w\|_{1,\Omega} + k\|w\|_{0,\Omega} + |w|_{1/2,\Gamma} \lesssim \|S_k^+(f, g)\|_{1,t,k} \lesssim k^{-1}\|f\|_{0,\Omega} + k^{-1/2}\|g\|_{0,\Gamma}$$

follows by Lax-Milgram. To show the shift theorem asserted in (AP2), we reformulate the problem solved by  $w = S_k^+(f, g)$ , i.e.,

$$-\nabla \cdot (A\nabla w) = f - k^2 w \quad \text{in } \Omega, \quad \partial_n w - \text{DtN}_0 w = g \quad \text{on } \Gamma.$$

as a transmission problem. We define the function

$$\tilde{w} := \begin{cases} w & \text{in } \Omega \\ w^{\text{ext}} := -\tilde{V}_0(\text{DtN}_0 \gamma^{\text{int}} w) + \tilde{K}_0 \gamma^{\text{int}} w & \text{in } \Omega^+. \end{cases}$$

By the mapping properties of the potentials  $\tilde{V}_0$ ,  $\tilde{K}_0$ , [SS11, Ste08], we have for any bounded domain  $\tilde{\Omega}$

$$(5.18) \quad \|\tilde{w}\|_{H^1(\tilde{\Omega})} \lesssim \|w\|_{1,\Omega} = \|S_k^+(f, g)\|_{1,t,k} \stackrel{(5.17)}{\lesssim} k^{-1} \|f\|_{0,\Omega} + k^{-1/2} \|g\|_{0,\Gamma}.$$

The function  $\tilde{w}$  satisfies

$$\begin{aligned} -\nabla \cdot (A \nabla \tilde{w}) + k^2 \tilde{w} &= f & \text{in } \Omega, & & -\Delta \tilde{w} &= 0 & \text{in } \Omega^+, \\ \llbracket \tilde{w} \rrbracket &= 0 & \text{on } \Gamma, & & \llbracket \partial_n \tilde{w} \rrbracket &= g & \text{on } \Gamma. \end{aligned}$$

Let  $\tilde{\Omega} \supset \supset \Omega$  with  $\partial \tilde{\Omega}$  smooth. Let  $\chi \in C_0^\infty(\tilde{\Omega})$  with  $\chi \equiv 1$  on a neighborhood of  $\bar{\Omega}$ . Upon writing  $\hat{A}$  for the extension of  $A$  by the identity matrix and the characteristic function  $\chi_\Omega$  of  $\Omega$ , we have

$$(5.19) \quad \begin{aligned} -\nabla \cdot (\hat{A} \nabla (\chi \tilde{w})) + k^2 \mathbb{1}_\Omega \chi \tilde{w} &= \hat{f} & \text{in } \tilde{\Omega}, \\ \llbracket \chi \tilde{w} \rrbracket &= 0 & \text{on } \Gamma, \\ \llbracket \partial_n (\chi \tilde{w}) \rrbracket &= g & \text{on } \Gamma, \\ \chi \tilde{w} &= 0 & \text{on } \partial \tilde{\Omega}. \end{aligned}$$

with  $\hat{f} = f$  in  $\Omega$  and  $\hat{f} = 2\nabla \tilde{w} \cdot \nabla \chi + \tilde{w} \Delta \chi$  in  $\tilde{\Omega} \setminus \bar{\Omega}$ . Since  $\hat{A}$  satisfies (5.3a) with  $\Omega$  replaced by  $\tilde{\Omega}$ , the Lax-Milgram Lemma gives for the solution  $\chi \tilde{w}$  of (5.19)

$$(5.20) \quad \begin{aligned} \|\nabla(\chi \tilde{w})\|_{L^2(\tilde{\Omega})} + \|(1 + k^2 \chi_\Omega)(\chi \tilde{w})\|_{L^2(\tilde{\Omega})} &\lesssim \|\hat{f}\|_{H^{-1}(\tilde{\Omega})} + \|g\|_{H^{-1/2}(\Gamma)} \\ &\lesssim \|\hat{f}\|_{L^2(\tilde{\Omega})} + \|g\|_{1/2,\Gamma} \lesssim \|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma} + \|\tilde{w}\|_{1,\tilde{\Omega} \setminus \Omega}. \end{aligned}$$

The function  $\chi \tilde{w}$  satisfies an elliptic transmission problem with piecewise smooth coefficients in the equation and smooth interface  $\Gamma \cup \Gamma_{\text{interf}}$ . Although the coefficient  $\hat{A}$  is complex-valued, it satisfies (5.3a) and, as discussed in the proof of Lemma 5.4, (piecewise)  $H^2$ -regularity as formulated in [Mel02, Prop. 5.4.8] is available. Such an  $H^2$ -regularity gives  $\chi \tilde{w} \in H^2(\tilde{\Omega} \setminus (\Gamma_{\text{interf}} \cup \Gamma))$  and consequently  $w = (\chi \tilde{w})|_\Omega \in H^2(\Omega \setminus \Gamma_{\text{interf}})$  together with

$$\begin{aligned} \|w\|_{2,\Omega \setminus \Gamma_{\text{interf}}} &= \|\chi \tilde{w}\|_{2,\Omega \setminus \Gamma_{\text{interf}}} \lesssim \|\hat{f} + k^2 \mathbb{1}_\Omega \chi \tilde{w}\|_{0,\tilde{\Omega}} + \|g\|_{1/2,\Gamma} + \|\chi \tilde{w}\|_{1,\tilde{\Omega}} \\ &\lesssim \|f\|_{0,\Omega} + \|2\nabla(\chi \tilde{w}) \cdot \nabla \chi + (\chi \tilde{w}) \Delta \chi\|_{0,\tilde{\Omega} \setminus \Omega} + k^2 \|w\|_{0,\Omega} + \|g\|_{1/2,\Gamma} \\ &\stackrel{(5.20)}{\lesssim} \|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma} + k^2 \|w\|_{0,\Omega} \stackrel{(5.17)}{\lesssim} \|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma} + k^{1/2} \|g\|_{0,\Gamma}. \end{aligned}$$

*Proof of (iii), (iv):* follows from Lemma 5.8.  $\square$

**Lemma 5.10** ((WP) for full space). *Let Assumption 5.2 and (5.3), (5.7) be valid. For  $L_k^-, L_k^+$  given by (5.2), (5.4) with  $T_{k,\Gamma}^- = \text{DtN}_k$  and  $T_{k,\Gamma}^+ = \text{DtN}_0$  the following holds with  $t = 1/2$  if (WP2) holds:*

- (i) (WP1) holds with  $C_{\text{cont},k}^- = O(k^\beta)$  with  $\beta$  given by Lemma 5.8, and (WP3) holds with  $C_{\text{ana},k}^- = O(1)$  (uniformly in  $k$ ).

- (ii) Let the operators  $B$  and  $[\![\partial_n \tilde{A}]\!]$  be given by Lemma 5.8(iii). Then **(WP4)** holds with  $R_{k,\Omega}^- u = k^2 n u$ ,  $A_{k,\Omega}^- = 0$  and the choices  $R_{k,\Gamma}^- = \text{DtN}_0 + kB$ ,  $A_{k,\Gamma}^- = [\![\partial_n \tilde{A}]\!]$ . In particular, for the constants in Assumption 4.6 we have  $C_{\text{cont},k}^- + C_{A,k}^- + C_{A,k}^\Delta = O(k^\beta)$  with  $\beta$  given by Lemma 5.8.
- (iii) For  $\Gamma = \partial B_1(0) \subset \mathbb{R}^3$  the assumption **(WP4)** holds with  $R_{k,\Omega}^- u = k^2 n u$ ,  $A_{k,\Omega}^- = 0$  and the choices  $R_{k,\Gamma}^- = \text{DtN}_k$  and  $A_{k,\Gamma}^- = 0$ . In particular, for the constants in Assumption 4.6 we have  $C_{\text{cont},k}^- + C_{A,k}^- + C_{A,k}^\Delta = O(1)$ .
- (iv) **(WP5)** holds.

*Proof.* Proof of (i): The full space problem corresponds to the choice  $T_{k,\Gamma}^- u = \text{DtN}_k u$ . As in the proof of Lemma 5.9, the function  $u$  is the restriction to  $\Omega$  of the solution of the following full-space transmission problem:

$$\begin{aligned} -\nabla \cdot (A \nabla u) - k^2 n^2 u &= f && \text{in } \Omega \cup \Omega^+, \\ \llbracket u \rrbracket &= 0 && \text{on } \Gamma, \\ \llbracket \partial_n u \rrbracket &= g && \text{on } \Gamma. \end{aligned}$$

$u$  satisfies the radiation condition.

From now on the proof is completely analogous to the Robin case of Lemma 5.5 replacing the use of Neumann conditions by transmission conditions on  $\Gamma$ .

*Proof of (ii):* We decompose  $T_{k,\Gamma}^- = \text{DtN}_k = \text{DtN}_0 + (\text{DtN}_k - \text{DtN}_0) = (\text{DtN}_0 + kB) + [\![\partial_n \tilde{A}]\!] =: R_{k,\Gamma}^- + A_{k,\Gamma}^-$ . The operator  $A_{k,\Gamma}^-$  is of the form given in **(WP4)**. For the operator  $R_{k,\Gamma}^-$ , we observe using the mapping properties of  $\text{DtN}_0$  and  $B$

$$\begin{aligned} \left| \langle R_{k,\Gamma}^- u, v \rangle \right| &\leq |\langle \text{DtN}_0 u, v \rangle| + k |\langle B u, v \rangle| \lesssim \|u\|_{1/2,\Gamma} \|v\|_{1/2,\Gamma} + k \|u\|_{0,\Gamma} \|v\|_{0,\Gamma} \\ &\stackrel{t=1/2}{\lesssim} \|u\|_{1,t,k} \|v\|_{1,t,k}. \end{aligned}$$

*Proof of (iv):* By inspection. □

Fixing the computational domain  $\Omega$  and the coupling boundary  $\Gamma$  on which the  $\text{DtN}$ -operator is employed, we have the following quasi-optimality result:

**Theorem 5.11** (full space). *Let  $t = 1/2$ . Let Assumption 5.2 and (5.3), (5.7) be valid. For problem (5.2) with  $T_{k,\Gamma}^- = \text{DtN}_k$  assume the polynomial well-posedness of the solution operator  $S_k^-$  of Assumption 4.6. In the discretization setting of Assumption 4.4, for each  $c_2 > 0$  there are constants  $c_1, C > 0$  independent of  $h, p$ , and  $k$  such that under the scale-resolution condition (4.21) the quasi-optimality result (4.22) holds.*

*Proof.* Combine Lemmas 5.9 and 5.10 with the stability Assumption 4.6. □

**Remark 5.12.** In practice, it is hard to realize the operator  $\text{DtN}_k$  exactly. Possible numerical realizations of  $\text{DtN}_k$  include FEM-BEM coupling [MMPR20] or to truncate the series in the case of spherical  $\Gamma$  [XY21]. An analysis of the additional variational crimes incurred is beyond the scope of the present analysis. ■

**5.3. Heterogeneous Helmholtz problem with second order ABCs.** Various boundary conditions that are formally of higher order than the Robin boundary conditions have been proposed in the literature, notably for case of a sphere in  $\mathbb{R}^2$ . Our theory covers those proposed by Bayliss-Gunzburger-Turkel [BGT82],

Bayliss-Gunzburger-Turkel	Engquist-Mayda	Feng
$\alpha = -\frac{1+ik}{2(1+k^2)}$	$\alpha = \frac{1+ik}{2k^2}$	$\alpha = -\frac{i}{2k}$
$\beta = \frac{-2k^2 - \frac{3ik}{2} + \frac{3}{4}}{2(ik-1)}$	$\beta = ik - \frac{1}{2}$	$\beta = ik - \frac{1}{2} - \frac{i}{8k}$

TABLE 5.1. Parameters for  $2^{nd}$ -order ABCs for  $\Gamma = \partial B_1(0) \subset \mathbb{R}^2$ ; taken from [Ihl98, Table 3.2].

Engquist-Majda [EM77, EM79] and Feng [Fen84], see [Ihl98, Sec. 3.3.3, Table 3.2] for a comprehensive comparison. We denote by  $\Delta_\Gamma$  and  $\nabla_\Gamma$  the surface Laplacian and the surface gradient on  $\Gamma$  (see, e.g., [Néd01, Sec. 2.5.6]). The model problem for a heterogeneous Helmholtz equation with second order absorbing boundary conditions is to find  $u$  such that

$$(5.21) \quad \begin{aligned} L_k^- u &:= -k^2 n^2 u - \nabla \cdot (A \nabla u) = f \quad \text{in } \Omega, \\ \partial_{n_A} u - T_{k,\Gamma}^- u &:= \partial_{n_A} u - (\beta u + \alpha \Delta_\Gamma u) = g \quad \text{on } \Gamma, \end{aligned}$$

for  $k$ -dependent parameters  $\alpha, \beta$  satisfying

$$(5.22) \quad \text{Im } \alpha \neq 0 \quad \text{and} \quad |\text{Im } \alpha| \sim \frac{1}{k} \quad \text{and} \quad |\text{Re } \alpha| \lesssim \frac{1}{k^2}, \quad |\beta| \sim k$$

for  $k \geq k_0 > 0$ . The choices of  $\alpha$  and  $\beta$  proposed by Bayliss-Gunzburger-Turkel, Engquist-Mayda, and Feng are given in Table 5.1 as presented in [Ihl98, Sec. 3.3.3, Table 3.2]. The coefficients  $A, n$  in (5.21) are assumed to satisfy (5.3b), (5.3c) and, instead of (5.3a),  $A$  is assumed to be pointwise SPD, viz.,

$$(5.23) \quad \xi^H A(x) \xi \geq a_{\min} |\xi|^2 > 0 \quad \forall \xi \in \mathbb{C}^d \setminus \{0\} \quad \forall x \in \Omega.$$

**Remark 5.13.** Our analysis covers operators  $T_{k,\Gamma}^-$  of the form (5.21) for smooth  $\Gamma$ , which is the form of the second order ABCs for spheres. It is expected that second order ABC for general  $\Gamma$  have a similar structure and  $k$ -scaling properties. ■

We select  $T_{k,\Gamma}^+ = \alpha \Delta_\Gamma$ . Then we have:

**Lemma 5.14** ((AP) for  $2^{nd}$  order ABCs). *Let Assumption 5.2 be valid. Let the coefficients  $A, n$  satisfy (5.3) and  $A$  additionally (5.23). For  $L_k^-, L_k^+$  of (5.2), (5.4) with  $T_{k,\Gamma}^-$  given by (5.21)–(5.22) and  $T_{k,\Gamma}^+ = \alpha \Delta_\Gamma$  the following is true for  $t = 1$ :*

- (i)  $b_k^+$  is bounded uniformly in  $k$ , and (AP1) holds for some  $\sigma$  depending on  $\alpha$  in (5.22).
- (ii)  $\|S_k^+(f, g)\|_{1,t,k} \leq C [k^{-1} \|f\|_{0,\Omega} + k^{-1/2} \|g\|_{0,\Gamma}]$ , and (AP2) holds for  $s_{\max} = 0$ .
- (iii) The splittings of  $T_{k,\Omega}^- - T_{k,\Omega}^+$  and  $T_{k,\Gamma}^- - T_{k,\Gamma}^+$  of (AP3) are given by  $R_{k,\Omega}^\Delta u = k^2(n^2 + 1)u$ ,  $A_{k,\Omega}^\Delta = 0$  and  $R_{k,\Gamma}^\Delta u = \beta u$ ,  $A_{k,\Gamma}^\Delta = 0$  and hence satisfy (AP3) for any fixed  $s_{\max} \geq 0$ .

*Proof.* *Proof of (i):* The Laplace-Beltrami operator  $\Delta_\Gamma: H^1(\Gamma) \rightarrow H^{-1}(\Gamma)$  is a bounded linear operator since  $\Gamma$  has no boundary. Furthermore, we have  $-\langle \alpha \Delta_\Gamma u, u \rangle = \alpha \langle \nabla_\Gamma u, \nabla_\Gamma u \rangle$ . Setting  $\sigma := \bar{\alpha}/|\alpha|$  we estimate

$$(5.24) \quad -\text{Re}(\sigma \langle T_{k,\Gamma}^+ u, u \rangle) = \text{Re}(\sigma \alpha) \langle \nabla_\Gamma u, \nabla_\Gamma u \rangle = \text{Re}(\sigma \alpha) |u|_{1,\Gamma}^2 \gtrsim k^{-1} |u|_{1,\Gamma}^2,$$

where, in the last step, we employed the assumptions  $\text{Im } \alpha \sim k^{-1}$  and  $|\text{Re } \alpha| \lesssim k^{-2}$  so that  $|\alpha| \sim 1/k$ . Since the coefficient  $A$  is assumed to be SPD and satisfy (5.23), the coercivity of  $b_k^+$  of (A P1) holds. The uniform-in- $k$  boundedness of  $b_k^+$  is easily seen.

*Proof of (ii):* The a priori estimate for  $S_k^+(f, g)$  follows from coercivity and continuity of  $b_k^+$ . The function  $w = S_k^+(f, g)$  satisfies

$$(5.25) \quad -\nabla \cdot (A\nabla w) = f - k^2 w \quad \text{in } \Omega, \quad \partial_{n_A} w = g + \alpha \Delta_\Gamma w \quad \text{on } \Gamma.$$

Note that for  $f \in L^2(\Omega)$  and  $g \in L^2(\Gamma)$  we have by Lax-Milgram, which follows from (i), and  $|\alpha| \sim k^{-1}$  (cf. (5.22))

$$(5.26) \quad \|\nabla w\|_{0,\Omega} + k\|w\|_{0,\Omega} + k^{-1/2}\|w\|_{1,\Gamma} \lesssim k^{-1}\|f\|_{0,\Omega} + k^{-1/2}\|g\|_{0,\Gamma}.$$

The following surface PDE is satisfied

$$\alpha \Delta_\Gamma w = -g + \partial_{n_A} w \quad \text{on } \Gamma,$$

which gives, since  $\Gamma$  is a smooth, compact manifold without boundary,

$$(5.27) \quad \|w\|_{2,\Gamma} \stackrel{|\alpha| \sim k^{-1}}{\lesssim} k\|g\|_{0,\Gamma} + k\|\partial_{n_A} w\|_{0,\Gamma} + \|w\|_{1,\Gamma} \stackrel{(5.26)}{\lesssim} k^{-1/2}\|f\|_{0,\Omega} + k\|g\|_{0,\Gamma} + k\|\partial_{n_A} w\|_{0,\Gamma}.$$

In order to estimate  $\|\partial_{n_A} w\|_{0,\Gamma}$  we introduce an auxiliary matrix-valued function  $\hat{A} \in C^\infty(\Omega, \text{GL}(\mathbb{C}^d))$  that satisfies (5.3a) as well as  $\hat{A} = A$  in a neighborhood of  $\Gamma$ . We introduce an auxiliary function  $\tilde{w} \in H^1(\Omega)$  as the solution of

$$(5.28) \quad -\nabla \cdot (\hat{A}\tilde{w}) = 0 \quad \text{in } \Omega, \quad \tilde{w} = w \quad \text{on } \Gamma.$$

By the smoothness of  $\Gamma$  and  $\hat{A}$ , the map  $w \mapsto \partial_{n_A} \tilde{w}$  maps  $H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$  and  $H^{3/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$  so that by interpolation it maps  $H^1(\Gamma) \rightarrow L^2(\Gamma)$  with the a priori estimate

$$(5.29) \quad \|\partial_{n_A} \tilde{w}\|_{0,\Gamma} \lesssim \|w\|_{1,\Gamma}.$$

By interior regularity, we have additionally that  $\tilde{w} \in C^\infty(\Omega)$  and for each open  $\Omega' \subset\subset \Omega$  there is  $C_{\Omega'} > 0$  with

$$(5.30) \quad \|\tilde{w}\|_{2,\Omega'} \leq C_{\Omega'} \|\tilde{w}\|_{1,\Omega} \lesssim \|w\|_{1/2,\Gamma}.$$

Let  $\chi \in C^\infty(\bar{\Omega})$  with  $\chi \equiv 1$  in a neighborhood of  $\Gamma$  and such that  $A|_{\text{supp } \chi} = \hat{A}|_{\text{supp } \chi}$ . The function  $w - \chi\tilde{w} \in H_0^1(\Omega)$  solves

$$(5.31) \quad \begin{aligned} -\nabla \cdot (A\nabla(w - \chi\tilde{w})) &= f - k^2 w + z \quad \text{in } \Omega, \\ w - \tilde{w} &= 0 \quad \text{on } \Gamma \end{aligned}$$

for the function  $z := \nabla \cdot (A(\chi\tilde{w}))$ . Since  $A = \hat{A}$  near  $\Gamma$  and therefore  $-\nabla \cdot (A\nabla(\chi\tilde{w})) = 0$  near  $\Gamma$ , we get in view of (5.30) that  $z \in L^2(\Omega)$  with

$$(5.32) \quad \|z\|_{0,\Omega} \lesssim \|w\|_{1/2,\Gamma}.$$

As discussed in the proof of Lemma 5.4, elliptic regularity implies  $w - \chi\tilde{w} \in H^2(\Omega \setminus \Gamma_{\text{interf}})$  with

$$(5.33) \quad \|w - \chi\tilde{w}\|_{2,\Omega \setminus \Gamma_{\text{interf}}} \lesssim \|f\|_{0,\Omega} + k^2\|w\|_{0,\Omega} + \|z\|_{0,\Omega} \stackrel{(5.26),(5.32)}{\lesssim} \|f\|_{0,\Omega} + k^{1/2}\|g\|_{0,\Gamma}.$$



Since  $\chi \equiv 1$  near  $\Gamma$ , we conclude

$$\begin{aligned} \|\partial_{n_A} w\|_{0,\Gamma} &\leq \|\partial_{n_A} \tilde{w}\|_{0,\Gamma} + \|\partial_{n_A} (w - \chi \tilde{w})\|_{0,\Gamma} \lesssim \|\partial_{n_A} \tilde{w}\|_{0,\Gamma} + C \|w - \chi \tilde{w}\|_{2,\Omega \setminus \Gamma_{\text{interf}}} \\ &\stackrel{(5.29),(5.26),(5.33)}{\lesssim} \|f\|_{0,\Omega} + k^{1/2} \|g\|_{0,\Gamma}. \end{aligned}$$

Inserting this estimate in (5.27) yields

$$\|\alpha \Delta_{\Gamma} w\|_{2,\Gamma} \lesssim \|f\|_{0,\Omega} + k^{1/2} \|g\|_{0,\Gamma}.$$

Noting that (5.25) is a Neumann problem for  $w$ , we get from elliptic regularity for Neumann problems the desired statement **(AP2)**.

We remark in passing that the above arguments can be bootstrapped to show **(AP2)** for  $s_{\max} > 0$ . We refer to [Ber21, Rem. 6.5.7] and in particular to the bootstrapping argument for the analytic regularity assertion **(WP3)** in Appendix B of [BCFM22] for more details.

*Proof of (iii):* By inspection.  $\square$

**Lemma 5.15** ((**WP**) for 2<sup>nd</sup> order ABC). *Let Assumption 5.2 be valid. Let the coefficients  $A$ ,  $n$  satisfy (5.3) and  $A$  additionally (5.23). For  $L_k^-, L_k^+$  of (5.2), (5.4) with  $T_{k,\Gamma}^-$  given by (5.21)–(5.22) and  $T_{k,\Gamma}^+ = \alpha \Delta_{\Gamma}$  the following is true for  $t = 1$  if **(WP2)** holds:*

- (i) **(WP3)** holds with  $C_{\text{ana},k}^- = O(1)$  (uniformly in  $k$ ).
- (ii) **(WP4)** holds with  $R_{k,\Omega}^- u = k^2(n^2 + 1)u$ ,  $A_{k,\Omega}^- = 0$  and  $R_{k,\Gamma}^- = \alpha \Delta_{\Gamma} u + \beta u$ ,  $A_{k,\Gamma}^- = 0$ . In particular, for the constants in Assumption 4.6 we have  $C_{\text{cont},k}^- = O(1)$  and  $C_{A,k}^- = C_{A,k}^{\Delta} = 0$ .
- (iii) **(WP5)** holds.

*Proof.* *Proof of (i):* See [BCFM22, App. B] for the proof. To give some details, by local changes of variables and the invariance of the analyticity classes under analytic changes of variables (see Lemma 2.2), the analysis is reduced to balls or half-balls. For the regularity of  $S_k^-(f, g)$  with  $f \in \mathfrak{A}(M_f, \gamma_f, \mathcal{P})$ ,  $g \in \mathfrak{A}(M_g, \gamma_g, T, \Gamma)$  one obtains from [BCFM22, Thm. B.5] (and the analytic changes of variables) for a tubular neighborhood  $T$  of  $\Omega$  for

$$\|\nabla^{p+2} u\|_{L^2(T \cap \Omega)} \lesssim \max(p, k)^{p+2} \gamma^p [k^{-2} M_f + k^{-1} M_g + k^{-1} \|u\|_{1,t,k}] \quad \forall p \in \mathbb{N}_0,$$

where  $\gamma$  depends on  $\gamma_f, \gamma_g$ , but is independent of  $k$ . Together with corresponding estimates in the interior of  $\Omega$ , this shows **(WP3)** with  $C_{\text{ana},k}^- = O(1)$ .

*Proof of (ii), (iii):* By inspection.  $\square$

**Theorem 5.16** (2<sup>nd</sup> order ABC). *Let  $t = 1$ . Let Assumption 5.2 and (5.3) be valid. For problem (5.2) with  $T_{k,\Gamma}^-$  given by (5.21)–(5.22) assume the polynomial well-posedness of the solution operator  $S_k^-$  of Assumption 4.6. In the discretization setting of Assumption 4.4, for each  $c_2 > 0$  there are constants  $c_1, C > 0$  independent of  $h, p$ , and  $k$  such that under the scale-resolution condition (4.21) the quasi-optimality result (4.22) holds.*

*Proof.* Combine Lemmas 5.14 and 5.15 with the stability Assumption 4.6.  $\square$

## APPENDIX A. DIRICHLET-TO-NEUMANN MAPS VIA BOUNDARY INTEGRAL OPERATORS

The main goal of the present section is prove Lemma 5.8(iii). To that end, we rewrite the Dirichlet-to-Neumann operators  $\text{DtN}_k$  and  $\text{DtN}_0$  in terms of boundary integral operators.

**A.1. Preliminaries.** Let  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , be a bounded Lipschitz domain with analytic boundary  $\Gamma := \partial\Omega$ . We denote by  $\Omega^+$  the exterior domain, i.e.,  $\Omega^+ := \mathbb{R}^d \setminus \bar{\Omega}$ . Throughout this section we assume  $\Omega^+$  to be non-trapping, see [BSW16, Def. 1.1]. Furthermore, we assume that the open ball  $B_R$  of radius  $R$  around the origin contains  $\bar{\Omega}$ , i.e.,  $\bar{\Omega} \subset B_R$ . We set  $\Omega_R := (\Omega \cup \Omega^+) \cap B_R = B_R \setminus \Gamma$ . Following standard notation we introduce the interior and exterior trace operators  $\gamma_0^{int}, \gamma_0^{ext}$  by restricting to  $\Gamma$  and the  $\gamma_1^{int}$  and  $\gamma_1^{ext}$  by setting, for sufficiently smooth functions  $v$ , we have  $\gamma_1^{int}v = \mathbf{n} \cdot \gamma_0^{int}(\nabla v)$  and  $\gamma_1^{ext}v = \mathbf{n} \cdot \gamma_0^{ext}(\nabla v)$ , where  $\mathbf{n}$  is the outer normal vector on  $\Gamma$ . Furthermore, we denote by  $V_k, K_k, K'_k$  and  $D_k$  the single layer, double layer, adjoint double layer and hypersingular boundary integral operators, see [Ste08, Sec. 6.9 and 7.9]. The single layer and double layer potentials are denoted  $\tilde{V}_k$  and  $\tilde{K}_k$ . Finally, given a coupling parameter  $\eta \in \mathbb{R} \setminus \{0\}$  we introduce the combined field operator  $A'_{k,\eta}$  by

$$A'_{k,\eta} := \frac{1}{2} + K'_k + i\eta V_k.$$

We remind the reader of the exterior Calderón identities

$$\begin{pmatrix} \gamma_0^{ext}u \\ \gamma_1^{ext}u \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + K_k & -V_k \\ -D_k & \frac{1}{2} - K'_k \end{pmatrix} \cdot \begin{pmatrix} \gamma_0^{ext}u \\ \gamma_1^{ext}u \end{pmatrix},$$

Given Dirichlet data  $u$ , the Dirichlet-to-Neumann operator  $\text{DtN}_k$  can be expressed for any  $k \geq 0$  by a complex linear combination of the two equations in the Calderón identity: For any  $\eta \in \mathbb{R} \setminus \{0\}$  we have

$$(A.1) \quad A'_{k,\eta}(\text{DtN}_k u) = \left( \frac{1}{2} + K'_k + i\eta V_k \right) \text{DtN}_k u = \left( -D_k + i\eta \left( -\frac{1}{2} + K_k \right) \right) u.$$

Our analysis relies on invertibility of the combined field operator  $A'_{k,\eta}$  as an operator mapping  $H^s(\Gamma)$  into itself. Wavenumber-explicit estimates for  $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma) \leftarrow L^2(\Gamma)}$  are discussed in [BSW16, Sec. 1.4]. For non-trapping  $\Omega^+ \subset \mathbb{R}^d$ ,  $d = 2, 3$  it is known that  $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma) \leftarrow L^2(\Gamma)} \lesssim k^{5/4} \left( 1 + \frac{k^{3/4}}{|\eta|} \right)$  for all  $k \geq k_0$  and  $\eta \in \mathbb{R} \setminus \{0\}$ , see [Spe14, Thm 1.11]. This bound can be sharpened assuming  $|\eta| \sim k$ . In fact, for non-trapping  $\Omega^+ \subset \mathbb{R}^d$ ,  $d = 2, 3$  and  $|\eta| \sim k$  there holds

$$(A.2) \quad \|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma) \leftarrow L^2(\Gamma)} \lesssim 1$$

for all  $k \geq k_0$ , see [BSW16, Thm. 1.13]. In Proposition A.1, we collect mapping properties of some boundary integral operators:

**Proposition A.1.** *Let  $\Omega^+$  be non-trapping,  $\Gamma$  be analytic and  $\eta \in \mathbb{R} \setminus \{0\}$  fixed. If  $d = 2$ , assume additionally  $\text{diam } \Omega < 1$ . Then*

- (i)  $A_{0,-\eta} := \frac{1}{2} + K_0 - i\eta V_0: H^s(\Gamma) \rightarrow H^s(\Gamma)$  is boundedly invertible for  $s \geq 0$ .
- (ii)  $A'_{0,\eta} = \frac{1}{2} + K'_0 + i\eta V_0: H^s(\Gamma) \rightarrow H^s(\Gamma)$  is boundedly invertible for  $s \geq -1$ .

(iii) For  $k > 0$  the combined field operator  $A'_{k,\eta} = \frac{1}{2} + K'_k + i\eta V_k: H^s(\Gamma) \rightarrow H^s(\Gamma)$  is boundedly invertible for  $s \geq -1$ .

(iv) For  $k \geq 0$  and  $s \geq -1/2$  the following operators are bounded:

$$(A.3) \quad \begin{aligned} V_k &: H^{-1/2+s}(\Gamma) \rightarrow H^{1/2+s}(\Gamma), & K_k &: H^{1/2+s}(\Gamma) \rightarrow H^{1/2+s}(\Gamma), \\ K'_k &: H^{-1/2+s}(\Gamma) \rightarrow H^{-1/2+s}(\Gamma), & D_k &: H^{1/2+s}(\Gamma) \rightarrow H^{-1/2+s}(\Gamma) \end{aligned}$$

(v) For  $k \geq k_0 > 0$  one can decompose

$$(A.4) \quad \begin{aligned} V_k - V_0 &= S_V + \gamma_0^{int} \tilde{A}_V, & K_k - K_0 &= S_K + \gamma_0^{int} \tilde{A}_K, \\ K'_k - K'_0 &= S_{K'} + \gamma_1^{int} \tilde{A}_V, & D_k - D_0 &= S_D - \gamma_1^{int} \tilde{A}_K \end{aligned}$$

with linear maps  $\tilde{A}_V: H^{-3/2}(\Gamma) \rightarrow C^\infty(\bar{\Omega})$  and  $\tilde{A}_K: H^{-1/2}(\Gamma) \rightarrow C^\infty(\bar{\Omega})$  and bounded linear operators  $S_V, S_K, S_{K'}$ , and  $S_D$  having the following mapping properties for  $s \geq -1$ :

$$(A.5) \quad \begin{aligned} \|S_V u\|_{-1/2+s',\Gamma} &\leq C_{s,s'} k^{-(1+s-s')} \|u\|_{-1/2+s,\Gamma}, & 1/2 \leq s' \leq s+3, \\ \|S_K u\|_{-1/2+s',\Gamma} &\leq C_{s,s'} k^{-(1+s-s')} \|u\|_{1/2+s,\Gamma}, & 1/2 \leq s' \leq s+3, \\ \|S_{K'} u\|_{-3/2+s',\Gamma} &\leq C_{s,s'} k^{-(1+s-s')} \|u\|_{-1/2+s,\Gamma}, & 3/2 \leq s' \leq s+3, \\ \|S_D u\|_{-3/2+s',\Gamma} &\leq C_{s,s'} k^{-(1+s-s')} \|u\|_{1/2+s,\Gamma}, & 3/2 \leq s' \leq s+3. \end{aligned}$$

The operators  $\tilde{A}_V$  and  $\tilde{A}_K$  have the mapping property

$$(A.6) \quad \tilde{A}_V f \in \mathfrak{A}(C_V k \|f\|_{-3/2,\Gamma}, \gamma_V, \Omega) \quad \forall f \in H^{-3/2}(\Gamma),$$

$$(A.7) \quad \tilde{A}_K f \in \mathfrak{A}(C_K k \|f\|_{-1/2,\Gamma}, \gamma_K, \Omega) \quad \forall f \in H^{-1/2}(\Gamma),$$

with constants  $C_V, \gamma_V, C_K, \gamma_K$  independent of  $k \geq k_0$ . For  $t \geq 0$  the following mapping properties hold true:

$$(A.8) \quad \begin{aligned} \|S_V u\|_{t,\Gamma} &\leq C_t k^{-1} \|u\|_{t,\Gamma}, & \|S_K u\|_{t,\Gamma} &\leq C_t \|u\|_{t,\Gamma}, \\ \|S_{K'} u\|_{t,\Gamma} &\leq C_t \|u\|_{t,\Gamma}, & \|S_D u\|_{t,\Gamma} &\leq C_t k \|u\|_{t,\Gamma}. \end{aligned}$$

*Proof.* For Item (i) see [Mel12, Lem. 3.5(ii)]. For Item (ii) in the case  $s \geq 0$  see [Mel12, Lemma 3.5(iv)]. We turn to the case  $s \in [-1, 0]$ . Note that the adjoint of  $A_{0,-\eta}$  is precisely the operator  $A'_{0,\eta}$ . Furthermore, by Item (i) the operator  $A_{0,-\eta}: H^t(\Gamma) \rightarrow H^t(\Gamma)$  is boundedly invertible in particular for  $t \in [0, 1]$ . Hence, due to the adjoint of  $A_{0,-\eta}$  being  $A'_{0,\eta}$ , we find that  $A'_{0,\eta}: H^s(\Gamma) \rightarrow H^s(\Gamma)$  is also boundedly invertible for  $s \in [-1, 0]$ . For Item (iii) see [CWGLS12, Thm. 2.27] in the case  $s \in [-1, 0]$  as well as [BSW16, Sec. 6.1]. Consequently, by [Mel12, Lem. 2.14] invertibility holds for any  $s \geq 0$ . The mapping properties (A.3) in Item (iv) are standard, see, e.g., [SS11, Rem. 3.1.18(c)]. For (A.4) and (A.5) in Item (v) see [MMPR20, Lem. A.1] for the cases  $1/2 < s'$  and  $3/2 < s'$ . The limiting cases  $s' = 1/2$  and  $s' = 3/2$  follow by inspection of the proof, a multiplicative trace estimate, and the estimates for the potentials  $\tilde{S}_V, \tilde{S}_K$  appearing in [MMPR20, Lem. A.1]. (A.7) is likewise shown in [MMPR20, Lem. A.1]. (A.8) is just a simplification of (A.5).  $\square$

**A.2. Decomposition of the Dirichlet-to-Neumann map.** Before proceeding with the proof of Lemma 5.8(iii) let us introduce the jumps of the trace operators:

$$(A.9) \quad [u] := \gamma_0^{ext} u - \gamma_0^{int} u, \quad [[\partial_n u]] := \gamma_1^{ext} u - \gamma_1^{int} u.$$

For linear operators  $\tilde{A}$  mapping into spaces of piecewise defined functions we define the operator  $[[\tilde{A}]]$  and  $[[\partial_n \tilde{A}]]$  analogously, e.g.,  $[[\tilde{A}]]u := [[\tilde{A}u]]$ .

We now collect further technical results from [Mel12]. We closely follow the notation and results of [Mel12]. As in [Mel12] we assume

$$(A.10) \quad C_\eta^{-1}k \leq |\eta| \leq C_\eta k$$

for some  $C_\eta > 0$  independent of  $k$ .

In the following Proposition A.2 we extend the results of [Mel12, Lem. 6.3] to a wider range of Sobolev spaces.

**Proposition A.2** ([Mel12, Lemma 6.3]). *Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain with an analytic boundary  $\Gamma$ . Let  $q \in (0, 1)$ . Then one can construct operators  $L_{\Gamma,q}^{neg}$ ,  $H_{\Gamma,q}^{neg}$  on  $H^{-1}(\Gamma)$  with the following properties:*

- (i)  $L_{\Gamma,q}^{neg}f + H_{\Gamma,q}^{neg}f = f$  for all  $f \in H^{-1}(\Gamma)$ .
- (ii) For  $-1 \leq s' \leq s$  there holds  $\|H_{\Gamma,q}^{neg}f\|_{s',\Gamma} \leq C_{s,s'}(q/k)^{s-s'}\|f\|_{s,\Gamma}$ .
- (iii)  $L_{\Gamma,q}^{neg}f$  is the restriction to  $\Gamma$  of a function that is analytic in a tubular neighborhood  $T$  of  $\Gamma$  and satisfies

$$\begin{aligned} \|\nabla^n L_{\Gamma,q}^{neg}f\|_{0,T} &\leq C_q k^{d/2} \gamma_q^n \max\{k, n\}^n \|f\|_{-1/2,\Gamma} \quad \forall n \in \mathbb{N}_0, \\ \|\nabla^n L_{\Gamma,q}^{neg}f\|_{0,T} &\leq C_q k^{d/2+1} \gamma_q^n \max\{k, n\}^n \|f\|_{-1,\Gamma} \quad \forall n \in \mathbb{N}_0. \end{aligned}$$

Here,  $C_{s,s'}$  is independent of  $q, k$ ; the constants  $C_q, \gamma_q > 0$  are independent of  $k$ .

*Proof.* For Items (i), (ii) in the case  $-1 \leq s' \leq s \leq 1$  and Item (iii) see [Mel12, Lem. 6.3 and Rem. 6.4]. The crucial extension is the estimate stated in Item (ii) in the case  $-1 \leq s' \leq s$  for  $s \geq 1$ . In the proof of [Mel12, Lem. 6.3] the operators  $H_{\Gamma,q}^{neg}$  and  $L_{\Gamma,q}^{neg}$  are explicitly constructed. We collect the important ingredients of the proof of [Mel12, Lem. 6.3] in the following. On the compact manifold  $\Gamma$  consider the eigenvalue problem for the Laplace-Beltrami operator

$$(A.11) \quad -\Delta_\Gamma \varphi = \lambda^2 \varphi \quad \text{on } \Gamma.$$

There are countably many eigenfunctions  $\varphi_m$ ,  $m \in \mathbb{N}_0$ , with corresponding eigenvalues  $\lambda_m^2 \geq 0$ , which we assume to be sorted in ascending order. Without loss of generality, these eigenfunctions are normalized in  $L^2(\Gamma)$ . The functions  $(\varphi_m)_{m \in \mathbb{N}_0}$  may be assumed to be an orthonormal basis of  $L^2(\Gamma)$  and an orthogonal basis of  $H^1(\Gamma)$ . With  $u_m := (u, \varphi_m)$  we have

$$(A.12) \quad \|u\|_{0,\Gamma}^2 = \sum_{m=0}^{\infty} |u_m|^2 \quad \text{and} \quad \|u\|_{1,\Gamma}^2 = \sum_{m=0}^{\infty} (1 + \lambda_m^2) |u_m|^2.$$

For  $s \in \mathbb{R}$  we introduce the sequence space  $h^s$  by

$$h^s := \left\{ (u_m)_{m \in \mathbb{N}} : \sum_{m=0}^{\infty} (1 + \lambda_m^2)^s |u_m|^2 < \infty \right\}.$$

The mapping  $\iota: u \mapsto ((u, \varphi_m))_{m \in \mathbb{N}_0}$  then provides an isomorphism between the Sobolev space  $H^s(\Gamma)$  and the sequence space  $h^s$  for  $s \in [-1, 1]$ , with corresponding norm equivalence, see [Mel12, Lem. C.3]. However, as we will see below  $\iota$  is in fact an isomorphism for all  $s \geq -1$ . Inspection of the proof of [Mel12, Lem. 6.3], in particular the proof of the estimate for  $H_{\Gamma,q}^{neg}$ , reveals that

$$\|H_{\Gamma,q}^{neg}f\|_{s',\Gamma} \leq C_{s,s'}(q/k)^{s-s'}\|f\|_{s,\Gamma}$$

holds for all  $-1 \leq s' \leq s$ , for which  $\iota: H^s(\Gamma) \rightarrow h^s$  and  $\iota: H^{s'}(\Gamma) \rightarrow h^{s'}$  are isomorphisms. Hence, the proof is complete once we establish that  $\iota: H^s(\Gamma) \rightarrow h^s$  is an isomorphism for all  $s > 1$ . We show the case  $s = 2$ .

**The inclusion  $h^2 \hookrightarrow H^2(\Gamma)$ :** Let  $u = \sum_{m=0}^{\infty} u_m \varphi_m$  be such that  $\sum_{m=0}^{\infty} (1 + \lambda_m^2)^2 |u_m|^2 < \infty$ . Let  $u^N = \sum_{m=0}^N u_m \varphi_m$ . By the above construction,  $u^N \rightarrow u$  in  $H^1(\Gamma)$  and  $\|u\|_{1,\Gamma} = \|(u_m)_{m \in \mathbb{N}}\|_{h^1}$ . Furthermore, we have

$$\|\Delta_{\Gamma}(u^N - u^{M-1})\|_{0,\Gamma}^2 = \left\| \sum_{m=M}^N u_m \Delta_{\Gamma} \varphi_m \right\|_{0,\Gamma}^2 = \left\| \sum_{m=M}^N u_m \lambda_m^2 \varphi_m \right\|_{0,\Gamma}^2 = \sum_{m=M}^N |u_m|^2 \lambda_m^4,$$

which converges to zero as  $N, M \rightarrow \infty$ . Here we used (A.11), the fact that the eigenfunctions are an orthonormal basis of  $L^2(\Gamma)$ , and the assumed convergence  $\sum_{m=0}^{\infty} (1 + \lambda_m^2)^2 |u_m|^2 < \infty$ . Therefore,  $(u^N)_{N \in \mathbb{N}_0}$  is a Cauchy sequence in  $H^1(\Gamma, \Delta_{\Gamma}) = \{u \in H^1(\Gamma) : \Delta_{\Gamma} u \in L^2(\Gamma)\}$ , endowed with the graph norm. Consequently  $(u^N)_{N \in \mathbb{N}_0}$  converges in  $H^1(\Gamma, \Delta_{\Gamma})$ . Since  $\Delta_{\Gamma} : H^1(\Gamma, \Delta_{\Gamma}) \rightarrow L^2(\Gamma)$  is continuous, we conclude  $\Delta_{\Gamma} u = \sum_{m \in \mathbb{N}_0} u_m \Delta_{\Gamma} \varphi_m = -\sum_{m \in \mathbb{N}_0} u_m \lambda_m^2 \varphi_m$ . Finally, by elliptic regularity we can now estimate

$$(A.13) \quad \|u\|_{2,\Gamma}^2 \lesssim \|\Delta_{\Gamma} u\|_{0,\Gamma}^2 = \sum_{m \in \mathbb{N}_0} |f_m|^2 \lambda_m^4 = \|(u_m)_{m \in \mathbb{N}_0}\|_{h^2}^2,$$

and  $\|u\|_{2,\Gamma}^2 \lesssim \|(u_m)_{m \in \mathbb{N}_0}\|_{h^2}^2$  follows by (A.13) together with (A.12).

**The inclusion  $H^2(\Gamma) \hookrightarrow h^2$ :** Let  $u \in H^2(\Gamma)$  be given with the representation  $u = \sum_{m=0}^{\infty} u_m \varphi_m$ , where the sum converges in  $H^1(\Gamma)$ . Since  $u \in H^2(\Gamma)$  we have  $-\Delta_{\Gamma} u =: f \in L^2(\Gamma)$ . In the following we express the coefficient  $u_m$  in terms of  $f_m$ . Note that

$$\lambda_m^2 u_m = \lambda_m^2 \langle u, \varphi_m \rangle = \langle \nabla_{\Gamma} u, \nabla_{\Gamma} \varphi_m \rangle = \langle f, \varphi_m \rangle = f_m.$$

Hence, we have  $\lambda_m^2 u_m = f_m$  and consequently

$$\sum_{m=0}^{\infty} \lambda_m^4 |u_m|^2 = \sum_{m=0}^{\infty} |f_m|^2 < \infty.$$

Finally, using (A.13) as well as the fact that  $\|u\|_{1,\Gamma} = \|(u_m)_{m \in \mathbb{N}_0}\|_{h^1}$ , we find

$$\|(u_m)_{m \in \mathbb{N}_0}\|_{h^2}^2 = \|(u_m)_{m \in \mathbb{N}_0}\|_{h^1}^2 + \|(u_m)_{m \in \mathbb{N}_0}\|_{h^2}^2 = \|u\|_{1,\Gamma}^2 + \|\Delta_{\Gamma} u\|_{0,\Gamma}^2 \lesssim \|u\|_{2,\Gamma}^2.$$

This concludes the proof for  $s = 2$ . Interpolation between  $s = 1$  and  $s = 2$  yields the result for  $s \in (1, 2)$ , see [Mel12, Lem. C.3]. Inductively one proceeds for the space  $H^{2n}(\Gamma)$  by similar arguments. Instead of  $\Delta_{\Gamma}$  one performs the same arguments for  $\Delta_{\Gamma}^n$ .  $\square$

In the following we will prove an extension of [Mel12, Thm. 2.9] in Theorem A.4 below. The proof of Theorem A.4 relies on a decomposition of the volume potential  $\tilde{V}_k$ , which we present below for the readers' convenience.

**Proposition A.3** ([Mel12, Thm. 5.3]). *Let  $\Gamma$  be analytic and  $q \in (0, 1)$ . Then*

$$\tilde{V}_k = \tilde{V}_0 + \tilde{S}_{V,pw} + \tilde{A}_{V,pw},$$

where the linear operators  $\tilde{S}_{V,pw}$  and  $\tilde{A}_{V,pw}$  satisfy the following for every  $s \geq -1$ :

(i)  $\tilde{S}_{V,pw} : H^{-1/2+s}(\Gamma) \rightarrow H^{3+s}(\Omega_R)$  with

$$\|\tilde{S}_{V,pw} \varphi\|_{s',\Omega_R} \leq C_{s',s} q^2 (qk^{-1})^{1+s-s'} \|\varphi\|_{-1/2+s,\Gamma}, \quad 0 \leq s' \leq s+3.$$

Here, the constant  $C_{s',s} > 0$  is independent of  $q$  and  $k \geq k_0$ .

(ii)  $\tilde{\mathcal{A}}_{V,pw} : H^{-1/2+s}(\Gamma) \rightarrow H^2(B_R) \cap C^\infty(\Omega_R)$  with

$$\|\nabla^n \tilde{\mathcal{A}}_{V,pw} \varphi\|_{0,\Omega_R} \leq C_q \gamma_q \max\{n+1, k\}^{n+1} \|\varphi\|_{-3/2,\Gamma} \quad \forall n \in \mathbb{N}_0.$$

Here,  $C_q, \gamma_q > 0$  are independent of  $k \geq k_0$  but may depend on  $q$ .

**Theorem A.4** (Extension of [Mel12, Thm. 2.9]). *Let  $\Gamma$  be analytic, and let  $s \geq 0$ . Fix  $q \in (0, 1)$ . Then the operator  $A'_{k,\eta}$  can be written in the form*

$$A'_{k,\eta} = A'_{0,1} + R_{A'} + k[\tilde{\mathcal{A}}_1] + [\partial_n \tilde{\mathcal{A}}_2],$$

where the linear operator  $R_{A'}$  satisfies

$$(A.14a) \quad \|R_{A'} u\|_{s+1,\Gamma} \leq Ck \|u\|_{s,\Gamma},$$

$$(A.14b) \quad \|R_{A'} u\|_{s,\Gamma} \leq Ck \|u\|_{s-1,\Gamma},$$

$$(A.14c) \quad \|R_{A'} u\|_{s,\Gamma} \leq q \|u\|_{s,\Gamma},$$

$$(A.14d) \quad \|R_{A'} u\|_{s-1,\Gamma} \leq q \|u\|_{s-1,\Gamma},$$

and the linear operators  $\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}_2 : H^{-1}(\Gamma) \rightarrow C^\infty(T)$  satisfy

$$(A.15a) \quad \tilde{\mathcal{A}}_1 \varphi \in \mathfrak{A}(C_q C_{1,\varphi}, \gamma_q, T), \quad C_{1,\varphi} = k \|\varphi\|_{-3/2,\Gamma} + k^{d/2} \|\varphi\|_{-1,\Gamma},$$

$$(A.15b) \quad \tilde{\mathcal{A}}_2 \varphi \in \mathfrak{A}(C_q C_{2,\varphi}, \gamma_q, T), \quad C_{2,\varphi} = k \|\varphi\|_{-3/2,\Gamma}.$$

The constant  $C$  and the tubular neighborhood  $T$  of  $\Gamma$  are independent of  $k \geq k_0$  and  $q$ ; the constants  $C_q, \gamma_q > 0$  are independent of  $k \geq k_0$  (but may depend of  $q$ ).

*Proof.* We perform a similar splitting as in the proof of [Mel12, Thm. 2.9]. The starting point of our analysis is the decomposition

$$A'_{k,\eta} = \frac{1}{2} + K'_0 + \gamma_1^{int} (\tilde{S}_{V,pw} + \tilde{\mathcal{A}}_{V,pw}) + i\eta \gamma_0^{int} (\tilde{V}_0 + \tilde{S}_{V,pw} + \tilde{\mathcal{A}}_{V,pw}),$$

with  $\tilde{S}_{V,pw}$  and  $\tilde{\mathcal{A}}_{V,pw}$  as in Proposition A.3, see [Mel12, Eq. (6.4)]. Adding and subtracting  $iV_0$  and noting  $V_0 = \gamma_0^{int} \tilde{V}_0$  we find

$$\begin{aligned} A'_{k,\eta} &= \frac{1}{2} + K'_0 + iV_0 + \gamma_1^{int} (\tilde{S}_{V,pw} + \tilde{\mathcal{A}}_{V,pw}) + i(\eta - 1) \gamma_0^{int} \tilde{V}_0 + i\eta \gamma_0^{int} (\tilde{S}_{V,pw} + \tilde{\mathcal{A}}_{V,pw}) \\ &= A'_{0,1} + \gamma_1^{int} (\tilde{S}_{V,pw} + \tilde{\mathcal{A}}_{V,pw}) + i(\eta - 1) \gamma_0^{int} \tilde{V}_0 + i\eta \gamma_0^{int} (\tilde{S}_{V,pw} + \tilde{\mathcal{A}}_{V,pw}). \end{aligned}$$

Using the filters  $H_{\Gamma,q}^{neg}$  and  $L_{\Gamma,q}^{neg}$  of Proposition A.2 we define

(A.16a)

$$\begin{aligned} R_{A'} &= H_{\Gamma,q}^{neg} \left( \gamma_1^{int} \tilde{S}_{V,pw} + i\eta \gamma_0^{int} \tilde{S}_{V,pw} + i(\eta - 1) V_0 \right), \\ \tilde{\mathcal{A}}_1 &= -k^{-1} \chi_\Omega \left( i\eta \tilde{\mathcal{A}}_{V,pw} + L_{\Gamma,q}^{neg} \left( \gamma_1^{int} \tilde{S}_{V,pw} + i\eta \gamma_0^{int} \tilde{S}_{V,pw} + i(\eta - 1) V_0 \right) \right), \\ \tilde{\mathcal{A}}_2 &= -\chi_\Omega \tilde{\mathcal{A}}_{V,pw}. \end{aligned}$$

The mapping properties of  $\tilde{\mathcal{A}}_1$  and  $\tilde{\mathcal{A}}_2$  stay the same as in [Mel12, Thm. 2.9]. We are left with the mapping properties of  $R_{A'}$ . In the following the parameter  $q$  appearing in Proposition A.2 and A.3 is still at our disposal<sup>1</sup>. We fix it at the end of the proof to ensure the estimates (A.14c) and (A.14d).

**Step 1:** We estimate the term  $i(\eta - 1) H_{\Gamma,q}^{neg} V_0$  in various norms. We heavily use the estimates for  $H_{\Gamma,q}^{neg}$  and  $V_0$  given in Proposition A.2 and (A.4) in Proposition A.1.

<sup>1</sup>Do not confuse this  $q$  with the one appearing in the statement of the present theorem.

First estimating  $\eta$ , then using the properties of  $H_{\Gamma,q}^{neg}$  in Proposition A.2 and finally the mapping properties of  $V_0$  we find

$$\begin{aligned} \|i(\eta-1)H_{\Gamma,q}^{neg}V_0u\|_{s+1,\Gamma} &\leq Ck\|H_{\Gamma,q}^{neg}V_0u\|_{s+1,\Gamma} \leq Ck\|V_0u\|_{s+1,\Gamma} \leq Ck\|u\|_{s,\Gamma}, \\ \|i(\eta-1)H_{\Gamma,q}^{neg}V_0u\|_{s,\Gamma} &\leq Ck\|H_{\Gamma,q}^{neg}V_0u\|_{s,\Gamma} \leq Ck\|V_0u\|_{s,\Gamma} \leq Ck\|u\|_{s-1,\Gamma}, \\ \|i(\eta-1)H_{\Gamma,q}^{neg}V_0u\|_{s,\Gamma} &\leq Ck\|H_{\Gamma,q}^{neg}V_0u\|_{s,\Gamma} \leq Ck(q/k)\|V_0u\|_{s+1,\Gamma} \leq Cq\|u\|_{s,\Gamma}, \\ \|i(\eta-1)H_{\Gamma,q}^{neg}V_0u\|_{s-1,\Gamma} &\leq Ck\|H_{\Gamma,q}^{neg}V_0u\|_{s-1,\Gamma} \leq Ck(q/k)\|V_0u\|_{s,\Gamma} \leq Cq\|u\|_{s-1,\Gamma}. \end{aligned}$$

In the Steps 2 and 3 below we will again heavily use the properties of  $H_{\Gamma,q}^{neg}$  given in Proposition A.2. Furthermore, we often apply the results of Proposition A.3, especially Item (i). Below, we will write certain exponents in a nonsimplified way to indicate the corresponding choices of Sobolev exponents when applying Proposition A.3.

**Step 2:** We claim:

$$(A.17) \quad \|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s+1,\Gamma} \leq Cqk\|u\|_{s,\Gamma},$$

$$(A.18) \quad \|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s,\Gamma} \leq Cqk\|u\|_{s-1,\Gamma},$$

$$(A.19) \quad \|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s,\Gamma} \leq Cq^2\|u\|_{s,\Gamma},$$

$$(A.20) \quad \|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s-1,\Gamma} \leq Cq^2\|u\|_{s-1,\Gamma}.$$

To see (A.17), we calculate

$$\begin{aligned} \|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s+1,\Gamma} &\leq C\|\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s+1,\Gamma} \leq C\|\tilde{S}_{V,pw}u\|_{s+5/2,\Omega} \\ &\leq Cq^2(qk^{-1})^{1+(s+1/2)-(s+5/2)}\|u\|_{s,\Gamma} = Cqk\|u\|_{s,\Gamma}. \end{aligned}$$

By a similar calculation, we obtain (A.19):

$$\|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s,\Gamma} \leq Cq/k\|\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s+1,\Gamma} \leq Cq^2\|u\|_{s,\Gamma}.$$

In the case  $s \in [0, 1/2)$ , we perform a multiplicative trace inequality and find

$$\begin{aligned} \|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s-1,\Gamma} &\leq C(q/k)^{-s+1}\|\gamma_1^{int}\tilde{S}_{V,pw}u\|_{0,\Gamma} \\ &\leq C(q/k)^{-s+1}\|\tilde{S}_{V,pw}u\|_{1,\Omega}^{1/2}\|\tilde{S}_{V,pw}u\|_{2,\Omega}^{1/2} \\ &\leq C(q/k)^{-s+1}\left[q^2(qk^{-1})^{1+(s-1/2)-1}\right]^{1/2}\left[q^2(qk^{-1})^{1+(s-1/2)-2}\right]^{1/2}\|u\|_{s-1,\Gamma} \\ &= Cq^2\|u\|_{s-1,\Gamma}. \end{aligned}$$

In the case  $s \geq 1/2$  we perform a standard trace estimate and find

$$\begin{aligned} \|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s-1,\Gamma} &\leq Cq/k\|\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s,\Gamma} \leq Cq/k\|\tilde{S}_{V,pw}u\|_{s+3/2,\Omega} \\ &\leq Cq/kq^2(qk^{-1})^{1+(s-1/2)-(s+3/2)}\|u\|_{s-1,\Gamma} \\ (A.21) \quad &= Cq^2\|u\|_{s-1,\Gamma}. \end{aligned}$$

The two previous estimates show (A.20). Analogously to (A.21), we find (A.18):

$$\|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s,\Gamma} \leq C\|\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s,\Gamma} \leq Cqk\|u\|_{s-1,\Gamma}.$$

**Step 3:** We claim:

$$(A.22) \quad \|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s+1,\Gamma} \leq Cq^2 k \|u\|_{s,\Gamma},$$

$$(A.23) \quad \|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s,\Gamma} \leq Cq^2 k \|u\|_{s-1,\Gamma},$$

$$(A.24) \quad \|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s,\Gamma} \leq Cq^3 \|u\|_{s,\Gamma},$$

$$(A.25) \quad \|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s-1,\Gamma} \leq Cq^3 \|u\|_{s-1,\Gamma}.$$

For (A.22) we estimate

$$\begin{aligned} \|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s+1,\Gamma} &\leq Ck \|\gamma_0^{int} \tilde{S}_{V,pw} u\|_{s+1,\Gamma} \leq Ck \|\tilde{S}_{V,pw} u\|_{s+3/2,\Omega} \\ &\leq Ckq^2 (qk^{-1})^{1+(s+1/2)-(s+3/2)} \|u\|_{s,\Gamma} = Cq^2 k \|u\|_{s,\Gamma}. \end{aligned}$$

By a similar calculation, we show (A.23):

$$\|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s,\Gamma} \leq Ckq/k \|\gamma_0^{int} \tilde{S}_{V,pw} u\|_{s+1,\Gamma} \leq Cq^3 \|u\|_{s,\Gamma}.$$

In the case  $s \in [0, 1/2)$ , we perform a multiplicative trace inequality and find

$$\begin{aligned} \|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s-1,\Gamma} &\leq Ck(q/k)^{-s+1} \|\gamma_0^{int} \tilde{S}_{V,pw} u\|_{0,\Gamma} \\ &\leq Ck(q/k)^{-s+1} \|\tilde{S}_{V,pw} u\|_{0,\Omega}^{1/2} \|\tilde{S}_{V,pw} u\|_{1,\Omega}^{1/2} \\ &\leq Ck(q/k)^{-s+1} \left[ q^2 (qk^{-1})^{1+(s-1/2)-0} \right]^{1/2} \left[ q^2 (qk^{-1})^{1+(s-1/2)-1} \right]^{1/2} \|u\|_{s-1,\Gamma} \\ &= Cq^3 \|u\|_{s-1,\Gamma}. \end{aligned}$$

In the case  $s \geq 1/2$  we perform a standard trace estimate and find

$$\begin{aligned} \|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s-1,\Gamma} &\leq Ckq/k \|\gamma_0^{int} \tilde{S}_{V,pw} u\|_{s,\Gamma} \leq Ckq/k \|\tilde{S}_{V,pw} u\|_{s+1/2,\Omega} \\ &\leq Ckq/kq^2 (qk^{-1})^{1+(s-1/2)-(s+1/2)} \|u\|_{s-1,\Gamma} \\ (A.26) \quad &= Cq^3 \|u\|_{s-1,\Gamma}. \end{aligned}$$

The two previous estimates show (A.25). Analogously to (A.26), we assert (A.24):

$$\|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s,\Gamma} \leq k \|\gamma_0^{int} \tilde{S}_{V,pw} u\|_{s,\Gamma} \leq Cq^2 k \|u\|_{s-1,\Gamma}.$$

**Step 4:** The definition of the operator  $R_{A'}$  in (A.16a), the triangle inequality, and appropriate choice of  $q$  yields mapping properties of  $R_{A'}$  as stated in (A.14).  $\square$

Finally, a simple application of [Mel12, Cor. 7.5] for non-trapping  $\Omega^+$  with analytic boundary is the following:

**Lemma A.5.** *Let  $\Omega^+$  be non-trapping ([BSW16, Def. 1.1]). Let  $\Gamma$  be analytic,  $T$  be a tubular neighborhood of  $\Gamma$  and  $C_{g_1}, C_{g_2}, \gamma_g > 0$ . Then there exist constants  $C, \gamma > 0$  independent of  $k \geq k_0$  such that for all  $g_1 \in \mathfrak{A}(C_{g_1}, \gamma_g, T)$ ,  $g_2 \in \mathfrak{A}(C_{g_2}, \gamma_g, T)$  the solution  $\varphi \in L^2(\Gamma)$  of*

$$A'_{k,\eta} \varphi = k \llbracket g_1 \rrbracket + \llbracket \partial_n g_2 \rrbracket$$

satisfies

$$\varphi = -\llbracket \partial_n v \rrbracket, \quad v \in \mathfrak{A}(Ck^{5/2}(C_{g_1} + C_{g_2}), \gamma, \Omega_R).$$

*Proof.* We apply [Mel12, Cor. 7.5] with  $s_A = 0$ . By Proposition A.1(iii) the operator  $A'_{\eta,k} : L^2(\Gamma) \rightarrow L^2(\Gamma)$  is boundedly invertible. The result follows immediately from [Mel12, Cor. 7.5] together with the bound (A.2).  $\square$



*Proof. Proof of Lemma 5.8(iii):*

**Step 1:** We derive a splitting of  $(A'_{k,\eta})^{-1}$  similar to the results of [Mel12, Thm. 2.11].

Fix  $\hat{q} \in (0, 1)$ . Let

$$q := \hat{q} \min \left\{ 1, \frac{1}{\|(A'_{0,1})^{-1}\|_{H^s(\Gamma) \leftarrow H^s(\Gamma)}}, \frac{1}{\|(A'_{0,1})^{-1}\|_{H^{s-1}(\Gamma) \leftarrow H^{s-1}(\Gamma)}} \right\}.$$

Note that by Proposition A.1 the operator  $A'_{0,1}: H^t(\Gamma) \rightarrow H^t(\Gamma)$  is boundedly invertible for  $t \geq -1$  and therefore  $q$  is well defined and  $q \in (0, 1)$ . Theorem A.4 applied with this  $q$  gives a decomposition

$$A'_{k,\eta} = A'_{0,1} + R + \llbracket \mathcal{A} \rrbracket,$$

with  $R = R_{A'}$  and  $\mathcal{A} = k\tilde{\mathcal{A}}_1 + \partial_n \tilde{\mathcal{A}}_2$ , as in Theorem A.4. By construction

$$(A.27) \quad \|(A'_{0,1})^{-1}R\|_{H^s(\Gamma) \leftarrow H^s(\Gamma)} \leq \hat{q} \quad \text{and} \quad \|(A'_{0,1})^{-1}R\|_{H^{s-1}(\Gamma) \leftarrow H^{s-1}(\Gamma)} \leq \hat{q}.$$

Hence,  $A'_{0,1} + R$  is boundedly invertible by a geometric series argument, since

$$(A.28) \quad (A'_{0,1} + R)^{-1} = (I + (A'_{0,1})^{-1}R)^{-1}(A'_{0,1})^{-1}$$

with the norm estimates

$$(A.29a) \quad \|(A'_{0,1} + R)^{-1}\|_{H^s(\Gamma) \leftarrow H^s(\Gamma)} \leq (1 - \hat{q})^{-1} \|(A'_{0,1})^{-1}\|_{H^s(\Gamma) \leftarrow H^s(\Gamma)},$$

$$(A.29b) \quad \|(A'_{0,1} + R)^{-1}\|_{H^{s-1}(\Gamma) \leftarrow H^{s-1}(\Gamma)} \leq (1 - \hat{q})^{-1} \|(A'_{0,1})^{-1}\|_{H^{s-1}(\Gamma) \leftarrow H^{s-1}(\Gamma)}.$$

By Proposition A.1 the operator  $A'_{k,\eta}: H^t(\Gamma) \rightarrow H^t(\Gamma)$  is boundedly invertible for  $t \geq -1$ . We may decompose  $(A'_{k,\eta})^{-1}$  as follows

$$(A.30) \quad (A'_{k,\eta})^{-1} = (A'_{0,1} + R)^{-1} + Q, \quad Q = -(A'_{k,\eta})^{-1} \llbracket \mathcal{A} \rrbracket (A'_{0,1} + R)^{-1},$$

since

$$\begin{aligned} I &= (A'_{k,\eta})(A'_{k,\eta})^{-1} = (A'_{k,\eta})(A'_{0,1} + R)^{-1} + (A'_{k,\eta})Q \\ &= (A'_{0,1} + R + \llbracket \mathcal{A} \rrbracket)(A'_{0,1} + R)^{-1} + (A'_{k,\eta})Q \\ &= I + \llbracket \mathcal{A} \rrbracket (A'_{0,1} + R)^{-1} + (A'_{k,\eta})Q. \end{aligned}$$

**Step 2** We rewrite the difference  $\text{DtN}_k - \text{DtN}_0$  using the combined field operators of (A.1). Using  $\eta$  as in (A.10) for  $\text{DtN}_k$  and  $\eta = 1$  for  $\text{DtN}_0$  we find

$$(A.31) \quad \text{DtN}_k - \text{DtN}_0 = (A'_{k,\eta})^{-1} [-D_k + i\eta(-1/2 + K_k)] - (A'_{0,1})^{-1} [-D_0 + i(-1/2 + K_0)].$$

Adding and subtracting  $D_0$  and  $K_0$  in (A.31), employing the splitting of  $D_k - D_0$  and  $K_k - K_0$  given in (A.4) in Proposition A.1, and applying the splitting of  $(A'_{k,1})^{-1}$

in (A.30) we find

$$\begin{aligned}
 \text{DtN}_k - \text{DtN}_0 &= -(A'_{k,\eta})^{-1} [D_k - D_0] - (A'_{k,\eta})^{-1} D_0 \\
 &\quad + i\eta(A'_{k,\eta})^{-1} [K_k - K_0] + i\eta(A'_{k,\eta})^{-1} [-1/2 + K_0] \\
 &\quad + (A'_{0,1})^{-1} D_0 - i(A'_{0,1})^{-1} [-1/2 + K_0] \\
 &= (A'_{0,1})^{-1} D_0 - (A'_{0,1} + R)^{-1} D_0 - QD_0 \\
 &\quad - (A'_{0,1} + R)^{-1} S_D - QS_D + (A'_{k,\eta})^{-1} \gamma_1^{int} \tilde{A}_K \\
 &\quad + i\eta(A'_{0,1} + R)^{-1} S_K + i\eta QS_K + i\eta(A'_{k,\eta})^{-1} \gamma_0^{int} \tilde{A}_K \\
 &\quad + i\eta(A'_{0,1} + R)^{-1} [-1/2 + K_0] + i\eta Q [-1/2 + K_0] \\
 &\quad - i(A'_{0,1})^{-1} [-1/2 + K_0] \\
 &= \text{FSO} + \text{ASO}
 \end{aligned}$$

with the Finite Shift Operators (FSO) and the Analytic Shift Operators (ASO) given by

$$\begin{aligned}
 \text{FSO} &:= (A'_{0,1})^{-1} D_0 - (A'_{0,1} + R)^{-1} D_0 - (A'_{0,1} + R)^{-1} S_D + i\eta(A'_{0,1} + R)^{-1} S_K \\
 &\quad + i\eta(A'_{0,1} + R)^{-1} [-1/2 + K_0] - i(A'_{0,1})^{-1} [-1/2 + K_0], \\
 \text{ASO} &:= -QD_0 - QS_D + (A'_{k,\eta})^{-1} \gamma_1^{int} \tilde{A}_K \\
 &\quad + i\eta QS_K + i\eta(A'_{k,\eta})^{-1} \gamma_0^{int} \tilde{A}_K + i\eta Q [-1/2 + K_0].
 \end{aligned}$$

**Step 3 (Analysis of FSO):** We claim

$$(A.32) \quad \text{FSO} = kB,$$

where  $B : H^s(\Gamma) \rightarrow H^s(\Gamma)$  is a bounded linear operator with  $\|Bu\|_{s,\Gamma} \lesssim \|u\|_{s,\Gamma}$ , as asserted in the present lemma. Using the mapping properties of  $(A'_{0,1} + R)^{-1}$  in (A.29a) as well as (A.3), (A.8) and Proposition A.1(ii) we find

$$\begin{aligned}
 \|(A'_{0,1} + R)^{-1} S_D u\|_{s,\Gamma} &\lesssim \|S_D u\|_{s,\Gamma} \lesssim k \|u\|_{s,\Gamma}, \\
 k \|(A'_{0,1} + R)^{-1} S_K u\|_{s,\Gamma} &\lesssim k \|S_K u\|_{s,\Gamma} \lesssim k \|u\|_{s,\Gamma}, \\
 k \|(A'_{0,1} + R)^{-1} [-1/2 + K_0] u\|_{s,\Gamma} &\lesssim k \|[-1/2 + K_0] u\|_{s,\Gamma} \lesssim k \|u\|_{s,\Gamma}, \\
 \|(A'_{0,1})^{-1} [-1/2 + K_0] u\|_{s,\Gamma} &\lesssim \|[-1/2 + K_0] u\|_{s,\Gamma} \lesssim \|u\|_{s,\Gamma}.
 \end{aligned}$$

The assertion (A.32) therefore follows once we have shown

$$\|(A'_{0,1})^{-1} D_0 - (A'_{0,1} + R)^{-1} D_0 u\|_{s,\Gamma} \lesssim k \|u\|_{s,\Gamma}.$$

To that end, we write using (A.28)

$$\begin{aligned}
 (A'_{0,1})^{-1} D_0 - (A'_{0,1} + R)^{-1} D_0 &= (A'_{0,1})^{-1} D_0 - (I + (A'_{0,1})^{-1} R)^{-1} (A'_{0,1})^{-1} D_0 \\
 &= [I - (I + (A'_{0,1})^{-1} R)^{-1}] (A'_{0,1})^{-1} D_0 = - \left[ \sum_{n=1}^{\infty} (-1)^n ((A'_{0,1})^{-1} R)^n \right] (A'_{0,1})^{-1} D_0.
 \end{aligned}$$

Applying the previous calculations, a geometric series argument with (A.27), the mapping properties of  $(A'_{0,1})^{-1}$  in Proposition A.1(ii), the estimate  $\|Ru\|_{s,\Gamma} \lesssim k \|u\|_{s-1,\Gamma}$  given by Theorem A.4, again the mapping properties of  $(A'_{0,1})^{-1}$ , and

finally the mapping properties of  $D_0$  given in (A.3) in Proposition A.1, we find

$$\begin{aligned} & \| (A'_{0,1})^{-1} D_0 - (A'_{0,1} + R)^{-1} D_0 u \|_{s,\Gamma} \\ &= \left\| \left[ \sum_{n=1}^{\infty} (-1)^n ((A'_{0,1})^{-1} R)^{n-1} \right] ((A'_{0,1})^{-1} R) (A'_{0,1})^{-1} D_0 u \right\|_{s,\Gamma} \\ &\leq \frac{1}{1-\hat{q}} \| (A'_{0,1})^{-1} R (A'_{0,1})^{-1} D_0 u \|_{s,\Gamma} \lesssim \| R (A'_{0,1})^{-1} D_0 u \|_{s,\Gamma} \\ &\lesssim k \| (A'_{0,1})^{-1} D_0 u \|_{s-1,\Gamma} \lesssim k \| D_0 u \|_{s-1,\Gamma} \lesssim k \| u \|_{s,\Gamma}. \end{aligned}$$

Hence, the assertion in (A.32) follows, which concludes the analysis of the finite shift operators FSO.

**Step 4 (Analysis of Analytic Shift Operators ASO):** We have

$$\text{ASO} = -Q D_0 - Q [S_D - i\eta S_K - i\eta (-1/2 + K_0)] + (A'_{k,\eta})^{-1} [i\eta \gamma_0^{int} \tilde{A}_K + \gamma_1^{int} \tilde{A}_K].$$

**Step 4a (Analysis of  $-Q D_0$ ):** With the definition of  $Q$  in (A.30) we have for  $f \in H^s(\Gamma)$

$$\begin{aligned} -Q D_0 f &= (A'_{k,\eta})^{-1} \llbracket \mathcal{A} \rrbracket (A'_{0,1} + R)^{-1} D_0 f \\ &= (A'_{k,\eta})^{-1} \{ k \llbracket \mathcal{A}_1 \rrbracket (A'_{0,1} + R)^{-1} D_0 f + \llbracket \partial_n \mathcal{A}_2 \rrbracket (A'_{0,1} + R)^{-1} D_0 f \}. \end{aligned}$$

In order to apply Lemma A.5, we use the mapping properties of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  given in Theorem A.4 and estimate

$$\begin{aligned} k \| (A'_{0,1} + R)^{-1} D_0 f \|_{-3/2,\Gamma} + k^{d/2} \| (A'_{0,1} + R)^{-1} D_0 f \|_{-1,\Gamma} \\ \lesssim k^{d/2} \| (A'_{0,1} + R)^{-1} D_0 f \|_{s-1,\Gamma} \lesssim k^{d/2} \| D_0 f \|_{s-1,\Gamma} \lesssim k^{d/2} \| f \|_{s,\Gamma}, \end{aligned}$$

where we used the trivial embedding  $H^{s-1}(\Gamma) \subset H^{-1}(\Gamma) \subset H^{-3/2}(\Gamma)$ , the fact that  $k + k^{d/2} \lesssim k^{d/2}$ , the mapping property (A.29b), and finally the mapping properties of  $D_0$  given in (A.3). Hence, for the tubular neighborhood  $T$  given in Theorem A.4 we find

$$\begin{aligned} \mathcal{A}_1 (A'_{0,1} + R)^{-1} D_0 f &\in \mathfrak{A}(C_1 k^{d/2} \| f \|_{s,\Gamma}, \gamma_1, T), \\ \mathcal{A}_2 (A'_{0,1} + R)^{-1} D_0 f &\in \mathfrak{A}(C_1 k \| f \|_{s,\Gamma}, \gamma_1, T), \end{aligned}$$

for constants  $C_1, \gamma_1 > 0$  independent of  $k$ . Lemma A.5 is applicable and yields, for constants  $\tilde{C}_1, \tilde{\gamma}_1 > 0$  independent of  $k$ , the representation

$$(A.33) \quad -Q D_0 f = \llbracket \partial_n v_f^1 \rrbracket, \quad v_f^1 \in \mathfrak{A}(\tilde{C}_1 k^{5/2+d/2} \| f \|_{s,\Gamma}, \tilde{\gamma}_1, \Omega_R).$$

**Step 4b (analysis of  $-Q [S_D - i\eta S_K - i\eta (-1/2 + K_0)]$ ):** We estimate

$$\begin{aligned} & k^{d/2} \| (A'_{0,1} + R)^{-1} [S_D - i\eta S_K - i\eta (-1/2 + K_0)] f \|_{-1,\Gamma} \\ &\lesssim k^{d/2} \| (A'_{0,1} + R)^{-1} [S_D - i\eta S_K - i\eta (-1/2 + K_0)] f \|_{s,\Gamma} \\ &\lesssim k^{d/2} \| [S_D - i\eta S_K - i\eta (-1/2 + K_0)] f \|_{s,\Gamma} \\ &\lesssim k^{d/2+1} \| f \|_{s,\Gamma}, \end{aligned}$$

where we first use the trivial embedding  $H^s(\Gamma) \subset H^{-1}(\Gamma)$ , the mapping property (A.29a), the mapping properties of  $S_D, S_K$ , and  $K_0$  given in (A.8) and (A.3) as well as  $|\eta| \lesssim k$ . Proceeding as in Step 4a we find the representation

$$(A.34) \quad -Q [S_D - i\eta S_K - i\eta (-1/2 + K_0)] = \llbracket \partial_n v_f^2 \rrbracket, \quad v_f^2 \in \mathfrak{A}(\tilde{C}_2 k^{5/2+d/2+1} \| f \|_{s,\Gamma}, \tilde{\gamma}_2, \Omega_R)$$

to hold true, for constants  $\tilde{C}_2, \tilde{\gamma}_2 > 0$  independent of  $k$ .

**Step 4c (analysis of  $(A'_{k,\eta})^{-1}[i\eta\gamma_0^{int}\tilde{A}_K + \gamma_1^{int}\tilde{A}_K]$ ):** For  $f \in H^s(\Gamma)$  the mapping properties of  $\tilde{A}_K$  imply  $\tilde{A}_K f \in \mathfrak{A}(C_K k \|f\|_{-1/2,\Gamma}, \gamma_K, \Omega)$ , see (A.7). Upon extending  $\tilde{A}_K f$  by zero outside of  $\Omega$ , we find Lemma A.5 to be applicable, which yields (A.35)

$$(A'_{k,\eta})^{-1}[i\eta\gamma_0^{int}\tilde{A}_K + \gamma_1^{int}\tilde{A}_K]f = \llbracket \partial_n v_f^3 \rrbracket, \quad v_f^3 \in \mathfrak{A}(\tilde{C}_3 k^{5/2+1} \|f\|_{-1/2,\Gamma}, \tilde{\gamma}_3, \Omega_R),$$

with constants  $\tilde{C}_3, \tilde{\gamma}_3 > 0$  independent of  $k$ .

**Step 5:** Collecting the representations (A.33), (A.34), and (A.35) gives

$$\text{ASO} = \llbracket \partial_n \tilde{A} \rrbracket, \quad \tilde{A}u \in \mathfrak{A}(Ck^{7/2+d/2} \|u\|_{s,\Gamma}, \gamma, \Omega_R)$$

with  $\tilde{A}$  as in the assertions of the present lemma. Hence, the splitting

$$\text{DtN}_k - \text{DtN}_0 = kB + \llbracket \partial_n \tilde{A} \rrbracket$$

with  $B$  and  $\tilde{A}$  as asserted, holds true. This concludes the proof.  $\square$

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