



HAL
open science

Wavenumber-explicit stability and convergence analysis of hp finite element discretizations of Helmholtz problems in piecewise smooth media

Maximilian Bernkopf, Théophile Chaumont-Frelet, Jens Markus Melenk

► To cite this version:

Maximilian Bernkopf, Théophile Chaumont-Frelet, Jens Markus Melenk. Wavenumber-explicit stability and convergence analysis of hp finite element discretizations of Helmholtz problems in piecewise smooth media. 2022. hal-03771988v1

HAL Id: hal-03771988

<https://inria.hal.science/hal-03771988v1>

Preprint submitted on 7 Sep 2022 (v1), last revised 2 Feb 2024 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

WAVENUMBER-EXPLICIT STABILITY AND CONVERGENCE ANALYSIS OF hp FINITE ELEMENT DISCRETIZATIONS OF HELMHOLTZ PROBLEMS IN PIECEWISE SMOOTH MEDIA

M. BERNKOPF*, T. CHAUMONT-FRELET†, AND J.M. MELENK*

ABSTRACT. We present a wavenumber-explicit convergence analysis of the hp Finite Element Method applied to a class of heterogeneous Helmholtz problems with piecewise analytic coefficients at large wavenumber k . Our analysis covers the heterogeneous Helmholtz equation with Robin, exact Dirichlet-to-Neumann, and second order absorbing boundary conditions, as well as perfectly matched layers.

1. INTRODUCTION

Time-harmonic wave propagation problems play a major role in a wide range of physical and industrial applications. For the numerical treatment of such problems in heterogeneous media, Galerkin discretization methods of finite element type constitute one of the most efficient approaches currently available. Still, in the high-wavenumber regime where the computational domain spans many wavelengths, the problem becomes highly indefinite, which leads to substantially increased computational costs due to increased dispersion errors. As a result, significant research efforts have focused in the past decades on understanding these dispersion errors and the question of how to design finite element methods (FEM) able to cope with them.

In the present work, as in [Ber21], we focus on the scalar Helmholtz equation, which is arguably the simplest, yet interesting, PDE model to analyze phenomena appearing in the high-wavenumber regime. A first milestone towards a better understanding of the behavior of finite element methods was achieved in the 90s in the seminal works [IB95, IB97]. There, the authors focus on homogeneous one-dimensional media, and present error estimates that are explicit in terms of the wavenumber k , the mesh size h , and the polynomial degree p . They also coined the term “pollution effect” for the dispersion errors observed in the numerics, which expresses the observation that in the high-wavenumber regime, quasi-optimality of the finite element solution is lost, unless the number of degrees of freedom per wavelength is increased as k is increased. Similar estimates were obtained for two and three-dimensional homogeneous media in the same period, but limited to the lowest-order case where $p = 1$ [Mel95].

The pollution effect has been further analyzed in the 2000s, in particular using dispersion analysis [Ain04]. While dispersion analysis is extremely insightful from a practical viewpoint, it is limited to homogeneous media discretized with translation-invariant grids, and right-hand sides as well as boundary conditions are not taken into account.

The direct extension of the first-order result of [Mel95] to higher-order elements is non-trivial. Indeed, the error analysis follows a duality technique commonly known as “Schatz argument”

*Institute for Analysis and Scientific Computing, TU Wien, A-1040 Vienna

†University Côte d’Azur, Inria, CNRS, LJAD, 2004 Rte des Lucioles, 06902 Valbonne

[Sch74], where the finite element error is abstractly used as a right-hand side of an adjoint problem. One is then led to study the regularity of the solution to the adjoint problem with an L^2 right-hand side, even if the actual right-hand side to the physical problem is more regular. This is an issue when analyzing high-order methods, since their high convergence rate do not apply to solutions with low regularity. The situation was unlocked in the 2010s thanks to the key concept of regularity splitting [MS10, MS11]. The idea is to identify two distinct contributions in the solution, namely, $u = u_F + u_A$, where u_F only possesses finite regularity, but behaves well in the high-wavenumber regime, whereas u_A is highly-oscillating, but analytic.

The splitting proposed in [MS10, MS11] is based on low- and high-pass filters defined via the Fourier transformation. For homogeneous media, it permits to establish the quasi-optimality of the finite element solution under the assumption that

$$(1.1) \quad \frac{kh}{p} + k^\theta \left(\frac{kh}{\sigma p} \right)^p \leq c,$$

for some problem-dependent θ and σ and a generic constant c that does not depend on k , h , or p . We can also rewrite (1.1) as

$$(1.2) \quad \frac{kh}{p} \leq c_1 \quad \text{and} \quad p \geq c_2 \log k,$$

where c_1 and c_2 are again independent of k , h , and p . Informally, (1.2) suggests that the finite element is stable if the number of degrees of freedom per wavelength is kept constant ($kh/p \leq c_1$) and the polynomial degree is mildly increased in the high-wavenumber regime ($p \geq c_2 \log k$). It also means that hp -FEMs are pollution free, provided the polynomial degree p is increased logarithmically with the wavenumber. Together with the dispersion analysis [Ain04], the stability condition (1.2) strongly encourages the use of high-order methods to solve high-wavenumber problems to reduce the pollution effect. In fact, the performance of high-order methods is also clearly observed numerically, and therefore, they are now widely used in industry [BPG16, TZNHD17].

The idea of regularity splitting is actually not limited to finite element methods, and has been widely used in the literature. Important examples include the sharp treatment of corner singularities [CFN18] and unfitted meshes [CF16], as well as other discretization techniques such as discontinuous Galerkin [MPS13], continuous interior penalty [DW15, ZW13] and multiscale [CFV20] methods. Besides, the technique is not only useful to derive *a priori* error estimates, but is also important in the context of *a posteriori* error estimation [CFEV21, DS13]. We finally mention that the idea can also be extended to more complicated wave propagation problems [MS21, CFV21].

Regularity splittings are thus an important ingredient of wavenumber-explicit analysis of FEMs. Yet, the technique proposed in [MS10, MS11] heavily relies on the Green's functions of the Helmholtz operator, and, as a result, it appears to be limited to homogeneous media. So far, two alternative procedures to build a regularity splitting in heterogeneous media have been proposed. On the one hand, an iterative argument based on elliptic regularity is proposed in [CFN19]. This technique can handle general wave operators in piecewise smooth media, but unfortunately, it only provides (1.1) with a constant c depending on p , which prevents the analysis of hp -FEMs, and do not lead to (1.2). On the other hand, novel ideas based on semi-classical analysis have been recently introduced in [LSW20, LSW21, GLSW22]. While this last splitting provides (1.1) and (1.2) with p -robust constants, it requires *globally* smooth

parameters. In particular, the important case of piecewise constant parameters, modelling a medium containing different materials, is not covered.

In this work, we modify the initial Fourier filtering approach of [MS10, MS11] and adapt it to be able to develop a regularity splitting of solutions of wave propagation problems in piecewise smooth media and subsequently apply a duality argument (“Schatz argument”) to Galerkin discretizations. We cover several classes of scalar Helmholtz problems in heterogeneous media with different types of boundary conditions, providing quasi-optimality under the resolution conditions (1.1) and (1.2) with p -robust constants. The approach taken in the present work is quite general and thus not limited to the scalar Helmholtz problems under consideration. The remainder of this work is organized as follows. Section 2 presents the settings and key notations for our work. We devise our regularity splitting in Section 3 and subsequently apply it to Galerkin discretizations in Section 4. The analysis of Sections 3 and 4 is performed in an abstract setting under general assumptions. In Section 5, we verify that these abstract assumptions indeed hold true for several classes of relevant scalar Helmholtz problems, namely, the heterogeneous Helmholtz equation with a) the classical impedance boundary conditions and perfectly matched layers on a circular/spherical domain; b) exact boundary conditions expressed in terms of the Dirichlet-to-Neumann operator; c) second order absorbing boundary conditions. We also refer to [Ber21] where in addition to the present examples the case of the elasticity equation on circular domains with exact Dirichlet-to-Neumann boundary conditions is analyzed with the present techniques. Appendix A presents technical results about Dirichlet-to-Neumann operators which are of independent interest. Appendix B shows analytic regularity for Helmholtz problems equipped with second order absorbing boundary conditions.

2. ASSUMPTIONS AND PROBLEM SPECIFIC NOTATION

2.1. Geometry. We consider a Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, with boundary $\Gamma := \partial\Omega$. We set $\Omega^+ := \mathbb{R}^d \setminus \bar{\Omega}$. We introduce the interior trace $\gamma_0^{int}u := (u|_{\Omega})|_{\Gamma}$ and the exterior trace $\gamma_0^{ext}u := (u|_{\Omega^+})|_{\Gamma}$. We assume that Ω is partitioned into a set \mathcal{P} of non-overlapping Lipschitz subdomains such that $\cup_{P \in \mathcal{P}} P = \bar{\Omega}$. The internal interface $(\cup_{P \in \mathcal{P}} \partial P) \setminus \Gamma$ will be denoted Γ_{interf} . While the abstract results in Sections 3 and 4 do not require much regularity of Γ and Γ_{interf} , we flag at this point already that in the applications discussed in Section 5 the assumptions on the geometry are such that Γ and the pieces ∂P , $P \in \mathcal{P}$, of Γ_{interf} are analytic (Assumption 5.2).

We use the letter k for the wavenumber. Since we are especially interested in the high-wavenumber regime, we assume for the sake of simplicity that k is bounded away from zero, i.e., $k \geq k_0 > 0$.

Throughout this work, C will denote a generic constant that is independent of the wavenumber k and other critical parameters such as the mesh size h and the polynomial degree p . The constant may, however, depend on the operators appearing in assumptions (WP) and (AP) below. It may also depend on the constant C_{affine} and C_{metric} introduced in Assumption 4.4 that describes the “shape regularity” of the elements of the mesh. If a, b, \dots are other quantities appearing in the analysis, we employ the notation $C_{a,b,\dots}$ for a constant that is additionally allowed to depend on a, b, \dots . Finally, we will employ explicit names together with a subscript k for constants allowed to depend on the wavenumber.

2.2. Function spaces. The notations $L^2(\Omega)$ and $L^2(\Gamma)$ stand for the usual Lebesgue spaces of square-integrable complex-valued functions defined on Ω and Γ , respectively. For $s, t \geq 0$,

$H^s(\Omega)$ and $H^t(\Gamma)$ are (possibly) fractional Sobolev spaces of functions defined on Ω and Γ . $\tilde{H}^{-s}(\Omega)$ and $H^{-t}(\Gamma)$ are then the (topological) antidual of $H^s(\Omega)$ and $H^t(\Gamma)$. For $0 \leq t \leq 1$, we introduce the ‘‘energy’’ space $H^{1,t}(\Omega, \Gamma) := \{u \in H^1(\Omega) : u|_{\Gamma} \in H^t(\Gamma)\}$ with k dependent norm

$$(2.1) \quad \|u\|_{1,t,k}^2 := k^2 \|u\|_{0,\Omega}^2 + k^{1-2t} |u|_{t,\Gamma}^2 + |u|_{1,\Omega}^2.$$

We introduce this space with $t = 1$ to handle second-order absorbing boundary conditions that involve surface differential operators. One easily sees that for $t \leq 1/2$ the space $H^{1,t}(\Omega, \Gamma)$ coincides with $H^1(\Omega)$, but we still employ the notation $H^{1,t}(\Omega, \Gamma)$ to unify our presentation. In addition to Sobolev spaces, we will employ spaces of (piecewise) analytic functions. For open sets $\omega \subset \mathbb{R}^d$ and $M, \gamma \geq 0$, we set

$$(2.2) \quad \mathfrak{A}(M, \gamma, \omega) := \{v \in L^2(\omega) \mid \|\nabla^n v\|_{0,\omega} \leq M\gamma^n \max\{n, k\}^n, \quad n \in \mathbb{N}_0\},$$

where $|\nabla^n v(x)|^2 = \sum_{|\alpha|=n} \frac{n!}{\alpha!} |D^\alpha v(x)|^2$. The analyticity class

$$(2.3) \quad \mathfrak{A}(M, \gamma, \mathcal{P}) := \{v \in L^2(\Omega) \mid v|_P \in \mathfrak{A}(M, \gamma, P) \quad \forall P \in \mathcal{P}\}$$

is a class of piecewise analytic functions on Ω . We will also employ analytic functions defined on the boundary Γ . For the sake of simplicity, analyticity classes on Γ are defined as traces of analytic functions. That is, for an open neighborhood T of Γ and $\gamma > 0$, we set

$$\mathfrak{A}(M, \gamma, T, \Gamma) := \{\gamma_0^{int} V \mid V \in \mathfrak{A}(M, \gamma, T \cap \Omega)\}.$$

By a slight abuse of notation, for a function $g \in L^2(\Gamma)$ we will write $g \in \mathfrak{A}(M, \gamma, T, \Gamma)$ to indicate the existence of a function $G \in \mathfrak{A}(M, \gamma, T \cap \Omega, \Gamma)$ with $\gamma_0^{int} G = g$.

Remark 2.1. (i) Functions $G \in \mathfrak{A}(M, \gamma, T \cap \Omega)$ are analytic on $\overline{T \cap \Omega}$ and therefore have a trace on Γ . Indeed, for $T \cap \Omega$ Lipschitz we have by the Sobolev embedding theorem for $t \in \mathbb{N}$ with $t > d/2$ that $H^t(T \cap \Omega) \subset C(\overline{T \cap \Omega})$. Hence, $\|\nabla^n G\|_{L^\infty(T \cap \Omega)} \leq C\gamma^{n+t} \max\{n+2, k\}^{n+t}$ for all $n \in \mathbb{N}_0$ for suitable C . In turn, this implies that the Taylor series of G about each point $x_0 \in T \cap \Omega$ converges with radius of convergence $1/(e\gamma)$.

(ii) The choice of interior traces in $\mathfrak{A}(M, \gamma, T, \Gamma)$ is arbitrary. One could equivalently define them as traces of functions analytic in subsets of Ω^+ : It is shown in [Mel12, Lemma B.5] that if $G \in \mathfrak{A}(M, \gamma, T \cap \Omega)$, then there is a neighborhood T' of Γ (depending only on γ), constants M', γ' and a $G' \in \mathfrak{A}(M', \gamma', T' \cap \Omega^+)$ with $\gamma_0^{int} G = \gamma_0^{ext} G'$.

(iii) Let T be an open neighborhood of Γ and $G \in \mathfrak{A}(M, \gamma, T \setminus \Gamma)$. By (i), the traces $\gamma_0^{int} G$ and $\gamma_0^{ext} G$ exist so that the jump $\llbracket G \rrbracket := \gamma_0^{ext} G - \gamma_0^{int} G$ exists. In view of the arguments given in (ii), we have $\llbracket G \rrbracket \in \mathfrak{A}(M', \gamma', T', \Gamma)$ for some γ' and T' depending only on γ and T and some M' depending additionally on M . ■

We note that if $\gamma \leq \gamma'$ and $M \leq M'$, then

$$\mathfrak{A}(M, \gamma, \mathcal{P}) \subset \mathfrak{A}(M', \gamma', \mathcal{P}), \quad \mathfrak{A}(M, \gamma, T, \Gamma) \subset \mathfrak{A}(M', \gamma', T, \Gamma)$$

We also notice that the trace inequality

$$(2.4) \quad k^{1/2} \|v\|_{s,\Gamma} + \|v\|_{s+1/2,\Gamma} \leq C_s (k \|v\|_{s,\mathcal{P}} + \|v\|_{s+1,\mathcal{P}})$$

holds true for every $s \geq 0$. The analyticity classes are invariant under analytic changes of variables and multiplication by analytic functions:

Lemma 2.2 ([MS21, Lemma 2.6]). *Let $\omega_1, \omega_2 \subset \mathbb{R}^d$ be bounded open and $g : \omega_1 \rightarrow \omega_2$ be a bijection and analytic on the closed set $\bar{\omega}_1$. Let f_1 be analytic on the closed set $\bar{\omega}_2$ and $f_2 \in \mathfrak{A}(M_f, \gamma_f, \omega_2)$. Then there are constants $C_1, \gamma_1 > 0$ depending only on $f_1, g, \omega_1, \omega_2$, and γ_f such that $f_1(f_2 \circ g) \in \mathfrak{A}(C_1 M_f, \gamma_1, \omega_1)$.*

2.3. Time-harmonic wave propagation problem. Given $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$, our model problem consists in finding $u \in H^{1,t}(\Omega, \Gamma)$ such that

$$(2.5) \quad b_k^-(u, v) = (f, v) + \langle g, v \rangle \quad \forall v \in H^{1,t}(\Omega, \Gamma)$$

with

$$(2.6) \quad b_k^-(u, v) := a(u, v) - (T_{k,\Omega}^- u, v) - \langle T_{k,\Gamma}^- u, v \rangle,$$

where $a(\cdot, \cdot)$ is a sesquilinear form over $H^{1,t}(\Omega, \Gamma)$ and the operators $T_{k,\Omega}^-$ and $T_{k,\Gamma}^-$ are differential operators acting on “volume” and “surface” functions. Before providing a detailed description of these operators, we first explain their “intuitive” meaning.

The sesquilinear form $a(\cdot, \cdot)$ is meant to represent the “elliptic part” of the wave operator. A typical example is the “grad-grad” form, but one may imagine other situation with the elasticity operator for instance. The operator $T_{k,\Omega}^-$ then accounts for the “frequency part” of the operator, i.e., the time-derivative transformed into ik in the frequency domain. The most basic situation is then $T_{k,\Omega}^- u := k^2 u$, but we allow for more generality. Finally, $T_{k,\Gamma}^-$ is an operator on Γ ; relevant examples are the Dirichlet-to-Neumann maps or some approximation of it.

We now summarize our assumptions **(WP)** on the wave propagation problem.

(WP1) The sesquilinear form $b_k^- : H^{1,t}(\Omega, \Gamma) \times H^{1,t}(\Omega, \Gamma) \rightarrow \mathbb{C}$ is continuous, i.e., there exists a constant $C_{\text{cont},k}^- \geq 0$, possibly depending on k , such that

$$|b_k^-(u, v)| \leq C_{\text{cont},k}^- \|u\|_{1,t,k} \|v\|_{1,t,k}, \quad \forall u, v \in H^{1,t}(\Omega, \Gamma).$$

(WP2) Problem (2.5) is well posed, i.e., for every $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$ it admits a unique weak solution $S_k^-(f, g) := u \in H^{1,t}(\Omega, \Gamma)$, and the stability estimate

$$(2.7) \quad \|S_k^-(f, g)\|_{1,t,k} \leq C_{\text{sol},k}^- (\|f\|_{0,\Omega} + k^{1/2} \|g\|_{0,\Gamma}),$$

holds true with a constant $C_{\text{sol},k}^-$, possibly depending on k , but independent of f and g (See also Remark 2.4).

Assumptions **(WP1)** and **(WP2)** simply state that the problem is well-posed and that the direct and inverse operators are bounded with continuity constants $C_{\text{cont},k}^-$ and $C_{\text{sol},k}^-$. For our quasi-optimality result we also need the solution to be (piecewise) analytic when the right-hand sides are (piecewise) analytic.

(WP3) For each tubular neighborhood T of Γ there exists a constant γ_0 and there exists a non-decreasing function $\vartheta : [\gamma_0, +\infty) \rightarrow \mathbb{R}$, both independent of k , and there exists a possibly k -dependent constant $C_{\text{ana},k}^-$ such that for all $\gamma \geq \gamma_0$ the following holds: for all piecewise analytic data $f \in \mathfrak{A}(M_f, \gamma, \mathcal{P})$ and all boundary data $g \in \mathfrak{A}(M_g, \gamma, T, \Gamma)$ the solution $S_k^-(f, g)$ to Problem (2.5) satisfies $S_k^-(f, g) \in \mathfrak{A}(M_u, \vartheta(\gamma), \mathcal{P})$, i.e., it is again piecewise analytic, and

$$(2.8) \quad M_u \leq C_{\text{ana},k}^- C_{\text{sol},k}^- k^{-1} (M_f + k M_g).$$

Remark 2.3. **(WP3)** permits the data f, g to depend on k . In particular, the constants M_f, M_g may depend on k but **(WP3)** requires the constant M_u to be bounded as given in (2.8). ■

For many model problems, the continuity constant $C_{\text{cont},k}^-$ of b_k^- does not depend on k , and the three above assumptions are sufficient to proceed with our analysis. However, $C_{\text{cont},k}^-$ may depend on k in general, for example, when considering Dirichlet-to-Neumann (DtN) operators on general surfaces Γ , or when dealing with Maxwell's equations [MS21]. As a result, we include an additional assumption to treat these cases.

(WP4) The linear operators $T_{k,\Omega}^-: H^{1,t}(\Omega, \Gamma) \rightarrow H^{1,t}(\Omega, \Gamma)'$ and $T_{k,\Gamma}^-: H^t(\Gamma) \rightarrow H^{-t}(\Gamma)$ admit splittings into linear operators

$$(2.9) \quad T_{k,\Omega}^- = R_{k,\Omega}^- + A_{\Omega}^-, \quad T_{k,\Gamma}^- = R_{k,\Gamma}^- + A_{\Gamma}^-,$$

such that the “ R part” is uniformly bounded in k , while the “ A part” maps into analytic functions. Specifically, we assume

$$(2.10) \quad | \langle R_{k,\Omega}^- u, v \rangle | + | \langle R_{k,\Gamma}^- u, v \rangle | \leq C \|u\|_{1,t,k} \|v\|_{1,t,k} \quad \forall u, v \in H^{1,t}(\Omega, \Gamma).$$

In addition, there exists a constant $C_{A,k}^-$ possibly depending on k , and there exist a constant γ_A^- and a tubular neighborhood T (both independent of k) such that for all $u \in H^{1,t}(\Omega, \Gamma)$ we have $A_{\Omega}^- u \in \mathfrak{A}(M_u, \gamma_A^-, \mathcal{P})$ and $A_{\Gamma}^-(u|_{\Gamma}) \in \mathfrak{A}(M_v, \gamma_A^-, T, \Gamma)$ with

$$M_u \leq C_{A,k}^- \|u\|_{1,k,t}, \quad kM_v \leq C_{A,k}^- \|u\|_{1,k,t}.$$

(WP5) We finally assume that $a(\cdot, \cdot)$ is uniformly-in- k continuous, i.e.,

$$(2.11) \quad |a(u, v)| \leq C \|u\|_{1,t,k} \|v\|_{1,t,k} \quad \forall u, v \in H^{1,t}(\Omega, \Gamma).$$

2.4. Auxiliary positive problem. Our analysis hinges on an auxiliary problem that is meant to be a “positive” version of the time-harmonic problem. We thus introduce the sesquilinear form

$$(2.12) \quad b_k^+(u, v) := a(u, v) - (T_{k,\Omega}^+ u, v) - \langle T_{k,\Gamma}^+ u, v \rangle \quad \forall u, v \in H^{1,t}(\Omega, \Gamma),$$

where $T_{k,\Omega}^+ : H^{1,t}(\Omega) \rightarrow (H^{1,t}(\Omega))'$ and $T_{k,\Gamma}^+ : H^t(\Omega) \rightarrow H^{-t}(\Omega)$ are problem specific abstract (continuous linear) operators specifically chosen to conduct the analysis. In the simplest case when $T_{k,\Omega}^- = +k^2$ and $T_{k,\Gamma}^- = -ik$, we can simply select $T_{k,\Omega}^+ = -k^2$ and $T_{k,\Gamma}^+ = 0$. However, allowing for more generality enables us to treat a much wider class of problems.

The assumptions **(AP)** on the auxiliary problem are listed below.

(AP1) b_k^+ is coercive: there exist $C_{\text{coer}}^+ > 0$ (independent of k) and $\sigma \in \mathbb{C}$ with $|\sigma| = 1$ such that

$$(2.13) \quad \text{Re}(\sigma b_k^+(u, u)) \geq C_{\text{coer}}^+ \|u\|_{1,t,k}^2 \quad \forall u \in H^{1,t}(\Omega, \Gamma).$$

Thus, for $f \in \tilde{H}^{-1}(\Omega)$ and $g \in H^{-t}(\Gamma)$, we may define $S_k^+(f, g)$ as the unique element of $H^{1,t}(\Omega, \Gamma)$ such that

$$(2.14) \quad b_k^+(S_k^+(f, g), v) = (f, v) + \langle g, v \rangle \quad \forall v \in H^{1,t}(\Omega, \Gamma).$$

(AP2) There exists an integer $s_{\max} \geq 0$ and for each $0 \leq s \leq s_{\max}$ there exists a $C_{\text{shift},s}^+ > 0$ independent of k such that if $f \in H^s(\mathcal{P})$ and $g \in H^{s+1/2}(\Gamma)$ we have $S_k^+(f, g) \in H^{s+2}(\mathcal{P})$ with

$$(2.15) \quad \|S_k^+(f, g)\|_{s+2, \mathcal{P}} \leq C_{\text{shift},s}^+ \left[k^s \|f\|_{0, \Omega} + \|f\|_{s, \mathcal{P}} + k^{s+1/2} \|g\|_{0, \Gamma} + \|g\|_{s+1/2, \Gamma} \right].$$

We further need the auxiliary problem to have similar Sobolev regularity properties as the original wave propagation problem. This is measured by the sesquilinear form

$$b_k^\Delta(u, v) := b_k^-(u, v) - b_k^+(u, v) = (T_{k, \Omega}^\Delta u, v) + \langle T_{k, \Gamma}^\Delta u, v \rangle \quad \forall u, v \in H^{1,t}(\Omega, \Gamma),$$

where $T_{k, \Omega}^\Delta = T_{k, \Omega}^- - T_{k, \Omega}^+$ and $T_{k, \Gamma}^\Delta = T_{k, \Gamma}^- - T_{k, \Gamma}^+$. Similar to the discussion on assumption **(WP4)** it is enough for many applications to assume that b_k^Δ is bounded uniformly in k . To be fully general, however, we need a last splitting assumption.

(AP3) We have the splittings

$$(2.16) \quad T_{k, \Omega}^\Delta = R_{k, \Omega}^\Delta + A_{k, \Omega}^\Delta, \quad T_{k, \Gamma}^\Delta = R_{k, \Gamma}^\Delta + A_{k, \Gamma}^\Delta.$$

The linear operator $R_{k, \Omega}^\Delta$ is of order at most one and satisfies for $0 \leq s \leq s_{\max}$

$$(2.17) \quad \|R_{k, \Omega}^\Delta u\|_{s, \mathcal{P}} \leq C_s (k^2 \|u\|_{s, \mathcal{P}} + k \|u\|_{s+1, \mathcal{P}}) \quad \forall u \in H^{s+1}(\mathcal{P}),$$

whereas the linear operator $R_{k, \Gamma}^\Delta$ is of order zero and satisfies

$$(2.18) \quad \|R_{k, \Gamma}^\Delta v\|_{s+1/2, \Gamma} \leq C \left(k^{3/2} \|v\|_{s, \Gamma} + k \|v\|_{s+1/2, \Gamma} \right) \quad \forall v \in H^{s+1/2}(\Gamma).$$

Finally, there exists a constant $C_{A,k}^\Delta$, possibly depending on k , and there exist a constant γ_A^Δ and a tubular neighborhood T of Γ (both independent of k) such that for all $u \in H^{1,t}(\Omega, \Gamma)$ we have $A_{k, \Omega}^\Delta u \in \mathfrak{A}(M_u, \gamma_A^\Delta, \mathcal{P})$ and $A_{k, \Gamma}^\Delta(u|_\Gamma) \in \mathfrak{A}(M_v, \gamma_A^\Delta, T, \Gamma)$ with

$$(2.19) \quad M_u \leq C_{A,k}^\Delta \|u\|_{1,k,t}, \quad k M_v \leq C_{A,k}^\Delta \|u\|_{1,k,t}$$

Remark 2.4 (on the stability assumption **(WP2)**). The stability estimate (2.7) does not make minimal assumptions on the data f, g . However, the stability constant $C_{\text{sol},k}^-$ (2.7) is closely related to the inf-sup constant

$$(2.20) \quad \gamma_{\text{inf-sup}} := \inf_{u \in H^{1,t}(\Omega, \Gamma)} \sup_{v \in H^{1,t}(\Omega, \Gamma)} \frac{|b_k^-(u, v)|}{\|u\|_{1,k,t} \|v\|_{1,k,t}}$$

if **(AP1)** and **(AP3)** hold. Let $B_k^- : H^{1,t}(\Omega, \Gamma) \rightarrow H^{1,t}(\Omega, \Gamma)'$ be the operator induced by b_k^- . We claim that if B_k^- is an isomorphism, in which case $1/\gamma_{\text{inf-sup}} = \|(B_k^-)'\|_{H^{1,t}(\Omega, \Gamma) \leftarrow H^{1,t}(\Omega, \Gamma)'}$, then (2.7) holds with $C_{\text{sol},k}^- = 1/\gamma_{\text{inf-sup}}$. To see this, note that for solution $u = S_k^-(f, g)$ of (2.5) there is $v \in H^{1,t}(\Omega, \Gamma)$ with $\|v\|_{1,k,t} = 1$ such that satisfies

$$\gamma_{\text{inf-sup}} \|u\|_{1,k,t} \leq |b_k^-(u, v)| = |(f, v) + \langle g, v \rangle| \lesssim (k^{-1} \|f\|_{0, \Omega} + k^{-1/2} \|g\|_{0, \Gamma}) \|v\|_{1,k,t}$$

so that (2.7) holds with $C_{\text{sol},k}^- = 1/\gamma_{\text{inf-sup}}$. Conversely, if (2.7) holds, then the inf-sup constant satisfies (2.21) below as we now show using the arguments used, e.g., in [Mel95, Prop. 8.2.7] or [EM12, Thm. 2.5]. The assumption **(AP3)** implies that B_k^- is a compact perturbation of a coercive operator. Since (2.7) implies uniqueness, the Fredholm alternative provides that B_k^- is an isomorphism. Since the norm of $(B_k^-)^{-1}$ equals the norm of $((B_k^-)')^{-1}$, we compute

the latter. To see this, let $u \in H^{1,t}(\Omega, \Gamma)$ be arbitrary and $z \in H^{1,t}(\Omega, \Gamma)$ solve, for all $w \in H^{1,t}(\Omega, \Gamma)$

$$\begin{aligned} b_k^-(z, w) &= b_k^-(u, w) - b_k^+(u, w) \\ &= (T_{k,\Omega}^\Delta u, w) + \langle T_{k,\Gamma}^\Delta u, w \rangle = (R_{k,\Omega}^\Delta u + A_{k,\Omega}^\Delta u, w) + \langle R_{k,\Gamma}^\Delta u + A_{k,\Gamma}^\Delta u, w \rangle. \end{aligned}$$

Noting

$$\begin{aligned} |\langle T_{k,\Omega}^\Delta u, w \rangle| &\lesssim (k + C_{A,k}^\Delta) \|u\|_{1,k,t} \|w\|_{0,\Omega}, \\ |\langle T_{k,\Gamma}^\Delta u, w \rangle| &\lesssim \|T_{k,\Gamma}^\Delta u\|_{0,\Gamma} \|w\|_{0,\Gamma} \lesssim \left(\|R_{k,\Gamma}^\Delta u\|_{1/2,\Gamma} + k^{-1/2} C_{A,k}^\Delta \|u\|_{1,k,t} \right) \|w\|_{0,\Gamma} \\ &\lesssim \left(k^{3/2} \|u\|_{0,\Gamma} + k \|u\|_{1/2,\Gamma} + C_{A,k}^\Delta k^{-1/2} \|u\|_{1,k,t} \right) \|w\|_{0,\Gamma} \\ &\lesssim (k + C_{A,k}^\Delta k^{-1/2}) \|u\|_{1,k,t} \|w\|_{0,\Gamma}. \end{aligned}$$

By (2.7) and these bounds, we get

$$\|z\|_{1,k,t} \lesssim (k^{3/2} + C_{A,k}^\Delta) C_{\text{sol},k}^- \|u\|_{1,k,t}.$$

We conclude

$$\sup_v \frac{|b_k^-(v, u)|}{\|u\|_{1,k,t} \|v\|_{1,k,t}} \stackrel{v=u-z}{\geq} \frac{|b_k^+(u, u)|}{\|u\|_{1,k,t} \|u-z\|_{1,k,t}} \stackrel{\text{(AP1)}}{\gtrsim} \frac{\|u\|_{1,k,t}^2}{\|u\|_{1,k,t} (\|u\|_{1,k,t} + \|z\|_{1,k,t})}$$

and arrive at

$$(2.21) \quad \gamma_{\text{infsup}} \gtrsim \frac{1}{C_{\text{sol},k}^- (k^{3/2} + C_{A,k}^\Delta)}.$$

■

3. ABSTRACT CONTRACTION ARGUMENT

3.1. Filtering operators. Following [MS10, MS11], a key ingredient of our splittings are filtering operators defined using cut-offs in the Fourier domain. We need to filter both surface and volume functions, leading to two pairs of filters.

On the boundary Γ , we can employ the surface filters introduced in [MS11, Lemmas 4.2, 4.3]. Proposition 3.1 sums up their essential properties.

Proposition 3.1 (Surface filters). *For each $\eta > 1$, there exist two operators $L_{k,\Gamma}^\eta, H_{k,\Gamma}^\eta: L^2(\Gamma) \rightarrow L^2(\Gamma)$ such that $H_{k,\Gamma}^\eta + L_{k,\Gamma}^\eta = \text{id}$. For $0 \leq s' \leq s$ the operator $H_{k,\Gamma}^\eta$ satisfies*

$$(3.1) \quad \|H_{k,\Gamma}^\eta g\|_{H^{s'}(\Gamma)} \leq C_{s,s'} (\eta k)^{s'-s} \|g\|_{H^s(\Gamma)} \quad \forall g \in H^s(\Gamma).$$

There exists a tubular neighborhood T of Γ that depends solely on the analyticity properties of Γ and there exists a constant $\eta' \geq \eta$ depending only on η and Γ such that $L_{k,\Gamma}^\eta g \in \mathfrak{A}(M, \eta', T, \Gamma)$ with

$$(3.2) \quad M \leq C_s \|g\|_{-1/2,\Gamma}.$$

Proof. The operators $H_{k,\Gamma}^\eta$ and $L_{k,\Gamma}^\eta$ are defined in [MS11, Lemmas 4.2, 4.3] and the properties of $H_{k,\Gamma}^\eta$ are shown there. For the properties of $L_{k,\Gamma}^\eta$, we note the following: Let T be a tubular neighborhood on which the analytic continuation \mathbf{n}^* of the normal vector \mathbf{n} exists and let, as in construction in [MS11], $G \in H^{s+3/2}(\Omega)$ be such that $\partial_n G = g$ on Γ and $\|G\|_{s'+3/2,\Omega} \lesssim \|g\|_{s',\Gamma}$

for $-1/2 \leq s' \leq s$. Then $L_{k,\Gamma}^\eta g$ is defined as the restriction to Γ of $\mathbf{n}^* \cdot \nabla L_{k,\Omega}^\eta G$. By construction, $L_{k,\Omega}^\eta G$ is analytic on \mathbb{R}^d with

$$\|\nabla^n L_{k,\Omega}^\eta G\|_{0,\mathbb{R}^d} \leq C(\eta k)^{n-1} \|G\|_{1,\Omega} \quad \forall n \in \mathbb{N}_0.$$

We conclude from Lemma 2.2 that $\mathbf{n}^* \cdot \nabla L_{k,\Omega}^\eta G \in \mathfrak{A}(C\|G\|_{1,\Omega}, \eta', T, \Gamma)$ for some C , η' depending only on η and Γ . Without loss generality, $\eta' \geq \eta$. We conclude with the estimate $\|G\|_{1,\Omega} \lesssim \|g\|_{-1/2,\Gamma}$. \square

Proposition 3.2 (Volume filters). *For each $\eta > 1$, there exist two operators $L_{k,\Omega}^\eta, H_{k,\Omega}^\eta: L^2(\Omega) \rightarrow L^2(\Omega)$ such that $H_{k,\Omega}^\eta + L_{k,\Omega}^\eta = \text{id}$. For $0 \leq s' \leq s$ and $0 \leq \varepsilon < 1/2$, the operator $H_{k,\Omega}^\eta$ satisfies*

$$(3.3) \quad \|H_{k,\Omega}^\eta f\|_{s',\mathcal{P}} \leq C_{s,s'}(\eta k)^{s'-s} \|f\|_{s,\mathcal{P}} \quad \forall f \in H^s(\mathcal{P}),$$

$$(3.4) \quad \|H_{k,\Omega}^\eta f\|_{\tilde{H}^{-\varepsilon}(\Omega)} \leq C_\varepsilon(\eta k)^{-\varepsilon} \|f\|_{0,\Omega} \quad \forall f \in L^2(\Omega)$$

for constants $C_{s,s'}$, C_ε independent of f , k , and η . Finally, we have $L_{k,\Omega}^\eta f \in \mathfrak{A}(M, \eta, \mathcal{P})$ with

$$(3.5) \quad M \leq C_\Omega \|f\|_{0,\Omega}.$$

Proof. For all $P \in \mathcal{P}$, we set

$$\left(H_{k,\Omega}^\eta f\right)\Big|_P = (H_{\eta k}(E_P f))\Big|_P \quad \left(L_{k,\Omega}^\eta f\right)\Big|_P = (L_{\eta k}(E_P f))\Big|_P,$$

where $E_P: H^s(P) \rightarrow H^s(\mathbb{R}^d)$ is the Stein extension operator, [Ste70, Chap. VI], and $H_{\eta k}$ and $L_{\eta k}$ are the ‘‘full space’’ high and low pass filters introduced in [MS11]. Then, the proof of (3.3) and (3.5) can be found in [MS11, Lemmas 4.2, 4.3].

Regarding (3.4), note that for $0 \leq \varepsilon < 1/2$ and a bounded Lipschitz domain Ω , the space of compactly supported smooth functions $\bigcup_{P \in \mathcal{P}} C_0^\infty(P)$ is dense in $H^\varepsilon(\Omega)$. It is easy to check that, for all $0 \leq s \leq 1$,

$$\|H_{\eta k} f\|_{-s,\mathbb{R}^d} \leq C_s(\eta k)^{-s} \|f\|_{0,\mathbb{R}^d} \quad \forall f \in L^2(\mathbb{R}^d),$$

[MS11, Section 4.1.1]. For $v \in C_0^\infty(P)$ let \tilde{v} denote the trivial extension by zero on all of \mathbb{R}^d of v . These observations give

$$\begin{aligned} |(H_{\eta k}^\Omega f, v)_\Omega| &= \left| \sum_{P \in \mathcal{P}} (H_{\eta k} E_P(f), v)_P \right| = \left| \sum_{P \in \mathcal{P}} (H_{\eta k} E_P(f), \tilde{v}_P)_{\mathbb{R}^d} \right| \\ &\leq \sum_{P \in \mathcal{P}} \|H_{\eta k} E_P(f)\|_{-\varepsilon,\mathbb{R}^d} \|\tilde{v}_P\|_{\varepsilon,\mathbb{R}^d}. \end{aligned}$$

Additionally, we have

$$\|H_{\eta k} E_P(f)\|_{-\varepsilon,\mathbb{R}^d} \leq C_\varepsilon(\eta k)^{-\varepsilon} \|E_P(f)\|_{0,\mathbb{R}^d} \leq C_\varepsilon(\eta k)^{-\varepsilon} \|f\|_{0,P}, \quad \|\tilde{v}\|_{\varepsilon,\mathbb{R}^d} \lesssim C_\varepsilon \|v\|_{\varepsilon,P}$$

which leads to

$$\left|(H_{k,\Omega}^\eta f, v)_\Omega\right| \leq C_\varepsilon(\eta k)^{-\varepsilon} \sum_{P \in \mathcal{P}} \|f\|_{0,P} \|v\|_{\varepsilon,P} \leq C_\varepsilon(\eta k)^{-\varepsilon} \|f\|_{0,\Omega} \|v\|_{\varepsilon,\mathcal{P}},$$

and the result follows since $\|v\|_{\varepsilon,\mathcal{P}} \leq \|v\|_{\varepsilon,\Omega}$. \square

Remark 3.3 (Other constructions of high and low pass filters). The presented high and low pass filters are by no means the only possible constructions. It is, e.g., possible to construct the low-frequency filter on bounded domains by truncating the expansion in eigenfunctions of the Laplacian endowed with Neumann boundary conditions. See also [Mel12, Sec. 6.1] for similar considerations. ■

3.2. A contraction argument. We employ the low and high pass filters to formulate a preliminary decomposition result. For given right-hand sides, the decomposition consists of the desired final splitting plus a reminder that is solution to the time-harmonic problem with new right-hand sides that are strictly smaller in norm than the original ones. We thus call the preliminary result a “contraction” argument.

Due to the duality between $H^\varepsilon(\Omega)$ and $\tilde{H}^{-\varepsilon}(\Omega)$, if $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$, we have with a constant C_ε only depending on ε ,

$$|(f, v) + \langle g, v \rangle| \leq C_\varepsilon \left(k^{\varepsilon-1} \|f\|_{\tilde{H}^{-\varepsilon}(\Omega)} + k^{-1/2} \|g\|_{0,\Gamma} \right) \|v\|_{1,k,\Omega} \quad \forall v \in H^1(\Omega).$$

Recalling that b_k^+ is coercive by **(AP1)**, this leads to

$$(3.6) \quad \|S_k^+(f, g)\|_{1,t,k} \leq C_\varepsilon \left(k^{\varepsilon-1} \|f\|_{\tilde{H}^{-\varepsilon}(\Omega)} + k^{-1/2} \|g\|_{0,\Gamma} \right).$$

Lemma 3.4 (Unified contraction argument). *Let **(WP)** and **(AP)** hold. Let $q \in (0, 1)$ and $s \in [0, s_{\max}]$ be given. Then, there exists a parameter $\eta > 1$ depending on q and s such that for all $f \in H^s(\mathcal{P})$ and $g \in H^{s+1/2}(\Gamma)$ we can write*

$$S_k^-(f, g) = u_F + u_A + S_k^-(\tilde{f}, \tilde{g}),$$

where

$$u_F := S_k^+(H_{k,\Omega}^\eta f, H_{k,\Gamma}^\eta g) \quad u_A := S_k^-(L_{k,\Omega}^\eta f, L_{k,\Gamma}^\eta g) + S_k^-(A_{k,\Omega}^\Delta u_F, A_{k,\Gamma}^\Delta u_F),$$

and the pair $(\tilde{f}, \tilde{g}) \in H^s(\mathcal{P}) \times H^{s+1/2}(\Gamma)$ satisfies

$$(3.7) \quad \|\tilde{f}\|_{s,\mathcal{P}} + \|\tilde{g}\|_{s+1/2,\Gamma} \leq q(\|f\|_{s,\mathcal{P}} + \|g\|_{s+1/2,\Gamma}).$$

In addition, we have $u_F \in H^{s+2}(\mathcal{P})$ with

$$(3.8) \quad \|u_F\|_{s+2,\mathcal{P}} \leq C_s (\|f\|_{s,\mathcal{P}} + \|g\|_{s+1/2,\Gamma}),$$

and $u_A \in \mathfrak{A}(M, \eta', \mathcal{P})$ for some $\eta' > 1$ depending only on η and Γ with

$$(3.9) \quad M \leq CC_{\text{ana},k}^- C_{\text{sol},k}^- (1 + k^{-2} C_{A,k}^\Delta) (\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma}).$$

Proof. Let $\eta > 1$ to be fixed later on. For $f \in H^s(\Omega)$ and $g \in H^{s+1/2}(\Gamma)$, we set $u := S_k^-(f, g)$. We introduce

$$u_F := S_k^+(H_{k,\Omega}^\eta f, H_{k,\Gamma}^\eta g), \quad u_{A,I} := S_k^-(L_{k,\Omega}^\eta f, L_{k,\Gamma}^\eta g)$$

so that $u = u_F + u_{A,I} + r_I$ with

$$r_I := S_k^-(T_{k,\Omega}^\Delta u_F, T_{k,\Gamma}^\Delta u_F) = S_k^-(R_{k,\Omega}^\Delta u_F, R_{k,\Gamma}^\Delta u_F) + S_k^-(A_{k,\Omega}^\Delta u_F, A_{k,\Gamma}^\Delta u_F),$$

where we employed **(AP3)**. We then introduce

$$u_{A,II} := S_k^-(A_{k,\Omega}^\Delta u_F, A_{k,\Gamma}^\Delta u_F), \quad \tilde{f} := R_{k,\Omega}^\Delta u_F, \quad \tilde{g} := R_{k,\Gamma}^\Delta u_F,$$

leading to the splitting $u = u_F + u_A + S_k^-(\tilde{f}, \tilde{g})$ with

$$u_A := u_{A,I} + u_{A,II} = S_k^-(L_{k,\Omega}^\eta f, L_{k,\Gamma}^\eta g) + S_k^-(A_{k,\Omega}^\Delta u_F, A_{k,\Gamma}^\Delta u_F).$$

Before establishing the contraction estimate (3.7), we prove the regularity estimate (3.8). Using (2.15) from (AP2) as well as the mapping properties (3.1) and (3.3) of the high pass filters, we have

$$\begin{aligned} \|u_F\|_{s+2, \mathcal{D}} &\leq C_s \left(\|H_{k, \Omega}^\eta f\|_{s, \mathcal{D}} + k^s \|H_{k, \Omega}^\eta f\|_{0, \Omega} + \|H_{k, \Gamma}^\eta g\|_{s+1/2, \Gamma} + k^{s+1/2} \|H_{k, \Gamma}^\eta g\|_{0, \Gamma} \right) \\ &\leq C_s \left(\|f\|_{s, \mathcal{D}} + k^s (k\eta)^{-s} \|f\|_{s, \mathcal{D}} + \|g\|_{s+1/2, \Gamma} + k^{s+1/2} (k\eta)^{-s-1/2} \|g\|_{s+1/2, \Gamma} \right) \\ &\leq C_s \left(\|f\|_{s, \mathcal{D}} + \|g\|_{s+1/2, \Gamma} \right), \end{aligned}$$

which proves (3.8).

We next establish (3.7). Recalling the definition of \tilde{f} and \tilde{g} , estimates (2.17) and (2.18) combined with trace inequality (2.4) show that

$$(3.10) \quad \|\tilde{f}\|_{s, \mathcal{D}} + \|\tilde{g}\|_{s+1/2, \Gamma} \leq C \left(k^2 \|u_F\|_{s, \mathcal{D}} + k \|u_F\|_{s+1, \mathcal{D}} \right).$$

The case $s = 0$ is treated separately. Fixing $\varepsilon = 1/4$, we have from (3.6) and the properties (3.1) and (3.4) of the high-pass filters

$$\begin{aligned} k \|u_F\|_{1, k, t} &\stackrel{(3.6)}{\leq} C_\varepsilon k \left(k^{\varepsilon-1} \|H_{k, \Omega}^\eta f\|_{\tilde{H}^{-\varepsilon}(\Omega)} + k^{-1/2} \|H_{k, \Gamma}^\eta g\|_{0, \Gamma} \right) \\ &\stackrel{(3.1), (3.4)}{\leq} C_\varepsilon k \left(k^{\varepsilon-1} (k\eta)^{-\varepsilon} \|f\|_{0, \Omega} + k^{-1/2} (k\eta)^{-1/2} \|g\|_{1/2, \Gamma} \right) \\ (3.11) \quad &\leq C \eta^{-\varepsilon} \left(\|f\|_{0, \Omega} + \|g\|_{1/2, \Gamma} \right), \end{aligned}$$

where, since $\varepsilon = 1/4$ is fixed, we absorbed the constant C_ε in a constant C independent of k and ε ; we also used that $\eta^{-1/2} \leq \eta^{-\varepsilon}$ since $\eta > 1$. We conclude from (3.10) and (3.11)

$$(3.12) \quad \|\tilde{f}\|_{0, \Omega} + \|\tilde{g}\|_{1/2, \Gamma} \leq C \eta^{-\varepsilon} \left(\|f\|_{0, \Omega} + \|g\|_{1/2, \Gamma} \right).$$

We turn to the case $0 < s \leq s_{\max}$. With $\varepsilon = 0$ in (3.6) we have together with the mapping properties (3.1) and (3.3) of the high-pass filters that

$$\begin{aligned} k^2 \|u_F\|_{0, \Omega} &\leq C k \left(k^{-1} \|H_{k, \Omega}^\eta f\|_{0, \Omega} + k^{-1/2} \|H_{k, \Gamma}^\eta g\|_{0, \Gamma} \right) \\ &\leq C_s (k\eta)^{-s} \|f\|_{s, \mathcal{D}} + k^{1/2} (k\eta)^{-s-1/2} \|g\|_{s+1/2, \Gamma}. \end{aligned}$$

Hence, we find

$$(3.13) \quad k^{s+2} \|u_F\|_{0, \Omega} \leq C_s \eta^{-s} \left(\|f\|_{s, \mathcal{D}} + \|g\|_{s+1/2, \Gamma} \right).$$

We note the multiplicative interpolation estimate for $0 \leq r \leq s+2$:

$$(3.14) \quad \|u_F\|_{r, \mathcal{D}} \leq C_r \|u_F\|_{0, \Omega}^{\frac{s+2-r}{s+2}} \|u_F\|_{s+2, \mathcal{D}}^{\frac{r}{s+2}}.$$

Inserting (3.13) and (3.8) into (3.14) we find

$$(3.15) \quad \|u_F\|_{r, \mathcal{D}} \leq C_r k^r \eta^{-s-2} \eta^{-s \frac{s+2-r}{s+2}} \left(\|f\|_{s, \mathcal{D}} + \|g\|_{s+1/2, \Gamma} \right).$$

Using (3.15) with $r = s$ and $r = s+1$, respectively, we find

$$\begin{aligned} (3.16) \quad \|\tilde{f}\|_{s, \Omega} + \|\tilde{g}\|_{s+1/2, \Gamma} &\leq C_s \left(k^2 \|u_F\|_{s, \mathcal{D}} + k \|u_F\|_{s+1, \mathcal{D}} \right) \\ &\leq C_s \left(\eta^{-s \frac{2}{s+2}} + \eta^{-s \frac{1}{s+2}} \right) \left(\|f\|_{s, \mathcal{D}} + \|g\|_{s+1/2, \Gamma} \right). \end{aligned}$$

At this point, since the constants appearing in (3.12) and (3.16) are independent of η , f , g , we can select η large enough (η may depend on s) to obtain (3.7) with $q < 1$. For the sake of simplicity, we additionally select η larger than the constants γ_0 of (WP4) and γ_A^Δ of (AP3). We next establish (3.9). For the first component $u_{A,I}$, properties (3.2) and (3.5) of the low pass filters show that

$$L_{k,\Omega}^\eta f \in \mathfrak{A}(M_f, \eta, \mathcal{P}), \quad L_{k,\Gamma}^\eta g \in \mathfrak{A}(M_g, \eta', T, \Gamma)$$

with

$$M_f \leq C\|f\|_{0,\Omega}, \quad M_g \leq C\|g\|_{-1/2,\Gamma}$$

and a tubular neighborhood T of Γ and a constant $\eta' \geq \eta$ that depend solely on η and Γ . Recalling (WP3) and that $\eta' \geq \eta \geq \gamma_0$, we have $u_{A,I} \in \mathfrak{A}(M_I, \vartheta(\eta'), \mathcal{P})$ with

$$M_I \leq CC_{\text{ana},k}^- C_{\text{sol},k}^- k^{-1} (M_f + kM_g) \leq CC_{\text{ana},k}^- C_{\text{sol},k}^- k^{-1} (\|f\|_{0,\Omega} + k\|g\|_{-1/2,\Gamma}).$$

We turn to the second component $u_{A,II}$. From (3.11) we get $\|u_F\|_{1,k,t} \lesssim k^{-1} (\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma})$. Using (AP3), we arrive at

$$f_{II} := A_{k,\Omega}^\Delta u_F \in \mathfrak{A}(M_{f,II}, \gamma_A^\Delta, \mathcal{P}) \quad g_{II} := A_{k,\Gamma}^\Delta u_F \in \mathfrak{A}(M_{g,II}, \gamma_A^\Delta, T, \mathcal{P})$$

with

$$M_{f,II} + kM_{g,II} \leq C_{A,k}^\Delta k^{-1} (\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma}).$$

Since we chose η larger than γ_0 and γ_A^Δ , we also have

$$f_{II} := A_{k,\Omega}^\Delta u_F \in \mathfrak{A}(M_{f,II}, \eta', \mathcal{P}), \quad g_{II} := A_{k,\Gamma}^\Delta u_F \in \mathfrak{A}(M_{g,II}, \eta', T, \Gamma),$$

and it follows from (WP3) that $u_{A,II} \in \mathfrak{A}(M_{II}, \vartheta(\eta'), \mathcal{P})$ with

$$M_{II} \leq CC_{\text{ana},k}^- C_{\text{sol},k}^- k^{-1} (M_{f,II} + kM_{g,II}) \leq CC_{\text{ana},k}^- C_{\text{sol},k}^- k^{-1} C_{A,k}^\Delta k^{-1} (\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma}).$$

Estimate (3.9) now follows since

$$u_A := u_{A,I} + u_{A,II} \in \mathfrak{A}(M_I + M_{II}, \vartheta(\eta'), \mathcal{P}).$$

Relabelling the parameter $\vartheta(\eta')$ as η' and estimating generously $\|g\|_{-1/2,\Gamma} \lesssim \|g\|_{1/2,\Gamma}$ gives the result. \square

3.3. Regularity splitting of time-harmonic solutions. The contraction result of Lemma 3.4 can be iterated to yield a splitting of the solution of Helmholtz problems into a part u_F with finite regularity and an analytic part u_A :

Theorem 3.5 (Abstract regularity splitting). *Assume (WP) and (AP). For all $0 \leq s \leq s_{\max}$, and for each $f \in H^s(\Omega)$ and $g \in H^{s+1/2}(\Gamma)$ there is a splitting*

$$S_k^-(f, g) = u_F + u_A$$

with $u_F \in H^{s+2}(\mathcal{P})$ satisfying

$$(3.17) \quad \|u_F\|_{s+2,\mathcal{P}} \leq C_s (\|f\|_{s,\mathcal{P}} + \|g\|_{s+1/2,\Gamma}),$$

and $u_A \in \mathfrak{A}(M, \gamma_s^*, \mathcal{P})$ with

$$(3.18) \quad M \leq C_{\text{ana},k}^- CC_{\text{sol},k}^- (1 + k^{-2} C_{A,k}^\Delta) (\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma}),$$

and a constant γ_s^* depending solely on s and the parameters appearing in (WP), (AP).

Proof. The proof iterates the contraction argument in Lemma 3.4. To fix ideas, we select $q := 1/2$ and denote by η_s, η'_s the parameters η, η' given by Lemma 3.4. Letting $f^{(0)} := f$ and $g^{(0)} := g$, the splitting

$$S_k^-(f^{(0)}, g^{(0)}) = u_F^{(0)} + u_A^{(0)} + S_k^-(f^{(1)}, g^{(1)}),$$

holds true with contracted data

$$\|f^{(1)}\|_{s, \mathcal{P}} + \|g^{(1)}\|_{s+1/2, \Gamma} \leq q(\|f^{(0)}\|_{s, \mathcal{P}} + \|g^{(0)}\|_{s+1/2, \Gamma});$$

the regular part $u_F^{(0)} \in H^{s+2}(\mathcal{P})$ satisfies

$$\|u_F^{(0)}\|_{s+2, \mathcal{P}} \leq C_s \left(\|f^{(0)}\|_{s, \mathcal{P}} + \|g^{(0)}\|_{s+1/2, \Gamma} \right)$$

and the analytic part $u_A^{(0)} \in \mathfrak{A}(M^{(0)}, \eta'_s, \mathcal{P})$ with

$$M^{(0)} \leq CC_{\text{sol}, k}^-(1 + k^{-2}C_{A, k}^\Delta) \left(\|f^{(0)}\|_{0, \Omega} + \|g^{(0)}\|_{1/2, \Gamma} \right).$$

We can then repeat this argument to split $S_k^-(f^{(1)}, g^{(1)})$, and so on, resulting in an inductive definition of $(f^{(i)}, g^{(i)}) \in H^s(\mathcal{P}) \times H^{s+1/2}(\Gamma)$, $u_F^{(i)} \in H^{s+2}(\mathcal{P})$ and $u_A^{(i)} \in \mathfrak{A}(M^{(i)}, \eta'_s, \mathcal{P})$ with

$$\begin{aligned} \|f^{(i)}\|_{s, \mathcal{P}} + \|g^{(i)}\|_{s+1/2, \Gamma} &\leq q^i (\|f^{(0)}\|_{s, \mathcal{P}} + \|g^{(0)}\|_{s+1/2, \Gamma}), \\ \|u_F^{(i)}\|_{s+2, \mathcal{P}} &\leq C_s q^i \left(\|f^{(0)}\|_{s, \mathcal{P}} + \|g^{(0)}\|_{s+1/2, \Gamma} \right) \\ M^{(i)} &\leq q^i CC_{\text{sol}, k}^-(1 + k^{-2}C_{A, k}^\Delta) \left(\|f^{(0)}\|_{0, \Omega} + \|g^{(0)}\|_{1/2, \Gamma} \right). \end{aligned}$$

It follows that we have

$$u = \underbrace{\sum_{i \in \mathbb{N}_0} u_F^{(i)}}_{=: u_F} + \underbrace{\sum_{i \in \mathbb{N}_0} u_A^{(i)}}_{=: u_A},$$

where the two series converge due to the exponential decrease of $\|f^{(i)}\|_{s, \mathcal{P}}$ and $\|g^{(i)}\|_{s+1/2, \Gamma}$. Then, introducing $\gamma_s^* := \eta'_s$, estimates (3.17) and (3.18) simply follows since $q = 1/2$ and $\sum_{q \in \mathbb{N}_0} q^i = 2$. \square

4. STABILITY AND CONVERGENCE OF ABSTRACT GALERKIN DISCRETIZATIONS

In this section, we utilize the splitting previously introduced to analyze Galerkin discretizations. We first establish quasi-optimality and error estimates for abstract discrete spaces. These results are then applied to the particular case of the hp finite element method, where explicit stability conditions are derived.

The proofs of this section hinge on duality techniques, and require properties of the adjoint wave propagation problem. For the sake of simplicity, we assume here that the considered wave propagation and auxiliary problems are symmetric in the sense that

$$(4.1) \quad b_k^-(v, u) = b_k^-(\bar{u}, \bar{v}), \quad b_k^+(v, u) = b_k^+(\bar{u}, \bar{v})$$

for all $u, v \in H^{1,t}(\Omega, \Gamma)$. We mention that (4.1) is not mandatory; however, without (4.1) one needs to assume **(WP)** and **(AP)** for the adjoint problems, which we avoid here to simplify the presentation.

4.1. Abstract Galerkin discretizations. We start by considering a generic finite-dimensional subspace $V_h \subset H^{1,t}(\Omega, \Gamma)$. The corresponding Galerkin approximation $u_h \in V_h$ to the exact solution $u \in H^{1,t}(\Omega, \Gamma)$ of (2.5) is given by

$$(4.2) \quad b_k^-(u_h, v_h) = (f, v_h) + \langle g, v_h \rangle \quad \forall v_h \in V_h.$$

The key approximation properties of V_h are captured by the following two quantities for $m \geq 1$ and $\gamma > 0$ and the (fixed) tubular neighborhood T of Γ given as the intersection of the two tubular neighborhoods given by (WP4) and (AP3):

$$(4.3a) \quad \eta^{(m)} := \sup_{\substack{(f,g) \in H^{m-1}(\mathcal{D}) \times H^{m-1/2}(\Gamma) \\ \|f\|_{m-1, \mathcal{D}} + \|g\|_{m-1/2, \Gamma} = 1}} \inf_{v_h \in V_h} \|S_k^-(f, g) - v_h\|_{1,t,k},$$

$$(4.3b) \quad \eta_\gamma^{(\text{exp})} := \sup_{\substack{(f,g) \in \mathfrak{A}(M_f, \gamma, \mathcal{D}) \times \mathfrak{A}(M_g, \gamma, T, \Gamma) \\ M_f + kM_g = 1}} \inf_{v_h \in V_h} \|S_k^-(f, g) - v_h\|_{1,t,k},$$

To ease the presentation, we introduce the projection

$$\Pi_h v := \arg \min_{v_h \in V_h} \|v - v_h\|_{1,t,k}.$$

Lemma 4.1. *Let $u \in H^{1,t}(\Omega, \Gamma)$ and assume that $u_h \in V_h$ satisfies*

$$(4.4) \quad b_k^-(u - u_h, v_h) = 0 \quad \forall v_h \in V_h.$$

Then, the estimate

$$(4.5) \quad |b_k^-(u - u_h, v)| \leq C \left(1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A)}^{(\text{exp})} \right) \|u - u_h\|_{1,t,k} \|v\|_{1,t,k}$$

holds true for all $v \in H^{1,t}(\Omega, \Gamma)$, where C is a constant independent of k , u , and v .

Proof. For the sake of brevity we write $e_h := u - u_h$. For an arbitrary $v \in H^{1,t}(\Omega, \Gamma)$, we have

$$\begin{aligned} |b_k^-(e_h, v)| &= |a(e_h, v) - (T_{k,\Omega}^- e_h, v) - \langle T_{k,\Gamma}^- e_h, v \rangle| \\ &\stackrel{(4.1)}{=} |a(e_h, v) - (T_{k,\Omega}^- \bar{v}, \bar{e}_h) - \langle T_{k,\Gamma}^- \bar{v}, \bar{e}_h \rangle| \\ &\stackrel{(2.9)}{\leq} |a(e_h, v)| + |(R_{k,\Omega}^- \bar{v}, \bar{e}_h)| + |\langle R_{k,\Gamma}^- \bar{v}, \bar{e}_h \rangle| + |(A_{k,\Omega}^- \bar{v}, \bar{e}_h) + \langle A_{k,\Gamma}^- \bar{v}, \bar{e}_h \rangle| \\ &\stackrel{(2.10), (2.11)}{\leq} C \|e_h\|_{1,t,k} \|v\|_{1,t,k} + |(A_{k,\Omega}^- \bar{v}, \bar{e}_h) + \langle A_{k,\Gamma}^- \bar{v}, \bar{e}_h \rangle|. \end{aligned}$$

The analytic part is now treated using a duality argument. Indeed, using (4.1), we have

$$(A_{k,\Omega}^- \bar{v}, \bar{e}_h) + \langle A_{k,\Gamma}^- \bar{v}, \bar{e}_h \rangle = b_k^-(S_k^-(A_{k,\Omega}^- \bar{v}, A_{k,\Gamma}^- \bar{v}), \bar{e}_h) = b_k^-(e_h, \phi) = b_k^-(e_h, \phi - \Pi_h \phi),$$

where $\bar{\phi} := S_k^-(A_{k,\Omega}^- \bar{v}, A_{k,\Gamma}^- \bar{v})$. Then, using (WP4), we have

$$\|\phi - \Pi_h \phi\|_{1,t,k} = \|\bar{\phi} - \Pi_h \bar{\phi}\|_{1,t,k} = \inf_{w_h \in V_h} \|S_k^-(A_{k,\Omega}^- \bar{v}, A_{k,\Gamma}^- \bar{v}) - w_h\|_{1,t,k} \leq C_{A,k}^- \eta_{\vartheta(\gamma_A)}^{(\text{exp})} \|v\|_{1,t,k},$$

and the conclusion follows from (WP1) since

$$|(A_{k,\Omega}^- \bar{v}, \bar{e}_h) + \langle A_{k,\Gamma}^- \bar{v}, \bar{e}_h \rangle| = |b_k^-(e_h, \phi - \Pi_h \phi)| \leq C_{\text{cont},k}^- \|e_h\|_{1,t,k} \|\phi - \Pi_h \phi\|_{1,t,k}.$$

□

Lemma 4.2. *Assume that $u \in H^{1,t}(\Omega, \Gamma)$ and $u_h \in V_h$ are such that*

$$(4.6) \quad b_k^-(u - u_h, v_h) = 0 \quad \forall v_h \in V_h.$$

Then, there exist two constants ρ and μ independent of k , u , and u_h such that

$$(4.7) \quad \left(\mu - k\eta^{(1)} - C_{A,k}^\Delta \eta_{\vartheta(\gamma_A^\Delta)}^{(\text{exp})} \right) \|u - u_h\|_{1,t,k} \leq \rho \left(1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})} \right) \|u - \Pi_h u\|_{1,t,k}.$$

Proof. We introduce the Galerkin error $e_h := u - u_h$. Recalling (2.13) and (4.6), there exists a constant C_0 independent of k , u , and u_h such that

$$\begin{aligned} \frac{1}{C_0} \|e_h\|_{1,t,k}^2 &\leq \text{Re } b_k^+(e_h, \sigma e_h) = |b_k^+(e_h, e_h)| \leq \\ &|b_k^\Delta(e_h, e_h)| + |b_k^-(e_h, e_h)| = |b_k^\Delta(e_h, e_h)| + |b_k^-(e_h, u - \Pi_h u)|. \end{aligned}$$

Recalling Lemma 4.1, we may then write

$$\|e_h\|_{1,t,k}^2 \leq C_0 |b_k^\Delta(e_h, e_h)| + C_1 \left(1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})} \right) \|e_h\|_{1,t,k} \|u - \Pi_h u\|_{1,t,k},$$

where C_1 does not depend on k , u or u_h . On the other hand, we have

$$\begin{aligned} b_k^\Delta(e_h, e_h) &\stackrel{(4.1)}{=} b_k^\Delta(\bar{e}_h, \bar{e}_h) \\ &\stackrel{(2.16)}{=} (R_{k,\Omega}^\Delta \bar{e}_h, \bar{e}_h) + \langle R_{k,\Gamma}^\Delta \bar{e}_h, \bar{e}_h \rangle + (A_{k,\Omega}^\Delta \bar{e}_h, \bar{e}_h) + \langle A_{k,\Gamma}^\Delta \bar{e}_h, \bar{e}_h \rangle \\ &= b_k^-(S_k^-(R_{k,\Omega}^\Delta \bar{e}_h, R_{k,\Gamma}^\Delta \bar{e}_h), \bar{e}_h) + b_k^-(S_k^-(A_{k,\Omega}^\Delta \bar{e}_h, A_{k,\Gamma}^\Delta \bar{e}_h), \bar{e}_h) \\ &= b_k^-(e_h, \phi_F) + b_k^-(e_h, \phi_A), \end{aligned}$$

where $\bar{\phi}_F := S_k^-(R_{k,\Omega}^\Delta \bar{e}_h, R_{k,\Gamma}^\Delta \bar{e}_h)$ and $\bar{\phi}_A := S_k^-(A_{k,\Omega}^\Delta \bar{e}_h, A_{k,\Gamma}^\Delta \bar{e}_h)$.

$$\begin{aligned} |b_k^-(e_h, \phi_F)| &= |b_k^-(e_h, \phi_F - \Pi_h \phi_F)| \\ &\leq \|e_h\|_{1,t,k} \|\phi_F - \Pi_h \phi_F\|_{1,t,k} \\ &\leq \eta^{(1)} \|e_h\|_{1,t,k} (\|R_{k,\Omega}^\Delta \bar{e}_h\|_{0,\Omega} + \|R_{k,\Gamma}^\Delta \bar{e}_h\|_{1/2,\Omega}) \\ &\leq C_2 k \eta^{(1)} \|e_h\|_{1,t,k}^2, \end{aligned}$$

where C_2 is independent of k , u and u_h . Similarly, we have

$$|b_k^-(e_h, \phi_A)| \leq \|e_h\|_{1,t,k} \|\phi_A - \Pi_h \phi_A\|_{1,t,k} \leq C_{A,k}^\Delta \eta_{\vartheta(\gamma_A^\Delta)}^{(\text{exp})} \|e_h\|_{1,t,k}^2,$$

which leads to

$$|b_k^\Delta(e_h, e_h)| \leq \left(C_2 k \eta^{(1)} + C_{A,k}^\Delta \eta_{\vartheta(\gamma_A^\Delta)}^{(\text{exp})} \right) \|e_h\|_{1,t,k}^2,$$

and

$$(4.8) \quad \left(1 - C_0 C_2 k \eta^{(1)} - C_0 C_{A,k}^\Delta \eta_{\vartheta(\gamma_A^\Delta)}^{(\text{exp})} \right) \|e_h\|_{1,t,k}^2 \leq C_1 \left(1 + C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})} \right) \|e_h\|_{1,t,k} \|u - \Pi_h u\|_{1,t,k}.$$

Estimate (4.7) then follows by setting $1/\mu := C_0 \max(1, C_2)$ and $\rho := \mu C_1$ and multiplying both sides by of (4.8) by μ . \square

Theorem 4.3 (abstract quasi-optimality). *Assume **(WP)** and **(AP)**. Let $s \in [0, s_{\max}]$, $f \in H^s(\mathcal{P})$ and $g \in H^{s+1/2}(\Gamma)$. Recall the constant μ from Lemma 4.2, and assume that*

$$(4.9) \quad k\eta^{(1)} + C_{A,k}^{\Delta} \eta_{\vartheta(\gamma_{\Delta}^{\Delta})}^{(\text{exp})} \leq \frac{\mu}{2}, \quad C_{\text{cont},k}^{-} C_{A,k}^{-} \eta_{\vartheta(\gamma_{\Delta}^{-})}^{(\text{exp})} \leq 1.$$

Then, there exists $C > 0$ independent of k, f, g , and there exists a unique $u_h \in V_h$ such that (4.2) holds, and for $u := S_k^{-}(f, g)$, we have

$$(4.10) \quad \|u - u_h\|_{1,t,k} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{1,t,k}.$$

Proof. Fix $s \in [0, s_{\max}]$, $f \in H^s(\mathcal{P})$, and $g \in H^{s+1/2}(\Gamma)$. Let u_h be any element of V_h such that (4.2) holds true. We easily see that $u := S_k^{-}(f, g)$ and u_h satisfy the assumptions of Lemma 4.2. Injecting (4.9) in (4.7), we have

$$\frac{\mu}{2} \|u - u_h\|_{1,t,k} \leq 2\rho \|u - \Pi_h u\|_{1,t,k} = 2\rho \inf_{v_h \in V_h} \|u - v_h\|_{1,t,k},$$

so that (4.10) holds for all u_h satisfying (4.2) with $C := 4\rho/\mu$.

Next, we observe that having established (4.10) for all u_h satisfying (4.2) implies, by linearity, the uniqueness of the solution to (4.10). Since (4.10) corresponds to finite-dimensional linear system, this proves the existence and uniqueness of u_h . \square

4.2. Application to hp -FEM. We now focus on the case where the discretization subspace V_h is a piecewise (mapped) polynomial finite element space. We consider the case where the mesh \mathcal{T}_h associated with the finite element space is aligned with the partition \mathcal{P} . Since we are working with analytic interfaces, dedicated assumptions are required [MS10].

Assumption 4.4 (Quasi-uniform regular fitted meshes). Let \widehat{K} be the reference simplex in spatial dimension $d = 2, 3$. The mesh \mathcal{T}_h is a partition of Ω into non-overlapping elements K such that $\cup_{K \in \mathcal{T}_h} \overline{K} = \overline{\Omega}$. For each $K \in \mathcal{T}_h$, there exist a bijective mapping $F_K: \widehat{K} \rightarrow K$ that can be written as $F_K = R_K \circ A_K$, where A_K and R_K are respectively an affine and analytic mappings. Specifically, there exists constants $C_{\text{affine}}, C_{\text{metric}}, \gamma > 0$ independent of K such that

$$\begin{aligned} \|A'_K\|_{L^\infty(\widehat{K})} &\leq C_{\text{affine}} h_K, & \|(A'_K)^{-1}\|_{L^\infty(\widehat{K})} &\leq C_{\text{affine}} h_K^{-1}, \\ \|(R'_K)^{-1}\|_{L^\infty(\widehat{K})} &\leq C_{\text{metric}}, & \|\nabla^n R_K\|_{L^\infty(\widehat{K})} &\leq C_{\text{metric}} \gamma^n n! \quad \forall n \in \mathbb{N}_0. \end{aligned}$$

Here, $\tilde{K} = A_K(\widehat{K})$ and $h_K > 0$ denotes the element diameter. Furthermore, we assume that mesh resolves the interfaces of \mathcal{P} , i.e., each element K entirely lies in on subdomain $P \in \mathcal{P}$.

We introduce, for $p \geq 1$, the space $S_p(\mathcal{T}_h)$ as the space of piecewise mapped polynomials of degree p :

$$S_p(\mathcal{T}_h) := \left\{ u \in H^1(\Omega) : u|_K \circ F_K \in \mathcal{P}_p(\widehat{K}) \text{ for all } K \in \mathcal{T}_h \right\}.$$

We note $S^p(\mathcal{T}_h) \subset H^{1,t}(\Omega, \Gamma)$. We will require approximation operators that approximate functions in the $\|\cdot\|_{1,t,k}$ -norm:

Proposition 4.5 (Interpolation errors). *Let Assumption 4.4 be valid. There is an operator $I_h : H^2(\mathcal{P}) \cap H^1(\Omega) \rightarrow S_p(\mathcal{T}_h)$ with the following properties:*

- (i) I_h is defined elementwise, i.e., $(I_h v)|_K$ depends solely on $v|_{\overline{K}}$.

(ii) For $1 \leq m \leq p$ and for all $v \in H^{m+1}(\mathcal{P})$, we have

$$(4.11) \quad \|v - I_h v\|_{1,t,k} \leq C_m \left(1 + \left(\frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left(\frac{h}{p} \right)^m \|v\|_{m+1,\mathcal{P}}.$$

(iii) Fix $\tilde{C} > 0$ and assume $kh/p \leq \tilde{C}$. For all γ , there exist constants σ_γ, C_γ solely depending on $\gamma, C_{\text{metric}}$ of Assumption 4.4, and \tilde{C} such that for all $v \in \mathfrak{A}(M, \gamma, \mathcal{P})$,

$$(4.12) \quad \|v - I_h v\|_{1,t,k} \leq C_\gamma \left(1 + \left(\frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left(\left(\frac{h}{h + \sigma_\gamma} \right)^p + k \left(\frac{kh}{\sigma_\gamma p} \right)^p \right) M.$$

(iv) Finally, if $0 \leq q \leq p$, we have

$$(4.13) \quad \|v - I_h v\|_{1,t,k} \leq C_{q,\gamma} \left(1 + \left(\frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left(\left(\frac{h}{p} \right)^q + k \left(\frac{kh}{\sigma_\gamma p} \right)^p \right) M.$$

Proof. The construction is essentially that of [MS10, Thm. 5.5], where the approximation in the norm $\|\nabla \cdot\|_{0,\mathcal{P}} + k\|\cdot\|_{0,\Omega}$ is considered instead of $\|\cdot\|_{1,k,t}$. We will therefore be brief. The operator I_h is defined on the reference simplex and taken to be the one of [MS22, Lemma 8.3], which in turn is actually the operator $\widehat{\Pi}_p^{\text{grad},3d}$ (for $d = 3$) or the operator $\widehat{\Pi}_p^{\text{grad},2d}$ (for $d = 2$) of [MR18].

Proof of (ii): The p -dependence of the approximation properties are spelled out in [MS22, Lemma 8.3] and the h -dependence follows from scaling arguments.

Proof of (iii): Denote by \widehat{I}_h the operator on the reference simplex, and introduce for $K \in \mathcal{T}_h$ and $t \in \{0, 1\}$ the norm $\|v\|_{1,k,t,K}^2 := \|\nabla v\|_{L^2(K)}^2 + k^2 \|v\|_{L^2(K)}^2 + k^{1-2t} |v|_{H^t(\partial K)}^2$. By standard scaling arguments and the approximation properties of \widehat{I}_h (see the discussion in the proof of [MS22, Lemma 8.3]) we get

$$\|v - I_h v\|_{1,k,t,K} \lesssim h^{d/2-1} p^{-1} \left(1 + \frac{kh}{p} + \left(\frac{kh}{p} \right)^{1/2-t} \right) |\widehat{v}|_{2,\widehat{K}},$$

where $\widehat{u} := u \circ F_K$ is the pull-back of u to \widehat{K} . Noting that \widehat{I}_h is a projection onto $\mathcal{P}_p(\widehat{K})$ (and thus I_h a projection onto $S_p(\mathcal{T}_h)$) we arrive at

$$\|v - I_h v\|_{1,k,t,K} \lesssim h^{d/2-1} p^{-1} \left(1 + \frac{kh}{p} + \left(\frac{kh}{p} \right)^{1/2-t} \right) \inf_{w \in \mathcal{P}_p(\widehat{K})} |\widehat{v} - w|_{2,\widehat{K}}.$$

For $v \in \mathfrak{A}(M, \gamma, \mathcal{P})$, we can now proceed as in the proof of [MS10, Thm. 5.5] using in particular the approximation result [MS10, Lemma C.2] for $\inf_{w \in \mathcal{P}_p(\widehat{K})} |\widehat{v} - w|_{2,\widehat{K}}$. This leads to the desired result for $t \in \{0, 1\}$. For $t \in (0, 1)$ the boundary estimate $|v - I_h v|_{t,\Gamma}$ then follows by the interpolation inequality.

Proof of (iv): If $q \leq p$, the function

$$h \rightarrow \left(\frac{h}{h + \sigma_\gamma} \right)^{p-q}$$

is non-decreasing. Hence, since $h \leq C$, we have

$$\begin{aligned} \left(\frac{h}{h+\sigma_\gamma}\right)^p &= \left(\frac{h}{h+\sigma_\gamma}\right)^q \left(\frac{h}{h+\sigma_\gamma}\right)^{p-q} \\ &\leq \sigma_\gamma^{-q} h^q \left(\frac{C}{C+\sigma_\gamma}\right)^{p-q} = \sigma_\gamma^{-q} h^q \rho_\gamma^{p-q} = (\rho_\gamma \sigma_\gamma)^{-q} h^q \rho_\gamma^p \end{aligned}$$

where $\rho_\gamma < 1$. It is then clear that $\rho_\gamma^p \leq C_{\gamma,q} p^{-q}$, and it follows that

$$\left(\frac{h}{h+\sigma_\gamma}\right)^p \leq C_{q,\gamma} \left(\frac{h}{p}\right)^q.$$

Then, estimate (4.13) follows from (4.12). \square

Our key result concerning finite element discretization is established under an additional assumption of polynomial well-posedness that we detail now:

Assumption 4.6 (Polynomial well-posedness). There exist constants $\alpha_a, \alpha_s, \alpha_c, C \geq 0$, independent of k such that

$$(4.14) \quad C_{\text{ana},k}^- \leq C k^{\alpha_a},$$

$$(4.15) \quad C_{\text{sol},k}^- \leq C k^{\alpha_s},$$

$$(4.16) \quad C_{\text{cont},k}^- + C_{A,k}^- + k^{-2} C_{A,k}^\Delta \leq C k^{\alpha_c}.$$

Lemma 4.7 (Approximation factors for hp finite elements). *Assume that $p \geq 1 + \alpha_a + 2\alpha_c + \alpha_s$. Then, we have*

$$(4.17) \quad \eta^{(m)} \leq C_{m,\alpha_s,\alpha_c} \left(1 + \left(\frac{kh}{p}\right)^{1/2-t} + \frac{kh}{p} \right) \left(\left(\frac{h}{p}\right)^m + \frac{h}{p} \left(\frac{kh}{p}\right)^{\alpha_a+\alpha_s+\alpha_c} + k^{\alpha_a+\alpha_s+\alpha_c+1} \left(\frac{kh}{\sigma_{\gamma_m^*} p}\right)^p \right)$$

for all $m \geq 1$, and

$$(4.18) \quad \eta_\gamma^{(\text{exp})} \leq C_{\gamma,\alpha_s} k^{-1} \left(1 + \left(\frac{kh}{p}\right)^{1/2-t} + \frac{kh}{p} \right) \left(\left(\frac{h}{p}\right)^{2\alpha_c+1} \left(\frac{kh}{p}\right)^{\alpha_a+\alpha_s} + k^{\alpha_a+\alpha_s+1} \left(\frac{kh}{\sigma_{\vartheta(\gamma)} p}\right)^p \right)$$

for all $\gamma > 0$.

Proof. Let $1 \leq m \leq p$, and fix $(f, g) \in H^{m-1}(\mathcal{P}) \times H^{m-1/2}(\Gamma)$ with

$$\|f\|_{m-1,\mathcal{P}} + \|g\|_{m-1/2,\Gamma} = 1.$$

Employing the notation $u := S_k^-(f, g)$, Theorem 3.5 provides the splitting $u = u_F + u_A$, where $u_F \in H^{m+1}(\Omega)$ with

$$\|u_F\|_{m+1,\mathcal{P}} \leq C_m$$

and $u_A \in \mathfrak{A}(M, \gamma_m^*, \mathcal{P})$ with

$$M \leq C C_{\text{ana},k}^- C_{\text{sol},k}^- (1 + k^{-2} C_{A,k}^\Delta) \leq C k^{\alpha_a+\alpha_s+\alpha_c}.$$

Applying (4.11), we have

$$\begin{aligned} \|u_F - I_h u_F\|_{1,t,k} &\leq C_m \left(1 + \left(\frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left(\frac{h}{p} \right)^m \|u_F\|_{m+1, \mathcal{P}} \\ &\leq C_m \left(1 + \left(\frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left(\frac{h}{p} \right)^m. \end{aligned}$$

On the other hand, since $p \geq 1 + \alpha_a + \alpha_s + 2\alpha_c$ by assumption we can use (4.13) with $q = \alpha_a + \alpha_s + \alpha_c + 1$ and $\gamma = \gamma_m^*$, showing that

$$\begin{aligned} \|u_A - I_h u_A\|_{1,t,k} &\leq C_q \left(1 + \left(\frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left(\left(\frac{h}{p} \right)^{\alpha_a + \alpha_s + \alpha_c + 1} + k \left(\frac{kh}{\sigma_{\gamma_m^*} p} \right)^p \right) M \\ &\leq C_q k^{\alpha_a + \alpha_s + \alpha_c} \left(1 + \left(\frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left(\left(\frac{h}{p} \right)^{\alpha_a + \alpha_s + \alpha_c + 1} + k \left(\frac{kh}{\sigma_{\gamma_m^*} p} \right)^p \right) \\ &\leq C_q \left(1 + \left(\frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left(\frac{h}{p} \left(\frac{kh}{p} \right)^{\alpha_a + \alpha_s + \alpha_c} + k^{\alpha_a + \alpha_s + \alpha_c + 1} \left(\frac{kh}{\sigma_{\gamma_m^*} p} \right)^p \right). \end{aligned}$$

Then, we have

$$\begin{aligned} \|u - I_h u\|_{1,t,k} &\leq \|u_F - I_h u_F\|_{1,t,k} + \|u_A - I_h u_A\|_{1,t,k} \\ &\leq C_{m,q} \left(1 + \left(\frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left(\left(\frac{h}{p} \right)^m + \frac{h}{p} \left(\frac{kh}{p} \right)^{\alpha_a + \alpha_s + \alpha_c} + k^{\alpha_a + \alpha_s + \alpha_c + 1} \left(\frac{kh}{\sigma_{\gamma_m^*} p} \right)^p \right), \end{aligned}$$

and (4.17) follows from the definition of $\eta^{(m)}$ in (4.3a).

Consider now $(f, g) \in \mathfrak{A}(M_f, \gamma, \mathcal{P}) \times \mathfrak{A}(M_g, \gamma, T, \Gamma)$ with $M_f + kM_g = 1$, and set $u := S_k^-(f, g)$. We know from (WP3) that $u \in \mathfrak{A}(M_u, \vartheta(\gamma), \mathcal{P})$ with

$$M_u \leq CC_{\text{ana},k}^- C_{\text{sol},k}^- k^{-1} (M_f + kM_g) = C_{\text{ana},k}^- CC_{\text{sol},k}^- k^{-1} \leq Ck^{\alpha_a + \alpha_s - 1}.$$

It then follows from (4.13) with $q = \alpha_a + \alpha_s + 2\alpha_c + 1$ and $\gamma = \vartheta(\gamma)$ that

$$\|u - I_h u\|_{1,t,k} \leq C_{\gamma,\alpha} k^{\alpha_a + \alpha_s - 1} \left(1 + \left(\frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left(\left(\frac{h}{p} \right)^{\alpha_a + \alpha_s + 2\alpha_c + 1} + k \left(\frac{kh}{\sigma_{\vartheta(\gamma)} p} \right)^p \right),$$

and (4.18) follows from the definition of $\eta_\gamma^{(\text{exp})}$ in (4.3b). \square

Our main result is then a direct consequence of Theorem 4.3 and Lemma 4.7.

Theorem 4.8 (discrete stability of hp -FEM). *Let (WP) and (AP) as well as Assumptions 4.4 and 4.6 hold true. Then, for all $c_2 > 0$ there exists a constant c_1 solely depending on c_2 , α and the constants appearing in (WP) and (AP) such that whenever*

$$(4.19) \quad \frac{kh}{p} \leq c_1 \quad \text{and} \quad p \geq 1 + 2\alpha_c + \alpha_a + \alpha_s + c_2 \log k$$

the Galerkin solution $u_{hp} \in S_p(\mathcal{T}_h)$ of (4.2) exists and is unique and satisfies

$$(4.20) \quad \|u - u_{hp}\|_{1,t,k} \leq C \inf_{v_{hp} \in S_p(\mathcal{T}_h)} \|u - v_{hp}\|_{1,t,k}.$$

Proof. Let $c_2 > 0$ and fix $p \geq 1 + 2\alpha_c + \alpha_a + \alpha_s + c_2 \ln k$. Then, we have in particular that $\ln k \leq 1/(c_2)(p - 1/2)$, so that for any $\sigma, b > 0$

$$k^b \leq \left(e^{b/c_2} \right)^{p-1/2},$$

and

$$(4.21) \quad k^b \left(\frac{kh}{\sigma p} \right)^{p-1/2} \leq \left(e^{b/c_2} \right)^{p-1/2} \left(\frac{c_1}{\sigma} \right)^{p-1/2} = \left(\frac{e^{b/c_2}}{\sigma} c_1 \right)^{p-1/2}.$$

Since $p \geq 1 + 3\alpha$ by assumption, we can use (4.17) with $m = 1$. Since $t \leq 1$, picking $\sigma = \sigma_{\gamma_1^*}$ and $b = \alpha_a + \alpha_s + \alpha_c + 2$ in (4.21), we have

$$\begin{aligned} k\eta^{(1)} &\leq C \left(1 + \left(\frac{kh}{p} \right)^{-1/2} + \frac{kh}{p} \right) \left(\frac{kh}{p} + \left(\frac{kh}{p} \right)^{\alpha_a + \alpha_s + \alpha_c + 1} + k^{\alpha_a + \alpha_s + \alpha_c + 2} \left(\frac{kh}{\sigma_{\gamma_1^*} p} \right)^p \right) \\ &\leq C \left(1 + \left(\frac{kh}{p} \right)^{1/2} + \left(\frac{kh}{p} \right)^{3/2} \right) \left(\left(\frac{kh}{p} \right)^{1/2} + \left(\frac{kh}{p} \right)^{\alpha_a + \alpha_s + \alpha_c + 1/2} + k^{\alpha_a + \alpha_s + \alpha_c + 2} \left(\frac{kh}{\sigma_{\gamma_1^*} p} \right)^{p-1/2} \right) \\ &\leq C \left(1 + c_1^{1/2} + c_1^{3/2} \right) \left(c_1^{1/2} + c_1^{\alpha_a + \alpha_s + \alpha_c + 1/2} + \left(\frac{e^{(\alpha_a + \alpha_s + \alpha_c + 2)/c_2}}{\sigma_{\gamma_1^*}} c_1 \right)^{p-1/2} \right). \end{aligned}$$

We then employ (4.18) with $\gamma = \vartheta(\gamma_A^-)$, leading to

$$\begin{aligned} &C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})} \\ &\leq C_{\alpha_s, \alpha_c} k^{2\alpha_c - 1} \left(1 + \left(\frac{kh}{p} \right)^{1/2-t} + \frac{kh}{p} \right) \left(\left(\frac{h}{p} \right)^{2\alpha_c + 1} \left(\frac{kh}{p} \right)^{\alpha_a + \alpha_s} + k^{\alpha_a + \alpha_s + 1} \left(\frac{kh}{\sigma_{\vartheta(\gamma)} p} \right)^p \right) \\ &\leq C \left(1 + \left(\frac{kh}{p} \right)^{1/2} + \left(\frac{kh}{p} \right)^{3/2} \right) \left(k^{-1} \left(\frac{h}{p} \right)^1 \left(\frac{kh}{p} \right)^{2\alpha_c + \alpha_a + \alpha_s - 1/2} + k^{2\alpha_c + \alpha_a + \alpha_s} \left(\frac{kh}{\sigma_{\vartheta(\gamma)} p} \right)^{p-1/2} \right) \\ &\leq C \left(1 + \left(\frac{kh}{p} \right)^{1/2} + \left(\frac{kh}{p} \right)^{3/2} \right) \left(\left(\frac{kh}{p} \right)^{2\alpha_c + \alpha_a + \alpha_s + 1/2} + k^{2\alpha_c + \alpha_a + \alpha_s} \left(\frac{kh}{\sigma_{\vartheta(\gamma_A^-)} p} \right)^{p-1/2} \right) \\ &\leq C \left(1 + c_1^{1/2} + c_1^{3/2} \right) \left(c_1^{2\alpha_c + \alpha_a + \alpha_s + 1/2} + \left(\frac{e^{(2\alpha_c + \alpha_a + \alpha_s)/c_2}}{\sigma_{\vartheta(\gamma_A^-)}} c_1 \right)^{p-1/2} \right), \end{aligned}$$

where we have used (4.21) with $\sigma = \sigma_{\vartheta(\gamma_A^-)}$ and $b = 2\alpha_c + \alpha_a + \alpha_s$. Finally, since $k \geq k_0$, a similar reasoning shows that

$$\begin{aligned} C_{A,k}^\Delta \eta_{\vartheta(\gamma_A^\Delta)}^{(\text{exp})} &\leq C k^{\alpha_c} \eta_{\vartheta(\gamma_A^\Delta)}^{(\text{exp})} \leq C k^{2\alpha_c} \eta_{\vartheta(\gamma_A^\Delta)}^{(\text{exp})} \\ &\leq C_\alpha \left(1 + c_1^{1/2} + c_1^{3/2} \right) \left(c_1^{2\alpha_c + \alpha_a + \alpha_s + 1/2} + \left(\frac{e^{(2\alpha_c + \alpha_a + \alpha_s)/c_2}}{\sigma_{\vartheta(\gamma_A^\Delta)}} c_1 \right)^{p-1/2} \right). \end{aligned}$$

At this point, we assume that

$$(4.22) \quad c_1 \leq 1 \quad \frac{e^{(\alpha_s+\alpha_c)/c_2}}{\sigma_{\gamma_1^*}} c_1 \leq 1, \quad \frac{e^{(2\alpha_c+\alpha_s)/c_2}}{\sigma_{\vartheta(\gamma_A^-)}} c_1 \leq 1, \quad \frac{e^{(2\alpha_c+\alpha_s)/c_2}}{\sigma_{\vartheta(\gamma_A^+)}} c_1 \leq 1,$$

leading to the simplified expressions

$$k\eta^{(1)} + C_{A,k}^{\Delta} \eta_{\vartheta(\gamma_A^+)}^{(\text{exp})} \leq C_{\alpha_s, \alpha_c} \left(1 + \sqrt{\frac{e^{(\alpha_c+\alpha_s)/c_2}}{\sigma_{\gamma_1^*}}} + \sqrt{\frac{e^{(2\alpha_c+\alpha_s)/c_2}}{\sigma_{\vartheta(\gamma_A^-)}}} \right) c_1^{1/2}$$

and

$$C_{\text{cont},k}^- C_{A,k}^- \eta_{\vartheta(\gamma_A^-)}^{(\text{exp})} \leq C_{\alpha_c, \alpha_s} \left(1 + \sqrt{\frac{e^{(2\alpha_c+\alpha_s)/c_2}}{\sigma_{\vartheta(\gamma_A^-)}}} \right) c_1^{1/2}.$$

Then, we see that requiring, in addition to (4.22), that

$$C_{\alpha_c, \alpha_s} \left(1 + \sqrt{\frac{e^{(\alpha_c+\alpha_s)/c_2}}{\sigma_{\gamma_1^*}}} + \sqrt{\frac{e^{(2\alpha_c+\alpha_s)/c_2}}{\sigma_{\vartheta(\gamma_A^-)}}} + \sqrt{\frac{e^{(2\alpha_c+\alpha_s)/c_2}}{\sigma_{\vartheta(\gamma_A^+)}}} \right) c_1^{1/2} \leq \min(\mu/2, 1),$$

the requirements of Theorem 4.3 are met, leading to the result. \square

5. APPLICATION TO WAVE PROAGATION PROBLEMS

In this section, we verify that the Assumptions **(WP)**, **(AP)** of Section 2 are in fact satisfied for a variety of time-harmonic wave propagation problems. We emphasize that the assumptions **(WP1)**, **(WP2)**, in particular, Assumption 4.6, are assumed and have to be proved separately.

We will consider the heterogeneous Helmholtz equation below and in all examples the sesquilinear form $a(\cdot, \cdot)$ will be of the form

$$(5.1) \quad a(u, v) = (A(x)\nabla u, \nabla v).$$

The strong form of problem (2.5) is

$$(5.2) \quad L_k^- u := -\nabla \cdot (A(x)\nabla u) - k^2 n^2 u = f \quad \text{in } \Omega, \quad \partial_{n_A} u - T_{k,\Gamma}^- u = g \quad \text{on } \Gamma,$$

with the co-normal derivative $\partial_{n_A} v := \mathbf{n} \cdot (A\nabla v)|_{\Gamma}$ and the outer normal vector \mathbf{n} . Hence, in all examples the volume operator is given by $T_{k,\Omega}^- = k^2 n^2$. The complex-valued coefficients $A \in L^\infty(\Omega, \text{GL}(\mathbb{C}^d))$ and $n \in L^\infty(\Omega, \mathbb{C})$ are assumed to satisfy the following:

$$(5.3a) \quad \text{Re}(\xi^H A(x)\xi) \geq a_{\min} |\xi|^2 \quad \forall \xi \in \mathbb{C}^d \quad \forall x \in \Omega,$$

$$(5.3b) \quad \|\nabla^p A\|_{L^\infty(\Omega \setminus \Gamma_{\text{interf}})} \leq C_A \gamma_A^p p! \quad \forall p \in \mathbb{N}_0,$$

$$(5.3c) \quad \|\nabla^p n\|_{L^\infty(\Omega \setminus \Gamma_{\text{interf}})} \leq C_n \gamma_n^p p! \quad \forall p \in \mathbb{N}_0.$$

Remark 5.1. The coefficients A and n are allowed to depend on k . We merely require that the constants in (5.3a)–(5.3c) not depend on k . Therefore, equations including volume damping, classically written in the form $-\nabla \cdot (A\nabla u) - k^2 n^2 u + ikmu$ with some m satisfying the condition that $\|\nabla^p m\|_{L^\infty(\Omega \setminus \Gamma_{\text{interf}})} \leq C_m \gamma_m^p p!$ for all $p \in \mathbb{N}_0$ are included in this setting. Also certain PML-variants lead to k -dependent matrices A , which are included in this setting. \blacksquare

The strong form of problem (2.14) will be

$$(5.4) \quad L_k^+ u := -\nabla \cdot (A(x)\nabla u) + k^2 u = f \quad \text{in } \Omega, \quad \partial_{n_A} u - T_{k,\Gamma}^+ u = g \quad \text{on } \Gamma$$

so that $T_{k,\Omega}^+ = k^2$. The problems (5.2), (5.4), which we discuss in more detail in Sections 5.1–5.3, thus differ in the boundary conditions, i.e., the operators $T_{k,\Gamma}^-$ and $T_{k,\Gamma}^+$.

Throughout this section, we will assume:

Assumption 5.2. The boundary Γ and the interface Γ_{interf} are analytic.

5.1. Heterogeneous Helmholtz problem with Robin boundary conditions and PML.

The heterogeneous Helmholtz problem with Robin boundary conditions corresponds to the choice

$$T_{k,\Gamma}^- u := ikGu$$

in (5.2) for some function G that is analytic in a fixed tubular neighborhood T of Γ . In (5.4), we select $T_{k,\Gamma}^+ := 0$. Note that the fact that we allow for complex-valued coefficients A and n , which covers forms of PML where the path deformation into the complex plane is based on the use of polar/spherical coordinate [CM98, GLSW22]. Specifically, it can be seen following the discussion in [CFV21, Remark 2.1] and the expression of A given in [GLSW22, Eq (1.11)] that (WP3) holds true if the damping parameter is small enough (the width of the PML, however, is not constrained).

Lemma 5.3 ((AP) for Robin b.c.). *Let Assumption 5.2 and (5.3) be valid. Let G be analytic in a tubular neighborhood T of Γ . For L_k^- , L_k^+ given by (5.2) and (5.4) with $T_{k,\Gamma}^- = ikG$ and $T_{k,\Gamma}^+ = 0$ the following holds with $t = 1/2$:*

- (i) b_k^+ is bounded uniformly in k , and (AP1) holds with $\sigma = 1$.
- (ii) $\|S_k^+(f, g)\|_{1,k,t} \leq C [k^{-1}\|f\|_{0,\Omega} + k^{-1/2}\|g\|_{0,\Gamma}]$, and (AP2) holds for any fixed $s_{\max} \geq 0$.
- (iii) The splittings of $T_{k,\Omega}^- - T_{k,\Omega}^+$ and $T_{k,\Gamma}^- - T_{k,\Gamma}^+$ of (AP3) are given by $R_{k,\Omega}^\Delta u = k^2(n^2 + 1)u$, $A_{k,\Omega}^\Delta = 0$ and $R_{k,\Gamma}^\Delta u = ikGu$, $A_{k,\Gamma}^\Delta = 0$ and hence satisfy (AP3) for any fixed $s_{\max} \geq 0$.

Proof. Proof of (i), (iii): Continuity of b_k^+ follows by inspection. The coercivity (AP1) of b_k^+ with $\sigma = 1$ follows from (5.3a) and $T_{k,\Gamma}^+ = 0$.

Proof of (ii): The coercivity of b_k^+ implies for $u := S_k^+(f, g)$

$$\|u\|_{1,t,k}^2 \lesssim |(f, u)| + |(g, u)_\Gamma| \lesssim k^{-1}\|f\|_{0,\Omega}\|u\|_{0,\Omega} + k^{-1/2}\|g\|_{0,\Gamma}k^{1/2}\|u\|_{0,\Gamma}.$$

The trace inequality $k\|v\|_{0,\Gamma} \lesssim \|v\|_{1,\Omega} + k\|v\|_{0,\Omega} \lesssim \|v\|_{1,t,k}$ then implies the a priori bound. The validity of (AP2) follows from standard elliptic regularity theory for Neumann problems, using that Γ and Γ_{interf} are smooth. The fact that A is $\mathbb{C}^{d \times d}$ -valued instead of pointwise SPD is discussed in more detail in the proof of Lemma 5.4 below.

Proof of (iii): follows by inspection. □

Lemma 5.4 ((WP) for Robin b.c.). *Let Assumption 5.2 and (5.3) be valid. Let G be analytic in a tubular neighborhood T of Γ . Assume (WP2). Then:*

- (i) (WP1) holds with $C_{\text{cont},k}^- = O(1)$, and (WP3) holds with $C_{\text{ana},k}^- = O(1)$ (both uniformly in k).
- (ii) (WP4) holds with $R_{k,\Omega}^- u = k^2 n^2 u$, $A_{k,\Omega}^- = 0$ and $R_{k,\Gamma}^- = ikGu$, $A_{k,\Gamma}^- = 0$. In particular, for the constants in Assumption 4.6, we have $C_{\text{cont},k}^- = O(1)$ (uniformly in k) and $C_{A,k}^- = 0$ and $C_{A,k}^\Delta = 0$.

(iii) **(WP5)** holds.

Proof. Proof of (i): $C_{\text{cont},k}^- = O(1)$ follows by inspection. To see **(WP3)**, we first discuss the case that the matrix A is pointwise SPD as this case is essentially covered by the literature. Specifically, we observe that the function $u = S_k^-(\tilde{f}, \tilde{g})$ with $\tilde{f} \in \mathfrak{A}(C_{\tilde{f}}, \gamma_{\tilde{f}}, \mathcal{P})$, $\tilde{g} \in \mathfrak{A}(C_{\tilde{g}}, \gamma_{\tilde{g}}, T, \Gamma)$, satisfies, upon setting $\varepsilon = 1/k$,

$$(5.5) \quad \begin{aligned} -\varepsilon^2 \nabla \cdot (A \nabla u) - n^2 u &= \varepsilon^2 f && \text{in } \Omega, \\ \varepsilon^2 \partial_{n_A} u &= \varepsilon(\varepsilon g + iGu) && \text{on } \Gamma, \end{aligned}$$

The regularity of solutions of these problems form has been addressed in [MS11, Lemma 4.13] and [Mel02, Prop. 5.4.5, Rem. 5.4.6]. The procedure is as follows: the regularity can be analyzed locally and the boundary or the interface may be flattened by (local) changes of variables so that one is reduced to the analysis on balls (interior estimates and transmission problems), half-balls (boundary estimates). Note that that Lemma 2.2 (and [Mel02, Lemma 4.3.4]) provide that the analyticity classes are invariant under bi-analytic changes of variables. Hence, one may apply [Mel02, Prop. 5.5.1] (interior analytic regularity), [Mel02, Prop. 5.5.3] (boundary analytic regularity for Neumann problems) as well as [Mel02, Prop. 5.5.4] (interface analytic regularity) if $\Gamma_{\text{interf}} \neq \emptyset$, to problem (5.5). Considering as an example the case of boundary regularity, we apply [Mel02, Prop. 5.5.3] with the following choices

$$\begin{aligned} C_A &= O(C_A), & C_b &= 0, & C_c &= O(C_{n^2}), & C_f &= O(\varepsilon^2 C_{\tilde{f}}), & C_{G_1} &= O(\varepsilon \tilde{C}_{\tilde{g}}), & C_{G_2} &= O(C_{n^2}), \\ \gamma_A &= O(\gamma_A), & \gamma_b &= 0, & \gamma_c &= O(\gamma_{n^2}), & \gamma_f &= O(\gamma_{\tilde{f}}), & \gamma_{G_1} &= O(\gamma_{\tilde{g}}), & C_{G_2} &= O(1), \end{aligned}$$

where γ_{n^2} and C_{n^2} are the analyticity constants of the coefficient n^2 and the $O(\cdot)$ -notation absorbs the modification of constants due to the analytic change of variables. By combining the finitely many local contributions, we arrive at

$$\begin{aligned} \|\nabla^p u\|_{L^2(\Omega \setminus \Gamma_{\text{interf}})} &\lesssim C \gamma^p \max\{k, p\}^p (k^{-2} C_{\tilde{f}} + k^{-1} C_{\tilde{g}} + k^{-1} \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \\ &\lesssim C \gamma^p \max\{k, p\}^p (k^{-2} C_{\tilde{f}} + k^{-1} C_{\tilde{g}} + C_{\text{sol},k}^- k^{-1} (C_{\tilde{f}} + k C_{\tilde{g}})) \\ &\lesssim C \gamma^p \max\{k, p\}^p C_{\text{sol},k}^- k^{-1} (C_{\tilde{f}} + k C_{\tilde{g}}) \quad \forall p \geq 2 \end{aligned}$$

where we applied the stability estimate (2.7) as well as $C_{\text{sol},k}^- \gtrsim 1$. The stability estimate (2.7) shows that the bound is valid for $p \in \{0, 1\}$ as well, so that $S_k^-(\tilde{f}, \tilde{g}) \in \mathfrak{A}(C C_{\text{sol},k}^- k^{-1} (C_{\tilde{f}} + k C_{\tilde{g}}), \gamma, \mathcal{P})$, i.e., **(WP3)** holds with $C_{\text{ana},k}^- = O(1)$.

For the case that the matrix A is not SPD but merely satisfies (5.3a), we note that the proof of [Mel02, Props. 5.5.1, 5.5.3, 5.5.4] relies on a (piecewise) H^2 -regularity result for the principal part of the operator (cf. [Mel02, Lemmas 5.5.5, 5.5.8, 5.5.9]), which in turns hinges on the difference quotient technique of Nirenberg. We note that the condition (5.3a) provides a coercivity that makes the difference quotient technique applicable for Neumann and transmission problems so that also [Mel02, Props. 5.5.1, 5.5.3, 5.5.4] holds for complex-valued coefficients A, n satisfying (5.3).

Proof of (ii), (iii): By inspection. □

Under the assumption **(WP2)** we have the following quasi-optimality result for the heterogeneous Helmholtz equation with Robin boundary conditions:

Theorem 5.5 (Robin b.c. and PML). *Let $t = 1/2$. Let Assumption 5.2 and (5.3) be valid. Let G be analytic on a tubular neighborhood of Γ . For the problem (5.2) with $T_{k,\Gamma}^- = ikG$ assume polynomial well-posedness of the solution operator S_k^- of Assumption 4.6. In the discretization setting of Assumption 4.4, for each $c_2 > 0$ there are constants $c_1, C > 0$ independent of h, p , and k such that under the scale-resolution condition (4.19) the quasi-optimality result (4.20) holds.*

Proof. Combine Lemmas 5.3 and 5.4 with the stability Assumption 4.6. \square

5.2. Heterogeneous Helmholtz problem in the full space. The heterogeneous Helmholtz problem with Sommerfeld radiation condition in full space \mathbb{R}^d is to find $U \in H_{\text{loc}}^1(\mathbb{R}^d)$ such that

$$(5.6) \quad \begin{aligned} -\nabla \cdot (A\nabla U) - k^2 n^2 U &= f && \text{in } \mathbb{R}^d, \\ |\partial_r U - ikU| &= o\left(\|x\|^{\frac{1-d}{2}}\right) && \text{for } \|x\| \rightarrow \infty, \end{aligned}$$

is satisfied in the weak sense. Here, ∂_r denotes the derivative in the radial direction. We assume f, A , and n (which are defined on \mathbb{R}^d) to be local in the sense that there exists a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, such that $\text{supp } f \subset \bar{\Omega}$, $\text{supp } (A - I) \subset \bar{\Omega}$ and $\text{supp } (n - 1) \subset \bar{\Omega}$. Problem (5.6) can be reformulated using the Dirichlet-to-Neumann operator $\text{DtN}_k : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$, which is given by $g \mapsto \text{DtN}_k g := \partial_n w := \mathbf{n} \cdot \nabla w$, and $w \in H_{\text{loc}}^1(\mathbb{R}^d \setminus \Omega)$ is the unique weak solution to

$$\begin{aligned} -\Delta w - k^2 w &= 0 && \text{in } \mathbb{R}^d \setminus \Omega, \\ w &= g && \text{on } \Gamma, \\ |\partial_r w - ikw| &= o\left(\|x\|^{\frac{1-d}{2}}\right) && \text{for } \|x\| \rightarrow \infty. \end{aligned}$$

The heterogeneous Helmholtz problem in the full space is to find $u \in H^1(\Omega)$ such that

$$(5.7) \quad \begin{aligned} L_k^- u &:= -\nabla \cdot (A\nabla u) - k^2 n^2 u = f && \text{in } \Omega, \\ \partial_{n_A} u - T_{k,\Gamma}^- u &:= \partial_{n_A} u - \text{DtN}_k u = g && \text{on } \Gamma. \end{aligned}$$

For $g = 0$, the solution u of (5.7) is then the restriction to Ω of the solution of (5.6). The coupling interface Γ between the interior and exterior domain may be chosen to be arbitrary as long as the exterior domain is non-trapping [BSW16, Def. 1.1] and Γ itself is analytic. If $d = 2$, we assume that $\text{diam } \Omega < 1$, which ensures that the single layer operator $V_0 : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$ appearing below is boundedly invertible.

We define the operator DtN_0 as the map $\text{DtN}_0 : g \mapsto \partial_n u^{\text{ext}}$, where u^{ext} solves

$$(5.8a) \quad -\Delta u^{\text{ext}} = 0 \quad \text{in } \mathbb{R}^d \setminus \bar{\Omega},$$

$$(5.8b) \quad u^{\text{ext}} = g \quad \text{on } \Gamma,$$

$$(5.8c) \quad u^{\text{ext}}(x) = \begin{cases} O(1/\|x\|) & \text{as } \|x\| \rightarrow \infty \text{ for } d = 3 \\ b \ln \|x\| + O(1/\|x\|) & \text{as } \|x\| \rightarrow \infty \text{ for some } b \in \mathbb{R} \text{ for } d = 2. \end{cases}$$

Remark 5.6. The reason for choosing the particular condition (5.8c) at ∞ is that it ensures the representation formula

$$u^{\text{ext}}(x) = -(\tilde{V}_0 \partial_n u^{\text{ext}})(x) + (\tilde{K}_0 u^{\text{ext}})(x)$$

with the usual single layer \tilde{V}_0 and double layer potential \tilde{K}_0 for the Laplacian; see, e.g., [McL00, Thm. 8.9] or [CS85, Lem. 3.5]. This particular choice will reappear in the analysis of the difference $\text{DtN}_k - \text{DtN}_0$ in Lemma 5.7. For our choice of condition at ∞ , one has the Calderón identity

$$(5.9) \quad \begin{pmatrix} u^{\text{ext}} \\ \partial_n u^{\text{ext}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + K_0 & -V_0 \\ -D_0 & \frac{1}{2} - K'_0 \end{pmatrix} \begin{pmatrix} u^{\text{ext}} \\ \partial_n u^{\text{ext}} \end{pmatrix},$$

where V_0 , K_0 , K'_0 , and D_0 denote the single layer, double layer, adjoint double layer, and the hypersingular operator, respectively (see, e.g., [SS11, Ste08] for the precise definition). By definition, we have $\partial_n u^{\text{ext}} = \text{DtN}_0 u^{\text{ext}}$. From the Calderón identity (5.9), one easily concludes

$$(5.10) \quad (\text{DtN}_0 u^{\text{ext}}, u^{\text{ext}})_\Gamma = -(D_0 u^{\text{ext}}, u^{\text{ext}})_\Gamma - (V_0 \partial_n u^{\text{ext}}, \partial_n u^{\text{ext}})_\Gamma \leq 0,$$

where we employed that D_0 is positive semidefinite and V_0 is positive definite, [Ste08, Thm. 6.23, Cor. 6.25]. \blacksquare

For the auxiliary problem (5.4), we select $T_{k,\Gamma}^+ = \text{DtN}_0$.

In order to show that Problem (5.7) fits into the framework of Section 4 we need to analyze the operators DtN_k and DtN_0 :

Lemma 5.7. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded Lipschitz domains with smooth boundary Γ . Then the following holds:*

- (i) $-\langle \text{DtN}_0 u, u \rangle \geq 0$ for all $u \in H^{1/2}(\Gamma)$.
- (ii) The operator DtN_0 is a bounded linear operator $H^{s+1/2}(\Gamma) \rightarrow H^{s-1/2}(\Gamma)$ for every $s \geq 0$.
- (iii) Let Ω^+ be nontrapping with analytic boundary Γ . Let the ball $B_R(0)$ satisfy $\bar{\Omega} \subset B_R(0)$, and set $\Omega_R := B_R(0) \setminus \Gamma$. Let $s \geq 0$ be given. Then

$$\text{DtN}_k - \text{DtN}_0 = kB + \llbracket \partial_n \tilde{A} \rrbracket$$

where the linear operators $B: H^s(\Gamma) \rightarrow H^s(\Gamma)$ and $\tilde{A}: H^s(\Gamma) \rightarrow C^\infty(\Omega_R)$ satisfy for all $u \in H^s(\Gamma)$

$$\|Bu\|_{s,\Gamma} \lesssim \|u\|_{s,\Gamma}, \quad \tilde{A}u \in \mathfrak{A}(Ck^\beta \|u\|_{s,\Gamma}, \gamma, \Omega_R)$$

with $\beta = 7/2 + d/2$, and constants C , $\gamma \geq 0$ independent of k . Here, the operator $\llbracket \partial_n \tilde{A} \rrbracket$ realizes the jump of the normal derivative of $\tilde{A}v$ and is given by $\llbracket \partial_n \tilde{A} \rrbracket v := \mathbf{n} \cdot \nabla(\tilde{A}v)|_{\Omega^+} - \mathbf{n} \cdot \nabla(\tilde{A}v)|_\Omega$ (see (A.9)).

- (iv) Let $\Gamma = \partial B_1(0) \subset \mathbb{R}^3$ be the unit ball in dimension $d = 3$. Then $\text{DtN}_k - \text{DtN}_0: H^s(\Gamma) \rightarrow H^s(\Gamma)$ satisfies, for every $s \geq 0$,

$$\|\text{DtN}_k u - \text{DtN}_0 u\|_{s,\Gamma} \lesssim k \|u\|_{s,\Gamma} \quad \forall u \in H^s(\Gamma).$$

Proof. Proof of (i): $-\langle \text{DtN}_0 u, u \rangle \geq 0$ is asserted in (5.10).

Proof of (ii): Follows from regularity properties of elliptic boundary value problems. Alternatively, it could be the representation of the (exterior) DtN_0 as $\text{DtN}_0 = V_0^{-1}(-1/2 + K_0)$ and the mapping properties of V_0 and K_0 as given in, e.g., [SS11, Thm. 3.2.2].

Proof of (iii): See Appendix A.

Proof of (iv): For $\Gamma = \partial B_1(0) \subset \mathbb{R}^3$, the operator DtN_k has an explicit series representation in terms of spherical harmonics. That is, denoting by Y_ℓ^m the standard spherical harmonics,

a function $u \in L^2(\Gamma)$ can be expanded as

$$u = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell}^m Y_{\ell}^m.$$

One has the norm equivalences $\|u\|_{H^s(\Gamma)}^2 \sim \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} |u_{\ell}^m|^2 (\ell+1)^{2s}$ for every $s \geq 0$, see, e.g., [Néd01, Sec. 2.5.1]. The operators DtN_0 and DtN_k can be written as

$$\text{DtN}_0 u = - \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell+1) u_{\ell}^m Y_{\ell}^m, \quad \text{DtN}_k u = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} z_{\ell}(k) u_{\ell}^m Y_{\ell}^m,$$

with explicit estimates for the symbol $z_{\ell}(k)$ given in [Néd01, Sec. 2.6.3]. The formulas for DtN_0 and DtN_k immediately give

$$(5.11) \quad \text{DtN}_k u - \text{DtN}_0 u = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (z_{\ell}(k) + \ell + 1) u_{\ell}^m Y_{\ell}^m,$$

From [DI01, Lem. 3.2, eqn. (3.28)], where the operator DtN_k has opposite sign, we have

$$(5.12) \quad \ell + 1 - k \leq -\text{Re } z_{\ell}(k) \leq \ell + 1 + k.$$

Consequently, we immediately have

$$(5.13) \quad |\text{Re } z_{\ell}(k) + \ell + 1| \leq k.$$

From [Néd01, Thm. 2.6.1, eqn. (2.6.24)] we have

$$(5.14) \quad 0 \leq \text{Im } z_{\ell}(k) \leq k.$$

Combining (5.13) and (5.14) yields

$$(5.15) \quad |z_{\ell}(k) + \ell + 1| \leq 2k.$$

For $u \in H^s(\Gamma)$ and with the previous estimate we have

$$\begin{aligned} \|\text{DtN}_k u - \text{DtN}_0 u\|_{s,\Gamma}^2 &\sim \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell+1)^{2s} |z_{\ell}(k) + \ell + 1|^2 |u_{\ell}^m|^2 Y_{\ell}^m \\ &\leq (2k)^2 \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell+1)^{2s} |u_{\ell}^m|^2 Y_{\ell}^m \sim (2k)^2 \|u\|_{s,\Gamma}^2, \end{aligned}$$

which yields the result. \square

Lemma 5.8 ((AP) for full space problem). *Let Assumption 5.2 and (5.3) be valid. Let Ω^+ be non-trapping. For L_k^-, L_k^+ given by (5.2), (5.4) with $T_{k,\Gamma}^- = \text{DtN}_k$ and $T_{k,\Gamma}^+ = \text{DtN}_0$ the following holds with $t = 1/2$:*

- (i) b_k^+ is bounded uniformly in k , and (AP1) holds with $\sigma = 1$.
- (ii) $\|S_k^+(f, g)\|_{1,k,t} \leq C [k^{-1} \|f\|_{0,\Omega} + k^{-1/2} \|g\|_{0,\Gamma}]$, and (AP2) holds for any fixed $s_{\max} \geq 0$.
- (iii) The splitting of $T_{k,\Omega}^- - T_{k,\Omega}^+$ in (AP3) is given by $R_{k,\Omega}^{\Delta} u = k^2(1+n^2)u$, $A_{k,\Omega}^{\Delta} = 0$. The splitting of $T_{k,\Gamma}^- - T_{k,\Gamma}^+$ in (AP3) is given by $R_{k,\Gamma}^{\Delta} u = kB$ and $A_{k,\Gamma}^{\Delta} = \llbracket \partial_n \tilde{A} \rrbracket$ with the operators B , $\llbracket \partial_n \tilde{A} \rrbracket$ given by Lemma 5.7(iii); we refer to Remark 2.1(iii) for the interpretation of $\llbracket \partial_n \tilde{A} \rrbracket$ mapping into the analyticity classes of the form $\mathfrak{A}(M, \gamma, T, \Gamma)$.

(iv) For the case $\Gamma = \partial B_1(0) \subset \mathbb{R}^3$, the statement (iii) holds true with the operator $A_{k,\Gamma}^\Delta = 0$ and the operator $R_{k,\Gamma}^\Delta = B = \text{DtN}_k - \text{DtN}_0$ of order zero bounded uniformly in k as described in Lemma 5.7(iv).

Proof. Proof of (i): Continuity of b_k^+ follows by inspection. The coercivity (AP1) of b_k^+ with $\sigma = 1$ follows from (5.3a) and the positivity of $-T_{k,\Gamma}^+ = -\text{DtN}_0$ by (5.10).

Proof of (ii): Set $w := S_k^+(f, g)$. The a priori bound

$$(5.16) \quad \|w\|_{1,\Omega} + k\|w\|_{0,\Omega} + |w|_{1/2,\Gamma} \lesssim \|S_k^+(f, g)\|_{1,k,t} \lesssim k^{-1}\|f\|_{0,\Omega} + k^{-1/2}\|g\|_{0,\Gamma}$$

follows by Lax-Milgram. To show the shift theorem asserted in (AP2), we reformulate the problem solved by $w = S_k^+(f, g)$, i.e.,

$$-\nabla \cdot (A\nabla w) = f - k^2 w \quad \text{in } \Omega, \quad \partial_n w - \text{DtN}_0 w = g \quad \text{on } \Gamma.$$

as a transmission problem. We define the function

$$\tilde{w} := \begin{cases} w & \text{in } \Omega \\ w^{\text{ext}} := -\tilde{V}_0(\text{DtN}_0 \gamma w) + \tilde{K}_0 \gamma w & \text{in } \Omega^+. \end{cases}$$

By the mapping properties of the potentials \tilde{V}_0, \tilde{K}_0 , [SS11, Ste08], we have for any bounded domain $\tilde{\Omega}$

$$(5.17) \quad \|\tilde{w}\|_{H^1(\tilde{\Omega})} \lesssim \|w\|_{1,\Omega} = \|S_k^+(f, g)\|_{1,k,t} \lesssim k^{-1}\|f\|_{0,\Omega} + k^{-1/2}\|g\|_{0,\Gamma}.$$

The function \tilde{w} satisfies

$$\begin{aligned} -\nabla \cdot (A\nabla \tilde{w}) + k^2 \tilde{w} &= f & \text{in } \Omega, & & -\Delta \tilde{w} &= 0 & \text{in } \Omega^+, \\ \llbracket \tilde{w} \rrbracket &= 0 & \text{on } \Gamma, & & \llbracket \partial_n \tilde{w} \rrbracket &= g & \text{on } \Gamma. \end{aligned}$$

Let $\tilde{\Omega} \supset \supset \Omega$ with $\partial\tilde{\Omega}$ smooth. Let $\chi \in C_0^\infty(\tilde{\Omega})$ with $\chi \equiv 1$ on a neighborhood of $\bar{\Omega}$. Upon writing \hat{A} for the extension of A by the identity matrix and the characteristic function χ_Ω of Ω

$$(5.18) \quad \begin{aligned} -\nabla \cdot (\hat{A}\nabla(\chi\tilde{w})) + k^2 \mathbb{1}_\Omega \chi\tilde{w} &= \hat{f} & \text{in } \tilde{\Omega}, \\ \llbracket \chi\tilde{w} \rrbracket &= 0 & \text{on } \Gamma, \\ \llbracket \partial_n(\chi\tilde{w}) \rrbracket &= g & \text{on } \Gamma, \\ \chi\tilde{w} &= 0 & \text{on } \partial\tilde{\Omega}. \end{aligned}$$

with $\hat{f} = f$ in Ω and $\hat{f} = 2\nabla\tilde{w} \cdot \nabla\chi + \tilde{w}\Delta\chi$ in $\tilde{\Omega} \setminus \Omega$. Since \hat{A} satisfies (5.3a) with Ω replaced by $\tilde{\Omega}$, the Lax-Milgram Lemma gives for the solution $\chi\tilde{w}$ of (5.18)

$$(5.19) \quad \begin{aligned} \|\nabla(\chi\tilde{w})\|_{L^2(\tilde{\Omega})} + \|(1 + k^2\chi_\Omega)(\chi\tilde{w})\|_{L^2(\tilde{\Omega})} &\lesssim \|\hat{f}\|_{H^{-1}(\tilde{\Omega})} + \|g\|_{H^{-1/2}(\Gamma)} \\ &\lesssim \|\hat{f}\|_{L^2(\tilde{\Omega})} + \|g\|_{1/2,\Gamma} \lesssim \|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma} + \|\tilde{w}\|_{1,\tilde{\Omega} \setminus \Omega} \end{aligned}$$

The function $\chi\tilde{w}$ satisfies an elliptic transmission problem with piecewise smooth coefficients in the equation and smooth interface $\Gamma \cup \Gamma_{\text{interf}}$. Although the coefficient \hat{A} is complex-valued, it satisfies (5.3a) and, as discussed in the proof of Lemma 5.3, (piecewise) H^2 -regularity as

formulated in [Mel02, Prop. 5.4.8] are available. Such an H^2 -regularity gives $\chi\tilde{w} \in H^2(\tilde{\Omega} \setminus (\Gamma_{\text{interf}} \cup \Gamma))$ and consequently $w = (\chi\tilde{w})|_{\Omega} \in H^2(\Omega \setminus \Gamma_{\text{interf}})$ together with

$$\begin{aligned} \|w\|_{2,\Omega \setminus \Gamma_{\text{interf}}} &= \|\chi\tilde{w}\|_{2,\Omega \setminus \Gamma_{\text{interf}}} \lesssim \|\hat{f} + k^2 \mathbb{1}_{\Omega} \chi\tilde{w}\|_{0,\tilde{\Omega}} + \|g\|_{1/2,\Gamma} + \|\chi\tilde{w}\|_{1,\tilde{\Omega}} \\ &\lesssim \|f\|_{0,\Omega} + \|2\nabla(\chi\tilde{w}) \cdot \nabla\chi + (\chi\tilde{w})\Delta\chi\|_{0,\tilde{\Omega} \setminus \Omega} + k^2 \|w\|_{0,\Omega} + \|g\|_{1/2,\Gamma} \\ &\stackrel{(5.19)}{\lesssim} \|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma} + k^2 \|w\|_{0,\Omega} \stackrel{(5.16)}{\lesssim} \|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma} + k^{1/2} \|g\|_{0,\Gamma}. \end{aligned}$$

Proof of (iii), (iv): follows from Lemma 5.7. \square

Lemma 5.9 ((WP) for full space). *Let Assumption 5.2 and (5.3) be valid. Let Ω^+ be non-trapping. For L_k^-, L_k^+ given by (5.2), (5.4) with $T_{k,\Gamma}^- = \text{DtN}_k$ and $T_{k,\Gamma}^+ = \text{DtN}_0$ the following holds with $t = 1/2$ if (WP2) holds:*

- (i) (WP1) holds with $C_{\text{cont},k}^- = O(k^\beta)$ with β given by Lemma 5.7, and (WP3) holds with $C_{\text{ana},k}^- = O(1)$ (uniformly in k).
- (ii) Let the operators B and $[[\partial_n \tilde{A}]]$ be given by Lemma 5.7(iii). Then (WP4) holds with $R_{k,\Omega}^- u = k^2 n u$, $A_{k,\Omega}^- = 0$ and the choices $R_{k,\Gamma}^- = \text{DtN}_0 + kB$, $A_{k,\Gamma}^- = [[\partial_n \tilde{A}]]$. In particular, for the constants in Assumption 4.6 we have $C_{\text{cont},k}^- + C_{A,k}^- + C_{A,k}^\Delta = O(k^\beta)$ with β given by Lemma 5.7.
- (iii) For $\Gamma = \partial B_1(0) \subset \mathbb{R}^3$ the assumption (WP4) holds with $R_{k,\Omega}^- u = k^2 n u$, $A_{k,\Omega}^- = 0$ and the choices $R_{k,\Gamma}^- = \text{DtN}_k$ and $A_{k,\Gamma}^- = 0$. In particular, for the constants in Assumption 4.6 we have $C_{\text{cont},k}^- + C_{A,k}^- + C_{A,k}^\Delta = O(1)$.
- (iv) (WP5) holds.

Proof. Proof of (i): The full space problem corresponds to the choice $T_{k,\Gamma}^- u = \text{DtN}_k u$. As in the proof of Lemma 5.8 u is the restriction to Ω of the solution of the following full-space transmission problem:

$$\begin{aligned} -\nabla \cdot (A\nabla u) - k^2 n^2 u &= f && \text{in } \Omega \cup \Omega^+, \\ [[u]] &= 0 && \text{on } \Gamma, \\ [[\partial_n u]] &= g && \text{on } \Gamma. \end{aligned}$$

u satisfies radiation condition.

From now on the proof is completely analogous to the Robin case of Lemma 5.4 replacing the use of Neumann conditions by transmission conditions on Γ .

Proof of (ii): We decompose $T_{k,\Gamma}^- = \text{DtN}_k = \text{DtN}_0 + (\text{DtN}_k - \text{DtN}_0) = (\text{DtN}_0 + kB) + [[\partial_n \tilde{A}]] =: R_{k,\Gamma}^- + A_{k,\Gamma}^-$. The operator $A_{k,\Gamma}^-$ is of the form given in (WP4). For the operator $R_{k,\Gamma}^-$, we observe using the mapping properties of DtN_0 and B

$$\begin{aligned} \left| \langle R_{k,\Gamma}^- u, v \rangle \right| &\leq |\langle \text{DtN}_0 u, v \rangle| + k |\langle B u, v \rangle| \lesssim \|u\|_{1/2,\Gamma} \|v\|_{1/2,\Gamma} + k \|u\|_{0,\Gamma} \|v\|_{0,\Gamma} \\ &\stackrel{t=1/2}{\lesssim} \|u\|_{1,t,k} \|v\|_{1,t,k} \end{aligned}$$

Proof of (iv): By inspection. \square

Bayliss-Gunzburger-Turkel	Engquist-Mayda	Feng
$\alpha = -\frac{1+ik}{2(1+k^2)}$ $\beta = \frac{-2k^2 - \frac{3ik}{2} + 3}{2(ik-1)}$	$\alpha = \frac{1+ik}{2k^2}$ $\beta = ik - \frac{1}{2}$	$\alpha = -\frac{i}{2k}$ $\beta = ik - \frac{1}{2} - \frac{i}{8k}$

TABLE 5.1. Parameters for 2^{nd} -order ABCs for $\Gamma = \partial B_1(0) \subset \mathbb{R}^2$; taken from [Ih198, Table 3.2].

Fixing the computational domain Ω and the coupling boundary Γ on which the DtN-operator is employed, we have the following quasi-optimality result:

Theorem 5.10 (full space). *Let $t = 1/2$. Let Assumption 5.2 and (5.3) be valid. For the problem (5.2) with $T_{k,\Gamma}^- = \text{DtN}_k$ assume polynomial well-posedness of the solution operator S_k^- of Assumption 4.6. In the discretization setting of Assumption 4.4, for each $c_2 > 0$ there are constants $c_1, C > 0$ independent of h, p , and k such that under the scale-resolution condition (4.19) the quasi-optimality result (4.20) holds.*

Proof. Combine Lemmas 5.8 and 5.9 with the stability Assumption 4.6. \square

5.3. Heterogeneous Helmholtz problem with second order ABCs. Various boundary conditions that are formally of higher order than the Robin boundary conditions have been proposed in the literature, notably for case of a sphere in \mathbb{R}^2 . Our theory covers those proposed by Bayliss-Gunzburger-Turkel [BGT82], Engquist-Majda [EM77, EM79] and Feng [Fen84], see [Ih198, Sec. 3.3.3, Table 3.2] for a comprehensive comparison. We denote by Δ_Γ and ∇_Γ the surface Laplacian and the surface gradient on Γ (see, e.g., [Néd01, Sec. 2.5.6]). The model problem for a heterogeneous Helmholtz equation with second order absorbing boundary conditions is to find u such that

$$(5.20) \quad \begin{aligned} L_k^- u &:= -\nabla \cdot (A \nabla u) - k^2 n^2 u = f \quad \text{in } \Omega, \\ \partial_{n_A} u - T_{k,\Gamma}^- u &:= \partial_{n_A} u - (\beta u + \alpha \Delta_\Gamma u) = g \quad \text{on } \Gamma, \end{aligned}$$

for k -dependent parameters α, β satisfying

$$(5.21) \quad \text{Im } \alpha \neq 0 \quad \text{and} \quad |\text{Im } \alpha| \sim \frac{1}{k} \quad \text{and} \quad |\text{Re } \alpha| \lesssim \frac{1}{k^2}, \quad |\beta| \sim k$$

for $k \geq k_0 > 0$. The choices of α and β proposed by Bayliss-Gunzburger-Turkel, Engquist-Mayda, and Feng are given in Table 5.1 as presented in [Ih198, Sec. 3.3.3, Table 3.2]. The coefficients A, n in (5.20) are assumed to satisfy (5.3b), (5.3c) and, instead of (5.3a), A is assumed to be pointwise SPD, viz.,

$$(5.22) \quad \xi^H A(x) \xi \geq a_{\min} |\xi|^2 > 0 \quad \forall \xi \in \mathbb{C}^d \setminus \{0\} \quad x \in \Omega.$$

Remark 5.11. Our analysis covers operators $T_{k,\Gamma}^-$ of the form (5.20) for smooth Γ , which is the form of the second order ABCs for spheres. It is expected that second order ABC for general Γ have similar structure and k -scaling properties. \blacksquare

We select $T_{k,\Gamma}^+ = \alpha \Delta_\Gamma$. Then we have:

Lemma 5.12 ((**AP**) for 2^{nd} order ABCs). *Let Assumption 5.2 be valid. Let the coefficients A , n satisfy (5.3) and A additionally (5.22). For L_k^-, L_k^+ of (5.2), (5.4) with $T_{k,\Gamma}^-$ given by (5.20)–(5.21) and $T_{k,\Gamma}^+ = \alpha\Delta_\Gamma$ the following is true for $t = 1$:*

- (i) b_k^+ is bounded uniformly in k , and (**AP1**) holds for some σ depending on α in (5.21).
- (ii) $\|S_k^+(f, g)\|_{1,k,t} \leq C [k^{-1}\|f\|_{0,\Omega} + k^{-1/2}\|g\|_{0,\Gamma}]$, and (**AP2**) holds for $s_{\max} = 0$.
- (iii) The splittings of $T_{k,\Omega}^- - T_{k,\Omega}^+$ and $T_{k,\Gamma}^- - T_{k,\Gamma}^+$ of (**AP3**) are given by $R_{k,\Omega}^\Delta u = k^2(n^2 + 1)u$, $A_{k,\Omega}^\Delta = 0$ and $R_{k,\Gamma}^\Delta u = \beta u$, $A_{k,\Gamma}^\Delta = 0$ and hence satisfy (**AP3**) for any fixed $s_{\max} \geq 0$.

Proof. Proof of (i): The Laplace-Beltrami operator $\Delta_\Gamma: H^1(\Gamma) \rightarrow H^{-1}(\Gamma)$ is a bounded linear operator since Γ has no boundary. Furthermore, we have $-\langle \alpha\Delta_\Gamma u, u \rangle = \alpha\langle \nabla_\Gamma u, \nabla_\Gamma u \rangle$. Setting $\sigma := \bar{\alpha}/|\alpha|$ we estimate

$$(5.23) \quad -\operatorname{Re}(\sigma\langle T_{k,\Gamma}^+ u, u \rangle) = \operatorname{Re}(\sigma\alpha)\langle \nabla_\Gamma u, \nabla_\Gamma u \rangle = \operatorname{Re}(\sigma\alpha)|u|_{1,\Gamma}^2 \gtrsim k^{-1}|u|_{1,\Gamma}^2,$$

where, in the last step, we employed the assumptions $\operatorname{Im} \alpha \sim k^{-1}$ and $|\operatorname{Re} \alpha| \lesssim k^{-2}$ so that $|\alpha| \sim 1/k$. Since the coefficient A is assumed to be SPD and satisfy (5.22), the coercivity of b_k^+ of (**AP1**) holds. The uniform-in- k boundedness of b_k^+ is easily seen.

Proof of (ii): The a priori estimate for $S_k^+(f, g)$ follows from coercivity and continuity of b_k^+ . The function $w = S_k^+(f, g)$ satisfies

$$(5.24) \quad -\nabla \cdot (A\nabla w) = f - k^2 w \quad \text{in } \Omega, \quad \partial_{n_A} w = g + \alpha\Delta_\Gamma w \quad \text{on } \Gamma.$$

Note that for $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$ we have by Lax-Migram, which follows from (i), and $|\alpha| \sim k^{-1}$ (cf. (5.21))

$$(5.25) \quad \|\nabla w\|_{0,\Omega} + k\|w\|_{0,\Omega} + k^{-1/2}|w|_{1,\Gamma} \lesssim k^{-1}\|f\|_{0,\Omega} + k^{-1/2}\|g\|_{0,\Gamma}.$$

The following surface PDE is satisfied

$$\alpha\Delta_\Gamma w = -g + \partial_{n_A} w \quad \text{on } \Gamma,$$

which gives, since Γ is a smooth, compact manifold without boundary,

$$(5.26) \quad \|w\|_{2,\Gamma} \stackrel{|\alpha| \sim k^{-1}}{\lesssim} k\|g\|_{0,\Gamma} + k\|\partial_{n_A} w\|_{0,\Gamma} + \|w\|_{1,\Gamma} \stackrel{(5.25)}{\lesssim} k^{-1/2}\|f\|_{0,\Omega} + k\|g\|_{0,\Gamma} + k\|\partial_{n_A} w\|_{0,\Gamma}.$$

In order to estimate $\|\partial_{n_A} w\|_{0,\Gamma}$ we introduce an auxiliary matrix-valued function $\hat{A} \in C^\infty(\Omega, \operatorname{GL}(\mathbb{C}^d))$ that satisfies (5.3a) as well as $\hat{A} = A$ in a neighborhood of Γ . We introduce an auxiliary function $\tilde{w} \in H^1(\Omega)$ as the solution of

$$(5.27) \quad -\nabla \cdot (\hat{A}\tilde{w}) = 0 \quad \text{in } \Omega, \quad \tilde{w} = w \quad \text{on } \Gamma.$$

By the smoothness of Γ and \hat{A} , the map $w \mapsto \partial_{n_A} \tilde{w}$ maps $H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ and $H^{3/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$ so that by interpolation it maps $H^1(\Gamma) \rightarrow L^2(\Gamma)$ with the a priori estimate

$$(5.28) \quad \|\partial_{n_A} \tilde{w}\|_{0,\Gamma} \lesssim \|w\|_{1,\Gamma}.$$

By interior regularity, we have additionally that $\tilde{w} \in C^\infty(\Omega)$ and for each open $\Omega' \subset\subset \Omega$ there is $C_{\Omega'} > 0$ with

$$(5.29) \quad \|\tilde{w}\|_{2,\Omega'} \leq C_{\Omega'} \|\tilde{w}\|_{1,\Omega} \lesssim \|w\|_{1/2,\Gamma}.$$

Let $\chi \in C^\infty(\bar{\Omega})$ with $\chi \equiv 1$ in a neighborhood of Γ and such that $A|_{\text{supp } \chi} = \hat{A}|_{\text{supp } \chi}$. The function $w - \chi\tilde{w} \in H_0^1(\Omega)$ solves

$$(5.30) \quad \begin{aligned} -\nabla \cdot (A\nabla(w - \chi\tilde{w})) &= f - k^2w + z && \text{in } \Omega, \\ w - \tilde{w} &= 0 && \text{on } \Gamma \end{aligned}$$

for the function $z := \nabla \cdot (A(\chi\tilde{w}))$. Since $A = \hat{A}$ near Γ and therefore $-\nabla \cdot (A\nabla(\chi\tilde{w})) = 0$ near Γ , we get in view of (5.29) that $z \in L^2(\Omega)$ with

$$(5.31) \quad \|z\|_{0,\Omega} \lesssim \|w\|_{1/2,\Gamma}.$$

As discussed in the proof of Lemma 5.3, elliptic regularity implies $w - \chi\tilde{w} \in H^2(\Omega \setminus \Gamma_{\text{interf}})$ with

$$(5.32) \quad \|w - \chi\tilde{w}\|_{2,\Omega \setminus \Gamma_{\text{interf}}} \lesssim \|f\|_{0,\Omega} + k^2\|w\|_{0,\Omega} + \|z\|_{0,\Omega} \stackrel{(5.25),(5.31)}{\lesssim} \|f\|_{0,\Omega} + k^{1/2}\|g\|_{0,\Gamma}.$$

Since $\chi \equiv 1$ near Γ , we conclude

$$\begin{aligned} \|\partial_{n_A} w\|_{0,\Gamma} &\leq \|\partial_{n_A} \tilde{w}\|_{0,\Gamma} + \|\partial_{n_A}(w - \chi\tilde{w})\|_{0,\Gamma} \lesssim \|\partial_{n_A} \tilde{w}\|_{0,\Gamma} + C\|w - \chi\tilde{w}\|_{2,\Omega \setminus \Gamma_{\text{interf}}} \\ &\stackrel{(5.28),(5.25),(5.32)}{\lesssim} \|f\|_{0,\Omega} + k^{1/2}\|g\|_{0,\Gamma}. \end{aligned}$$

Inserting this estimate in (5.26) yields

$$\|\alpha\Delta_\Gamma w\|_{2,\Gamma} \lesssim \|f\|_{0,\Omega} + k^{1/2}\|g\|_{0,\Gamma}.$$

Noting that (5.24) is a Neumann problem for w , we get from elliptic regularity for Neumann problems the desired statement **(AP2)**.

We remark in passing that the above arguments can be bootstrapped to show **(AP2)** for $s_{\max} > 0$. We refer to [Ber21, Rem. 6.5.7] and in particular to the bootstrapping argument for the analytic regularity assertion **(WP3)** in Appendix B for more details.

Proof of (iii): By inspection. \square

Lemma 5.13 (**(WP)** for 2nd order ABC). *Let Assumption 5.2 be valid. Let the coefficients A , n satisfy (5.3) and A additionally (5.22). For L_k^-, L_k^+ of (5.2), (5.4) with $T_{k,\Gamma}^-$ given by (5.20)–(5.21) and $T_{k,\Gamma}^+ = \alpha\Delta_\Gamma$ the following is true for $t = 1$ if **(WP2)** holds:*

- (i) **(WP3)** holds with $C_{\text{ana},k}^- = O(1)$ (uniformly in k).
- (ii) **(WP4)** holds with $R_{k,\Omega}^- u = k^2(n^2 + 1)u$, $A_{k,\Omega}^- = 0$ and $R_{k,\Gamma}^- = \alpha\Delta_\Gamma u + \beta u$, $A_{k,\Gamma}^- = 0$. In particular, for the constants in Assumption 4.6 we have $C_{\text{cont},k}^- = O(1)$ and $C_{A,k}^- = C_{A,k}^\Delta = 0$.
- (iii) **(WP5)** holds.

Proof. Proof of (i): See Appendix B. To give more details, by local changes of variables and the invariance of the analyticity classes under analytic changes of variables (see Lemma 2.2), the analysis is reduced to balls or half-balls. For the regularity of $S_k^-(f, g)$ with $f \in \mathfrak{A}(M_f, \gamma_f, \mathscr{P})$, $g \in \mathfrak{A}(M_g, \gamma_g, T)\Gamma$ one obtains from Theorem B.5 (and the analytic changes of variables) for a tubular neighborhood T of Ω for

$$\|\nabla^{p+2}u\|_{L^2(T \cap \Omega)} \lesssim \max(p, k)^{p+2}\gamma^p [k^{-2}M_f + k^{-1}M_g + k^{-1}\|u\|_{1,k,t}] \quad \forall p \in \mathbb{N}_0,$$

where γ depends on γ_f, γ_g , but is independent of k . Together with corresponding estimates in the interior of Ω , this shows **(WP3)** with $C_{\text{ana},k}^- = O(1)$.

Proof of (ii), (iii): By inspection. \square

Theorem 5.14 (*2nd order ABC*). *Let $t = 1$. Let Assumption 5.2 and (5.3) be valid. For the problem (5.2) with $T_{k,\Gamma}^- = \text{DtN}_k$ given by (5.20)–(5.21) assume polynomial well-posedness of the solution operator S_k^- of Assumption 4.6. In the discretization setting of Assumption 4.4, for each $c_2 > 0$ there are constants $c_1, C > 0$ independent of h, p , and k such that under the scale-resolution condition (4.19) the quasi-optimality result (4.20) holds.*

Proof. Combine Lemmas 5.12 and 5.13 with the stability Assumption 4.6. \square

APPENDIX A. DIRICHLET-TO-NEUMANN MAPS VIA BOUNDARY INTEGRAL OPERATORS

The main goal of the present section is prove Lemma 5.7(iii). To that end, we rewrite the Dirichlet-to-Neumann operators DtN_k and DtN_0 in terms of boundary integral operators.

A.1. Preliminaries. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain with analytic boundary $\Gamma := \partial\Omega$. We denote by Ω^+ the exterior domain, i.e., $\Omega^+ := \mathbb{R}^d \setminus \bar{\Omega}$. Throughout this section we assume Ω^+ to be non-trapping, see [BSW16, Def. 1.1]. Furthermore, we assume that the open ball B_R of radius R around the origin contains $\bar{\Omega}$, i.e., $\bar{\Omega} \subset B_R$. We set $\Omega_R := (\Omega \cup \Omega^+) \cap B_R = B_R \setminus \Gamma$. Following standard notation we introduce the interior and exterior trace operators $\gamma_0^{int}, \gamma_0^{ext}$ by restricting to Γ and the γ_1^{int} and γ_1^{ext} by setting, for sufficiently smooth functions v , we have $\gamma_1^{int}v = \mathbf{n} \cdot \gamma_0^{int}(\nabla v)$ and $\gamma_1^{ext}v = \mathbf{n} \cdot \gamma_0^{ext}(\nabla v)$, where \mathbf{n} is the outer normal vector on Γ . Furthermore, we denote by V_k, K_k, K'_k and D_k the single layer, double layer, adjoint double layer and hypersingular boundary integral operators, see [Ste08, Sec. 6.9 and 7.9]. The single layer and double layer potentials are denoted \tilde{V}_k and \tilde{K}_k . Finally, given a coupling parameter $\eta \in \mathbb{R} \setminus \{0\}$ we introduce the combined field operator $A'_{k,\eta}$ by

$$A'_{k,\eta} := \frac{1}{2} + K'_k + i\eta V_k.$$

We remind the reader of the exterior Calderón identities

$$\begin{pmatrix} \gamma_0^{ext}u \\ \gamma_1^{ext}u \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + K_k & -V_k \\ -D_k & \frac{1}{2} - K'_k \end{pmatrix} \cdot \begin{pmatrix} \gamma_0^{ext}u \\ \gamma_1^{ext}u \end{pmatrix},$$

Given Dirichlet data u the Dirichlet-to-Neumann operator DtN_k can be expressed for any $k \geq 0$ by a complex linear combination of the two equations in the Calderón identity: For any $\eta \in \mathbb{R} \setminus \{0\}$ we have

$$(A.1) \quad A'_{k,\eta}(\text{DtN}_k u) = \left(\frac{1}{2} + K'_k + i\eta V_k \right) \text{DtN}_k u = \left(-D_k + i\eta \left(-\frac{1}{2} + K_k \right) \right) u.$$

Our analysis relies on invertibility of the combined field operator $A'_{k,\eta}$ as an operator mapping $H^s(\Gamma)$ into itself. Wavenumber-explicit estimates for $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma) \leftarrow L^2(\Gamma)}$ are discussed in [BSW16, Sec. 1.4]. For non-trapping $\Omega^+ \subset \mathbb{R}^d$, $d = 2, 3$ it is known that

$$\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma) \leftarrow L^2(\Gamma)} \lesssim k^{5/4} \left(1 + \frac{k^{3/4}}{|\eta|} \right) \text{ for all } k \geq k_0 \text{ and } \eta \in \mathbb{R} \setminus \{0\}, \text{ see [Spe14, Thm 1.11].}$$

This bound can be sharpened assuming $|\eta| \sim k$. In fact, for non-trapping $\Omega^+ \subset \mathbb{R}^d$, $d = 2, 3$ and $|\eta| \sim k$ there holds

$$(A.2) \quad \|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma) \leftarrow L^2(\Gamma)} \lesssim 1$$

for all $k \geq k_0$, see [BSW16, Thm. 1.13]. In Proposition A.1, we collect mapping properties of some boundary integral operators:

Proposition A.1. *Let Γ be analytic and $\eta \in \mathbb{R} \setminus \{0\}$ fixed. If $d = 2$, assume additionally $\text{diam } \Omega < 1$. Then*

- (i) $A_{0,-\eta} = \frac{1}{2} + K_0 - i\eta V_0: H^s(\Gamma) \rightarrow H^s(\Gamma)$ is boundedly invertible for $s \geq 0$.
- (ii) $A'_{0,\eta} = \frac{1}{2} + K'_0 + i\eta V_0: H^s(\Gamma) \rightarrow H^s(\Gamma)$ is boundedly invertible for $s \geq -1$.
- (iii) For $k > 0$ the combined field operator $A'_{k,\eta} = \frac{1}{2} + K'_k + i\eta V_k: H^s(\Gamma) \rightarrow H^s(\Gamma)$ is boundedly invertible for $s \geq -1$.
- (iv) For $k \geq 0$ and $s \geq -1/2$ the following operators are bounded:

$$(A.3) \quad \begin{aligned} V_k: H^{-1/2+s}(\Gamma) &\rightarrow H^{1/2+s}(\Gamma), & K_k: H^{1/2+s}(\Gamma) &\rightarrow H^{1/2+s}(\Gamma), \\ K'_k: H^{-1/2+s}(\Gamma) &\rightarrow H^{-1/2+s}(\Gamma), & D_k: H^{1/2+s}(\Gamma) &\rightarrow H^{-1/2+s}(\Gamma) \end{aligned}$$

- (v) For $k \geq k_0 > 0$ one can decompose

$$(A.4) \quad \begin{aligned} V_k - V_0 &= S_V + \gamma_0^{\text{int}} \tilde{A}_V, & K_k - K_0 &= S_K + \gamma_0^{\text{int}} \tilde{A}_K, \\ K'_k - K'_0 &= S_{K'} + \gamma_1^{\text{int}} \tilde{A}_V, & D_k - D_0 &= S_D + \gamma_1^{\text{int}} \tilde{A}_K \end{aligned}$$

with linear maps $\tilde{A}_V: H^{-3/2}(\Gamma) \rightarrow C^\infty(\bar{\Omega})$ and $\tilde{A}_K: H^{-1/2}(\Gamma) \rightarrow C^\infty(\bar{\Omega})$ and bounded linear operators $S_V, S_K, S_{K'}$, and S_D having the following mapping properties for $s \geq -1$:

$$(A.5) \quad \begin{aligned} \|S_V u\|_{-1/2+s,\Gamma} &\leq C_{s,s'} k^{-(1+s-s')} \|u\|_{-1/2+s',\Gamma}, & 1/2 \leq s' \leq s+3, \\ \|S_K u\|_{1/2+s,\Gamma} &\leq C_{s,s'} k^{-(1+s-s')} \|u\|_{-1/2+s',\Gamma}, & 1/2 \leq s' \leq s+3, \\ \|S_{K'} u\|_{-1/2+s,\Gamma} &\leq C_{s,s'} k^{-(1+s-s')} \|u\|_{-3/2+s',\Gamma}, & 3/2 \leq s' \leq s+3, \\ \|S_D u\|_{1/2+s,\Gamma} &\leq C_{s,s'} k^{-(1+s-s')} \|u\|_{-3/2+s',\Gamma}, & 3/2 \leq s' \leq s+3. \end{aligned}$$

The operators \tilde{A}_V and \tilde{A}_K have the mapping property

$$(A.6) \quad \tilde{A}_V f \in \mathfrak{A}(C_V k \|f\|_{-3/2,\Gamma}, \gamma_V, \Omega) \quad \forall f \in H^{-3/2}(\Gamma),$$

$$(A.7) \quad \tilde{A}_K f \in \mathfrak{A}(C_K k \|f\|_{-1/2,\Gamma}, \gamma_K, \Omega) \quad \forall f \in H^{-1/2}(\Gamma),$$

with constants $C_V, \gamma_V, C_K, \gamma_K$ independent of $k \geq k_0$. For $t \geq 0$ the following mapping properties hold true:

$$(A.8) \quad \begin{aligned} \|S_V u\|_{t,\Gamma} &\leq C_t k^{-1} \|u\|_{t,\Gamma}, & \|S_K u\|_{t,\Gamma} &\leq C_t \|u\|_{t,\Gamma}, \\ \|S_{K'} u\|_{t,\Gamma} &\leq C_t \|u\|_{t,\Gamma}, & \|S_D u\|_{t,\Gamma} &\leq C_t k \|u\|_{t,\Gamma}. \end{aligned}$$

Proof. For Item (i) see [Mel12, Lem. 3.5(ii)]. For Item (ii) in the case $s \geq 0$ see [Mel12, Lemma 3.5(iv)]. We turn to the case $s \in [-1, 0]$. Note that the adjoint of $A_{0,-\eta}$ is precisely the operator $A'_{0,\eta}$. Furthermore, by Item (i) the operator $A_{0,-\eta}: H^t(\Gamma) \rightarrow H^t(\Gamma)$ is boundedly invertible in particular for $t \in [0, 1]$. Hence, due to the adjoint of $A_{0,-\eta}$ being $A'_{0,\eta}$, we find that $A'_{0,\eta}: H^s(\Gamma) \rightarrow H^s(\Gamma)$ is also boundedly invertible for $s \in [-1, 0]$. For Item (iii) see [CWGLS12, Thm. 2.27] in the case $s \in [-1, 0]$ as well as [BSW16, Sec. 6.1]. Consequently, by [Mel12, Lem. 2.14] invertibility holds for any $s \geq 0$. The mapping properties (A.3) in Item (iv) are standard, see, e.g., [SS11, Rem. 3.1.18(c)]. For (A.4) and (A.5) in Item (v) see [MMPR20, Lem. A.1] for the $1/2 < s'$ and $3/2 < s'$, respectively. The limiting cases follow by inspection of the proof, a multiplicative trace estimate, and the estimates for the

potentials given in [MMPR20, Lem. A.1]. (A.7) is likewise shown in [MMPR20, Lem. A.1]. (A.8) is just a simplification of (A.5). \square

A.2. Decomposition of the Dirichlet-to-Neumann map. Before proceeding with the proof of Lemma 5.7(iii) let us introduce the jumps of the trace operators:

$$(A.9) \quad \llbracket u \rrbracket := \gamma_0^{ext} u - \gamma_0^{int} u, \quad \llbracket \partial_n u \rrbracket := \gamma_1^{ext} u - \gamma_1^{int} u.$$

For linear operators \tilde{A} mapping into spaces of piecewise defined functions we define the operator $\llbracket \tilde{A} \rrbracket$ and $\llbracket \partial_n \tilde{A} \rrbracket$ analogously, e.g., $\llbracket \tilde{A} \rrbracket u := \llbracket \tilde{A} u \rrbracket$.

We now collect further technical results of [Mel12]. We closely follow the notation and results of [Mel12]. As in [Mel12] we assume

$$(A.10) \quad C_\eta^{-1} k \leq |\eta| \leq C_\eta k$$

for some $C_\eta > 0$ independent of k .

In the following Proposition A.2 we extend the results of [Mel12, Lem. 6.3] to a wider range of Sobolev spaces.

Proposition A.2 ([Mel12, Lemma 6.3]). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with an analytic boundary Γ . Let $q \in (0, 1)$. Then one can construct operators $L_{\Gamma,q}^{neg}$, $H_{\Gamma,q}^{neg}$ on $H^{-1}(\Gamma)$ with the following properties:*

- (i) $L_{\Gamma,q}^{neg} f + H_{\Gamma,q}^{neg} f = f$ for all $f \in H^{-1}(\Gamma)$.
- (ii) For $-1 \leq s' \leq s$ there holds $\|H_{\Gamma,q}^{neg} f\|_{s',\Gamma} \leq C_{s,s'} (q/k)^{s-s'} \|f\|_{s,\Gamma}$.
- (iii) $L_{\Gamma,q}^{neg} f$ is the restriction to Γ of a function that is analytic on a tubular neighborhood T of Γ and satisfies

$$\begin{aligned} \|\nabla^n L_{\Gamma,q}^{neg} f\|_{0,T} &\leq C_q k^{d/2} \gamma_q^n \max\{k, n\}^n \|f\|_{-1/2,\Gamma} & \forall n \in \mathbb{N}_0, \\ \|\nabla^n L_{\Gamma,q}^{neg} f\|_{0,T} &\leq C_q k^{d/2+1} \gamma_q^n \max\{k, n\}^n \|f\|_{-1,\Gamma} & \forall n \in \mathbb{N}_0. \end{aligned}$$

Here, $C_{s,s'}$ is independent of q and k ; the constants $C_q, \gamma_q > 0$ are independent of k .

Proof. For Items (i), (ii) in the case $-1 \leq s' \leq s \leq 1$ and Item (iii) see [Mel12, Lem. 6.3 and Rem. 6.4]. The crucial extension is the estimate stated in Item (ii) in the case $-1 \leq s' \leq s$ for $s \geq 1$. In the proof of [Mel12, Lem. 6.3] the operators $H_{\Gamma,q}^{neg}$ and $L_{\Gamma,q}^{neg}$ are explicitly constructed. We collect the important ingredients of the proof of [Mel12, Lem. 6.3] in the following. On the compact manifold Γ consider the eigenvalue problem for the Laplace-Beltrami operator

$$(A.11) \quad -\Delta_\Gamma \varphi = \lambda^2 \varphi \quad \text{on } \Gamma.$$

There are countably many eigenfunctions φ_m , $m \in \mathbb{N}_0$, with corresponding eigenvalues $\lambda_m^2 \geq 0$, which we assume to be sorted in ascending order. Without loss of generality, these eigenfunctions are normalized in $L^2(\Gamma)$. The functions $(\varphi_m)_{m \in \mathbb{N}_0}$ may be assumed to be an orthonormal basis of $L^2(\Gamma)$ and an orthogonal basis of $H^1(\Gamma)$. With $u_m := (u, \varphi_m)$ we have

$$(A.12) \quad \|u\|_{0,\Gamma}^2 = \sum_{m=0}^{\infty} |u_m|^2 \quad \text{and} \quad \|u\|_{1,\Gamma}^2 = \sum_{m=0}^{\infty} (1 + \lambda_m^2) |u_m|^2.$$

For $s \in \mathbb{R}$ we introduce the sequence space h^s by

$$h^s := \left\{ (u_m)_{m \in \mathbb{N}} : \sum_{m=0}^{\infty} (1 + \lambda_m^2)^s |u_m|^2 < \infty \right\}.$$

The mapping $\iota: u \mapsto ((u, \varphi_m))_{m \in \mathbb{N}_0}$ then provides an isomorphism between the Sobolev space $H^s(\Gamma)$ and the sequence space h^s for $s \in [-1, 1]$, with corresponding norm equivalence, see [Mel12, Lem. C.3]. However, as we will see below ι is in fact an isomorphism for all $s \geq -1$. Inspection of the proof of [Mel12, Lem. 6.3], in particular the proof of the estimate for $H_{\Gamma, q}^{neg}$, reveals that

$$\|H_{\Gamma, q}^{neg} f\|_{s', \Gamma} \leq C_{s, s'} (q/k)^{s-s'} \|f\|_{s, \Gamma}$$

holds for all $-1 \leq s' \leq s$, for which $\iota: H^s(\Gamma) \rightarrow h^s$ and $\iota: H^{s'}(\Gamma) \rightarrow h^{s'}$ are isomorphisms. Hence, the proof is complete once we establish that $\iota: H^s(\Gamma) \rightarrow h^s$ is an isomorphism for all $s > 1$. We show the case $s = 2$.

The inclusion $h^2 \hookrightarrow H^2(\Gamma)$: Let $u = \sum_{m=0}^{\infty} u_m \varphi_m$ be such that $\sum_{m=0}^{\infty} (1 + \lambda_m^2)^2 |u_m|^2 < \infty$. Let $u^N = \sum_{m=0}^N u_m \varphi_m$. By the above construction, $u^N \rightarrow u$ in $H^1(\Gamma)$ and $\|u\|_{1, \Gamma} = \|(u_m)_{m \in \mathbb{N}}\|_{h^1}$. Furthermore, we have

$$\|\Delta_{\Gamma}(u^N - u^{M-1})\|_{0, \Gamma}^2 = \left\| \sum_{m=M}^N u_m \Delta_{\Gamma} \varphi_m \right\|_{0, \Gamma}^2 = \left\| \sum_{i=M}^N u_m \lambda_m^2 \varphi_m \right\|_{0, \Gamma}^2 = \sum_{i=M}^N |u_m|^2 \lambda_m^4,$$

which converges to zero as $N, M \rightarrow \infty$. Here we used (A.11), the fact that the eigenfunctions are an orthonormal basis of $L^2(\Gamma)$, and the assumed convergence $\sum_{m=0}^{\infty} (1 + \lambda_m^2)^2 |u_m|^2 < \infty$. Therefore, u^N is a Cauchy sequence in $H^1(\Gamma, \Delta_{\Gamma}) = \{u \in H^1(\Gamma) : \Delta_{\Gamma} u \in L^2(\Gamma)\}$, endowed with the graph norm. Consequently u^N converges in $H^1(\Gamma, \Delta_{\Gamma})$. Since $\Delta_{\Gamma} : H^1(\Gamma, \Delta_{\Gamma}) \rightarrow L^2(\Gamma)$ is continuous, we conclude $\Delta_{\Gamma} u = \sum_{m \in \mathbb{N}_0} u_m \Delta_{\Gamma} \varphi_m = -\sum_{m \in \mathbb{N}_0} u_m \lambda_m^2 \varphi_m$. Finally, by elliptic regularity we can now estimate

$$(A.13) \quad \|u\|_{2, \Gamma}^2 \lesssim \|\Delta_{\Gamma} u\|_{0, \Gamma}^2 = \sum_{m \in \mathbb{N}_0} |f_m|^2 \lambda_m^4 = \|(u_m)_{m \in \mathbb{N}_0}\|_{h^2}^2,$$

and $\|u\|_{2, \Gamma}^2 \lesssim \|(u_m)_{m \in \mathbb{N}_0}\|_{h^2}^2$ follows by (A.13) together with (A.12).

The inclusion $H^2(\Gamma) \hookrightarrow h^2$: Let $u \in H^2(\Gamma)$ be given with the representation $u = \sum_{m=0}^{\infty} u_m \varphi_m$, where the sum converges in $H^1(\Gamma)$. Since $u \in H^2(\Gamma)$ we have $-\Delta_{\Gamma} u =: f \in L^2(\Gamma)$. In the following we express the coefficient u_m in terms of f_m . Note that

$$\lambda_m^2 u_m = \lambda_m^2 (u, \varphi_m) = (\nabla_{\Gamma} u, \nabla_{\Gamma} \varphi_m) = (f, \varphi_m) = f_m.$$

Hence, we have $\lambda_m^2 u_m = f_m$ and consequently

$$\sum_{m=0}^{\infty} \lambda_m^4 |u_m|^2 = \sum_{m=0}^{\infty} |f_m|^2 < \infty.$$

Finally, using (A.13) as well as the fact that $\|u\|_{1, \Gamma} = \|(u_m)_{m \in \mathbb{N}}\|_{h^1}$, we find

$$\|(u_m)_{m \in \mathbb{N}_0}\|_{h^2}^2 = \|(u_m)_{m \in \mathbb{N}_0}\|_{h^1}^2 + \|(u_m)_{m \in \mathbb{N}_0}\|_{h^2}^2 = \|u\|_{1, \Gamma}^2 + \|\Delta_{\Gamma} u\|_{0, \Gamma}^2 \leq \|u\|_{2, \Gamma}^2.$$

This concludes the proof for $s = 2$. Interpolation between $s = 1$ and $s = 2$ yields the result for $s \in (1, 2)$, see [Mel12, Lem. C.3]. Inductively one proceeds for the space $H^{2n}(\Gamma)$ by similar arguments. Instead of Δ_{Γ} one performs the same arguments for Δ_{Γ}^n . \square

In the following we will prove an extension of [Mel12, Thm. 2.9] in Theorem A.4 below. The proof of Theorem A.4 relies on a decomposition of the volume potential \tilde{V}_k , which we present below for the readers' convenience.

Proposition A.3 ([Mel12, Thm. 5.3]). *Let Γ be analytic and $q \in (0, 1)$. Then*

$$\tilde{V}_k = \tilde{V}_0 + \tilde{S}_{V,pw} + \tilde{\mathcal{A}}_{V,pw},$$

where the linear operators $\tilde{S}_{V,pw}$ and $\tilde{\mathcal{A}}_{V,pw}$ satisfy the following for every $s \geq -1$:

(i) $\tilde{S}_{V,pw}: H^{-1/2+s}(\Gamma) \rightarrow H^{3+s}(\Omega_R)$ with

$$\|\tilde{S}_{V,pw}\varphi\|_{s',\Omega_R} \leq C_{s',s} q^2 (qk^{-1})^{1+s-s'} \|\varphi\|_{-1/2+s,\Gamma}, \quad 0 \leq s' \leq s+3.$$

Here, the constant $C_{s',s} > 0$ is independent of q and $k \geq k_0$.

(ii) $\tilde{\mathcal{A}}_{V,pw}: H^{-1/2+s}(\Gamma) \rightarrow H^2(B_R) \cap C^\infty(\Omega_R)$ with

$$\|\nabla^n \tilde{\mathcal{A}}_{V,pw}\varphi\|_{0,\Omega_R} \leq C_q \gamma_q \max\{n+1, k\}^{n+1} \|\varphi\|_{-3/2,\Gamma} \quad \forall n \in \mathbb{N}_0.$$

Here, the constants $C_q, \gamma_q > 0$ are independent of $k \geq k_0$ but may depend on q .

Theorem A.4 (Extension of [Mel12, Thm. 2.9]). *Let Γ be analytic and let $s \geq 0$. Fix $q \in (0, 1)$. Then the operator $A'_{k,\eta}$ can be written in the form*

$$A'_{k,\eta} = A'_{0,1} + R_{A'} + k[\tilde{\mathcal{A}}_1] + [\partial_n \tilde{\mathcal{A}}_2],$$

where the linear operator $R_{A'}$ satisfies

$$(A.14a) \quad \|R_{A'}u\|_{s+1,\Gamma} \leq Ck \|u\|_{s,\Gamma},$$

$$(A.14b) \quad \|R_{A'}u\|_{s,\Gamma} \leq Ck \|u\|_{s-1,\Gamma},$$

$$(A.14c) \quad \|R_{A'}u\|_{s,\Gamma} \leq q \|u\|_{s,\Gamma},$$

$$(A.14d) \quad \|R_{A'}u\|_{s-1,\Gamma} \leq q \|u\|_{s-1,\Gamma},$$

and the linear operators $\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}_2: H^{-1}(\Gamma) \rightarrow C^\infty(T)$ satisfy

$$(A.15a) \quad \tilde{\mathcal{A}}_1\varphi \in \mathfrak{A}(C_q C_{1,\varphi}, \gamma_q, T), \quad C_{1,\varphi} = k \|\varphi\|_{-3/2,\Gamma} + k^{d/2} \|\varphi\|_{-1,\Gamma},$$

$$(A.15b) \quad \tilde{\mathcal{A}}_2\varphi \in \mathfrak{A}(C_q C_{2,\varphi}, \gamma_q, T), \quad C_{2,\varphi} = k \|\varphi\|_{-3/2,\Gamma}.$$

The constant C and the tubular neighborhood T of Γ are independent of $k \geq k_0$ and q ; the constants $C_q, \gamma_q > 0$ are independent of $k \geq k_0$ (but may depend of q).

Proof. We perform a similar splitting as in the proof of [Mel12, Thm. 2.9]. The starting point of our analysis is the decomposition

$$A'_{k,\eta} = \frac{1}{2} + K'_0 + \gamma_1^{int}(\tilde{S}_{V,pw} + \tilde{\mathcal{A}}_{V,pw}) + i\eta\gamma_0^{int}(\tilde{V}_0 + \tilde{S}_{V,pw} + \tilde{\mathcal{A}}_{V,pw}),$$

with $\tilde{S}_{V,pw}$ and $\tilde{\mathcal{A}}_{V,pw}$ as in Proposition A.3, see [Mel12, Eq. (6.4)]. Adding and subtracting iV_0 and noting $V_0 = \gamma_0^{int}\tilde{V}_0$ we find

$$\begin{aligned} A'_{k,\eta} &= \frac{1}{2} + K'_0 + iV_0 + \gamma_1^{int}(\tilde{S}_{V,pw} + \tilde{\mathcal{A}}_{V,pw}) + i(\eta-1)\gamma_0^{int}\tilde{V}_0 + i\eta\gamma_0^{int}(\tilde{S}_{V,pw} + \tilde{\mathcal{A}}_{V,pw}) \\ &= A'_{0,1} + \gamma_1^{int}(\tilde{S}_{V,pw} + \tilde{\mathcal{A}}_{V,pw}) + i(\eta-1)\gamma_0^{int}\tilde{V}_0 + i\eta\gamma_0^{int}(\tilde{S}_{V,pw} + \tilde{\mathcal{A}}_{V,pw}). \end{aligned}$$

Using the filters $H_{\Gamma,q}^{neg}$ and $L_{\Gamma,q}^{neg}$ of Proposition A.2 we define

$$(A.16a) \quad \begin{aligned} R_{A'} &= H_{\Gamma,q}^{neg} \left(\gamma_1^{int}\tilde{S}_{V,pw} + i\eta\gamma_0^{int}\tilde{S}_{V,pw} + i(\eta-1)V_0 \right), \\ \tilde{\mathcal{A}}_1 &= -k^{-1}\chi_\Omega \left(i\eta\tilde{\mathcal{A}}_{V,pw} + L_{\Gamma,q}^{neg} \left(\gamma_1^{int}\tilde{S}_{V,pw} + i\eta\gamma_0^{int}\tilde{S}_{V,pw} + i(\eta-1)V_0 \right) \right), \\ \tilde{\mathcal{A}}_2 &= -\chi_\Omega \tilde{\mathcal{A}}_{V,pw}. \end{aligned}$$

The mapping properties of $\tilde{\mathcal{A}}_1$ and $\tilde{\mathcal{A}}_2$ stay the same as in [Mel12, Thm. 2.9]. We are left with the mapping properties of $R_{A'}$. In the following the parameter q appearing in Proposition A.2 and A.3 is still at our disposal¹. We fix it at the end of the proof to ensure the estimates (A.14c) and (A.14d).

Step 1: We estimate the term $i(\eta - 1)H_{\Gamma,q}^{neg}V_0$ in various norms. We heavily use the estimates for $H_{\Gamma,q}^{neg}$ and V_0 given in Proposition A.2 and (A.4) in Proposition A.1. First estimating η , then using the properties of $H_{\Gamma,q}^{neg}$ in Proposition A.2 and finally the mapping properties of V_0 we find

$$\begin{aligned} \|i(\eta - 1)H_{\Gamma,q}^{neg}V_0u\|_{s+1,\Gamma} &\leq Ck\|H_{\Gamma,q}^{neg}V_0u\|_{s+1,\Gamma} \leq Ck\|V_0u\|_{s+1,\Gamma} \leq Ck\|u\|_{s,\Gamma}, \\ \|i(\eta - 1)H_{\Gamma,q}^{neg}V_0u\|_{s,\Gamma} &\leq Ck\|H_{\Gamma,q}^{neg}V_0u\|_{s,\Gamma} \leq Ck\|V_0u\|_{s,\Gamma} \leq Ck\|u\|_{s-1,\Gamma}, \\ \|i(\eta - 1)H_{\Gamma,q}^{neg}V_0u\|_{s,\Gamma} &\leq Ck\|H_{\Gamma,q}^{neg}V_0u\|_{s,\Gamma} \leq Ck(q/k)\|V_0u\|_{s+1,\Gamma} \leq Cq\|u\|_{s,\Gamma}, \\ \|i(\eta - 1)H_{\Gamma,q}^{neg}V_0u\|_{s-1,\Gamma} &\leq Ck\|H_{\Gamma,q}^{neg}V_0u\|_{s-1,\Gamma} \leq Ck(q/k)\|V_0u\|_{s,\Gamma} \leq Cq\|u\|_{s-1,\Gamma}. \end{aligned}$$

In the Steps 2 and 3 below we again heavily use the properties of $H_{\Gamma,q}^{neg}$ given in Proposition A.2. Furthermore, we often apply the results of Proposition A.3, especially Item (i). Below, we will write certain exponents nonsimplified in order to indicate the corresponding choices of Sobolev exponents when applying Proposition A.3.

Step 2: We claim:

$$(A.17) \quad \|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s+1,\Gamma} \leq Cqk\|u\|_{s,\Gamma},$$

$$(A.18) \quad \|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s,\Gamma} \leq Cqk\|u\|_{s-1,\Gamma},$$

$$(A.19) \quad \|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s,\Gamma} \leq Cq^2\|u\|_{s,\Gamma},$$

$$(A.20) \quad \|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s-1,\Gamma} \leq Cq^2\|u\|_{s-1,\Gamma}.$$

To see (A.17), we calculate

$$\begin{aligned} \|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s+1,\Gamma} &\leq C\|\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s+1,\Gamma} \leq C\|\tilde{S}_{V,pw}u\|_{s+5/2,\Omega} \\ &\leq Cq^2(qk^{-1})^{1+(s+1/2)-(s+5/2)}\|u\|_{s,\Gamma} = Cqk\|u\|_{s,\Gamma}. \end{aligned}$$

By a similar calculation, we obtain (A.19):

$$\|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s,\Gamma} \leq Cq/k\|\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s+1,\Gamma} \leq Cq^2\|u\|_{s,\Gamma}.$$

In the case $s \in [0, 1/2)$, we perform a multiplicative trace inequality and find

$$\begin{aligned} \|H_{\Gamma,q}^{neg}\gamma_1^{int}\tilde{S}_{V,pw}u\|_{s-1,\Gamma} &\leq C(q/k)^{-s+1}\|\gamma_1^{int}\tilde{S}_{V,pw}u\|_{0,\Gamma} \\ &\leq C(q/k)^{-s+1}\|\tilde{S}_{V,pw}u\|_{1,\Omega}^{1/2}\|\tilde{S}_{V,pw}u\|_{2,\Omega}^{1/2} \\ &\leq C(q/k)^{-s+1}\left[q^2(qk^{-1})^{1+(s-1/2)-1}\right]^{1/2}\left[q^2(qk^{-1})^{1+(s-1/2)-2}\right]^{1/2}\|u\|_{s-1,\Gamma} \\ &= Cq^2\|u\|_{s-1,\Gamma}. \end{aligned}$$

¹Do not confuse this q with the one appearing in the statement of the present theorem.

In the case $s \geq 1/2$ we perform a standard trace estimate and find

$$\begin{aligned}
\|H_{\Gamma,q}^{neg} \gamma_1^{int} \tilde{S}_{V,pw} u\|_{s-1,\Gamma} &\leq Cq/k \|\gamma_1^{int} \tilde{S}_{V,pw} u\|_{s,\Gamma} \leq Cq/k \|\tilde{S}_{V,pw} u\|_{s+3/2,\Omega} \\
&\leq Cq/k q^2 (qk^{-1})^{1+(s-1/2)-(s+3/2)} \|u\|_{s-1,\Gamma} \\
(A.21) \qquad \qquad \qquad &= Cq^2 \|u\|_{s-1,\Gamma}.
\end{aligned}$$

The two previous estimates show (A.20). Analogously to (A.21), we find (A.18):

$$\|H_{\Gamma,q}^{neg} \gamma_1^{int} \tilde{S}_{V,pw} u\|_{s,\Gamma} \leq C \|\gamma_1^{int} \tilde{S}_{V,pw} u\|_{s,\Gamma} \leq Cqk \|u\|_{s-1,\Gamma}.$$

Step 3: We claim:

$$(A.22) \qquad \qquad \qquad \|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s+1,\Gamma} \leq Cq^2 k \|u\|_{s,\Gamma},$$

$$(A.23) \qquad \qquad \qquad \|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s,\Gamma} \leq Cq^2 k \|u\|_{s-1,\Gamma},$$

$$(A.24) \qquad \qquad \qquad \|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s,\Gamma} \leq Cq^3 \|u\|_{s,\Gamma},$$

$$(A.25) \qquad \qquad \qquad \|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s-1,\Gamma} \leq Cq^3 \|u\|_{s-1,\Gamma}.$$

For (A.22) we estimate

$$\begin{aligned}
\|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s+1,\Gamma} &\leq Ck \|\gamma_0^{int} \tilde{S}_{V,pw} u\|_{s+1,\Gamma} \leq Ck \|\tilde{S}_{V,pw} u\|_{s+3/2,\Omega} \\
&\leq Ck q^2 (qk^{-1})^{1+(s+1/2)-(s+3/2)} \|u\|_{s,\Gamma} = Cq^2 k \|u\|_{s,\Gamma}.
\end{aligned}$$

By a similar calculation, we show (A.23):

$$\|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s,\Gamma} \leq Ckq/k \|\gamma_0^{int} \tilde{S}_{V,pw} u\|_{s+1,\Gamma} \leq Cq^3 \|u\|_{s,\Gamma}.$$

In the case $s \in [0, 1/2)$, we perform a multiplicative trace inequality and find

$$\begin{aligned}
\|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s-1,\Gamma} &\leq Ck(q/k)^{-s+1} \|\gamma_0^{int} \tilde{S}_{V,pw} u\|_{0,\Gamma} \\
&\leq C(q/k)^{-s+1} \|\tilde{S}_{V,pw} u\|_{0,\Omega}^{1/2} \|\tilde{S}_{V,pw} u\|_{1,\Omega}^{1/2} \\
&\leq C(q/k)^{-s+1} \left[q^2 (qk^{-1})^{1+(s-1/2)-0} \right]^{1/2} \left[q^2 (qk^{-1})^{1+(s-1/2)-1} \right]^{1/2} \|u\|_{s-1,\Gamma} \\
&= Cq^3 \|u\|_{s-1,\Gamma}.
\end{aligned}$$

In the case $s \geq 1/2$ we perform a standard trace estimate and find

$$\begin{aligned}
\|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s-1,\Gamma} &\leq Ckq/k \|\gamma_0^{int} \tilde{S}_{V,pw} u\|_{s,\Gamma} \leq Ckq/k \|\tilde{S}_{V,pw} u\|_{s+1/2,\Omega} \\
&\leq Ckq/k q^2 (qk^{-1})^{1+(s-1/2)-(s+1/2)} \|u\|_{s-1,\Gamma} \\
(A.26) \qquad \qquad \qquad &= Cq^3 \|u\|_{s-1,\Gamma}.
\end{aligned}$$

The two previous estimates show (A.25). Analogously to (A.26), we assert (A.24):

$$\|\eta H_{\Gamma,q}^{neg} \gamma_0^{int} \tilde{S}_{V,pw} u\|_{s,\Gamma} \leq k \|\gamma_0^{int} \tilde{S}_{V,pw} u\|_{s,\Gamma} \leq Cq^2 k \|u\|_{s-1,\Gamma}.$$

Step 4: The definition of the operator $R_{A'}$ in (A.16a), the triangle inequality, and appropriate choice of q yields mapping properties of $R_{A'}$ as stated in (A.14). \square

Finally, a simple application of [Mel12, Cor. 7.5] for non-trapping Ω^+ with analytic boundary is the following

Lemma A.5. *Let Ω^+ be non-trapping ([BSW16, Def. 1.1]). Let Γ be analytic, T be a tubular neighborhood of Γ and $C_{g_1}, C_{g_2}, \gamma_g > 0$. Then there exist constants $C, \gamma > 0$ independent of $k \geq k_0$ such that for all $g_1 \in \mathfrak{A}(C_{g_1}, \gamma_g, T)$, $g_2 \in \mathfrak{A}(C_{g_2}, \gamma_g, T)$ the solution $\varphi \in L^2(\Gamma)$ of*

$$A'_{k,\eta}\varphi = k[[g_1]] + [[\partial_n g_2]]$$

satisfies

$$\varphi = -[[\partial_n v]], \quad v \in \mathfrak{A}(Ck^{5/2}(C_{g_1} + C_{g_2}), \gamma, \Omega_R).$$

Proof. We apply [Mel12, Cor. 7.5] with $s_A = 0$. By Proposition A.1(iii) the operator $A'_{\eta,k}: L^2(\Gamma) \rightarrow L^2(\Gamma)$ is boundedly invertible. The result follows immediately from [Mel12, Cor. 7.5] together with the bound (A.2). \square

Proof. Proof of Lemma 5.7(iii):

Step 1: We derive a splitting of $(A'_{k,\eta})^{-1}$, similar to the results of [Mel12, Thm. 2.11]. Fix $\hat{q} \in (0, 1)$. Let

$$q := \hat{q} \min \left\{ 1, \frac{1}{\|(A'_{0,1})^{-1}\|_{H^s(\Gamma) \leftarrow H^s(\Gamma)}}, \frac{1}{\|(A'_{0,1})^{-1}\|_{H^{s-1}(\Gamma) \leftarrow H^{s-1}(\Gamma)}} \right\}.$$

Note that by Proposition A.1 the operator $A'_{0,1}: H^t(\Gamma) \rightarrow H^t(\Gamma)$ is boundedly invertible for $t \geq -1$ and therefore q is well defined and $q \in (0, 1)$. Theorem A.4 applied with this q gives a decomposition

$$A'_{k,\eta} = A'_{0,1} + R + [[\mathcal{A}]],$$

with $R = R_{A'}$ and $\mathcal{A} = k\tilde{\mathcal{A}}_1 + \partial_n \tilde{\mathcal{A}}_2$, as in Theorem A.4. Note that by construction

$$(A.27) \quad \|(A'_{0,1})^{-1}R\|_{H^s(\Gamma) \leftarrow H^s(\Gamma)} \leq \hat{q} \quad \text{and} \quad \|(A'_{0,1})^{-1}R\|_{H^{s-1}(\Gamma) \leftarrow H^{s-1}(\Gamma)} \leq \hat{q}.$$

Hence, $A'_{0,1} + R$ is boundedly invertible by a geometric series argument, since

$$(A.28) \quad (A'_{0,1} + R)^{-1} = (I + (A'_{0,1})^{-1}R)^{-1}(A'_{0,1})^{-1}$$

with norm estimates

$$(A.29a) \quad \|(A'_{0,1} + R)^{-1}\|_{H^s(\Gamma) \leftarrow H^s(\Gamma)} \leq (1 - \hat{q})^{-1} \|(A'_{0,1})^{-1}\|_{H^s(\Gamma) \leftarrow H^s(\Gamma)},$$

$$(A.29b) \quad \|(A'_{0,1} + R)^{-1}\|_{H^{s-1}(\Gamma) \leftarrow H^{s-1}(\Gamma)} \leq (1 - \hat{q})^{-1} \|(A'_{0,1})^{-1}\|_{H^{s-1}(\Gamma) \leftarrow H^{s-1}(\Gamma)}.$$

By Proposition A.1 the operator $A'_{k,\eta}: H^t(\Gamma) \rightarrow H^t(\Gamma)$ is boundedly invertible for $t \geq -1$. We may decompose $(A'_{k,\eta})^{-1}$ as follows

$$(A.30) \quad (A'_{k,\eta})^{-1} = (A'_{0,1} + R)^{-1} + Q, \quad Q = -(A'_{k,\eta})^{-1}[[\mathcal{A}]](A'_{0,1} + R)^{-1},$$

since

$$\begin{aligned} I &= (A'_{k,\eta})(A'_{k,\eta})^{-1} = (A'_{k,\eta})(A'_{0,1} + R)^{-1} + (A'_{k,\eta})Q \\ &= (A'_{0,1} + R + [[\mathcal{A}]])^{-1}(A'_{0,1} + R)^{-1} + (A'_{k,\eta})Q \\ &= I + [[\mathcal{A}]](A'_{0,1} + R)^{-1} + (A'_{k,\eta})Q. \end{aligned}$$

Step 2 We rewrite the difference $\text{DtN}_k - \text{DtN}_0$ using the combined field equations. Using the combined field equations (A.1) with for η as in (A.10) (for DtN_k) and $\eta = 1$ (for DtN_0) we find

$$(A.31) \quad \text{DtN}_k - \text{DtN}_0 = (A'_{k,\eta})^{-1}[-D_k + i\eta(-1/2 + K_k)] - (A'_{0,1})^{-1}[-D_0 + i(-1/2 + K_0)].$$

Adding and subtracting D_0 and K_0 in (A.31), employing the splitting of $D_k - D_0$ and $K_k - K_0$ given in (A.4) in Proposition A.1, and applying the splitting of $(A'_{k,1})^{-1}$ in (A.30) we find

$$\begin{aligned}
\text{DtN}_k - \text{DtN}_0 &= -(A'_{k,\eta})^{-1} [D_k - D_0] - (A'_{k,\eta})^{-1} D_0 \\
&\quad + i\eta(A'_{k,\eta})^{-1} [K_k - K_0] + i\eta(A'_{k,\eta})^{-1} [-1/2 + K_0] \\
&\quad + (A'_{0,1})^{-1} D_0 - i(A'_{0,1})^{-1} D_0 [-1/2 + K_0] \\
&= (A'_{0,1})^{-1} D_0 - (A'_{0,1} + R)^{-1} D_0 - QD_0 \\
&\quad - (A'_{0,1} + R)^{-1} S_D - QS_D - (A'_{k,\eta})^{-1} \gamma_1^{\text{int}} \tilde{A}_K \\
&\quad + i\eta(A'_{0,1} + R)^{-1} S_K + i\eta QS_K + i\eta(A'_{k,\eta})^{-1} \gamma_0^{\text{int}} \tilde{A}_K \\
&\quad + i\eta(A'_{0,1} + R)^{-1} [-1/2 + K_0] + i\eta Q [-1/2 + K_0] \\
&\quad - i(A'_{0,1})^{-1} [-1/2 + K_0] \\
&= \text{FSO} + \text{ASO}
\end{aligned}$$

with the Finite Shift Operators (FSO) and the Analytic Shift Operators (ASO) given by

$$\begin{aligned}
\text{FSO} &:= (A'_{0,1})^{-1} D_0 - (A'_{0,1} + R)^{-1} D_0 - (A'_{0,1} + R)^{-1} S_D + i\eta(A'_{0,1} + R)^{-1} S_K \\
&\quad + i\eta(A'_{0,1} + R)^{-1} [-1/2 + K_0] - i(A'_{0,1})^{-1} [-1/2 + K_0], \\
\text{ASO} &:= -QD_0 - QS_D - (A'_{k,\eta})^{-1} \gamma_1^{\text{int}} \tilde{A}_K \\
&\quad + i\eta QS_K + i\eta(A'_{k,\eta})^{-1} \gamma_0^{\text{int}} \tilde{A}_K + i\eta Q [-1/2 + K_0].
\end{aligned}$$

Step 3 (Analysis of FSO): We claim

$$(A.32) \quad \text{FSO} = kB,$$

where $B : H^s(\Gamma) \rightarrow H^s(\Gamma)$ is a bounded linear operator with $\|Bu\|_{s,\Gamma} \lesssim \|u\|_{s,\Gamma}$, as in the asserted of the present lemma. Using the mapping properties of $(A'_{0,1} + R)^{-1}$ in (A.29a) as well as (A.3), (A.8) and Proposition A.1(ii) we find

$$\begin{aligned}
\|(A'_{0,1} + R)^{-1} S_D u\|_{s,\Gamma} &\lesssim \|S_D u\|_{s,\Gamma} \lesssim k \|u\|_{s,\Gamma}, \\
k \|(A'_{0,1} + R)^{-1} S_K u\|_{s,\Gamma} &\lesssim k \|S_K u\|_{s,\Gamma} \lesssim k \|u\|_{s,\Gamma}, \\
k \|(A'_{0,1} + R)^{-1} [-1/2 + K_0] u\|_{s,\Gamma} &\lesssim k \|[-1/2 + K_0] u\|_{s,\Gamma} \lesssim k \|u\|_{s,\Gamma}, \\
\|(A'_{0,1})^{-1} [-1/2 + K_0] u\|_{s,\Gamma} &\lesssim \|[-1/2 + K_0] u\|_{s,\Gamma} \lesssim \|u\|_{s,\Gamma}.
\end{aligned}$$

The assertion (A.32) therefore follows once we have shown

$$\|(A'_{0,1})^{-1} D_0 - (A'_{0,1} + R)^{-1} D_0 u\|_{s,\Gamma} \lesssim k \|u\|_{s,\Gamma}.$$

To that end, we find using (A.28)

$$\begin{aligned}
(A'_{0,1})^{-1} D_0 - (A'_{0,1} + R)^{-1} D_0 &= (A'_{0,1})^{-1} D_0 - (I + (A'_{0,1})^{-1} R)^{-1} (A'_{0,1})^{-1} D_0 \\
&= [I - (I + (A'_{0,1})^{-1} R)^{-1}] (A'_{0,1})^{-1} D_0 = - \left[\sum_{n=1}^{\infty} (-1)^n ((A'_{0,1})^{-1} R)^n \right] (A'_{0,1})^{-1} D_0.
\end{aligned}$$

Applying the previous calculations, a geometric series argument with (A.27), the mapping properties of $(A'_{0,1})^{-1}$ in Proposition A.1(ii), the estimate $\|Ru\|_{s,\Gamma} \lesssim k \|u\|_{s-1,\Gamma}$ given by

Theorem A.4, again the mapping properties of $(A'_{0,1})^{-1}$ and finally the mapping properties of D_0 given in (A.3) in Proposition A.1 we find

$$\begin{aligned} & \| (A'_{0,1})^{-1} D_0 - (A'_{0,1} + R)^{-1} D_0 u \|_{s,\Gamma} \\ &= \left\| \left[\sum_{n=1}^{\infty} (-1)^n ((A'_{0,1})^{-1} R)^{n-1} \right] ((A'_{0,1})^{-1} R) (A'_{0,1})^{-1} D_0 u \right\|_{s,\Gamma} \\ &\leq \frac{1}{1-\hat{q}} \| (A'_{0,1})^{-1} R (A'_{0,1})^{-1} D_0 u \|_{s,\Gamma} \lesssim \| R (A'_{0,1})^{-1} D_0 u \|_{s,\Gamma} \\ &\lesssim k \| (A'_{0,1})^{-1} D_0 u \|_{s-1,\Gamma} \lesssim k \| D_0 u \|_{s-1,\Gamma} \lesssim k \| u \|_{s,\Gamma}. \end{aligned}$$

Hence, the assertion in (A.32) follows, which concludes the analysis of the finite shift operators FSO.

Step 4 (Analysis of Analytic Shift Operators ASO): We have

$$\text{ASO} = -QD_0 - Q[S_D - i\eta S_K - i\eta[-1/2 + K_0]] + (A'_{k,\eta})^{-1} [i\eta\gamma_0^{\text{int}} \tilde{A}_K - \gamma_1^{\text{int}} \tilde{A}_K].$$

Step 4a (Analysis of $-QD_0$): With the definition of Q in (A.30) we have for $f \in H^s(\Gamma)$

$$\begin{aligned} -QD_0 f &= (A'_{k,\eta})^{-1} \llbracket \mathcal{A} \rrbracket (A'_{0,1} + R)^{-1} D_0 f \\ &= (A'_{k,\eta})^{-1} [k \llbracket \mathcal{A}_1 \rrbracket (A'_{0,1} + R)^{-1} D_0 f + \llbracket \partial_n \mathcal{A}_2 \rrbracket (A'_{0,1} + R)^{-1} D_0 f]. \end{aligned}$$

In order to apply Lemma A.5, we use the mapping properties of \mathcal{A}_1 and \mathcal{A}_2 given in Theorem A.4 and estimate

$$\begin{aligned} & k \| (A'_{0,1} + R)^{-1} D_0 f \|_{-3/2,\Gamma} + k^{d/2} \| (A'_{0,1} + R)^{-1} D_0 f \|_{-1,\Gamma} \\ & \lesssim k^{d/2} \| (A'_{0,1} + R)^{-1} D_0 f \|_{s-1,\Gamma} \lesssim k^{d/2} \| D_0 f \|_{s-1,\Gamma} \lesssim k^{d/2} \| f \|_{s,\Gamma}, \end{aligned}$$

where we used the trivial embedding $H^{s-1}(\Gamma) \subset H^{-1}(\Gamma) \subset H^{-3/2}(\Gamma)$, the fact that $k + k^{d/2} \lesssim k^{d/2}$, the mapping property (A.29b), and finally the mapping properties of D_0 given in (A.3). Hence, for the tubular neighborhood T given in Theorem A.4 we find

$$\begin{aligned} \mathcal{A}_1 (A'_{0,1} + R)^{-1} D_0 f &\in \mathfrak{A}(C_1 k^{d/2} \| f \|_{s,\Gamma}, \gamma_1, T), \\ \mathcal{A}_2 (A'_{0,1} + R)^{-1} D_0 f &\in \mathfrak{A}(C_1 k \| f \|_{s,\Gamma}, \gamma_1, T), \end{aligned}$$

for constants $C_1, \gamma_1 > 0$ independent of k . We find Lemma A.5 to be applicable, which yields the representation, for constants $\tilde{C}_1, \tilde{\gamma}_1 > 0$ independent of k ,

$$(A.33) \quad -QD_0 f = \llbracket \partial_n v_f^1 \rrbracket, \quad v_f^1 \in \mathfrak{A}(\tilde{C}_1 k^{5/2+d/2} \| f \|_{s,\Gamma}, \tilde{\gamma}_1, \Omega_R).$$

Step 4b (analysis of $-Q[S_D - i\eta S_K - i\eta[-1/2 + K_0]]$): We estimate

$$\begin{aligned} & k^{d/2} \| (A'_{0,1} + R)^{-1} [S_D - i\eta S_K - i\eta[-1/2 + K_0]] f \|_{-1,\Gamma} \\ & \lesssim k^{d/2} \| (A'_{0,1} + R)^{-1} [S_D - i\eta S_K - i\eta[-1/2 + K_0]] f \|_{s,\Gamma} \\ & \lesssim k^{d/2} \| [S_D - i\eta S_K - i\eta[-1/2 + K_0]] f \|_{s,\Gamma} \\ & \lesssim k^{d/2+1} \| f \|_{s,\Gamma}, \end{aligned}$$

where we first use the trivial embedding $H^s(\Gamma) \subset H^{-1}(\Gamma)$, the mapping property (A.29a), the mapping properties of S_D, S_K , and K_0 given in (A.8) and (A.3) as well as $|\eta| \lesssim k$. Proceeding as in Step 4a we find the representation

$$(A.34) \quad -Q[S_D - i\eta S_K - i\eta[-1/2 + K_0]] = \llbracket \partial_n v_f^2 \rrbracket, \quad v_f^2 \in \mathfrak{A}(\tilde{C}_2 k^{5/2+d/2+1} \| f \|_{s,\Gamma}, \tilde{\gamma}_2, \Omega_R)$$

to hold true, for constants $\tilde{C}_2, \tilde{\gamma}_2 > 0$ independent of k .

Step 4c (analysis of $(A'_{k,\eta})^{-1}[i\eta\gamma_0^{int}\tilde{A}_K - \gamma_1^{int}\tilde{A}_K]$): For $f \in H^s(\Gamma)$ the mapping properties of \tilde{A}_K are such that $\tilde{A}_K f \in \mathfrak{A}(C_K k \|f\|_{-1/2,\Gamma}, \gamma_K, \Omega)$, see (A.7). Upon extending $\tilde{A}_K f$ with zero outside of Ω , we find Lemma A.5 to be applicable, which yields the representation

$$(A.35) \quad (A'_{k,\eta})^{-1}[i\eta\gamma_0^{int}\tilde{A}_K - \gamma_1^{int}\tilde{A}_K]f = \llbracket \partial_n v_f^3 \rrbracket, \quad v_f^3 \in \mathfrak{A}(\tilde{C}_3 k^{5/2+1} \|f\|_{-1/2,\Gamma}, \tilde{\gamma}_3, \Omega_R),$$

with constants $\tilde{C}_3, \tilde{\gamma}_3 > 0$ independent of k .

Step 5: Collecting the representations derived in (A.33), (A.34), and (A.35), we find

$$\text{ASO} = \llbracket \partial_n \tilde{A} \rrbracket, \quad \tilde{A}u \in \mathfrak{A}(Ck^{7/2+d/2} \|u\|_{s,\Gamma}, \gamma, \Omega_R)$$

with \tilde{A} as in the assertions of the present lemma. Hence, the splitting

$$\text{DtN}_k - \text{DtN}_0 = kB + \llbracket \partial_n \tilde{A} \rrbracket$$

with B and \tilde{A} as asserted, holds true. This concludes the proof. \square

APPENDIX B. ANALYTIC REGULARITY FOR SECOND ORDER ABSORBING BOUNDARY CONDITIONS

The present section develops similar results as [Mel02, Sec. 5.5] for a model problem with second order boundary conditions. We start by introducing general notation: For $d \geq 2$ and $R > 0$ let $B_R \subset \mathbb{R}^d$ denote the ball of radius R with center in the origin. Let $B_R^+ \subset \mathbb{R}^d$ be a half ball with radius R , i.e., $B_R^+ = \{x \in B_R : x_d > 0\}$. Furthermore, let $\Gamma_R := \{x \in B_R : x_d = 0\}$. We consider functions u that satisfy

$$(B.1) \quad \begin{aligned} -\nabla \cdot (A\nabla u) &= f && \text{in } B_R^+, \\ \partial_{n_A} u + \alpha \nabla_\Gamma \cdot (A_\Gamma \nabla_\Gamma u) &= \alpha^{1/2} g + G && \text{on } \Gamma_R, \end{aligned}$$

with the conormal derivative $\partial_{n_A} u = \mathbf{n} \cdot (A\nabla u)$. We assume that solutions u of (B.1) satisfy

$$(B.2) \quad \begin{aligned} a(u, v) &:= \int_{B_R^+} (A\nabla u) \cdot \nabla \bar{v} + \alpha \int_{\Gamma_R} (A_\Gamma \nabla_\Gamma u) \cdot \nabla_\Gamma \bar{v} \\ &= \int_{B_R^+} f \bar{v} + \int_{\Gamma_R} (\alpha^{1/2} g + G) \bar{v} \quad \forall v \in C_0^\infty(B_R^+). \end{aligned}$$

To describe the parameter α , it is convenient to introduce, for fixed $\theta > 0$, the sector

$$\mathcal{S}_\theta := \{\alpha \in \mathbb{C} \mid |\arg \alpha| \leq \pi - \theta\}.$$

B.1. The shift theorem in tangential direction. The proof of Lemma B.2 uses the well-established difference quotients method of Nirenberg that can be found, e.g., in [Eva98, Sec. 6.3]. For $j \in \{1, \dots, d-1\}$, the j -th unit vector $e_j \in \mathbb{R}^d$, and $h \in \mathbb{R} \setminus \{0\}$ we introduce the translation operator τ_j^h by $(\tau_j^h v)(x) = v(x + he_j)$ and the difference quotient $(D_j^h v)(x) = h^{-1}(v(x + he_j) - v(x))$. Inspection of the proof of [Eva98, Sec. 5.8.2, Thm. 3] shows that for fixed $0 < r_1 < r_2$

$$(B.3) \quad \partial_{x_j} v \in L^2(B_{r_2}) \implies \|D_j^h v\|_{L^2(B_{r_1})} \leq \|\partial_{x_j} v\|_{L^2(B_{r_2})} \quad \forall |h| < r_2 - r_1,$$

as well as

$$(B.4) \quad \|D_j^h v\|_{L^2(B_{r_1})} \leq C_v \quad \forall |h| \leq r_2 - r_1 \implies \|\partial_{x_j} v\|_{L^2(B_{r_1})} \leq C_v.$$

Lemma B.1. *Let $G, u \in H^1(B_R^+)$. Let $r, \delta > 0$ with $r + \delta < R$ be given. Let $\chi \in C_0^\infty(\mathbb{R}^d; \mathbb{R})$ be a cut-off function with $\text{supp } \chi \subset B_{r+\delta/2}$ and $\chi \equiv 1$ on B_r . Then there exists a constant $C > 0$ depending only on the spatial dimension, such that*

$$\left| \int_{\Gamma_R} GD_{x_j}^{-h}(\chi D_{x_j}^h u) \right| \leq C \left(\delta^{-1} \|G\|_{L^2(B_{r+\delta}^+)} + \|\nabla G\|_{L^2(B_{r+\delta}^+)} \right) \left(\delta^{-1} \|\nabla u\|_{L^2(B_{r+\delta}^+)} + \|\chi \nabla D_{x_j}^h u\|_{L^2(B_{r+\delta}^+)} \right).$$

Proof. Let χ' be another cut-off function with $\text{supp } \chi' \subset B_{r+\delta}$, $\chi' \equiv 1$ on $\text{supp } \chi$ and $\|\nabla \chi'\|_{L^\infty} \leq C\delta^{-1}$. Then, for h sufficiently small (depending only on r, δ)

$$\int_{\Gamma_R} GD_j^{-h}(\chi D_j^h u) = \int_{\Gamma_R} \chi' GD_j^{-h}(\chi D_j^h u).$$

Let $v = D_j^{-h}(\chi D_j^h u)$. Scaling B_R^+ to a half ball of radius 1, we denote by \widehat{G} the scaled version of $\chi'G$ and by \widehat{v} the scaled version of v . It is also convenient to define ψ as the solution of the Neumann problem $-\Delta \psi = 0$ in B_1^+ and $\partial_n \psi = \widehat{v}$ on ∂B_1^+ . Note that $\widehat{v} \equiv 0$ on $\partial B_1^+ \setminus \Gamma_1$. Furthermore, $\int_{\Gamma_1} \widehat{v} = 0$ and therefore the solvability condition for the above Neumann problem is satisfied. We then have

$$\begin{aligned} R^{-(d-1)} \int_{\Gamma_R} \chi' G v &= \int_{\partial B_1^+} \widehat{G} \widehat{v} = \int_{\partial B_1^+} \widehat{G} \partial_n \psi = \int_{B_1^+} \nabla \widehat{G} \cdot \nabla \psi \\ &\leq \|\nabla \widehat{G}\|_{L^2(B_1^+)} \|\nabla \psi\|_{L^2(B_1^+)} \\ &\leq R^{(1-d)/2} \|\nabla(\chi' G)\|_{L^2(B_R^+)} \|\widehat{v}\|_{H^{-1/2}(\partial B_1^+)}. \end{aligned}$$

Since $\widehat{v} \equiv 0$ on $\partial B_1^+ \setminus \Gamma_1$, we estimate $\|\widehat{v}\|_{H^{-1/2}(\partial B_1^+)} \leq C \|\widehat{v}\|_{H^{-1/2}(\Gamma_1)}$. To estimate this last norm, we write $w = \chi D_j^h u$ and denote by \widehat{w} the corresponding scaled function. We observe for $\varphi \in C_0^\infty(\Gamma_1)$ and h sufficiently small (depending only on δ, r, R)

$$\begin{aligned} \left| \int_{\Gamma_1} (D_j^{-h/R} \widehat{w}) \varphi \right| &= \left| \int_{\Gamma_1} \widehat{w} D_j^{h/R} \varphi \right| \\ &\leq \|\widehat{w}\|_{L^2(\Gamma_1)} \|D_j^{h/R} \varphi\|_{L^2(\Gamma_2)} \\ &\stackrel{\text{(B.3)}}{\leq} \|\widehat{w}\|_{L^2(\Gamma_1)} \|\partial_{x_j} \varphi\|_{L^2(\Gamma_1)} \\ &\leq \|\widehat{w}\|_{L^2(\Gamma_1)} \|\varphi\|_{H^1(\Gamma_1)}. \end{aligned}$$

Hence, $\|D_j^{-h/R} \widehat{w}\|_{H^{-1}(\Gamma_1)} \leq C \|\widehat{w}\|_{L^2(\Gamma_1)}$ uniformly in h . Similarly, (B.3) shows

$$\|D_{x_j}^{-h/R} \widehat{w}\|_{L^2(\Gamma_1)} \leq C \|\partial_{x_j} \widehat{w}\|_{L^2(\Gamma_1)} \leq C \|\widehat{w}\|_{H^1(\Gamma_1)}$$

uniformly in h . By interpolation, we arrive at

$$\|D_{x_j}^{-h/R} \widehat{w}\|_{H^{-1/2}(\Gamma_1)} \leq C \|\widehat{w}\|_{H^{1/2}(\Gamma_1)} \leq C \|\widehat{w}\|_{H^1(B_1^+)} \leq C \|\nabla \widehat{w}\|_{L^2(B_1^+)},$$

where the penultimate estimate follows from the trace inequality and the last one by a Poincaré inequality, which is applicable due to the support properties of \widehat{w} . In conclusion, we arrive at

$$\begin{aligned} \left| \int_{\Gamma_R} GD_{x_j}^{-h}(\chi D_{x_j}^h u) \right| &\leq C \|\nabla(\chi' G)\|_{L^2(B_R^+)} \|\nabla(\chi D_{x_j}^h u)\|_{L^2(B_R^+)} \\ &\leq C \left(\delta^{-1} \|G\|_{L^2(B_{r+\delta}^+)} + \|\nabla G\|_{L^2(B_{r+\delta}^+)} \right) \left(\delta^{-1} \|\nabla u\|_{L^2(B_{r+\delta}^+)} + \|\chi \nabla D_{x_j}^h u\|_{L^2(B_{r+\delta}^+)} \right). \end{aligned}$$

□

Lemma B.2. *Let $A \in C^1(\overline{B_R^+})$, $A_\Gamma \in C^1(\overline{\Gamma_R^+})$ be matrix-valued functions that are pointwise symmetric positive definite with lower bound on the eigenvalues $\lambda_{\min} > 0$. Let $\alpha \in \mathcal{S}_\theta$. Let $f \in L^2(B_R^+)$, $g \in L^2(\Gamma_R)$ and $G \in H^1(B_R^+)$. Then there exists $C_{\text{stab}} > 0$ depending only on θ , a lower bound on λ_{\min} , and an upper bound on $\|A\|_{L^\infty} + R\|\nabla A\|_{L^\infty} + \|A_\Gamma\|_{L^\infty} + R\|\nabla A_\Gamma\|_{L^\infty}$ such that any solution u of (B.1) satisfies for all $r, \delta > 0$ with $r + \delta < R$*

$$(B.5) \quad \begin{aligned} &\|\nabla^2 u\|_{L^2(B_r)} + |\alpha|^{1/2} \|\nabla_\Gamma^2 u\|_{L^2(\Gamma_r)} \leq \\ &C_{\text{stab}} \left(\|f\|_{L^2(B_{r+\delta}^+)} + \|g\|_{L^2(\Gamma_{r+\delta})} + \delta^{-1} \|G\|_{L^2(B_{r+\delta}^+)} + \|\nabla G\|_{L^2(B_{r+\delta}^+)} \right. \\ &\quad \left. + \delta^{-1} \|\nabla u\|_{L^2(B_{r+\delta}^+)} + |\alpha|^{1/2} \delta^{-1} \|\nabla_\Gamma u\|_{L^2(\Gamma_{r+\delta})} \right). \end{aligned}$$

Proof. Step 1: A calculation reveals that $\alpha \in \mathcal{S}_\theta$ implies the existence of $c_{\text{coer}} > 0$ such that

$$(B.6) \quad |y_1 + \alpha y_2| \geq c_{\text{coer}}(y_1 + |\alpha| y_2) \quad \forall y_1, y_2 \geq 0.$$

Step 2: Let $\chi \in C_0^\infty(\mathbb{R}^d; \mathbb{R})$ be a cut-off function with $\text{supp } \chi \subset B_{r+\delta/2}$ and $\chi \equiv 1$ on B_r . We assume furthermore $\|\nabla \chi\|_{L^\infty} \leq C\delta^{-1}$, and $\partial_{n_A} \chi = 0$ on Γ_R , see [Mel02, Lemma 5.5.21] for a similar construction. For h sufficiently small, we select the test function $v = -D_j^{-h} \chi^2 D_j^h u$ in (B.2) and get

$$(B.7) \quad a(u, -D_j^{-h} \chi^2 D_j^h u) = - \int_{B_R^+} f D_j^{-h} \chi^2 D_j^h \bar{u} - \int_{\Gamma_R} (\alpha^{1/2} g + G) D_j^{-h} \chi^2 D_j^h \bar{u}.$$

We treat the left-hand and the right-hand side separately. We proceed as in the proof of [Eva98, Sec. 6.3, Thm. 1]. We have

$$\begin{aligned} - \int_{B_R^+} (A \nabla u) \cdot \nabla (D_j^{-h} \chi^2 D_j^h \bar{u}) &= \int_{B_R^+} D_j^h (A \nabla u) \cdot \nabla (\chi^2 D_j^h \bar{u}) \\ &= \int_{B_R^+} ((\tau_j^h A)(D_j^h \nabla u) + (D_j^h A) \nabla u) \cdot \nabla (\chi^2 D_j^h \bar{u}) \\ &= \int_{B_R^+} ((\tau_j^h A)(D_j^h \nabla u) + (D_j^h A) \nabla u) \cdot (\chi^2 \nabla D_j^h \bar{u} + 2\chi \nabla \chi D_j^h \bar{u}) \\ &= \int_{B_R^+} \chi^2 ((\tau_j^h A) \nabla D_j^h u) \cdot \nabla D_j^h \bar{u} + R_{\text{vol}} \end{aligned}$$

with R_{vol} given by

$$R_{\text{vol}} = \int_{B_R^+} (\tau_j^h A)(D_j^h \nabla u) 2\chi \nabla \chi D_j^h \bar{u} + (D_j^h A) \nabla u \cdot (\chi^2 \nabla D_j^h \bar{u} + 2\chi \nabla \chi D_j^h \bar{u}).$$

Hence, we can estimate

$$\begin{aligned} |R_{\text{vol}}| &\leq C\delta^{-1}\|A\|_{L^\infty}\|\chi\nabla D_j^h u\|_{L^2(B_R^+)}\|D_j^h u\|_{L^2(B_{r+\delta/2}^+)} \\ &\quad + C\|\nabla A\|_{L^\infty}\|\nabla u\|_{L^2(B_{r+\delta/2}^+)}\|\chi\nabla D_j^h u\|_{L^2(B_R^+)} \\ &\quad + C\delta^{-1}\|\nabla A\|_{L^\infty}\|\nabla u\|_{L^2(B_{r+\delta/2}^+)}\|D_j^h u\|_{L^2(B_{r+\delta/2}^+)}. \end{aligned}$$

Analogously, we get

$$-\int_{\Gamma_R} (A_\Gamma \nabla_\Gamma) \cdot \nabla_\Gamma D_j^{-h} \chi^2 D_j^h \bar{u} = \int_{\Gamma_R} \chi^2 ((\tau_j^h A_\Gamma) \nabla_\Gamma D_j^h u) \cdot \nabla_\Gamma D_j^h u + R_{\text{bnd}}$$

with

$$\begin{aligned} |R_{\text{bnd}}| &\leq C\delta^{-1}\|A_\Gamma\|_{L^\infty}\|\chi\nabla_\Gamma D_j^h u\|_{L^2(\Gamma_R)}\|D_j^h u\|_{L^2(\Gamma_{r+\delta/2})} \\ &\quad + C\|\nabla A_\Gamma\|_{L^\infty}\|\nabla_\Gamma u\|_{L^2(\Gamma_{r+\delta})}\|\chi\nabla_\Gamma D_j^h u\|_{L^2(\Gamma_R)} \\ &\quad + C\delta^{-1}\|\nabla A_\Gamma\|_{L^\infty}\|\nabla_\Gamma u\|_{L^2(\Gamma_{r+\delta/2})}\|D_j^h u\|_{L^2(\Gamma_{r+\delta/2})}. \end{aligned}$$

For the right-hand side of (B.7), we get with (B.3) applied to $v = -D_j^{-h} \chi^2 D_j^h u$ and Lemma B.1

$$\begin{aligned} \left| \int_{B_R^+} f D_j^{-h} \chi^2 D_j^h \bar{u} \right| &\leq C\|f\|_{L^2(B_{r+\delta/2}^+)} \left(\delta^{-1}\|D_j^h u\|_{L^2(B_{r+\delta/2}^+)} + \|\chi\nabla D_j^h u\|_{L^2(B_R^+)} \right), \\ \left| \int_{\Gamma_R} g D_j^{-h} \chi^2 D_j^h \bar{u} \right| &\leq C\|g\|_{L^2(\Gamma_{r+\delta/2})} \left(\delta^{-1}\|D_j^h u\|_{L^2(\Gamma_{r+\delta/2})} + \|\chi\nabla_\Gamma D_j^h u\|_{L^2(\Gamma_R)} \right), \\ \left| \int_{\Gamma_R} G D_j^{-h} \chi^2 D_j^h \bar{u} \right| &= \left| \int_{\Gamma_R} D_j^h G \chi^2 D_j^h \bar{u} \right| \\ &\leq C \left(\delta^{-1}\|G\|_{L^2(B_{r+\delta}^+)} + \|\nabla G\|_{L^2(B_{r+\delta}^+)} \right) \left(\delta^{-1}\|\nabla u\|_{L^2(B_{r+\delta}^+)} + \|\chi\nabla D_j^h u\|_{L^2(B_{r+\delta}^+)} \right). \end{aligned}$$

Step 3: Using (B.6) we get

$$\begin{aligned} c_{\text{coer}} &\left(\int_{B_R^+} \chi^2 ((\tau_j^h A) \nabla D_j^h u) \cdot \nabla D_j^h \bar{u} + |\alpha| \int_{\Gamma_R} \chi^2 ((\tau_j^h A) \nabla_\Gamma D_j^h u) \cdot \nabla_\Gamma D_j^h \bar{u} \right) \\ &\leq |a(u, -D_j^{-h} \chi^2 D_j^h u) - R_{\text{vol}} - \alpha R_{\text{bnd}}| \\ &\leq \left| \int_{B_R^+} f D_j^{-h} \chi^2 D_j^h \bar{u} \right| + \left| \int_{\Gamma_R} (\alpha^{1/2} g + G) D_j^{-h} \chi^2 D_j^h \bar{u} \right| + |R_{\text{vol}}| + |\alpha| |R_{\text{bnd}}|. \end{aligned}$$

Using the lower bound for A and A_Γ and the above estimates together with (B.3) we find

$$\begin{aligned}
& c_{\text{coer}} \lambda_{\min} \left(\|\chi \nabla D_j^h u\|_{L^2(B_r^+)}^2 + |\alpha| \|\chi \nabla_\Gamma D_j^h u\|_{L^2(\Gamma_R)}^2 \right) \leq \\
& C \left(\|f\|_{L^2(B_{r+\delta}^+)} \delta^{-1} \|\partial_{x_j} u\|_{L^2(B_{r+\delta})} + \|f\|_{L^2(B_{r+\delta}^+)} \|\chi \nabla D_j^h u\|_{L^2(B_r^+)} \right. \\
& \quad + \|g\|_{L^2(B_{r+\delta}^+)} \delta^{-1} |\alpha|^{1/2} \|\partial_{x_j} u\|_{L^2(\Gamma_{r+\delta})} + \|g\|_{L^2(\Gamma_{r+\delta})} |\alpha|^{1/2} \|\chi \nabla_\Gamma D_j^h u\|_{L^2(\Gamma_R)} \\
& \quad + \left(\delta^{-1} \|G\|_{L^2(B_{r+\delta}^+)} + \|\nabla G\|_{L^2(B_{r+\delta}^+)} \right) \left(\delta^{-1} \|\nabla u\|_{L^2(B_{r+\delta}^+)} + \|\chi \nabla D_j^h u\|_{L^2(B_{r+\delta}^+)} \right) \\
& \quad + (\|A\|_{L^\infty} + \delta \|\nabla A\|_{L^\infty}) \delta^{-1} \|\nabla u\|_{L^2(B_{r+\delta}^+)} \|\chi \nabla D_j^h u\|_{L^2(B_r^+)} \\
& \quad + \delta \|\nabla A\|_{L^\infty} \delta^{-2} \|\nabla u\|_{L^2(B_{r+\delta}^+)}^2 \\
& \quad + (\|A_\Gamma\|_{L^\infty} + \delta \|\nabla A_\Gamma\|_{L^\infty}) |\alpha| \delta^{-1} \|\nabla_\Gamma u\|_{L^2(\Gamma_{r+\delta})} \|\chi \nabla_\Gamma D_j^h u\|_{L^2(\Gamma_R)} \\
& \quad \left. + \delta \|\nabla A_\Gamma\|_{L^\infty} |\alpha| \delta^{-2} \|\nabla_\Gamma u\|_{L^2(\Gamma_{r+\delta})}^2 \right).
\end{aligned}$$

The Cauchy-Schwarz inequality with epsilon allows us to absorb the terms $\|\chi \nabla D_j^h u\|_{L^2(B_r^+)}$ and $\|\chi \nabla_\Gamma D_j^h u\|_{L^2(\Gamma_R)}$ from the right-hand side into the left-hand side and we get

$$\begin{aligned}
& \frac{1}{2} c_{\text{coer}} \lambda_{\min} \left(\|\chi \nabla D_j^h u\|_{L^2(B_r^+)}^2 + |\alpha| \|\chi \nabla_\Gamma D_j^h u\|_{L^2(\Gamma_R)}^2 \right) \leq \\
& C \left(\|f\|_{L^2(B_{r+\delta}^+)} \delta^{-1} \|\partial_{x_j} u\|_{L^2(B_{r+\delta})} + \lambda_{\min}^{-1} \|f\|_{L^2(B_{r+\delta}^+)}^2 \right. \\
& \quad + \|g\|_{L^2(B_{r+\delta}^+)} \delta^{-1} |\alpha|^{1/2} \|\partial_{x_j} u\|_{L^2(\Gamma_{r+\delta})} + \lambda_{\min}^{-1} \|g\|_{L^2(\Gamma_{r+\delta})}^2 \\
& \quad + \left(\delta^{-1} \|G\|_{L^2(B_{r+\delta}^+)} + \|\nabla G\|_{L^2(B_{r+\delta}^+)} \right) \delta^{-1} \|\nabla u\|_{L^2(B_{r+\delta}^+)} \\
& \quad + R^2 \lambda_{\min}^{-1} \left(\delta^{-1} \|G\|_{L^2(B_{r+\delta}^+)} + \|\nabla G\|_{L^2(B_{r+\delta}^+)} \right)^2 \\
& \quad + \lambda_{\min}^{-1} (\|A\|_{L^\infty} + \delta \|\nabla A\|_{L^\infty})^2 \delta^{-2} \|\nabla u\|_{L^2(B_{r+\delta}^+)}^2 \\
& \quad + \delta \|\nabla A\|_{L^\infty} \delta^{-2} \|\nabla u\|_{L^2(B_{r+\delta}^+)}^2 \\
& \quad + \lambda_{\min}^{-1} (\|A_\Gamma\|_{L^\infty} + \delta \|\nabla A_\Gamma\|_{L^\infty})^2 |\alpha| \delta^{-2} \|\nabla_\Gamma u\|_{L^2(\Gamma_{r+\delta})}^2 \\
& \quad \left. + \delta \|\nabla A_\Gamma\|_{L^\infty} |\alpha| \delta^{-2} \|\nabla_\Gamma u\|_{L^2(\Gamma_{r+\delta})}^2 \right).
\end{aligned}$$

From (B.4), we get in the limit $h \rightarrow 0$ that $\nabla \partial_{x_j} u \in L^2(B_r^+)$ and $\nabla_\Gamma \partial_{x_j} u \in L^2(\Gamma_r^+)$ together with

$$\begin{aligned}
& \|\nabla \nabla_{x'} u\|_{L^2(B_r)} + |\alpha|^{1/2} \|\nabla_\Gamma \nabla_{x'} u\|_{L^2(\Gamma_r)} \leq \\
\text{(B.8)} \quad & C_{\text{stab}} \left(\|f\|_{L^2(B_{r+\delta}^+)} + \|g\|_{L^2(\Gamma_{r+\delta})} + \delta^{-1} \|G\|_{L^2(B_{r+\delta}^+)} + \|\nabla G\|_{L^2(B_{r+\delta}^+)} \right. \\
& \quad \left. + \delta^{-1} \|\nabla u\|_{L^2(B_{r+\delta}^+)} + |\alpha|^{1/2} \delta^{-1} \|\nabla_\Gamma u\|_{L^2(\Gamma_{r+\delta})} \right)
\end{aligned}$$

with $C_{\text{stab}} > 0$ depending only on θ , a lower bound on λ_{\min} , and an upper bound on $\|A\|_{L^\infty} + R\|\nabla A\|_{L^\infty} + \|A_\Gamma\|_{L^\infty} + R\|\nabla A_\Gamma\|_{L^\infty}$.

Step 4: We complete the proof by controlling $\|\partial_{x_d}^2 u\|_{L^2(B_r^+)}$. This follows from the differential

equation

$$-A_{dd}\partial_{x_d}^2 u = f + \sum_{i,j=1}^d (\partial_{x_i} A_{ij}) \partial_{x_j} u - \sum_{(i,j) \neq (d,d)} A_{ij} \partial_{x_i} \partial_{x_j} u.$$

We have $A_{dd} = e_d^\top A e_d \geq \lambda_{\min}$ and therefore

$$\lambda_{\min} \|\partial_{x_d}^2 u\|_{L^2(B_r^+)} \leq \|f\|_{L^2(B_r^+)} + C\delta \|\nabla A\|_{L^\infty} \delta^{-1} \|\nabla u\|_{L^2(B_r^+)} + \|A\|_{L^\infty} \|\nabla \nabla_{x'} u\|_{L^2(B_r^+)}.$$

Noting $\frac{\|A\|_{L^\infty}}{\lambda_{\min}} \geq 1$ together with (B.8) concludes the proof. \square

B.2. Control of the tangential derivatives. For functions $v = (v_i)_{i \in \mathbf{I}}$, (\mathbf{I} some finite index set) defined on \mathbb{R}^d we introduce the notation

$$(B.9) \quad |\nabla^p v|^2 = \sum_{i \in \mathbf{I}} \sum_{\alpha \in \mathbb{N}_0^d: |\alpha|=p} \frac{|\alpha|!}{\alpha!} |D^\alpha v_i|^2.$$

Analogously, the notation $|\nabla_{x'}^p v|$ for $x' \in \mathbb{R}^{d-1}$ indicates that $\alpha \in \mathbb{N}^{d-1}$ in the sum. We proceed as described, e.g., in [Mel02, Sec. 5.5.3 and 5.5.4]. We write $\nabla_{x'}$ for the tangential derivatives in contrast to [Mel02, Sec. 5.5.3], where ∇_x denotes the tangential derivatives and introduce

$$\begin{aligned} M'_{R,p}(f) &:= \frac{1}{p!} \sup_{R/2 \leq r < R} (R-r)^{p+2} \|\nabla_{x'}^p f\|_{L^2(B_r^+)}, \\ M'_{R,p,\Gamma}(g) &:= \frac{1}{p!} \sup_{R/2 \leq r < R} (R-r)^{p+2} \|\nabla_{x'}^p g\|_{L^2(\Gamma_r)}, \\ N'_{R,p}(v) &:= \begin{cases} \frac{1}{p!} \sup_{R/2 \leq r < R} (R-r)^{p+2} \|\nabla^2 \nabla_{x'}^p v\|_{L^2(B_r^+)} & \text{if } p \geq 0, \\ \sup_{R/2 \leq r < R} (R-r)^{p+2} \|\nabla^{2+p} v\|_{L^2(B_r^+)} & \text{if } p = -1, -2, \end{cases} \\ N'_{R,p,\Gamma}(v) &:= \begin{cases} \frac{1}{p!} \sup_{R/2 \leq r < R} (R-r)^{p+2} \|\nabla_{x'}^{p+2} v\|_{L^2(\Gamma_r)} & \text{for } p \geq 0, \\ \sup_{R/2 \leq r < R} (R-r)^{p+2} \|\nabla_{x'}^{p+2} v\|_{L^2(\Gamma_r)} & \text{for } p = -2, -1, \end{cases} \\ H_{R,p}(v) &:= \frac{1}{[p-1]!} \sup_{R/2 \leq r < R} (R-r)^{p+1} \left[\|\nabla_{x'}^p v\|_{L^2(B_r^+)} + \frac{R-r}{[p]} \|\nabla_{x'}^p \nabla v\|_{L^2(B_r^+)} \right], \end{aligned}$$

where

$$[p]! := \max\{p, 1\}!.$$

Analogous to [Mel02, Lemmas 5.5.15 and 5.5.23] is

Lemma B.3. *Let $p \in \mathbb{N}_0$. Let $A \in C^{p+1}(\overline{B}_R^+)$, $A_\Gamma \in C^{p+1}(\overline{\Gamma}_R^+)$ be matrix-valued functions that are pointwise symmetric positive definite with lower bound on the eigenvalues $\lambda_{\min} > 0$. Let $f \in H^p(B_R^+)$, $g \in H^p(\Gamma_R^+)$, $G \in H^{p+1}(B_R^+)$ and $\alpha \in \mathcal{S}_\theta$. There exists a constant $C_B > 0$*

depending only on the same quantities as the constant C_{stab} in Lemma B.2 such that

$$\begin{aligned}
N'_{R,p}(u) + |\alpha|^{1/2} N'_{R,p,\Gamma}(u) &\leq C_B \left\{ M'_{R,p}(f) + M'_{R,p,\Gamma}(g) + H_{R,p}(G) \right. \\
&+ \sum_{q=1}^{p+1} \binom{p+1}{q} \left(\left(\frac{R}{2} \right)^q \|\nabla^q A\|_{L^\infty(B_R^+)} + \left(\frac{R}{2} \right)^{q-1} q \|\nabla^{q-1} A\|_{L^\infty(B_R^+)} \right) \frac{[p-q]!}{p!} N'_{R,p-q}(u) \\
&+ N'_{R,p-1}(u) + N'_{R,p-2}(u) \\
&+ |\alpha|^{1/2} \left(\sum_{q=1}^{p+1} \binom{p+1}{q} \left(\frac{R}{2} \right)^q \|\nabla_{x'}^q A_\Gamma\|_{L^\infty(\Gamma_R)} \frac{[p-q]!}{p!} N'_{R,p-q,\Gamma}(u) \right. \\
&\left. \left. + N'_{R,p-1,\Gamma}(u) + N'_{R,p-2,\Gamma}(u) \right) \right\}.
\end{aligned}$$

Proof. The proof follows the lines of the proofs of [Mel02, Lemmas 5.5.12, 5.5.15 and 5.5.23]. We abbreviate $\mathbf{a} = \mathbf{A}\mathbf{n}$, where \mathbf{n} is the outer normal vector on Γ_R . The derivative $D_{x'}^{\alpha'} u$ with $\alpha' \in \mathbb{N}_0^{d-1}$ satisfies

$$\begin{aligned}
-\nabla \cdot (A \nabla D_{x'}^{\alpha'} u) &= D_{x'}^{\alpha'} f - \{\nabla \cdot (A \nabla D_{x'}^{\alpha'} u) - D_{x'}^{\alpha'} (\nabla \cdot (A \nabla u))\}, \\
\partial_{n_A} D_{x'}^{\alpha'} u + \alpha \nabla_\Gamma \cdot (A_\Gamma \nabla_\Gamma D_{x'}^{\alpha'} u) &= D_{x'}^{\alpha'} (\alpha^{1/2} g_1 + g_2) \\
&+ \alpha \{\nabla_{x'} \cdot (A_\Gamma \nabla_{x'} D_{x'}^{\alpha'} u) - D_{x'}^{\alpha'} (\nabla_{x'} \cdot (A_\Gamma \nabla_{x'} u))\} \\
&+ \{(\mathbf{a} \cdot \nabla D_{x'}^{\alpha'} u) - D_{x'}^{\alpha'} (\mathbf{a} \cdot \nabla u)\}.
\end{aligned}$$

The remainder of the proof proceeds analogously to the proof of [Mel02, Lemmas 5.5.12 and 5.5.23] and uses Lemma B.2 with $\delta = \frac{R-r}{p+2}$. Specifically, the terms arising from A , f , G are obtained directly as in the proofs of [Mel02, Lemmas 5.5.12 and 5.5.23]. The terms arising from the Laplace-Beltrami are treated with the same arguments as in the proof of [Mel02, Lemma 5.5.12]. \square

B.3. Tangential control for k -dependent problem. We consider the problem

$$\begin{aligned}
-\nabla \cdot (A \nabla u) - ck^2 u &= f && \text{in } B_R^+, \\
\partial_{n_A} u + \alpha \nabla_\Gamma \cdot (A_\Gamma \nabla_\Gamma u) &= \alpha^{1/2} g + G + bku && \text{on } \Gamma.
\end{aligned}
\tag{B.10}$$

For the data, we assume

$$\|\nabla^p f\|_{L^2(B_R^+)} \leq C_f \gamma_f^p R^{-p} \max\{p+1, Rk\}^p \quad \forall p \in \mathbb{N}_0,
\tag{B.11a}$$

$$\|\nabla_{x'}^p g\|_{L^2(\Gamma_R)} \leq C_g \gamma_g^p R^{-p} \max\{p+1, Rk\}^p \quad \forall p \in \mathbb{N}_0,
\tag{B.11b}$$

$$\|\nabla^p G\|_{L^2(B_R^+)} \leq C_G \gamma_G^p R^{-p} \max\{p+1, Rk\}^p \quad \forall p \in \mathbb{N}_0,
\tag{B.11c}$$

$$\|\nabla^p A\|_{L^\infty(B_R^+)} \leq C_A \gamma_A^p p! \quad \forall p \in \mathbb{N}_0,
\tag{B.11d}$$

$$\|\nabla_{x'}^p A_\Gamma\|_{L^\infty(\Gamma_R)} \leq C_{A_\Gamma} \gamma_{A_\Gamma}^p p! \quad \forall p \in \mathbb{N}_0,
\tag{B.11e}$$

$$\|\nabla^p b\|_{L^\infty(B_R^+)} \leq C_b \gamma_b^p p! \quad \forall p \in \mathbb{N}_0,
\tag{B.11f}$$

$$\|\nabla^p c\|_{L^\infty(B_R^+)} \leq C_c \gamma_c^p p! \quad \forall p \in \mathbb{N}_0.
\tag{B.11g}$$

Additionally, we assume that

$$(B.12) \quad |\alpha|^{1/2} \leq C_\alpha k^{-1/2}$$

as well as $\alpha \in S_\theta$ for some $\theta > 0$ uniformly in k .

Theorem B.4. *Let $R \leq 1$. Let f, g, G, A, A_Γ, b and α satisfy (B.11) and (B.12), respectively. Let u solve (B.10). Then there exists a $K \geq 1$ depending only on the coefficients A, A_Γ of the differential operator, θ , and the constants in (B.11) and (B.12) such that for all $p \geq -1$*

$$(B.13) \quad N'_{R,p}(u) + k^{-1/2} N'_{R,p,\Gamma}(u) \leq C_u K^{p+2} \frac{\max\{p+3, Rk\}^{p+2}}{[p]!}$$

where

$$(B.14) \quad \begin{aligned} C_u = & \min\{1, Rk\}^2 (k^{-2} C_f + k^{-2} C_g + C_c \|u\|_{L^2(B_R^+)}) \\ & + \min\{1, Rk\} k^{-1} C_G (1 + \gamma_G R) \\ & + \min\{1, Rk\} (1 + \gamma_b R) C_b \|u\|_{L^2(B_R^+)} \\ & + \min\{1, Rk\} (1 + C_b \min\{1, Rk\}) k^{-1} \|\nabla u\|_{L^2(B_R^+)} \\ & + \min\{1, Rk\} k^{-3/2} C_\alpha \|\nabla_\Gamma u\|_{L^2(\Gamma_R)}. \end{aligned}$$

Proof. We will frequently use the elementary property

$$\min\{a, Rk\} \max\{a, Rk\} = a Rk \quad a > 0.$$

We first verify that (B.13) is correct for $p = -1$, if $K \geq (1 + C_\alpha^{-1})/2$. To that end, note that

$$\begin{aligned} N'_{R,-1}(u) & \leq \frac{R}{2} \|\nabla u\|_{L^2(B_R^+)} = \frac{1}{2} k R k^{-1} \|\nabla u\|_{L^2(B_R^+)} \\ & = \frac{1}{2} \max\{1, Rk\} \min\{1, Rk\} k^{-1} \|\nabla u\|_{L^2(B_R^+)} \\ & \leq \frac{1}{2} \max\{1, Rk\} C_u, \\ k^{-1/2} N'_{R,-1,\Gamma}(u) & \leq k^{-1/2} \frac{R}{2} \|\nabla_\Gamma u\|_{L^2(\Gamma_R)} \\ & \leq \frac{C_\alpha^{-1}}{2} \max\{1, Rk\} \min\{1, Rk\} k^{-1} k^{-1/2} C_\alpha \|\nabla_\Gamma u\|_{L^2(\Gamma_R)} \\ & \leq \frac{C_\alpha^{-1}}{2} \max\{1, Rk\} C_u. \end{aligned}$$

Hence, we have

$$N'_{R,-1}(u) + k^{-1/2} N'_{R,-1,\Gamma}(u) \leq \frac{1 + C_\alpha^{-1}}{2} \max\{1, Rk\} C_u,$$

which concludes the case $p = -1$. We next show that it is correct for $p = 0$ and then by induction for all $p \geq 1$. To that end, we rewrite the equation as

$$\begin{aligned} -\nabla \cdot (A \nabla u) & = f + ck^2 u & \text{in } B_R^+, \\ \partial_{n_A} u + \alpha \nabla_\Gamma \cdot (A_\Gamma \nabla_\Gamma u) & = \alpha^{1/2} g + G + bku & \text{on } \Gamma. \end{aligned}$$

For $p = 0$, an application of Lemma B.2 (used with $\delta \sim R - r$) gives

$$\begin{aligned}
& N'_{R,p}(u) + k^{-1/2}N'_{R,p,\Gamma}(u) \\
& \leq C_{\text{stab}} \left(\frac{R^2}{4} (\|f + ck^2u\|_{L^2(B_R^+)}) + \|g\|_{L^2(\Gamma_R)} \right) \\
& \quad + \frac{R}{2} \|G + bku\|_{L^2(B_R^+)} + \frac{R^2}{4} \|\nabla(G + bku)\|_{L^2(B_R^+)} \\
& \quad + \frac{R}{2} \|\nabla u\|_{L^2(B_R^+)} + |\alpha|^{1/2} \frac{R}{2} \|\nabla_{\Gamma} u\|_{L^2(\Gamma_R)} \\
& \leq C_{\text{stab}} \left(R^2 (\|f\|_{L^2(B_R^+)}) + C_c k^2 \|u\|_{L^2(B_R^+)} + \|g\|_{L^2(\Gamma_R)} \right) \\
& \quad + R \|G\|_{L^2(B_R^+)} + R^2 \|\nabla G\|_{L^2(B_R^+)} \\
& \quad + Rk(1 + \gamma_b R) C_b \|u\|_{L^2(B_R^+)} + R^2 k C_b \|\nabla u\|_{L^2(B_R^+)} \\
& \quad + R \|\nabla u\|_{L^2(B_R^+)} + C_\alpha k^{-1/2} R \|\nabla_{\Gamma} u\|_{L^2(\Gamma_R)} \\
& \leq C_{\text{stab}} \left(R^2 (C_f + C_g + C_c k^2 \|u\|_{L^2(B_R^+)}) \right) \\
& \quad + RC_G + R^2 \gamma_G C_G \\
& \quad + Rk(1 + \gamma_b R) C_b \|u\|_{L^2(B_R^+)} + R^2 k C_b \|\nabla u\|_{L^2(B_R^+)} \\
& \quad + R \|\nabla u\|_{L^2(B_R^+)} + Rk^{-1/2} C_\alpha \|\nabla_{\Gamma} u\|_{L^2(\Gamma_R)} \\
& \leq C_{\text{stab}} C_u \max\{3, Rk\}^2,
\end{aligned}$$

where the last estimate follows by similar estimates as in the case $p = -1$. Let us assume that (B.13) is correct for all $-1 \leq p' \leq p-1$ for some $p \geq 1$. We show that it is correct for p if K is chosen sufficiently large. In fact, we will implicitly assume $K > \max\{\gamma_f, \gamma_g, \gamma_G, \gamma_c, \gamma_b\}$, so that the various geometric series below converge.

It is convenient to abbreviate

$$(B.15) \quad m(p) := \max\{p + 3, Rk\}^{p+2}.$$

In order to apply Lemma B.3, we have to estimate $M'_{R,p}(f - k^2cu)$, $M'_{R,p,\Gamma}(g)$ and $H'(kbu)$. Since $M'_{R,p}(f - k^2cu) \leq M'_{R,p}(f) + M'_{R,p}(k^2cu)$, we first estimate $M'_{R,p}(f)$.

$$\begin{aligned}
M'_{R,p}(f) & \leq \frac{1}{p!} \left(\frac{R}{2} \right)^{p+2} C_f \gamma_f^p R^{-p} \max\{p + 1, Rk\}^p \\
& \leq \frac{C_f}{4} \left(\frac{\gamma_f}{2} \right)^p R^2 \frac{1}{p!} \max\{p + 3, Rk\}^p \\
& \leq \frac{C_f}{4} \left(\frac{\gamma_f}{2} \right)^p k^{-2} \min\{1, Rk\}^2 \frac{m(p)}{p!}.
\end{aligned}$$

From [Mel02, Lemma 5.5.13] and the induction hypothesis, we have

$$\begin{aligned}
M'_{R,p}(k^2cu) &\leq k^2 C_c \sum_{q=0}^p \left(\gamma_c \frac{R}{2}\right)^q \left(\frac{R}{2}\right)^2 \frac{[p-q-2]!}{(p-q)!} N'_{R,p-q-2}(u) \\
&= \frac{C_c}{4} \min\{1, Rk\}^2 \max\{1, Rk\}^2 \sum_{q=0}^p \left(\frac{\gamma_c R}{2}\right)^q \frac{[p-q-2]!}{(p-q)!} N'_{R,p-q-2}(u) \\
&\leq \frac{C_c}{4} \min\{1, Rk\}^2 \max\{1, Rk\}^2 \sum_{q=0}^p \left(\frac{\gamma_c R}{2}\right)^q \frac{1}{(p-q)!} K^{p-q} \max\{p-q+1, Rk\}^{p-q} C_u \\
&= \frac{C_c}{4} \min\{1, Rk\}^2 K^p \max\{1, Rk\}^2 \sum_{q=0}^p \left(\frac{\gamma_c R}{2K}\right)^q \frac{1}{(p-q)!} \max\{p-q+1, Rk\}^{p-q} C_u \\
&\leq \frac{C_c}{4} \min\{1, Rk\}^2 K^p \max\{1, Rk\}^2 \sum_{q=0}^p \left(\frac{\gamma_c R}{2K}\right)^q \frac{1}{p!} p^q \max\{p-q+1, Rk\}^{p-q} C_u \\
&\leq \frac{C_c}{4} \min\{1, Rk\}^2 K^{p+2} \frac{m(p)}{p!} K^{-2} \sum_{q=0}^p \left(\frac{\gamma_c R}{2K}\right)^q C_u \\
&\leq \frac{C_c}{4} \min\{1, Rk\}^2 K^{p+2} \frac{m(p)}{p!} \frac{K^{-2}}{1 - \gamma_c R/(2K)} C_u,
\end{aligned}$$

where the last step follows from a geometric series argument and assumes $K > \gamma_c R/2$. Similarly, we find

$$M'_{R,p,\Gamma}(g) \leq \frac{C_g}{4} \left(\frac{\gamma_g}{2}\right)^p k^{-2} \min\{1, Rk\}^2 \frac{m(p)}{p!}.$$

From [Mel02, Lemma 5.5.24], similar estimates to the above and the induction hypothesis, we have

$$\begin{aligned}
H_{R,p}(kbu) &\leq C_b \frac{1}{[p]!} \frac{1}{2} \min\{1, Rk\} \times \\
&\quad \left\{ \max\{p+1, Rk\}^2 \sum_{q=0}^p \binom{p}{q} \left[\left(\frac{\gamma_b R}{2}\right)^q + \left(\frac{\gamma_b R}{2}\right)^{q+1} \right] q! [p-q-2]! N'_{R,p-q-2}(u) \right. \\
&\quad \left. + \max\{p+1, Rk\} \sum_{q=0}^p \binom{p}{q} \left(\frac{\gamma_b R}{2}\right)^q q! [p-q-1]! N'_{R,p-q-1}(u) \right\} \\
&\leq \frac{C_b}{2} \min\{1, Rk\} K^{p+2} \frac{m(p)}{p!} \left(\frac{K^{-2}}{1 - \gamma_b R/(2K)} + \frac{K^{-2} \gamma_b R/2}{1 - \gamma_b R/(2K)} + \frac{K^{-1}}{1 - \gamma_b R/(2K)} \right) C_u.
\end{aligned}$$

Finally, we estimate $H_{R,p}(G)$ as at the end of the proof of [Mel02, Prop. 5.5.25]

$$(B.16) \quad H_{R,p}(G) \leq C_G \gamma_G^p k^{-1} \min\{1, Rk\} (1 + \gamma_G R) \frac{m(p)}{p!}.$$

We are now in the place to perform the induction argument. Lemma B.3 gives

$$\begin{aligned}
& N'_{R,p}(u) + k^{-1/2}N'_{R,p,\Gamma}(u) \\
& \leq C_B \left[M'_{R,p}(f - k^2cu) + M'_{R,p,\Gamma}(g) + H_{R,p}(G + bku) \right. \\
& \quad + C_A \sum_{q=1}^{p+1} \binom{p+1}{q} q! \left(\left(\frac{R\gamma_A}{2} \right)^q + \left(\frac{R\gamma_A}{2} \right)^{q-1} \right) \frac{[p-q]!}{p!} N'_{R,p-q}(u) \\
& \quad + N'_{R,p-1}(u) + N'_{R,p-2}(u) \\
& \quad + |\alpha|^{1/2} C_{A_\Gamma} \sum_{q=1}^{p+1} \binom{p+1}{q} q! \left(\frac{R\gamma_{A_\Gamma}}{2} \right)^q \frac{[p-q]!}{p!} N'_{R,p-q,\Gamma}(u) \\
& \quad \left. + |\alpha|^{1/2} (N'_{R,p-1,\Gamma}(u) + N'_{R,p-2,\Gamma}(u)) \right] \\
& \leq C_B \left[C_f k^{-2} \min\{1, Rk\}^2 \gamma_f^p \frac{m(p)}{p!} + C_g k^{-2} \min\{1, Rk\}^2 \gamma_g^p \frac{m(p)}{p!} \right. \\
& \quad + \frac{C_c}{4} \min\{1, Rk\}^2 K^{p+2} \frac{m(p)}{p!} \frac{K^{-2}}{1 - \gamma_c R/(2K)} C_u \\
& \quad + C_G \gamma_G^p k^{-1} \min\{1, Rk\} (1 + \gamma_G R) \frac{m(p)}{p!} \\
& \quad + \frac{C_b}{2} \min\{1, Rk\} K^{p+2} \frac{m(p)}{p!} \left(\frac{K^{-2}}{1 - \gamma_b R/(2K)} + \frac{K^{-2} \gamma_b R/2}{1 - \gamma_b R/(2K)} + \frac{K^{-1}}{1 - \gamma_b R/(2K)} \right) C_u \\
& \quad + C_A K^{p+2} \frac{m(p)}{p!} \left(\frac{K^{-2}}{1 - \gamma_A R/(2K)} + \frac{K^{-2} \gamma_A R/2}{1 - \gamma_A R/(2K)} \right) C_u \\
& \quad + K^{p+2} \frac{m(p)}{p!} (K^{-2} + K^{-1}) C_u \\
& \quad + C_\alpha C_{A_\Gamma} K^{p+2} \frac{m(p)}{p!} \frac{K^{-2}}{1 - \gamma_{A_\Gamma} R/(2K)} C_u \\
& \quad \left. + C_\alpha K^{p+2} \frac{m(p)}{p!} (K^{-2} + K^{-1}) C_u \right] \\
& \leq C_B K^{p+2} \frac{m(p)}{p!} \left[C_f k^{-2} \min\{1, Rk\}^2 K^{-2} \left(\frac{\gamma_f}{K} \right)^p + C_g k^{-2} \min\{1, Rk\}^2 K^{-2} \left(\frac{\gamma_g}{K} \right)^p \right. \\
& \quad + \frac{C_c}{4} \min\{1, Rk\}^2 \frac{K^{-2}}{1 - \gamma_c R/(2K)} C_u + C_G k^{-1} \min\{1, Rk\} (1 + \gamma_G R) K^{-2} \left(\frac{\gamma_G}{K} \right)^p \\
& \quad + \frac{C_b}{2} \min\{1, Rk\} \left(\frac{K^{-2}}{1 - \gamma_b R/(2K)} + \frac{K^{-2} \gamma_b R/2}{1 - \gamma_b R/(2K)} + \frac{K^{-1}}{1 - \gamma_b R/(2K)} \right) C_u \\
& \quad + C_A \left(\frac{K^{-2}}{1 - \gamma_A R/(2K)} + \frac{K^{-2} \gamma_A R/2}{1 - \gamma_A R/(2K)} \right) C_u + (K^{-2} + K^{-1}) C_u \\
& \quad \left. + C_\alpha C_{A_\Gamma} \frac{K^{-2}}{1 - \gamma_{A_\Gamma} R/(2K)} C_u + C_\alpha (K^{-2} + K^{-1}) C_u \right] \\
& \leq C_u K^{p+2} \frac{m(p)}{p!} C_B [\dots].
\end{aligned}$$

The induction argument is complete, once we established that there exists a sufficiently large K , which can be chosen uniformly in p such that $C_B[\dots] \leq 1$. This can be easily seen, since

$$\begin{aligned} C_B[\dots] &= C_B \left[K^{-2} \left(\frac{\gamma_f}{K} \right)^p + K^{-2} \left(\frac{\gamma_g}{K} \right)^p + \frac{C_c}{4} \frac{K^{-2}}{1 - \gamma_c R / (2K)} + K^{-2} \left(\frac{\gamma_G}{K} \right)^p \right. \\ &\quad \left. + \frac{C_b}{2} \left(\frac{K^{-2}}{1 - \gamma_b R / (2K)} + \frac{K^{-2} \gamma_b R / 2}{1 - \gamma_b R / (2K)} + \frac{K^{-1}}{1 - \gamma_b R / (2K)} \right) \right. \\ &\quad \left. + C_A \left(\frac{K^{-2}}{1 - \gamma_A R / (2K)} + \frac{K^{-2} \gamma_A R / 2}{1 - \gamma_A R / (2K)} \right) + (K^{-2} + K^{-1}) \right. \\ &\quad \left. + C_\alpha C_{A_\Gamma} \frac{K^{-2}}{1 - \gamma_{A_\Gamma} R / (2K)} + C_\alpha (K^{-2} + K^{-1}) \right], \end{aligned}$$

which concludes the proof. \square

B.4. Control of all derivatives for k -dependent problems. Control of the derivatives $\partial_{x_d}^q$ is achieved using the differential equation. For that purpose, we introduce

$$N'_{p,q,R}(u) := \frac{1}{[p+q]!} \sup_{R/2 \leq r < R} (R-r)^{p+q+2} \|\nabla_{x'}^p \partial_{x_d}^{q+2} u\|_{L^2(B_r^+)}.$$

Theorem B.5. *Assume the hypotheses of Theorem B.4. Then there exist constants K_1, K_2 depending only on the coefficients A, A_Γ of the differential operator, θ , and the constants in (B.11) and (B.12) such that for all $p, q \in \mathbb{N}_0 \cup \{-1, -2\}$ with $p+q \neq -2$*

$$(B.17) \quad N'_{R,p,q}(u) \leq C_u K_1^{p+2} K_2^{q+2} \frac{\max\{p+q+3, Rk\}^{p+q+2}}{[p+q]!},$$

where C_u is given in Theorem B.4.

Proof. The proof follows directly from the proof of [Mel02, Prop. 5.5.2]. Namely, [Mel02, Prop. 5.5.2] proceeds by induction on q . The induction step relies on a) the fact that an elliptic equation in B_R^+ of the form studied here is considered and b) control of the tangential derivatives, which is provided in Theorem B.4. Hence, the proof of [Mel02, Prop. 5.5.2] applies. \square

Acknowledgement: MB is grateful for financial support by the Austrian Science Fund (FWF) through the doctoral school *Dissipation and dispersion in nonlinear PDEs* (grant W1245). The authors thank Joachim Schöberl (TU Wien) and his group for their help regarding the numerical experiments.

REFERENCES

- [Ain04] Mark Ainsworth. Discrete dispersion relation for hp -version finite element approximation at high wave number. *SIAM J. Numer. Anal.*, 42(2):553–575, 2004.
- [Ber21] M. Bernkopf. *Finite Element Analysis of the Heterogeneous Helmholtz Equation and Least Squares Methods*. PhD thesis, Technische Universität Wien, 2021.
- [BGT82] Alvin Bayliss, Max Gunzburger, and Eli Turkel. Boundary conditions for the numerical solution of elliptic equations in exterior regions. *SIAM Journal on Applied Mathematics*, 42(2):430–451, 1982.
- [BPG16] H. Bériot, A. Prinn, and G. Gabard. Efficient implementation of high-order finite elements for Helmholtz problems. *Int. J. Numer. Meth. Engng.*, 106:213–240, 2016.

- [BSW16] Dean Baskin, Euan A. Spence, and Jared Wunsch. Sharp high-frequency estimates for the Helmholtz equation and applications to boundary integral equations. *SIAM J. Math. Anal.*, 48(1):229–267, 2016.
- [CF16] T. Chaumont-Frelet. On high order methods for the heterogeneous Helmholtz equation. *Comp. Math. Appl.*, 72:2203–2225, 2016.
- [CFEV21] T. Chaumont-Frelet, A. Ern, and M. Vohralík. On the derivation of guaranteed and p -robust a posteriori error estimates for the helmholtz equation. accepted in *Numer. Math.*, 2021.
- [CFN18] T. Chaumont-Frelet and S. Nicaise. High-frequency behaviour of corner singularities in Helmholtz problems. *ESAIM Math. Model. Numer. Anal.*, 5:1803–1845, 2018.
- [CFN19] T. Chaumont-Frelet and S. Nicaise. Wavenumber explicit convergence analysis for finite element discretizations of general wave propagation problems. *IMA J. Numer. Anal.*, 40:1503–1543, 2019.
- [CFV20] T. Chaumont-Frelet and F. Valentin. A multiscale hybrid-mixed method for the Helmholtz equation in heterogeneous domains. *SIAM J. Numer. Anal.*, 58(2):1029–1067, 2020.
- [CFV21] T. Chaumont-Frelet and P. Vega. Frequency-explicit a posteriori error estimates for finite element discretizations of maxwell’s equations. hal-02943386, 2021.
- [CM98] Francis Collino and Peter Monk. The perfectly matched layer in curvilinear coordinates. *SIAM J. Sci. Comput.*, 19(6):2061–2090, 1998.
- [CS85] Martin Costabel and Ernst Stephan. A direct boundary integral equation method for transmission problems. *J. Math. Anal. Appl.*, 106(2):367–413, 1985.
- [CWGLS12] Simon N. Chandler-Wilde, Ivan G. Graham, Stephen Langdon, and Euan A. Spence. Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering. *Acta Numer.*, 21:89–305, 2012.
- [DI01] Leszek Demkowicz and Frank Ihlenburg. Analysis of a coupled finite-infinite element method for exterior Helmholtz problems. *Numer. Math.*, 88(1):43–73, 2001.
- [DS13] W. Dörfler and S. Sauter. A posteriori error estimation for highly indefinite Helmholtz problems. *Comput. Meth. Appl. Math.*, 13:333–347, 2013.
- [DW15] Y. Du and H. Wu. Preasymptotic error analysis of higher order FEM and CIP-FEM for Helmholtz equation with high wave number. *SIAM J. Numer. Anal.*, 53(2):782–804, 2015.
- [EM77] Bjorn Engquist and Andrew Majda. Absorbing boundary conditions for the numerical simulation of waves. *Mathematics of Computation*, 31(139):629–651, July 1977.
- [EM79] Bjorn Engquist and Andrew Majda. Radiation boundary conditions for acoustic and elastic wave calculations. *Communications on Pure and Applied Mathematics*, 32(3):313–357, 1979.
- [EM12] S. Esterhazy and J. M. Melenk. On stability of discretizations of the Helmholtz equation. In *Numerical analysis of multiscale problems*, volume 83 of *Lect. Notes Comput. Sci. Eng.*, pages 285–324. Springer, Heidelberg, 2012.
- [Eva98] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.
- [Fen84] Kang Feng. Finite element method and natural boundary reduction. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 1439–1453. PWN, Warsaw, 1984.
- [GLSW22] Jeffrey Galkowski, D. Lafontaine, Euan A. Spence, and J. Wunsch. The hp -FEM applied to the Helmholtz equation with PML truncation does not suffer from the pollution effect, 2022. arXiv:2207.05542v2.
- [IB95] F. Ihlenburg and I. Babuška. Finite element solution to the Helmholtz equation with high wave number. Part I: The h -version of the FEM. *Comput. Math. Appl.*, 30:9–37, 1995.
- [IB97] F. Ihlenburg and I. Babuška. Finite element solution to the Helmholtz equation with high wave number. Part II: The hp -version of the FEM. *SIAM J. Numer. Anal.*, 34:315–358, 1997.
- [Ihl98] Frank Ihlenburg. *Finite element analysis of acoustic scattering*, volume 132 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1998.
- [LSW20] David Lafontaine, Euan A. Spence, and Jared Wunsch. Wavenumber-explicit convergence of the hp -fem for the full-space heterogeneous helmholtz equation with smooth coefficients, 2020.
- [LSW21] David Lafontaine, Euan A. Spence, and Jared Wunsch. Decompositions of high-frequency Helmholtz solutions via functional calculus, and application to the finite element methods, 2021.
- [McL00] William McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

- [Mel95] J.M. Melenk. *On generalized finite element methods*. PhD thesis, University of Maryland, 1995.
- [Mel02] J. M. Melenk. *hp-finite element methods for singular perturbations*, volume 1796 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.
- [Mel12] J. M. Melenk. Mapping properties of combined field Helmholtz boundary integral operators. *SIAM J. Math. Anal.*, 44(4):2599–2636, 2012.
- [MMPR20] Lorenzo Mascotto, Jens M. Melenk, Ilaria Perugia, and Alexander Rieder. FEM-BEM mortar coupling for the Helmholtz problem in three dimensions. *Comput. Math. Appl.*, 80(11):2351–2378, 2020.
- [MPS13] J. M. Melenk, A. Parsania, and S. Sauter. General dg-methods for highly indefinite helmholtz problems. *Journal of Scientific Computing*, 57(3):536–581, Dec 2013.
- [MR18] Jens Markus Melenk and Claudio Rojik. On commuting p -version projection-based interpolation on tetrahedra, 2018.
- [MS10] J. M. Melenk and S. Sauter. Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions. *Math. Comp.*, 79(272):1871–1914, 2010.
- [MS11] J. M. Melenk and S. Sauter. Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation. *SIAM J. Numer. Anal.*, 49(3):1210–1243, 2011.
- [MS21] Jens M. Melenk and Stefan A. Sauter. Wavenumber-explicit hp -FEM analysis for Maxwell’s equations with transparent boundary conditions. *Found. Comput. Math.*, 21(1):125–241, 2021.
- [MS22] Jens M. Melenk and Stefan A. Sauter. Wavenumber-explicit hp -fem analysis for maxwell’s equations with impedance boundary conditions, 2022. arXiv:2201.02602.
- [Néd01] Jean-Claude Nédélec. *Acoustic and electromagnetic equations*, volume 144 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2001. Integral representations for harmonic problems.
- [Sch74] A.H. Schatz. An observation concerning Ritz-Galerkin methods with indefinite bilinear forms. *Math. Comp.*, 28(128):959–962, 1974.
- [Spe14] E. A. Spence. Wavenumber-explicit bounds in time-harmonic acoustic scattering. *SIAM J. Math. Anal.*, 46(4):2987–3024, 2014.
- [SS11] Stefan A. Sauter and Christoph Schwab. *Boundary element methods*, volume 39 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2011. Translated and expanded from the 2004 German original.
- [Ste70] E.M. Stein. *Singular integrals and differentiability properties of functions*. Princeton University Press, 1970.
- [Ste08] O. Steinbach. *Numerical approximation methods for elliptic boundary value problems*. Springer, New York, 2008. Finite and boundary elements, Translated from the 2003 German original.
- [TZNHD17] M. Taus, L. Zepeda-Nunez, R. Hewett, and L. Demanet. Pollution-free and fast hybridizable discontinuous Galerkin solvers for the high-frequency Helmholtz equation. In *Proc. SEG annual meeting*, Houston, 2017.
- [ZW13] Lingxue Zhu and Haijun Wu. Preasymptotic error analysis of CIP-FEM and FEM for Helmholtz equation with high wave number. Part II: hp version. *SIAM J. Numer. Anal.*, 51(3):1828–1852, 2013.