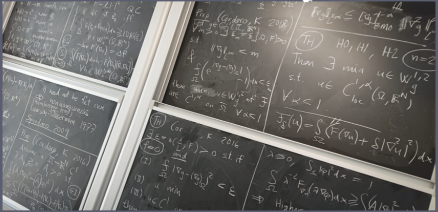
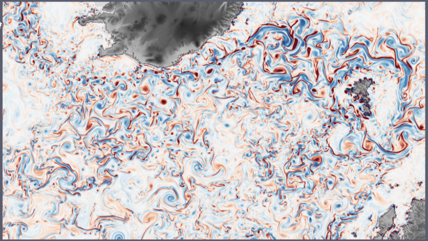
Potentiels apports de l'analyse mathématique à la modélisation du climat

Anne-Laure Dalibard, Florian Lemarié





Potentiels apports de l'analyse mathématique à la modélisation du climat

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Oceanic primitive equations

Energy consistent seawater Boussinesq system (e.g. Young, 2010)

In Cartesian geopotential coordinate

$$
\begin{aligned}
\partial_t \mathbf{v} + 2\boldsymbol{\Omega} \times \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{v}) &= -\frac{1}{\rho_0}\nabla p - \frac{\rho}{\rho_0}\mathbf{g} + \nu \Delta \mathbf{v} \\
\nabla \cdot \mathbf{u} &= 0 \\
\partial_t \varphi + \nabla \cdot (\mathbf{u}\varphi) &= F_\varphi, \qquad (\varphi = \Theta, S_A) \\
\rho &= \rho_{\mathrm{eos}}(\Theta, S_A, p_0(z)) \\
+\ \text{initial} \quad &\text{and} \quad \text{boundary conditions}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{u} &= (u, v, w)^T \\
\mathbf{v} &= (u, v, 0)^T \\
\boldsymbol{\Omega} &= (0, 0, f/2)^T \quad \text{shallow ocean traditional assumption} \\
p_0 &= -\rho_0 g z \quad \text{background hydrostatic pressure} \\
p &= p_h + \rho_0 g \eta \\
\mathbf{g} &= (0, 0, g) \quad \text{spherical geoid assumption}
\end{aligned}
$$

Common approximations:

- Geometrical assumptions (spherical geoid approximation, traditional shallow-ocean approximation)
- Dynamical assumptions (Hydrostatic and Boussinesq assumptions)

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+ initial and boundary conditions

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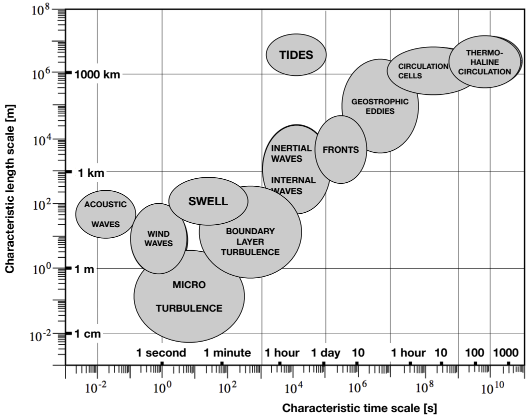
$$

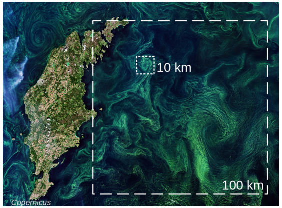
P. Korn (2021) : *Global Well-Posedness of the Ocean Primitive Equations with Nonlinear Thermodynamics*, J. Math. Fluid Mech.

Model resolution is finite

Scales of motion and processes present in the ocean occur on a wide range of scales from global $(\mathcal{O}(10^7 \text{ m}, 10^{10} \text{ s}))$ to micro $(\mathcal{O}(10^{-3} \text{ m}, 10^{-3} \text{ s}))$ scales



Various important processes occur on scales that are too fine to be resolved



Processes smaller than the grid size must be homogenized (i.e. averaged)

$\Rightarrow$ **subgrid models (a.k.a. "parameterizations")** represent subgrid-scale processes in *parametric* manner

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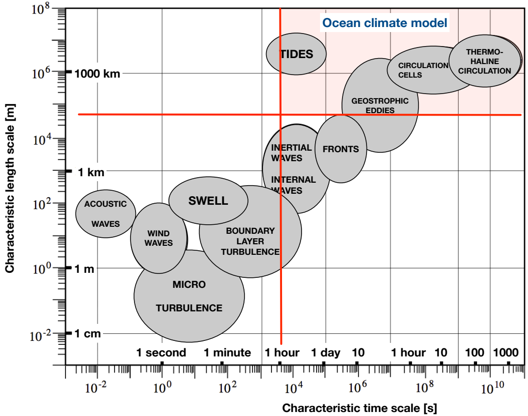

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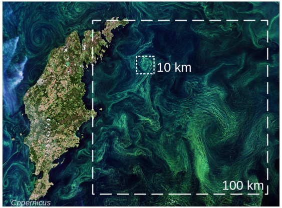


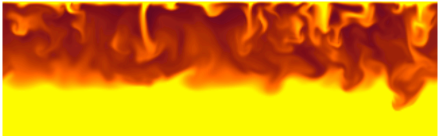
Processes smaller than the grid size must be homogenized (i.e. averaged)

$\Rightarrow$ **subgrid models (a.k.a. "parameterizations")** represent subgrid-scale processes in *parametric* manner

Subgrid models

A great variety of processes need to be parameterized:

- Unresolved waves (e.g. breaking internal waves)
- Local turbulence (e.g. boundary layer turbulence)
- Mesoscale eddies
- Intermittent coherent structures (e.g. convective plumes)



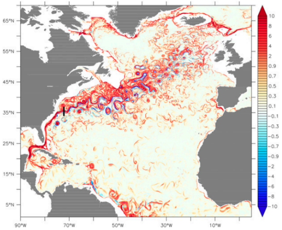
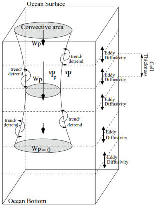
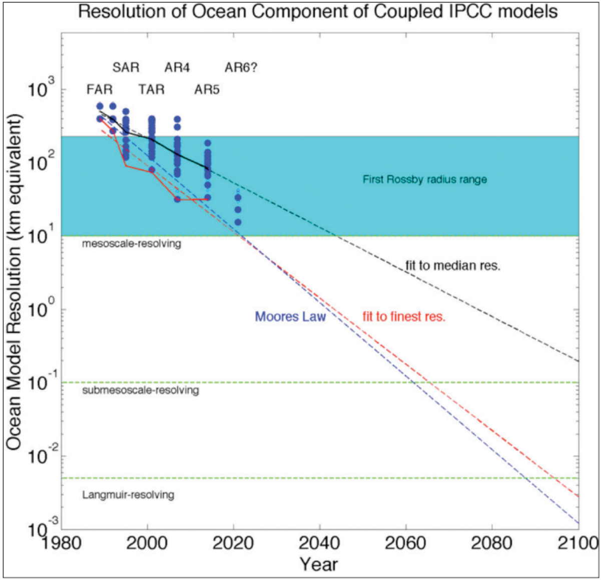


FIG. 2. Instantaneous energy transfer from mean flow to waves (10^{-6} W m^{-3}) at 300 m depth in a high-resolution model of the North Atlantic Ocean. The position of the section shown in Fig. 3 at



Challenges :

- ▷ **Appropriate choice of an averaging operator to separate the resolved and unresolved scales**
- ▷ Variety of processes $\rightarrow$ structurally different subgrid models (diffusive/viscous term, advective term, drag, etc.)



⇒ *discretizing fluid equations without further approximations and parameterizations would require computers about 10^{10}* ***times faster and bigger in storage*** *than present supercomputers*

(Fox-Kemper et al., 2014)

⇒ Complexity of climate models progresses very quickly compared to the mathematical formalization and the analysis of these highly complex systems.

Content

1. Construct the coarse-grained equations
   - Practical point of view : Reynolds decomposition
   - Theoretical point of view : homogeneisation of conservation laws
2. Boundary layers methods
   - Ocean/atmosphere context
   - Wall laws for fluid flows
3. Coupling two turbulent Ekman boundary layers
   - Motivations
   - Mathematical analysis
4. Analysis with nonlinear thermodynamics

1

Construct the coarse-grained equations

Reynolds decomposition

Any physical variable designated by φ can be decomposed in a statistical mean $\overline{\varphi} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \varphi_n(\mathbf{x},t)$ and fluctuating components (**Reynolds decomposition**)

$$\varphi = \overline{\varphi} + \varphi', \qquad \overline{\varphi'} = 0$$

For any physical variable designated by φ

$$\partial_t \varphi + \nabla \cdot (\mathbf{u}_h \varphi) + \partial_z (w\varphi) = F_\varphi$$

Apply the decomposition and use properties of ensemble averaging

$$\partial_t \overline{\varphi} + \nabla \cdot (\overline{\mathbf{u}}_h \overline{\varphi}) + \partial_z (\overline{w}\,\overline{\varphi}) = \overline{F}_\varphi - \nabla \cdot (\overline{\mathbf{u}'_h \varphi'}) - \partial_z \overline{w'\varphi'}$$

- **Subgrid-scale parameterization problem** = find a closed expression for $\overline{\mathbf{u}'_h \varphi'}$ and $\overline{w'\varphi'}$ in terms of the known resolved-scale variables
- Because of the stratification, in many oceanic turbulent problems, ***the dominant term is the vertical eddy transport***

Theoretical point of view : homogeneisation of conservation laws

First case: vanishing viscosity

Consider the equation

$$\begin{aligned} \partial_t \varphi^\epsilon + \operatorname{div}\left(u\left(x, \frac{x}{\epsilon}\right)\varphi^\epsilon\right) = \epsilon \Delta \varphi^\epsilon \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \varphi^\epsilon(t=0) = \varphi^0\left(x, \frac{x}{\epsilon}\right) \end{aligned} \qquad (\mathrm{T}\epsilon)$$

where $u(x, z)$, $\varphi^0(x, z)$ are periodic w.r.t. z.
Question: asymptotic behavior of φ^ϵ?

Theoretical point of view : homogeneisation of conservation laws

First case: vanishing viscosity

Consider the equation

$$\partial_t \varphi^\epsilon + \operatorname{div}\left(u\left(x, \frac{x}{\epsilon}\right)\varphi^\epsilon\right) = \epsilon \Delta \varphi^\epsilon \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d,$$
$$\varphi^\epsilon(t=0) = \varphi^0\left(x, \frac{x}{\epsilon}\right) \qquad (\mathrm{T}\epsilon)$$

where $u(x,z)$, $\varphi^0(x,z)$ are periodic w.r.t. z.
Question: asymptotic behavior of φ^ϵ?
Ansatz: Look for φ^ϵ in the form

$$\varphi^\epsilon = \overline{\varphi}(t,x) + \varphi'\left(t, x, \frac{x}{\epsilon}\right) + \epsilon\varphi^1\left(t, x, \frac{x}{\epsilon}\right) + O(\epsilon^2))$$

Order ϵ^{-1}: Cell problem (sub-grid):

$$-\Delta_z \varphi' + \operatorname{div}_z(u(\overline{\varphi} + \varphi)) = 0, \quad \overline{\varphi'(t,x,\cdot)} = 0.$$

Theoretical point of view : homogeneisation of conservation laws

First case: vanishing viscosity

$$\begin{aligned} \partial_t \varphi^\epsilon + \operatorname{div}\left(u\left(x, \frac{x}{\epsilon}\right)\varphi^\epsilon\right) &= \epsilon \Delta \varphi^\epsilon \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \varphi^\epsilon(t=0) &= \varphi_i\left(x, \frac{x}{\epsilon}\right) \end{aligned} \tag{Tϵ}$$

Order ϵ^0:

$$\partial_t \overline{\varphi} + \partial_t \varphi' + \operatorname{div}_x(u(\overline{\varphi} + \varphi')) + \operatorname{div}_z(u\varphi^1) = \Delta_z \varphi^1 + 2\nabla_x \cdot \nabla_z \varphi'.$$

Average w.r.t. $z \rightsquigarrow$ Homogenized problem/Subgrid-scale parameterization:

$$\partial_t \overline{\varphi} + \operatorname{div}_x (U(x)\overline{\varphi}) = 0;$$

Vector field $U(x) := \overline{u(x,\cdot)(1+\psi'(x,\cdot)}(\neq \overline{u(x)})$.

Remark: can be extended to nonlinear settings, conservation laws...

Theoretical point of view : homogeneisation of conservation laws

Second case: without viscosity

$$\partial_t \varphi^\epsilon + \operatorname{div}\left(u\left(x, \frac{x}{\epsilon}\right)\varphi^\epsilon\right) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \qquad (\mathrm{T}\epsilon)$$
$$\varphi^\epsilon(t=0) = \varphi_i\left(x, \frac{x}{\epsilon}\right)$$

Same Ansatz:

$$\varphi^\epsilon = \overline{\varphi}(t,x) + \varphi'\left(t, x, \frac{x}{\epsilon}\right) + \epsilon\varphi^1\left(t, x, \frac{x}{\epsilon}\right) + O(\epsilon^2))$$

Order ϵ^{-1}: Cell problem:

$$\operatorname{div}_z\left(u\left(\overline{\varphi} + \varphi'\right)\right) = 0.$$

Difficulty: does not allow to identify φ' ! (no uniqueness).
Consequence: no homogenized problem, i.e. no parametrization.

Theoretical point of view : homogeneisation of conservation laws

Second case: without viscosity

$$\partial_t \varphi^\epsilon + \operatorname{div}\left(u\left(x, \frac{x}{\epsilon}\right)\varphi^\epsilon\right) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d,$$
$$\varphi^\epsilon(t=0) = \varphi_i\left(x, \frac{x}{\epsilon}\right) \qquad (\mathrm{T}\epsilon)$$

But... a limit problem with a unique solution can be identified!

If $\operatorname{div}_z u = 0$, this problem is

$$\partial_t \varphi(t,x,z) + \operatorname{div}_x(\widetilde{u}(x,z)\varphi(t,x,z)) = 0,$$
$$\operatorname{div}_z(u(x,z)\varphi(t,x,z)) = 0,$$

where $\widetilde{u}$ = projection of u onto $\{v, \operatorname{div}_z(uv) = 0\}$.

Remarks:

- Coupling of micro/macro scales, no parametrization;
- If $\operatorname{div}_z u \neq 0$, in general the limit problem is a kinetic system (in $2d+2$ variables). No associated conservation law.

Averaged primitive equations and eddy-viscosity hypothesis

$$\begin{aligned}
\partial_t \overline{\mathbf{v}} + 2\boldsymbol{\Omega} \times \overline{\mathbf{u}} + \nabla \cdot (\overline{\mathbf{u}} \otimes \overline{\mathbf{v}}) &= -\frac{1}{\rho_0}\nabla \overline{p} - \frac{\overline{\rho}}{\rho_0}\mathbf{g} + \nu \Delta \mathbf{v} - \nabla \cdot (\overline{\mathbf{v}'\varphi'}) - \partial_z(\overline{w'\mathbf{v}'}) \\
\nabla \cdot \overline{\mathbf{u}} &= 0 \\
\partial_t \overline{\varphi} + \nabla \cdot (\overline{\mathbf{u}}\overline{\varphi}) &= \overline{F}_\varphi - \nabla \cdot (\overline{\mathbf{v}'\varphi'}) - \partial_z(\overline{w'\varphi'}), \qquad (\varphi = \Theta, S_A) \\
\overline{\rho} &= \rho_{\mathrm{eos}}(\overline{\Theta}, \overline{S_A}, p_0(z))
\end{aligned}$$

Eddy-viscosity model conjectures (early work of Boussinesq & St Venant)

- Turbulent fluctuations have a dissipative/diffusive effect on the mean flow (energy source for the fluctuations and sink for the means)
- This dissipation/diffusion can be modelled by an eddy viscosity/diffusivity depending on mean flow quantities (about 10^6 times larger than molecular diffusion coefficient)

$$\overline{w'\mathbf{v}'} = -K_m(\overline{\varphi}, \overline{\mathbf{u}})\partial_z \overline{\mathbf{v}}, \qquad \overline{w'\varphi'} = -K_h(\overline{\varphi}, \overline{\mathbf{u}})\partial_z \overline{\varphi}$$

Omits kinetic energy flow from fluctuations back to the mean velocity (backscatter), non-local mixing, ...

Mathematical foundations of eddy viscosity models of turbulence

Layton, W. (2014) ***The 1877 Boussinesq Conjecture: Turbulent Fluctuations are Dissipative on the Mean Flow*****, TR 14-07**
Jiang et al. (2021) ***On the Foundations of Eddy Viscosity Models of Turbulence*****, Fluids**

Theorem 1. *Suppose that each realization is a strong solution of the NSE, the ensemble is generated by different initial data (and not different body forces or physical parameters) and that $u^0(x;\omega_j) \in L^2(\Omega)$, $f(x,t) \in L^\infty(0,\infty;L^2(\Omega))$. Then, for any sequence $T_j \to \infty$ for which the limits exist*

$$\underbrace{\lim_{T_j\to\infty} \frac{1}{T_j}\int_0^{T_j}\int_\Omega \overline{\mathbf{u}'\otimes\mathbf{u}'} : \nabla\overline{\mathbf{u}}\,\mathrm{d}\mathbf{x}\mathrm{d}t}_{\text{Time averaged effects of fluctuations on the KE of the mean flow}} = \underbrace{\lim_{T_j\to\infty} \frac{1}{T_j}\int_0^{T_j} \overline{\nu\|\nabla\mathbf{u}'\|^2}\,\mathrm{d}\mathbf{x}\mathrm{d}t}_{\text{dissipation of ''turbulent'' KE}} \geq 0$$

$\Rightarrow$ Reynolds stress act on average as an energy source for the fluctuations and sink for the means

Additional remarks

- An eddy-viscosity model consistent with the Prandtl-Kolmogorov relation (i.e. $K_m = c_m l\sqrt{E_K}$) has the correct aggregate effect
- In the space-time-average, fluctuations dissipate energy. EV models dissipate energy at every instant of time.

2

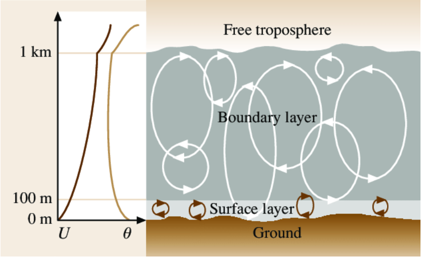
Boundary layers methods

Notion of "boundary layer" in OA context

Lemone et al. (2019) :

- Atmospheric boundary layer : "*part of the troposphere that is directly influenced by the presence of the earth's surface, and responds to surface boundary conditions with a timescale of about an hour or less*"

- In the surface layer the velocity passes from $\mathcal{O}(0)$ to $\mathcal{O}(1)$ over a short distance



Surface layer representation

Question : boundary condition for $\overline{\mathbf{v}'w'}$ and $\overline{w'\Theta'}$?

- In the near-wall region : constant flux layer

$$-\overline{\mathbf{v}'w'}_0 = u_\star^2$$

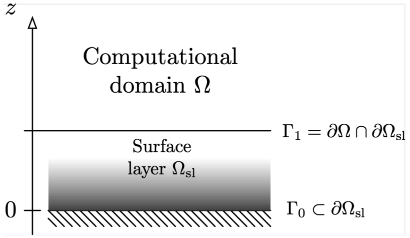
- Surface layer profile (flux-profile relationship)

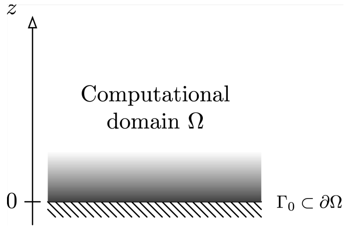
$$\overline{u}(z) = \frac{u_\star}{\kappa}\ln(z) + C$$

- The constant of integration C is used to characterize the geometrical properties of the underlying surface
$C = -\frac{u_\star}{\kappa}\ln(z_0)$

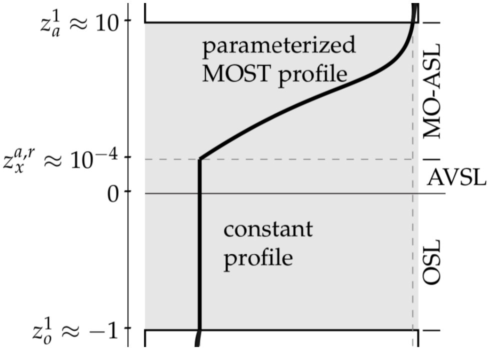
- Extended to moist stratified case by Monin-Obukhov (1954)

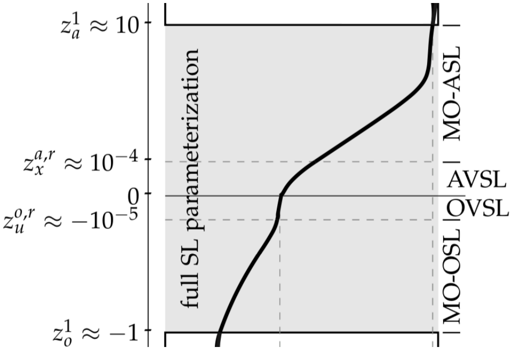
$$-\overline{w'\Theta'}_0 = u_\star\Theta_\star$$





Two-sided surface layer representation



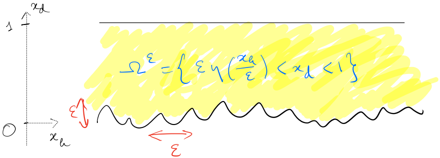


Pelletier et al. (2021), QJRMS

Wall laws for fluid flows

Example: Navier-Stokes in a rough channel

$$\begin{cases} -\Delta u^\epsilon + u^\epsilon \cdot \nabla u^\epsilon + \nabla p^\epsilon = f, \ x \in \Omega^\epsilon, \\ \mathrm{div} u^\epsilon = 0, \ x \in \Omega^\epsilon, \\ u^\epsilon_{|\partial\Omega^\epsilon} = 0. \end{cases} \qquad (\mathrm{NS}^\epsilon)$$



Question: effective boundary condition on a flat wall $x_d = 0$?

Answer:

- Main order: $u^\varepsilon \approx u^0$ such that $u^0_{|x_d=0} = 0$;
- Next order: $u^\varepsilon - u^N = O(\epsilon^{3/2})$ in L^2 (periodic case), where

$$u_d^N = 0, \ u_h^N = \epsilon \, M \partial_{x_d} u_h^N \quad \text{on } x_d = 0, \quad M \in \mathcal{M}_{d-1}(\mathbb{R}).$$

Derivation of the Dirichlet and Navier wall laws

Ansatz: $u^\varepsilon(x) \approx u^0(x) + \epsilon u_{\text{BL}}\left(x_h, \frac{x}{\epsilon}\right) + \epsilon u^1(x) + O(\epsilon^2)$ where

- u^0, u^1 are smooth;
- u_{BL} is a boundary layer term.

On the boundary $x_d = \epsilon\eta(x_h/\epsilon)$, expand the condition $u^\varepsilon = 0$:

- **Order ϵ^0:** $u^0(x_h, 0) = 0$;
- **Order ϵ^1:** Taylor expand u^0: with $y_h = x_h/\epsilon$,

$$u_{\text{BL}}(x_h; y_h, y_d) = -y_d \partial_{x_d} u^0(x_h, 0) \text{ on } y_d = \eta(y_h).$$

Boundary layer term u_{BL}: Stokes system in $\{y_d > \eta(y_h)\}$
$\rightsquigarrow \exists M \in \mathcal{M}_{d-1}(\mathbb{R})$,

$$\lim_{y_d \to \infty} u_{\text{BL}}(y_h, y_d) = (M\partial_d u_{0,h}(x_h, 0), 0).$$

3

Coupling two turbulent Ekman boundary layers

Motivation # 1 : Mathematical stability of parameterizations

- **An example :** analogy with a local Ri-dependent model ($\mathrm{Ri} = \frac{(g/\rho_0)\partial_z \rho}{\|\partial_z \mathbf{u}_h\|^2}$)

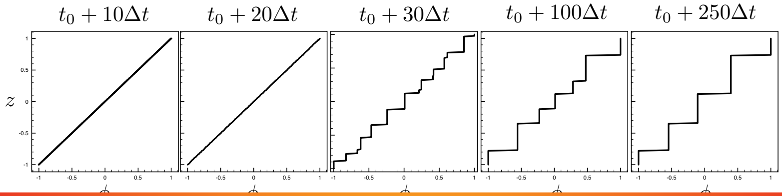
$$\partial_t \varphi = \partial_z \left(K_m(z) \partial_z \varphi \right), \qquad K_m(z) = (\partial_z \varphi)^{-2}$$

  - ▷ $K_m(z) > 0 \ \rightarrow \ \varphi$ remains bounded
  - ▷ Original equation can be re-expressed as

$$\partial_t \left(\partial_z \varphi \right) = \partial_z \left(\widetilde{K}_m(z) \partial_z \left(\partial_z \varphi \right) \right), \qquad \widetilde{K}_m(z) = -(\partial_z \varphi)^{-2}$$

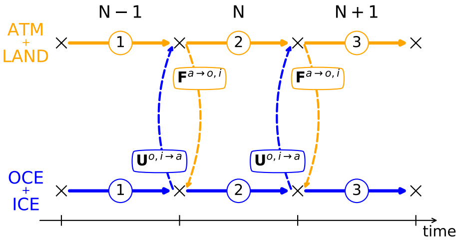
$\rightarrow$ the gradient can grow unbounded

- **Numerical test :** $\varphi(z, t = 0) = z$, $\varphi(z = -1, t) = -1$, $\varphi(z = 1, t) = 1$



Motivation # 2 : Ocean-atmosphere coupling

Standard algorithmic approach (e.g. in IPSL-CM and CNRM-CM) $\rightarrow$ loosely coupled solution



- **[Keyes et al. (2013)]**: *" Using an approach that ignores strong couplings between components gives a false sense of completion"*

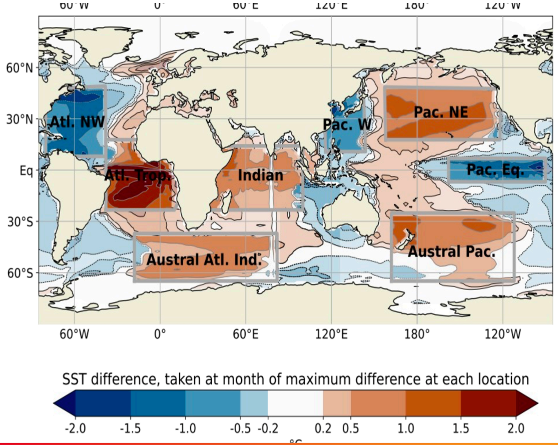
  If the iterations actually fail to converge, using only one iteration won't reveal this fact but in this case numerical results would be questionable

$\Rightarrow$ **BUT:** tight coupling between components requires "smoothness"

Motivation # 2 : ocean-atmosphere coupling

Marti et al. (2021) ***A Schwarz iterative method to evaluate ocean–atmosphere coupling schemes: implementation and diagnostics in IPSL-CM6-SW-VLR*****, Geosci. Model Dev.**

Use a global-in-time Schwarz iterative method to assess current coupling strategy in IPSL-CM6



Sea-surface temperature monthly averaged difference between loosely coupled and tightly coupled solutions from ensemble simulations (50 members)

Model problem

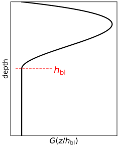
Ekman boundary layer theory's three-term balance equation

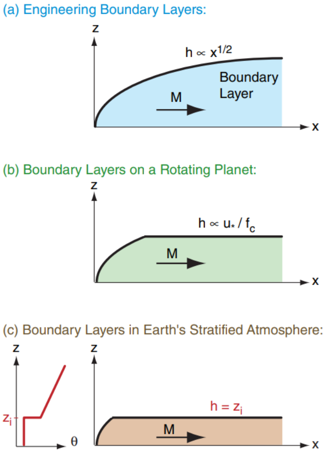
$$f\mathbf{e}_z \times (\mathbf{v} - \mathbf{v}_G) + \partial_z \overline{w'\mathbf{v}'} = 0$$

See Klein (2004) for a rigorous multiple scales derivation from the 3D compressible equations

- Original Ekman problem was laminar : $\partial_z \overline{w'\mathbf{v}'} \quad \rightarrow \quad -\nu \Delta \mathbf{v}$
- Turbulent Ekman problem with KPP eddy-viscosity model (Arya (1981); McWilliams & Huckle (2005)) : $\partial_z \overline{w'\mathbf{v}'} \quad \rightarrow \quad -\partial_z \left(K_m \partial_z \mathbf{v} \right)$

$$K_m(z) = \kappa \, u_\star \left(h_{\rm bl} G(z/h_{\rm bl}) \right), \qquad h_{\rm bl} = c_1 \frac{u_\star}{f}$$





Coupling two turbulent Ekman boundary layers

z_∞^{a}: $\mathbf{u}_h^{\mathrm{a}} = \mathbf{u}_G^{\mathrm{a}}$

$$f\mathbf{e}_z \times (\mathbf{u}_h^{\mathrm{a}} - \mathbf{u}_G^{\mathrm{a}}) = \partial_z \left(K_m^{\mathrm{a}}(z, u_\star)\partial_z \mathbf{u}_h^{\mathrm{a}}\right)$$

$$K_m^{\mathrm{a}}\partial_z \mathbf{u}_h^{\mathrm{a}}(\delta^{\mathrm{a}}) = u_\star^2 \frac{\delta\mathbf{u}}{\|\delta\mathbf{u}\|}$$

δ^{a}

$$u_\star^2 = C_D \|\mathbf{u}_h^{\mathrm{a}}(\delta^{\mathrm{a}}) - \mathbf{u}_h^{\mathrm{o}}(\delta^{\mathrm{o}})\|^2 = C_D\|\delta\mathbf{u}\|^2$$

δ^{o}

$$K_m^{\mathrm{o}}\partial_z \mathbf{u}_h^{\mathrm{o}}(\delta^{\mathrm{o}}) = \frac{\rho^{\mathrm{a}}}{\rho^{\mathrm{o}}} u_\star^2 \frac{\delta\mathbf{u}}{\|\delta\mathbf{u}\|}$$

$$f\mathbf{e}_z \times (\mathbf{u}_h^{\mathrm{o}} - \mathbf{u}_G^{\mathrm{o}}) = \partial_z \left(K_m^{\mathrm{o}}(z, u_\star)\partial_z \mathbf{u}_h^{\mathrm{o}}\right)$$

z_∞^{o}: $\mathbf{u}_h^{\mathrm{o}} = \mathbf{u}_G^{\mathrm{o}}$

Preliminary results : (Thery, 2021, PhD; in coll. with V. Martin)

Increasing the complexity :

- Unsteady case
- Stratification effects

Mathematical tools/strategy to approach the system

Fixed point argument: Construct an application $\mathcal{F} : v^* \mapsto \widetilde{v^*}$, where

$$
\begin{aligned}
\partial_t u_h^\alpha + f e_z \times (u_h^\alpha - u_G^\alpha) - \partial_z (K_m^\alpha(z, v^*) \partial_z u_h^\alpha), \quad z \in (\delta^\alpha, z_\infty^\alpha), \\
u_h^\alpha = u_G^\alpha \quad z = z_\infty^\alpha, \\
K_m^a \partial_z u_h^a(\delta^a) = \lambda^2 K_m^o \partial_z u_h^o(\delta^o), \\
K_m^a \partial_z u_h^a(\delta^a) = v^* \sqrt{C_D(v^*)}\,(u_h^a(\delta_a) - u_h^o(\delta_o))
\end{aligned}
$$

and

$$
\frac{(\widetilde{v^*})^2}{C_D(\widetilde{v^*})} = \|u_h^a(\delta_a) - u_h^o(\delta_o)\|^2.
$$

Mathematical issues:

- Definition of $\mathcal{F}$;
- Identification of invariant set for $\mathcal{F}$;
- Identification of a regime (+conditions on K_m^α, C_D) in which $\mathcal{F}$ is a contraction.

4

Analysis with nonlinear thermodynamics

Role of the seawater equation of state

- The Equation of State of seawater is an empirical thermodynamic relationship relating in situ density to temperature, salinity and pressure

$$\rho = \rho_{\text{eos}}(\Theta, S_A, p)$$

Essential link between the distribution of active tracers and the fluid dynamics

Most of the effort is to ensure an accurate fit of density measurements under the conditions of different temperature, salinity, and pressure.

Standard formulations :

- UNESCO-80
- TEOS-10
  - consistent derivation of the thermodynamics properties (e.g. density, heat capacity, internal energy, entropy, etc) based on a Gibbs function
  - take the form of a 48-term rational function of temperature, salinity and pressure

State of the art from the theoretical perspective

- **Compressible dynamics:**
  - Many works on Navier-Stokes-Fourier, but none with TEOS-10;
  - Work in progress with Didier Bresch and Marc Briant, with UNESCO-80 eq. of state, using asymptotic analysis (fast rotation, small aspect ratio, strong stratification...); need for specific relationships between diffusion coefficients for temperature/salinity.
- **Incompressible dynamics:** primitive equations:
  - Seminal works with truncated equation of state [Cao-Titi, Kukavica-Ziane]
  - Recent work by Peter Korn with (a simplification of) TEOS-10.
- **Key points:**
  - Obtain estimates on the derivatives of ρ;
  - For compressible models: existence of solutions on large time intervals.

Other potential topics of interest

- ”Super”-parameterization formulation via multiscale asymptotic analysis (e.g. Majda (2007))
- Analysis of multi-fluid models (e.g. hydrostatic and Non-hydrostatic fluids to represent oceanic convection)
- Derivation of thermodynamically consistent evolution equations for multi-component fluids (e.g. work of F. Gay-Balmaz)
- ...