Stability and accuracy of Runge-Kutta based split-explicit time-stepping algorithms for free-surface ocean models

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Runge-Kutta methods

Advantages of RK methods:

- 2-time-level schemes (no computational mode)
- Good stability properties for advection and diffusion
- Abundant literature (e.g. RK-IMEX, RK-SSP, DIRK, etc)
- Easy to implement (convex combination of Euler steps)

Main questions:

- Many variants of RK schemes
- RK with mode-splitting?

Existing work:

- Many RK-based atmospheric models (e.g. WRF, Cosmo, Homme-NH, etc)
Runge-Kutta methods in the oceanic context

Many variants of RK schemes

- Low-storage and « order=stage » method for advection [Wicker & Skamarock, 2002]

\[
\begin{align*}
\phi^{n+1/3} &= \phi^n + \frac{\Delta t}{3} F(\phi^n) \\
\phi^{n+1/2} &= \phi^n + \frac{\Delta t}{2} F(\phi^{n+1/3}) \\
\phi^{n+1} &= \phi^n + \Delta t F(\phi^{n+1/2})
\end{align*}
\]

- Introduce Shuman averaging for internal gravity waves (« order=stage-1 »)

Additional difficulty inherent to oceanic models

- 2D barotropic / 3D baroclinic mode splitting to improve computational efficiency (fast and slow dynamics are split into separate subproblems)

- Separation of slow and fast modes is non-orthogonal (depth-independent barotropic mode assumption)

⇒ Some form of filtering is required to stabilize the splitting

\[
\begin{align*}
\partial_t \bar{\mu} &= -g\partial_x \left( \eta + \frac{1}{\rho_0 g H} \int_{-H}^{0} p_h \, dz \right) \\
\partial_t \eta &= -H \partial_x \bar{\mu}
\end{align*}
\]
Robust mode-splitting method based on RK3

**Peculiarity of oceanic models:** 2D barotropic / 3D baroclinic mode splitting to improve computational efficiency

**Single-first approach:**
- Only one integration of the 2D within the RK3 time-step
- Split-explicit approach with a dissipative time-stepping for the barotropic mode
- Linear stability analysis can be conducted following Demange et al. (2019)

\[ \mu_0 = c_0(k_x \Delta t) \]
Robust mode-splitting method based on RK3

**Peculiarity of oceanic models:** 2D barotropic / 3D baroclinic mode splitting to improve computational efficiency

\[
\begin{align*}
\pi^{n+1} &= \pi^n - g\Delta t_0 \partial_x \eta^n \\
\eta^{n+1} &= \eta^n - H\Delta t_0 \partial_x \left( (1+\theta)\bar{u}^{n+1} - \theta \bar{u}^n \right)
\end{align*}
\]

- Linear stability analysis provides the minimal dissipation to stabilize the splitting
- Stability constrained by barotropic mode and not by baroclinic ones
Conclusion and future work

Summary:

- Design of an RK3 based time-stepping algorithm for split-explicit free-surface ocean models
- The single-first strategy appears to be more robust than other alternatives
- Thought for a quasi-Eulerian vertical coordinate
- A non-linear \( z^* \) 2D \( x-z \) model has been developed to check results from stability analysis

Ongoing work:

- Implementation in NEMO (see next talk)
- Accuracy analysis and consequences of mode-splitting