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► **To cite this version:**

Santiago Bautista, Thomas Jensen, Benoît Montagu. Lifting Numeric Relational Domains to Algebraic Data Types (extended version). [Research Report] Centre Inria de l'Université de Rennes; Univ Rennes. 2022. hal-03765357v2

HAL Id: hal-03765357

<https://inria.hal.science/hal-03765357v2>

Submitted on 1 Sep 2022 (v2), last revised 26 Jun 2023 (v4)

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Lifting Numeric Relational Domains to Algebraic Data Types (extended version)

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Abstract. We present `RAND`, a relational abstract domain that expresses relations between values of non-recursive algebraic data types (ADTs), and numeric relations between their scalar parts. `RAND` is parametrised on a user-provided numeric relational domain, that we lift to pairs of variables and projection paths. It is constructed as a disjunctive completion of a reduced product of domains for numeric relations, for equalities, and for cases of variant constructors. Using `RAND`, we define a modular, inter-procedural, relational analysis for a `while` language with ADTs and function calls. The analysis computes function summaries, that describe relations between the inputs of programs and their outputs.

Keywords: Static Analysis · Abstract interpretation · Relational abstract domains · Algebraic data types · Input-output relations · Function summaries.



1 Introduction

Research in static analysis has successfully developed automatic techniques to ensure the safety and security of programs, by detecting bugs *before* a program actually runs. In particular, there exists a substantial number of analyses that target programs with numeric or pointer-based computations and which can detect frequent bugs that arise from arithmetic overflows or memory safety issues. Another important class of programs are those manipulating algebraic data types (ADTs). ADTs form the core of modern programming languages—such as OCaml, Haskell, Scala, Rust or Swift—that have been adopted by the software industry. The static analysis of this class of programs has seen important progress too,

with the development of type systems [37, 38] or by leveraging tree automata techniques [6] for approximating the tree structures described by ADTs [17, 29, 36].

In this paper, we focus on the automatic analysis of programs that perform numeric operations *and* manipulate ADTs. So far, few works [15, 24, 25] have put emphasis on the analysis of such programs. They provide additional safety guarantees specifically related to this combination—such as the unreachability of branches of a pattern matching. Such static analyses can also alleviate the interactive verification of large, critical programs that compute over ADTs, by automatically discharging a substantial number of proof obligations [1].

To this end, we develop a novel relational abstract domain that can express relations between numeric-algebraic values of a program state. We build this abstract domain in a generic way, by taking as a parameter *any* relational abstract domain that fulfils an Apron-like interface [23] to handle the numeric properties. One difficulty in designing this domain is to handle soundly and precisely the mutually exclusive cases that an algebraic value may take. We tackle this issue using *projection paths* that point inside algebraic values, and by devising a notion of *compatibility* between paths: two paths are compatible when they make consistent assumptions over the constructors of variant values. The resulting abstract domain can describe sets of states of algebraic data structures with scalar data.

Then, we show how to turn our abstract domain into RAND—the Relational Algebraic-Numeric Domain—an abstract domain that can express relations between *different* states. For an example process management program from an idealised operating system (Fig. 1), RAND can express that the input and output processes p and p' satisfy the constraint $p'.\text{status@Running.count} = p.\text{status@Asleep.count} + 1$, meaning that the `status` fields of p and p' differ by 1, whenever the process p has a *running* status, and p' a *sleeping* status. This is indicated by the projections on constructor cases `@Running` and `@Sleeping`. We discuss this example further in the paper (§2.3).

Using RAND, we define a *relational* analysis for a `while` language that features non-recursive ADTs. Our analysis infers relations between the inputs and the outputs of programs. In particular, we explain how a standard static analysis for reachable states can be turned into an analysis for input-output relations. This relational analysis is well suited for designing an inter-procedural analysis based on function summaries.

Our work offers the following contributions:

- We present RAND: a novel abstract domain that expresses relations between values of non-recursive ADTs (§ 3 and 4). Our abstract domain can be instantiated with any numeric relational domain. This offers the possibility to choose domains with different precision *vs* cost balances, and allows to capture numeric inequalities, which improves upon the correlation domain [1], that is restricted to information about equality and reachability.
- Our abstract domain uses a form of *disjunctive completion* (§ 3.6), where we limit the number of disjuncts by *merging* some of them. Our merging strategy is guided by observing the different *cases* of algebraic values.

- We give a formal justification to the folklore assertion that “*a static analysis can be made relational by duplicating variables*”, by showing that a non-relational and a relational analysis actually share the same *structure* (Lem. 1) and by showing how any relational domain can express relations between different stores (§4.2).
- We formally define a relational analysis (§5) that infers relations between inputs and outputs of programs, and propose a modular inter-procedural extension that is based on function summaries. We illustrate the analyser’s results on a running example taken from an idealised operating system.
- We provide an OCaml implementation [3] of our analyser, for a `while` language with algebraic types; together with 42 test cases, some of which are inspired from an operating system code.

2 Syntax and Semantics

Our programming language is an extension of a classic `while` language with algebraic data types (products and sums). §2.1 presents algebraic types, §2.2 presents the language and its semantics, and §2.3 introduces our running example.

2.1 Algebraic Types and Values

ADTs are pervasively used in functional languages like OCaml, Haskell, Coq, or F*, and have become a central feature of more recent programming languages, such as Swift or Rust, just to name a few. We briefly recall the definitions of algebraic types, and of the *structured values* that inhabit them.

Definition 1 (Algebraic types and structured values). Algebraic types and structured values are inductively defined as follows:

$$\begin{aligned} \tau \in \text{Types} & ::= \mathbb{N} \quad | \quad \{\overline{f_i \rightarrow \tau_i}^{i \in I}\} \quad | \quad [\overline{A_i \rightarrow \tau_i}^{i \in I}] \\ v \in \text{Values} & ::= \underline{n} \quad | \quad \{\overline{f_i = v_i}^{i \in I}\} \quad | \quad A(v) \end{aligned}$$

Here, \mathbb{N} is the type of numbers, the $(f_i)_{i \in I}$ are field names, the $(A_i)_{i \in I}$ are constructor names, and I ranges over finite sets. The compound type $\{\overline{f_i \rightarrow \tau_i}^{i \in I}\}$ is a *record type*, in which a type τ_i is associated to each field f_i . The type $[\overline{A_i \rightarrow \tau_i}^{i \in I}]$ is a *sum type* containing values formed with a head constructor that must be one of the A_i , and whose argument must be of type τ_i . $\{\overline{f_i = v_i}^{i \in I}\}$ denotes a *record value* where each field f_i has value v_i for every $i \in I$. $A(v)$ denotes a *variant value*, built by applying the constructor A to the value v .

We use *projection paths* to refer to a part of a structured value (*i.e.*, to a value embedded *inside* another structured value). A path is either the empty path ε , or the path $p.f$, that first accesses the value at path p and then accesses the record field f , or the path $p@A$, that first accesses the value at path p and then accesses the argument of variant constructor A .

Definition 2 (Paths). *Paths are inductively defined as follows:*

$$p \in \text{Paths} \quad ::= \quad \varepsilon \quad | \quad p.f \quad | \quad p@A$$

The *projection of the value v on the path p* , written $v \Downarrow^{\text{val}} p$, is the value pointed to by p inside v . It is defined as follows:

Definition 3 (Projection of a value on a path).

$$v \Downarrow^{\text{val}} p = \begin{cases} v & \text{if } p = \varepsilon \\ v' \Downarrow^{\text{val}} p' & \text{if } p = @Ap' \text{ and } v = A(v') \\ v_i \Downarrow^{\text{val}} p' & \text{if } p = .f_i p' \text{ and } v = \{\overline{f_j = v_j}^{j \in I}\} \text{ and } i \in I \\ \text{Undef} & \text{otherwise} \end{cases}$$

Our definition returns Undef when a path does not make sense for some value.

2.2 A Language With Algebraic Data Types

The syntax of the language consists of expressions t , boolean conditions b , and commands c . Vars denotes the set of variables that may appear in commands. Expressions include the projection of a variable $x \in \text{Vars}$ over a path $p \in \text{Paths}$, written $x.p$. The expression $t_1 \boxplus t_2$ denotes some arithmetic operations on the expressions t_1 and t_2 , and $t_1 \boxtimes t_2$ ranges over arithmetic comparisons.

$$\begin{aligned} t \in \text{Exp} & ::= \underline{n} \quad | \quad A(t) \quad | \quad \{\overline{f_i = t_i}^{i \in I}\} \quad | \quad x.p \quad | \quad t_1 \boxplus t_2 \\ b \in \text{BExp} & ::= t_1 \boxtimes t_2 \quad | \quad b_1 \wedge b_2 \quad | \quad b_1 \vee b_2 \quad | \quad \neg b \\ c \in \text{Cmd} & ::= \text{skip} \quad | \quad c_1 ; c_2 \quad | \quad \text{branch } c_1 \text{ or } \dots \text{ or } c_n \text{ end} \quad | \\ & \quad \text{while } b \text{ do } c \text{ end} \quad | \quad \text{assert } b \quad | \quad x := t \end{aligned}$$

We restrict our attention to well-typed commands (that we call *programs*), following a standard structural type system [38]. For instance, well-typedness ensures that arithmetic tests and operations receive arguments of integer type, and that every projection $x.p$ is consistent with the type of the variable x .

Programs operate on *stores*, denoted by s , that are finite maps from Vars to Values. We define the semantics of programs by means of a standard small-step semantics that specifies the effects of commands on stores. The relation $(c, s) \rightarrow (c', s')$ tells that the command c transforms the store s into a store s' , and that the command c' is to be executed next. We briefly explain the semantics of each commands, and refer the reader to appendix §A.1 for technical details.

The command **skip** performs no operation, whereas the sequence $c_1 ; c_2$ executes c_1 followed by c_2 . The branching command **branch c_1 or \dots or c_n end** non-deterministically chooses one of the commands c_i and executes it, discarding the other branches. The command **while b do c end** executes the command c as long as the condition b holds, and successfully terminates otherwise.

The command **assert(b)** tests whether the condition b holds, in which case it successfully terminates without performing any action. When b is not satisfied

`assert(b)` fails, *i.e.*, the program remains stuck. We can express the conditional construct `if b then c_1 else c_2` as `branch assert(b); c_1 or assert($\neg b$); c_2 end`.

Finally, the assignment command `$x := t$` evaluates t to some value v and updates the variable x with v . We write $s(x \mapsto v)$ to denote the store s in which the variable x is associated to the value v . If there was an entry for x in s already, then it is replaced with the value v . Otherwise, a new entry is created.

The evaluation $\llbracket t \rrbracket_s^{\text{exp}}$ of an expression t in a store s proceeds by induction on the structure of t to evaluate sub-expressions, and reads in the store s the values of variables. $\llbracket t \rrbracket_s^{\text{exp}}$ is either a singleton, which denotes normal execution, or the empty set, which denotes a failure, such as an invalid projection $x.p$. For example, $\llbracket x@A \rrbracket_s^{\text{exp}} = \emptyset$ when $s(x) = B(v)$, because the constructors A and B are different. The evaluation of booleans $\llbracket b \rrbracket_s^{\text{bool}}$ is standard.

Importantly, records and variants are *immutable* in our language: it is not possible to update some field f of a record *in-place*, for example. Instead, the programmer must follow the functional idiom, and create a new record value, that contains a different value for the field f .

We recover the *pattern matching* construct `match t with $A_1(x_1) \rightarrow c_1 \mid \dots \mid A_n(x_n) \rightarrow c_n$ end` as a syntactic sugar for command `$z := t$; branch $x_1 := z@A_1$; c_1 or \dots or $x_n := z@A_n$; c_n end` for a freshly chosen variable z .

2.3 Running Example

Fig. 1 shows an example program for which we would like to infer precise input-output properties. This program features algebraic data type definitions that represent the meta-data of a process, as usually found in operating system implementations. Here, a process is a record composed of an identifier, some incoming message that was sent by another process and finally a piece of data that describes the status of the process. The message is a record that contains some payload and whether it needs a reply (and to whom). The process status is either running, in which case it records how many times the process has been activated, or it is asleep, in which case it also records how many seconds the process should remain asleep before waking up again. The function `do_ticks(p , n)` simulates the action of n clock ticks on a process p : a clock tick leaves the process p unchanged if p is already running, or, if it is asleep, decrements the sleeping budget of p . If that budget is already equal to zero, the clock tick promotes p into a running process.

The important properties of `do_ticks(p , n)` that we intend to infer *automatically* are the following:

1. If p is initially running, then it remains unchanged;
2. If p is initially sleeping, then it *might* wake up: in this case, its original sleeping budget was less than n , and its number of activations `count` has been incremented by one;
3. If p is initially sleeping, then it *might* remain sleeping: in this case, its sleeping budget decreased by n , and its number of activations remains the same;
4. The field `id`, of integer type, of the process p has not changed;

```

type status = [
  (* Scheduling status *)
  | Running of { count: int }
    (* Running: wake-up times *)
  | Asleep of { secs: int; count: int }
    (* Sleeping: remaining seconds, wake-up times *)
]
type msg = {
  (* Messages *)
  data : int ;
    (* Payload *)
  reply : [
    (* Whether to reply or not *)
    | Reply of int
      (* Who to reply to *)
    | DontReply of {}
      (* No reply expected *)
  ]
}
type process = { id: int; msg: msg; status: status }
(* Process structure *)

def do_ticks(process p, int n) : process = {
  (* Performs n clock ticks on the process p *)
  int count; int secs; int i
  assert (n > 0)
  i = 0
  while (i < n) do (* loop n times, i.e.: perform n clock ticks *)
    branch (* case where p is running *)
      count = p.status@Running.count
    or (* case where p is asleep and can sleep longer *)
      assert (p.status@Asleep.secs > 0)
      count = p.status@Asleep.count
      secs = p.status@Asleep.secs
      p = { id = p.id; msg = p.msg;
        status = Asleep { secs = secs - 1; count = count } }
    or (* case where p is asleep and has no more sleeping budget *)
      assert (p.status@Asleep.secs = 0)
      count = p.status@Asleep.count
      p = { id = p.id; msg = p.msg;
        status = Running { count = count + 1 } }
    end
    i = i + 1
  end
  return p
}

```

Fig. 1: Example: performing clock ticks on a process' meta-data.

- The field `msg`, of record type, of the process `p` has not changed either.

Sections §3 to 5 explain in detail how we express and capture these properties by presenting the structure of the RAND abstract domain. Fig. 4 shows the output we obtain by running our analysis on our example. The correlation abstract domain [1] was also designed to handle programs that manipulate algebraic data types, but cannot express numeric properties other than equality. Using the correlation domain, we could infer all the properties listed above, *except* the ones that involve arithmetics.

3 Extending Numeric Domains to Algebraic Types

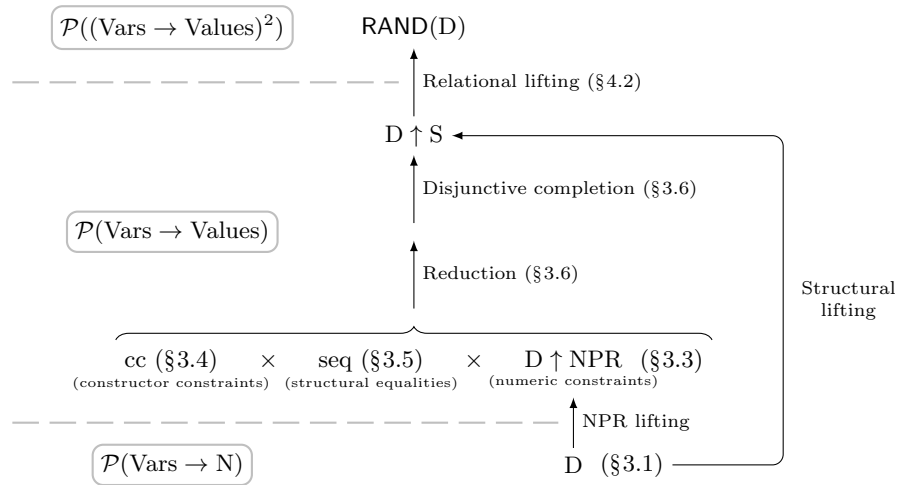


Fig. 2: The construction of the RAND abstract domain. The frame-enclosed sets are the sets the abstract domains concretize to.

In this section, we introduce an abstract domain, that is able to express equality and numeric constraints between parts of structured values. Our construction is summarised in Fig. 2. It is parametric with respect to a numeric abstract domain D , so that we can instantiate it on domains with different precision versus cost trade-offs. We expect the numeric domain D to provide the operations described in §3.1, which are a subset of the API offered by Apron [23]. An essential ingredient of our construction is the use of *extended variables* (§3.2), *i.e.*, regular program variables equipped with a projection path. Our example `do_ticks` on Fig. 1 features extended variables, such as `p.status@Asleep.secs`. Using extended variables, we define in §3.3 a first way to lift numeric domains to languages with algebraic types: the *Numeric Path Relations* lifting, or *NPR lifting* for short. This first lifting builds on the ideas of [2], but achieves a better

precision. It can express that a call to `do_ticks` can only decrease the value in the field `secs` of processes (that denotes the number of seconds for which a process should remain asleep), thanks to the constraint on extended variables $p.\text{status@Asleep.secs} \geq p'.\text{status@Asleep.secs}$. We improve the precision of the *NPR lifting* by combining it with two other domains in a product domain. A first domain of *constructor constraints* tracks which constructors are used for values of variant types (§3.4). Constructor constraints allow us to distinguish different cases, by stating which extended variables are valid in each case. For the `do_ticks` program, a possible case is when the input process `p` is sleeping—*i.e.*, `p.status@Asleep` is valid—and the output process `p'` is running—*i.e.*, `p'.status@Running` is valid. Another domain (§3.5) uses equality constraints between extended variables to express equalities that *must* hold between parts of structured values. With this domain, we can tell for the `do_ticks` program that the `msg` field of processes cannot change, by saying that the extended variables `p.msg` and `p'.msg` are equal. Finally, in order to obtain additional precision when analysing pattern-matching, we use a *disjunctive completion* of the product of these domains (§3.6): we obtain the *structural lifting* of the numeric abstract domain. Each value of the structural lifting can contain multiple cases, and each case has three components: one that expresses constructor constraints, one that expresses structural equalities, and one that expresses numeric constraints. §3.3 to 3.5 also define abstractions for assignments and conditionals, that are needed in §5 to define the analysis of our language.

3.1 Background: Numeric Abstract Domains

We first review the structure of traditional numeric domains [34] such as intervals, octagons and polyhedra. The domains are parametrised by a set of variables, and describe sets of *numeric* stores over those variables, *i.e.*, sets of maps from variables to numbers.

Given a set of variables V , we expect a numeric abstract domain $D(V)$ to provide the operations listed below (which are included in the user interface of the Apron library [23]) in such a way that the standard soundness properties of abstract interpretation [8, 9] are met: A set of abstract values $D(V)$ with a concretisation function $\gamma^{D(V)} \in D(V) \rightarrow \mathcal{P}(V \rightarrow \mathbb{N})$, a pre-order on abstract values $\sqsubseteq^{D(V)}$, abstract union $\sqcup^{D(V)}$ and intersection $\sqcap^{D(V)}$, and a widening operator $\nabla^{D(V)}$. The domain must also offer abstractions for boolean conditions $\text{Cond}^{D(V)} \in \text{BExp} \rightarrow D(V) \rightarrow D(V)$ and for assignment $\text{Assign}^{D(V)} \in V \times \text{Arith}(V) \rightarrow D(V) \rightarrow D(V)$ (where $\text{Arith}(V)$ is the set of arithmetic expressions over the variables V), satisfying the soundness properties:

$$\begin{aligned} \gamma^{D(V)}(\text{Assign}^{D(V)}(x := t)(d)) &\supseteq \{s(x \mapsto v) \mid s \in \gamma^{D(V)}(d) \wedge v \in \llbracket t \rrbracket_s^{\text{exp}}\} \\ \gamma^{D(V)}(\text{Cond}^{D(V)}(b)(d)) &\supseteq \{s \in \gamma^{D(V)}(d) \mid \mathbf{tt} \in \llbracket b \rrbracket_s^{\text{bool}}\} \end{aligned}$$

We also assume the existence of “variable management” operators for removing, adding and renaming variables. The operator $\text{Rem}_{V'}^{D(V)}$ projects an element of $D(V)$ onto $D(V \setminus V')$. The operator $\text{Add}_{V'}^{D(V)}$ embeds an element into the domain

$D(V \cup V')$. Given a bijection $r : V_1 \rightarrow V_2$, the operator $\text{Rename}_r^{\text{D}(V_1)}$ translates an element of $D(V_1)$ into $D(V_2)$. These operators satisfy the soundness properties:

$$\begin{aligned} \gamma^{\text{D}(V \setminus V')}(\text{Rem}_{V'}^{\text{D}(V)}(d)) &\supseteq \{s|_{(V \setminus V')} \mid s \in \gamma^{\text{D}(V)}(d)\} \\ \gamma^{\text{D}(V \cup V')}(\text{Add}_{V'}^{\text{D}(V)}(a)) &\supseteq \{s : (V \cup V') \rightarrow \mathbb{N} \mid s|_V \in \gamma^{\text{D}(V)}(a)\} \\ \gamma^{\text{D}(V_2)}(\text{Rename}_r^{\text{D}(V_1)}(d)) &\supseteq \{s \mid s \circ r \in \gamma^{\text{D}(V_1)}(d)\} \end{aligned}$$

3.2 Extended Variables

We call *extended variable* the pair of a variable and a path. Extended variables designate some values that are located *inside* a structured value. We only consider paths that make sense for the given variables, *i.e.*, paths whose projections on a variable's type are valid in the following sense:

Definition 4 (Projection of a type on a path). *The judgement $\tau \Downarrow^{\text{typ}} p$ defines when a path p is consistent with a type τ , and is inductively defined by:*

$$\frac{}{\tau \Downarrow^{\text{typ}} \varepsilon} \quad \frac{\tau_i \Downarrow^{\text{typ}} p \quad i \in I}{\{f_j \rightarrow \tau_j^{j \in I}\} \Downarrow^{\text{typ}} .f_i p} \quad \frac{\tau_i \Downarrow^{\text{typ}} p \quad i \in I}{[A_j \rightarrow \tau_j^{j \in I}] \Downarrow^{\text{typ}} @A_i p}$$

Typing contexts, written Γ , are mappings from variables to types. We write $\mathcal{E}(\Gamma) = \{x.p \mid x \in \text{dom } \Gamma \wedge \Gamma(x) \Downarrow^{\text{typ}} p\}$ for the set of extended variables $x.p$ such that p is consistent with the type of x in Γ .

We say that two extended variables $x.p_1$ and $x.p_2$ are *incompatible*—written $x.p_1 \langle \rangle x.p_2$ —if they would force a value (or part of a value) to be in two different variants of a sum type. Def. 5 formalises this notion of incompatibility, using the prefix order \preceq on extended variables ($x.p \preceq y.q$ iff $x = y$ and p is a prefix of q).

Definition 5 (Incompatibility and inconsistency). *Two extended variables $x_1.p_1$ and $x_2.p_2$ are incompatible, written $x_1.p_1 \langle \rangle x_2.p_2$, if and only if $x_1 = x_2$ and there is a path p and two distinct constructors A_1 and A_2 , such that $x_1.p @ A_1 \preceq x_1.p_1$ and $x_2.p @ A_2 \preceq x_2.p_2$. A set of extended variables E is inconsistent if it contains two or more incompatible extended variables. Two sets of extended variables are incompatible, written $E_1 \langle \rangle E_2$, if their union is inconsistent.*

We will see in §3.4 that inconsistent sets of extended variables denote empty sets of stores. Such inconsistent sets correspond to unreachable program points, and can be safely removed from the disjunctive completion of §3.6.

Assignment decomposition. To easily define the abstract transfer functions for assignment in the next subsections, it is useful to decompose an assignment command $x := t$ —where t can be a compound expression—into an equivalent set of *parallel* assignments of the form $x.p := t'$, where t' is either an expression of numeric type or an extended variable. The idea is to model the effect of the assignment as a set of parallel assignments on the paths of variable x .

Definition 6. *The decomposition of the assignment $x := t$ is defined by:*

$$\text{Decomp}(x.p := t) = \begin{cases} \bigcup_{i \in I} \text{Decomp}(x.p.f_i := t_i) & \text{if } t = \overline{\{f_i = t_i\}}^{i \in I} \\ \text{Decomp}(x.p@A := t') & \text{if } t = A(t') \\ \{x.p := t\} & \text{if } \Gamma \vdash t : \mathbb{N} \vee t \in \mathcal{E} \end{cases}$$

We write $\text{Decomp}(x := t)$ as a shorthand for $\text{Decomp}(x.\varepsilon := t)$.

3.3 Numeric Domains Over Extended Variables

In this section, extending ideas from [2], we define the *Numeric Path Relations lifting* $\text{D} \uparrow \text{NPR}$ as a generic way to lift a domain D that is numeric—*i.e.*, that denotes sets of stores that map variables to numbers—to a domain that denotes sets of stores that map variables to *structured* values. The main idea is to use *extended variables* as the variables of the underlying numeric domain.

For a typing context Γ , the abstract values of $\text{D} \uparrow \text{NPR}(\Gamma)$ are pairs of a set E of extended variables that are valid in Γ , and a numeric abstract value from $\text{D}(E)$ —*i.e.*, whose variables are the extended variables in E .

Definition 7. $\text{D} \uparrow \text{NPR}(\Gamma) = \{(d, E) \mid E \in \mathcal{P}(\mathcal{E}(\Gamma)) \wedge d \in \text{D}(E)\}$

In this definition, an abstract numeric value d can refer to any extended variable $x.p$ declared in E , and does not need to reason on whether $x.p$ is a valid projection. In practice, though, the complete domain of §3.6 will only consider sets E that are consistent. When writing examples in the rest of the paper, we may omit the set E when it can be deduced from context, for example when E is exactly the set of extended variables used in d .

Intuitively, an abstract value (d, E) denotes a set of stores that map regular variables to structured values, such that the paths listed in E point to integer values, and such that those integers are related by the numeric abstract value d . Using the projection function for values (Def. 3), it is easy to transform a store whose indices are variables into a store whose indices are *extended variables* :

Definition 8 (Projection of a store). *The projection of a store $s \in \text{Vars} \rightarrow \text{Values}$ on a set of extended variables $E \in \mathcal{P}(\mathcal{E})$ is a store in $E \rightarrow (\text{Values} \cup \{\text{Undef}\})$, written $s \Downarrow^{\text{sto}} E$, and is defined by: $(s \Downarrow^{\text{sto}} E)(x.p) = s(x) \Downarrow^{\text{val}} p$.*

The concretisation of an element $(d, E) \in \text{D} \uparrow \text{NPR}(\Gamma)$ easily follows: it is the set of well-typed stores whose projections on E satisfy the numeric constraints d . The typing judgement $\Gamma \vdash s$ means that $s(x)$ has type $\Gamma(x)$ for every x .

Definition 9. $\gamma^{\text{D} \uparrow \text{NPR}(\Gamma)}(d, E) = \{s \mid \Gamma \vdash s \wedge s \Downarrow^{\text{sto}} E \in \gamma^{\text{D}(E)}(d)\}$

We briefly explain how to define the abstract intersection in $\text{D} \uparrow \text{NPR}(\Gamma)$. The intersection of (d_1, E_1) and (d_2, E_2) denotes the conjunction of the constraints d_1 and d_2 . Therefore, the extended variables that appear in the conjunction

are in $E_1 \cup E_2$. Thus, one must inject d_1 and d_2 in $E_1 \cup E_2$ using the $\text{Add}_{E_j}^{\text{D}(E_i)}$ operators, before actually taking their intersection in the numeric domain:

$$(d_1, E_1) \sqcap^{\text{D}\uparrow\text{NPR}(\Gamma)} (d_2, E_2) = \left(\text{Add}_{E_2}^{\text{D}(E_1)}(d_1) \sqcap^{\text{D}(E_1 \cup E_2)} \text{Add}_{E_1}^{\text{D}(E_2)}(d_2), E_1 \cup E_2 \right)$$

The pre-order, union and widening are defined in a similar way :

$$\begin{aligned} (d_1, E_1) \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} (d_2, E_2) &\text{ iff } E_2 \subseteq E_1 \wedge \text{Rem}_{E_1 \setminus E_2}^{\text{D}(E_1)}(d_1) \sqsubseteq^{\text{D}(E_2)} d_2 \\ (d, E) \sqcap^{\text{D}\uparrow\text{NPR}(\Gamma)} (d', E') &= \left(\text{Add}_{E'}^{\text{D}(E)}(d) \sqcap^{\text{D}(E \cup E')} \text{Add}_E^{\text{D}(E')}(d'), E \cup E' \right) \\ (d, E) \sqcup^{\text{D}\uparrow\text{NPR}(\Gamma)} (d', E') &= \left(\text{Rem}_{E \setminus E'}^{\text{D}(E)}(d) \sqcup^{\text{D}(E \cap E')} \text{Rem}_{E' \setminus E}^{\text{D}(E')}(d'), E \cap E' \right) \\ (d, E) \nabla^{\text{D}\uparrow\text{NPR}(\Gamma)} (d', E') &= \left(\text{Rem}_{E \setminus E'}^{\text{D}(E)}(d) \nabla^{\text{D}(E \cap E')} \text{Rem}_{E' \setminus E}^{\text{D}(E')}(d'), E \cap E' \right) \end{aligned}$$

Transfer Functions. The transfer function for assignment $x := t$ works by temporarily introducing a new variable x' (that represents the value of x *after* assignment). First, it applies the transfer function for assignment on every numeric assignment in the decomposition of $x' := t$ (Def. 6). Then, it removes the references to the paths of x , and finally renames x' into x . The auxiliary variable x' is introduced to avoid clashes between the paths that are valid for x *before* the assignment and those that are valid *after* the assignment.

$$\text{Assign}^{\text{D}\uparrow\text{NPR}(\Gamma)}(x := t)(d, E) = \text{Rename}_{[x' \mapsto x]}^{\text{D}\uparrow\text{NPR}(\Gamma)}(\text{Rem}_{E_x}^{\text{D}\uparrow\text{NPR}(\Gamma)}(d_1, E_1))$$

where $\begin{cases} E_x = \{y.p \in E \mid y = x\} \\ d_1 = \prod_{x'.p := u \in \text{Decomp}(x' := t), \Gamma \vdash u : \text{N}} \text{Assign}^{\text{D}(E_1)}(x'.p := u)(\text{Add}_{E_0}^{\text{D}(E)}(d)) \\ E_1 = E \cup E_0 \\ E_0 = \{y.p \in \text{Env}(\text{Decomp}(x' := t)) \mid \Gamma \vdash y.p : \text{N}\} \end{cases}$

The transfer function for conditionals is simpler: negation is eliminated using De Morgan laws, whereas conjunctions and disjunctions are handled by abstract intersection and union, respectively. The remaining case is the one of a numeric test $t_1 \bowtie t_2$: it suffices to call the transfer function of domain D for conditionals, and to extend the extended variables with those that occur in the test:

$$\text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)}(b)(d, E) = \left(\text{Cond}^{\text{D}(E \cup \text{Env}(b))}(b)(\text{Add}_{\text{Env}(b)}^{\text{D}(E)}(d)), E \cup \text{Env}(b) \right) \text{ if } b = t_1 \bowtie t_2$$

3.4 Constructor Constraints

We introduce the abstract domain of *constructor constraints*, that intuitively describes in which *cases* the values of a store might be, *i.e.*, which are the allowed variant constructors of the values of a store. We write $\text{cc}(\Gamma)$ for the set of constructor constraints for a typing environment Γ . An element $E \in \text{cc}(\Gamma)$ is a set of extended variables, that restricts the possible sets of stores to those that are compatible with *every* path in E . In other words, if a path in E mentions

some constructor, then the corresponding value in any store of the concretisation must be built using that constructor.

Constructor constraints are a key ingredient of the disjunctive completion of §3.6, as they serve as hints for which disjuncts need to be kept separate, and which should be merged.

An element $E \in \text{cc}(\Gamma)$ is either the bottom value $\perp^{\text{cc}(\Gamma)}$, or must be a *consistent* set of extended variables, that is *closed under the prefix order* \preceq .

Definition 10 (Constructor constraints). *The domain of constructor constraints is defined by $\text{cc}(\Gamma) = \{E \subseteq \mathcal{E}(\Gamma) \mid E \preceq\text{-closed and consistent}\} \cup \{\perp^{\text{cc}(\Gamma)}\}$ and is equipped with the ordering $\sqsubseteq^{\text{cc}(\Gamma)}$ defined as $E_1 \sqsubseteq^{\text{cc}(\Gamma)} E_2$ iff $E_1 = \perp^{\text{cc}(\Gamma)}$ or $E_1 \supseteq E_2$.*

We write $\text{clos}^{\text{cc}(\Gamma)}(E)$ to denote the prefix-closure of E , *i.e.*, the smallest \preceq -closed set that contains E . For a given Γ , the domain $\text{cc}(\Gamma)$ is *finite*: because our types are not recursive, the valid paths necessarily have finite lengths.

The concretisation $\gamma^{\text{cc}(\Gamma)}$ defines the stores denoted by constructor constraints.

Definition 11 (Concretisation for constructor constraints).

$$\gamma^{\text{cc}(\Gamma)}(\perp^{\text{cc}(\Gamma)}) = \emptyset \quad \gamma^{\text{cc}(\Gamma)}(E) = \{s \mid \Gamma \vdash s \wedge \forall x.p \in E, s(x) \Downarrow^{\text{val}} p \neq \text{Undef}\}$$

The concretisation of a set E produces a set of well-typed stores such that the values in the stores can be projected along the paths in E .

The abstract union and intersection for the $\text{cc}(\Gamma)$ domain are easily obtained:

$$\begin{aligned} \perp^{\text{cc}(\Gamma)} \sqcup^{\text{cc}(\Gamma)} E &= E \sqcup^{\text{cc}(\Gamma)} \perp^{\text{cc}(\Gamma)} = E & E_1 \sqcup^{\text{cc}(\Gamma)} E_2 &= E_1 \cap E_2 \text{ otherwise} \\ E_1 \sqcap^{\text{cc}(\Gamma)} E_2 &= \begin{cases} \perp^{\text{cc}(\Gamma)} & \text{if } E_1 = \perp^{\text{cc}(\Gamma)} \text{ or } E_2 = \perp^{\text{cc}(\Gamma)} \text{ or } E_1 \prec E_2 \\ E_1 \cup E_2 & \text{otherwise} \end{cases} \end{aligned}$$

Because the domain is finite, there is no issue with infinite ascending chains, and we can simply define the widening as the abstract union.

Transfer Functions. We express the abstract transfer function for assignment in the $\text{cc}(\Gamma)$ domain in a standard “*kill-gen*” form as follows:

$$\text{Assign}^{\text{cc}(\Gamma)}(x := t)(E) = (E \setminus \text{Kill}^{\text{cc}(\Gamma)}(x)(E)) \sqcap^{\text{cc}(\Gamma)} \text{Gen}^{\text{cc}(\Gamma)}(x := t)(E)$$

$$\text{where } \text{Kill}^{\text{cc}(\Gamma)}(x)(E) = \{y.p \in E \mid y = x\}$$

$$\text{and } \text{Gen}^{\text{cc}(\Gamma)}(x := t)(E) =$$

$$\text{clos}^{\text{cc}(\Gamma)}(\{y.q \in \text{Env}(t) \mid y \neq x\} \cup \{x.p \mid \exists t', x.p := t' \in \text{Decomp}(x := t)\})$$

The extended variables that must be removed are those that have x as root, since the new value for x might be modified by the assignment. The newly added extended variables are those of t that are still live after x is updated, and the ones that are effectively assigned, as given by the decomposition of the assignment. We ensure that the added variables remain prefix-closed thanks to a call to $\text{clos}^{\text{cc}(\Gamma)}$.

The transfer function for conditionals is straightforward: it only adds the extended variables of the boolean expression:

$$\text{Cond}^{\text{cc}(\Gamma)}(b)(E) = E \sqcap^{\text{cc}(\Gamma)} \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(b))$$

3.5 Structural Equalities

The NPR lifting of §3.3 can only express relations between the *numeric* parts of values. It can't record whether some non-numeric part of a value has not changed. In our example of Fig. 1, this is the case of the `msg` field of processes, that is not modified, and is of record type. We introduce in this section the relational domain $\text{seq}(\Gamma)$, that tracks the *structural equalities*, *i.e.*, which parts of the values of a store *must* be identical.

One could argue that any equality between structured values could be replaced with a conjunction of equalities between the integer fields of those values, and, consequently, that the $\text{seq}(\Gamma)$ domain is hardly useful. Such a decomposition could lead, however, to more verbose abstract values, and could also introduce extra disjunctions when dealing with values of sum types. Thus, our choice of handling equality constraints between structured values is beneficial, as it helps keep our abstract values small in size.

The abstract values of $\text{seq}(\Gamma)$ are either the bottom element—denoting the empty set of stores—or a finite set of pairs of extended variables $(x.p, y.q)$ —denoting a set of stores s in which the value at path p in the value $s(x)$ is equal to the one at path q in $s(y)$. Def. 12 gives a simplified definition of the domain. In practice, our implementation uses a map from extended variables to equivalence class indices, to ensure we remain closed by reflexivity, symmetry and transitivity.

Definition 12 (Domain of structural equalities). *The domain of structural equalities $\text{seq}(\Gamma) = \mathcal{P}(\mathcal{E}(\Gamma) \times \mathcal{E}(\Gamma)) \cup \{\perp^{\text{seq}(\Gamma)}\}$ is equipped with the concretisation function $\gamma^{\text{seq}(\Gamma)} \in \text{seq}(\Gamma) \rightarrow \mathcal{P}(\text{Vars} \rightarrow \text{Values})$ that is defined as follows:*

$$\gamma^{\text{seq}(\Gamma)}(\perp^{\text{seq}(\Gamma)}) = \emptyset$$

$$\gamma^{\text{seq}(\Gamma)}(C) = \{s \mid \Gamma \vdash s \wedge \forall (x.p, y.q) \in C, s(x) \Downarrow^{\text{val}} p = s(y) \Downarrow^{\text{val}} q \neq \text{Undef}\}$$

Abstract values in this domain might carry some implicit information. For example, if x and y have a type $\{f \rightarrow \mathbb{N}; g \rightarrow \mathbb{N}\}$, the abstract value $\{(x, y)\}$ also *implicitly* implies that $x.f = y.f$ and $x.g = y.g$. To avoid losing precision, it is sometimes necessary to *saturate* an abstract value by congruence with all the equalities it implies *with respect to a given set of extended variables*. For this purpose, we define a closure operator.

Definition 13 (Closure of structural equalities). *The closure of a set of structural equalities C with respect to a set of extended variables E is written $\text{clos}^{\text{seq}(\Gamma)}_E(C)$ and defined as the smallest set larger than C , mentioning the variables in E , closed by symmetry, reflexivity and transitivity, and verifying the following property of congruence:*

$$\left. \begin{array}{l} (x.p, y.q) \in \text{clos}^{\text{seq}(\Gamma)}_E(C) \\ (x.pr) \in \text{Env} \left(\text{clos}^{\text{seq}(\Gamma)}_E(C) \right) \end{array} \right\} \Rightarrow (x.pr, y.qr) \in \text{clos}^{\text{seq}(\Gamma)}_E(C)$$

The need for a closure operator is not surprising, as it occurs in other relational domains, like octagons [33]. We use this closure operator to gain precision in the

transfer function for assignment, and in the reduction operator of the product domain of §3.6.

Transfer Functions. The transfer function for assignment $x := t$ exploits the decomposition of assignments from Def. 6, but considers only the assignments of the form $x.p := y.q$, where the right-hand side is an extended variable. We express the transfer function in a “kill-gen” form, where we kill every equality that involves x , and add the new equalities $x.p = y.q$ where we are careful to avoid any use of x that refers to the state *before* the assignment.

$$\begin{aligned} \text{Assign}^{\text{seq}(\Gamma)}(x := t)(C) &= \left(C \setminus \text{Kill}^{\text{seq}(\Gamma)}(x)(C) \right) \cup \text{Gen}^{\text{seq}(\Gamma)}(x := t)(C) \\ \text{where } \text{Kill}^{\text{seq}(\Gamma)}(x)(C) &= \{(y.p, z.q) \in C \mid y = x \vee z = x\} \\ \text{and } \text{Gen}^{\text{seq}(\Gamma)}(x := t)(C) &= \\ &\bigcup_{x.p := y.q \in \text{Decomp}(x := t)} \{(x.p, z.r) \mid z \neq x \wedge (y.q, z.r) \in \text{clos}^{\text{seq}(\Gamma)}_{\{y.q\}}(C)\} \end{aligned}$$

The transfer functions for conditionals can only exploit equality tests between extended variables: $\text{Cond}^{\text{seq}(\Gamma)}(b)(C) = C \sqcap^{\text{seq}(\Gamma)} \{(x.p, y.q)\}$ if b is $x.p = y.q$.

3.6 Bringing Everything Together: Product Domain and Disjunctive Completion

In this section, we describe the remaining steps of our construction, that lead to the *structural lifting* $D \uparrow S$ of the numeric domain D that we have considered. We combine the domains we have defined in the previous sections—the constructor constraints (§3.4), the structural equalities (§3.5), and the NPR lifting (§3.3)—into a *reduced product*, and then add a disjunctive completion layer on top of that product. We will ultimately obtain the domain of relations **RAND** once we apply the relational lifting defined in §4.2.

Reduced Product Our reduced product is based on a reduction operator ρ , that enables information transfer between the different domains of the product.

Definition 14 (Reduction operator). *The reduction operator ρ for the product of constructor constraints, structural equalities and the NPR lifting is defined as follows:*

$$\rho(E, C, N) = \left(\begin{array}{l} E \sqcap^{\text{cc}(\Gamma)} \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(C')), \\ C', \\ \text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)}(\bigwedge_{(x.p, y.q) \in C' \wedge \Gamma \vdash x.p : N} x.p = y.q)(N) \end{array} \right)$$

where $C' = \frac{\text{clos}^{\text{seq}(\Gamma)}}{\text{clos}^{\text{cc}(\Gamma)}(\text{Env}((E, C, N)))}(C)$

The reduction operator ρ transfers the following pieces of information between the three components of the product:

- Structural equalities are completed with additional constraints, so that all the extended variables that are used in the constructor constraints and the numeric constraints (this is the role of C') are mentioned.
- If some equalities between integers are deduced from the structural equalities, then they are added to the numeric constraints.
- The extended variables from the structural equalities and the numeric constraints are added to the constructor constraints, which may reveal some inconsistent cases.

Union, intersection and widening for the reduced product domain are computed by first closing structural equalities and reducing the product, then proceeding component-wise. For widening, closure is only applied to the right-hand side argument to avoid interfering with convergence. We invite the reader to look at appendix §A.7 for further details. The transfer functions for assignment and conditionals use the transfer functions of each component.

Using Disjunctions to Handle Incompatible Cases Pattern matching performs a case analysis on the different constructors a value may start with: these cases are pairwise incompatible. To analyse pattern matching with precision, we add disjunctions to our abstract domain by means of a disjunctive completion, so that each pattern matching case has a distinct disjunct. Hence, for any numeric domain D , we take the disjunctive completion [8] of the reduced product of constructor constraints, structural equalities and the NPR lifting of D . We call this the *structural lifting* of D , written $D \uparrow S(\Gamma)$, and defined as $D \uparrow S(\Gamma) = \mathcal{P}(\text{cc}(\Gamma) \times \text{seq}(\Gamma) \times D \uparrow \text{NPR}(\Gamma))$. To control the number of disjuncts, however, we *merge* some cases together: merging is performed when the constructor constraints of two abstract values concretise to the same sets of stores—*i.e.*, when they impose the same constraints on the constructors used for variant values. The technical details are provided in appendix §A.8.

4 A Collecting Semantics of Relations

We define a *relational* semantics of programs, that determines the relation that relates the input stores of a program with its output stores, *i.e.*, the stores that are obtained when there is no more command to evaluate.

Definition 15 (Relational semantics). *The relational semantics of a command c is defined as follows: $\mathbb{S} \llbracket c \rrbracket = \{(s_1, s_2) \mid (c, s_1) \rightarrow^* (\text{skip}, s_2)\}$.*

$\mathbb{S} \llbracket c \rrbracket$ is a binary relation that may be employed to derive fully compositional static analyses, such as CRA [13, 27]. Indeed, it enjoys equations (*e.g.*, $\mathbb{S} \llbracket c_1 ; c_2 \rrbracket = \mathbb{S} \llbracket c_1 \rrbracket ; \mathbb{S} \llbracket c_2 \rrbracket$) that help defining the analysis of a compound command from the *independent* analyses of its constituents. A drawback of this approach, however, is its inability to exploit any information about the states that have been reached so far, which may degrade the precision of an analysis. The following piece of code illustrates this issue:


```

assert (x > 1 && y > 1);
x := y * x

```

If we analyse the assignment $x := y * x$ with no knowledge that the preceding assertion succeeded, then, using a linear relational domain—*e.g.*, octagons or polyhedra—we will not obtain any precise information about how the value of x has changed, as the domain cannot express non-linear relations.

The relational collecting semantics of the next section waives this limitation, as it allows to exploit the information that has so far been obtained for the current program point.

4.1 Relational Collecting Semantics

In this section, we build a collecting relational semantics on top of $\mathbb{S}[[c]]$, that can exploit the information about the states that have been reached. Let us consider again the example from the previous paragraph: with the knowledge that the assertion succeeded, then a linear relational domain will be able to express that x has strictly increased, for example. Our collecting semantics $\mathbb{P}[[c]]$ is a function from relations to relations: given some initial relation a that holds between initial stores s_i and the stores s_b at the current program point (*before* the execution of c), $\mathbb{P}[[c]](a)$ computes a relation between the initial stores s_i and the final stores s_f that are produced by evaluating the command c from the stores s_b . Thus, $\mathbb{P}[[c]]$ *extends* the relations *in time* by composing on the right-hand side with the behaviour of command c . Our collecting semantics is defined as follows:

Definition 16 (Collecting semantics). $\mathbb{P}[[c]](a) = a; \mathbb{S}[[c]]$

$\mathbb{P}[[c]]$ is an abstraction of a semantics of computation traces [7], that collects the intermediate stores that a program may reach. The collecting semantics $\mathbb{P}[[c]]$ enjoys the equations listed in the next lemma, that shows how it decomposes by following the syntax of commands.

Lemma 1. *The following equations hold:*

$$\begin{aligned}
\mathbb{P}[[\text{skip}]](a) &= a \\
\mathbb{P}[[c_1; c_2]](a) &= \mathbb{P}[[c_2]](\mathbb{P}[[c_1]](a)) \\
\mathbb{P}[[\text{branch } c_1 \text{ or } \dots \text{ or } c_n \text{ end}]](a) &= \bigcup_{1 \leq i \leq n} \mathbb{P}[[c_i]](a) \\
\mathbb{P}[[\text{while } b \text{ do } c \text{ end}]](a) &= \mathbb{P}[[\text{assert}(\neg b)]](\text{lfp } f_a) \\
&\quad \text{where } f_a(r) = a \cup \mathbb{P}[[c]](\mathbb{P}[[\text{assert}(b)]](r)) \\
\mathbb{P}[[\text{assert}(b)]](a) &= \{(s_1, s_2) \mid (s_1, s_2) \in a \wedge \llbracket b \rrbracket_{s_2}^{\text{bool}} = \{\mathbf{tt}\}\} \\
\mathbb{P}[[x := t]](a) &= \{(s_1, s_2(x \mapsto v)) \mid (s_1, s_2) \in a \wedge v \in \llbracket t \rrbracket_{s_2}^{\text{exp}}\}
\end{aligned}$$

The proof of Lem. 1 is available in appendix §A.4.

Lem. 1 shows that the syntax-directed decomposition of the relation transformer $\mathbb{P}[[c]]$ follows the same structure as the standard set-based collecting semantics, that collects the set of reachable states. The transfer functions of our collecting semantics are the same, but they operate on different objects (binary relations on stores instead of sets of stores). An important difference between the two styles of analyses is the choice of the most precise starting point for the

initial store, when no assumption is made on that store. In set-based analysis, that starting point is \top (the set of all stores), whereas in the relational analysis, the starting point is Id (the identity relation on stores).

Lem. 1 will serve as the semantic basis for the analysis that we describe in §5.2. Its equations show that the set-based and the relation-based collecting semantics share the same *structure*: the only parts that differ are the transfer functions for the assertion and for the assignment commands. We show in §4.2 how to define those two transfer functions—that transform relations that relate stores in *different* program points—using *any* relational abstract domain that represents sets of stores for one program point. Using these two results, we can turn a folklore technique into a formal claim: transforming a non-relational analysis into a relational one is “as simple as” duplicating variables [4, 18].

4.2 Leveraging Relations in Space to Express Relations in Time

In this section we show that any domain for sets of stores that can express binary relations between different variables of a single store, can be lifted to a domain for pairs of stores, able to express relations between input stores and output stores. The main idea is simple: a pair of stores $(s_1, s_2) \in (\text{Vars} \rightarrow \text{Values})^2$ can be represented as a single store, provided we can distinguish the variables in s_1 from those in s_2 .

Formally, this is achieved by assuming two bijections $prime : \text{Vars} \rightarrow \text{Vars}'$ and $second : \text{Vars} \rightarrow \text{Vars}''$ where Vars' and Vars'' are disjoint “copies” of Vars , that intuitively contain the “primed” and “seconded” versions of the variables of Vars . We write x' as a shorthand for $prime(x)$, and x'' for $second(x)$, use the same convention as in [13], *i.e.*, we use regular variables for the left-hand sides of relations—the input stores—and primed variables for the right-hand sides—the output stores. For any map f , we write f' as a shorthand for $f \circ prime^{-1}$, and we write $f \cup g$ for the union of maps with disjoint domains. This allows us to represent any pair (s_1, s_2) of stores as a single store $s_1 \cup s_2'$. We use this encoding to transform any relational domain that represents a set of stores into a domain that represents a binary relation over stores.

With this encoding, if some variable x is effectively present both in the input store s_1 and in the output store s_2 , it will occur twice—as x and as x' —in the store $s_1 \cup s_2'$. This gives a formal justification to the folklore knowledge that to design an input-output analysis, “it suffices to duplicate the variables”.

Definition 17 (Relational lifting). *Let A be an abstract domain, such that for any typing context Γ , $A(\Gamma)$ is equipped with concretisation function $\gamma^{A(\Gamma)} \in A(\Gamma) \rightarrow \mathcal{P}(\text{Vars} \rightarrow \text{Values})$. For any two typing contexts Γ_1 and Γ_2 , the relational lifting $A \uparrow R(\Gamma_1, \Gamma_2)$ of A and its concretisation function are defined as follows:*

$$A \uparrow R(\Gamma_1, \Gamma_2) = A(\Gamma_1 \cup \Gamma_2')$$

$$\gamma^{A \uparrow R(\Gamma_1, \Gamma_2)}(a) = \{(s_1, s_2) \mid s_1 \cup s_2' \in \gamma^{A(\Gamma_1 \cup \Gamma_2')}(a)\}$$

The relational lifting expects two typing contexts—one for the input stores, and one for the output stores. The fact of allowing different typing environments

for the store on the left and the store on the right will prove useful when analysing function calls. Indeed, the same variable name can represent variables with different types for the caller and the callee. Hence, the abstract values representing the transitions from caller to callee and from callee to caller will simultaneously handle two different typing contexts.

The lifted domain $A \uparrow R(\Gamma_1, \Gamma_2)$ is naturally equipped with order relation, abstract union, intersection and widening, by reusing those of $A(\Gamma_1 \cup \Gamma_2')$.

As we remarked in § 4.1, only two pieces are missing to get a relational input-output analysis: now that we can express relations on stores, the question remains of how to express the transfer functions for conditionals and assignments. We show in Fig. 3 how to do so in a generic way, by exploiting the transfer functions of the underlying domain.

$$\begin{aligned} \text{Cond}^{A \uparrow R(\Gamma_1, \Gamma_2)}(b)(a) &= \text{Cond}^{A(\Gamma_1 \cup \Gamma_2')}(b')(a) \\ \text{Assign}^{A \uparrow R(\Gamma_1, \Gamma_2)}(x := t)(a) &= \text{Rem}_{\Gamma_2''}^{A(\Gamma)} \left(\text{Assign}^{A(\Gamma)}(x' := t'') \left(\text{Add}_{\Gamma_2'}^{A(\Gamma_1 \cup \Gamma_2'')} c \right) \right) \\ &\quad \text{where } \Gamma = \Gamma_1 \cup \Gamma_2' \cup \Gamma_2'' \text{ and } c = \text{Rename}_{\text{second} \circ \text{prime}^{-1}}(a) \end{aligned}$$

Fig. 3: Relational transfer functions for conditionals and assignment.

The transfer function for conditionals $\text{Cond}^{A \uparrow R(\Gamma_1, \Gamma_2)}(b)(a)$ constrains the right-hand side of the relation a to satisfy the boolean expression b . This is achieved by calling the transfer function for conditions of the underlying domain on b' , to enforce that the variables of b refer to the outputs of a .

The transfer function for assignment $\text{Assign}^{A \uparrow R(\Gamma_1, \Gamma_2)}(x := t)(a)$ first renames the output variables y' of a into y'' . The variables y'' belong to the state that lies just *before* the assignment. Then, we call the transfer function for assignment from the underlying domain for the command $x' := t''$. This has the effect of extending a with relations that express the link between t and the new variable x . Then, we add the equalities $y'' = y'$ for all the variables other than x , because none of them was modified by the assignment. Finally, we eliminate the auxiliary variables y'' . This effectively builds a relation between the input state and the state that is obtained after the assignment.

This concludes our justification of the folklore claim that, “*to turn a static analysis for the sets of final states into an analysis for input-output relations, it suffices to duplicate variables*”. We have built our justification on the following remarks: 1. Duplicating variables turns a relational domain—*i.e.*, a relation between variables of the stores of a *single* program point—into a domain of binary relations between stores of *two* different program points. 2. A relational analysis has the same structure as an analysis for final states. 3. The transfer functions that are specific to the relational analysis can be defined in a generic way, using those of the analysis for final states.

```

Function summary for function do_ticks(p, n) returning p' :
(Constructor constraints : p.status@Running; p'.status@Running ...
 with structural equalities : p = p' ; ...
 and numeric constraints : n >= 1 ; ... )
Or (Constructor constraints : p.status@Asleep; p'.status@Running ...
 with structural equalities : p.msg = p'.msg
 and numeric constraints :
   p.id = p'.id; p'.status@Running.count = p.status@Asleep.count + 1;
   p.status@Asleep.secs >= 0; n >= p.status@Asleep.secs + 1 )
Or (Constructor constraints : p.status@Asleep; p'.status@Asleep ...
 with structural equalities : p.msg = p'.msg
 and numeric constraints :
   p.id = p'.id; p.status@Asleep.secs >= n; n >= 1;
   p.status@Asleep.count = p'.status@Asleep.count;
   p'.status@Asleep.secs = p.status@Asleep.secs - n )

```

Fig. 4: Result of our analysis on the example of Fig. 1. Ellipses mark information that is also present in other components of the same case and is elided.

In the rest of this article, we use the relational lifting of the abstract domain from §3, that we call *RAND*—short for *Relational Algebraic and Numeric Domain*.

5 Analysis

This section explains how to use the abstract domain built in §3 and 4, to analyse the language described in §2.2. After providing an example that illustrates what the analysis computes (§5.1), we first describe an intra-procedural analysis (§5.2) and then extend it to support function calls, yielding a modular, summary-based, inter-procedural analysis (§5.3).

The inter-procedural version of our analysis does not currently handle recursive or mutually recursive functions, as we chose to focus solely on the topic of handling algebraic values and arithmetic relations. Nevertheless, this can be achieved by a widened fixpoint iteration sequence at the level of function summaries.

5.1 Analysis Result For the `do_ticks` Function

Before giving the formal description of the analysis we give an example of the properties that it can infer. Fig. 4 shows the result of running our analyser on the example from §2.3 using polyhedra as a numeric domain. We see that our disjunctive completion considers three different cases, and contains all five properties that we wanted to infer automatically. In the first case, both the input and the output are running processes and the structural equality $p = p'$ tells us that the process remained unchanged (property 1). In the two other cases, structural equality $p.msg = p'.msg$ conveys that the `msg` field has not changed (property 5) while numeric constraints indicate that the `id` field has not changed

(property 4). In the second case, the input process is asleep while the output process is running. The numeric properties tell us that the wake up count has increased by one and the sleeping budget of the input process is lower than argument n (property 2). In the third case, both the input and output process are asleep. The numeric relations tell us that the initial sleeping budget was greater than n and has decreased by n ; also, the wake up count remains unchanged (property 3).

5.2 Intra-Procedural Analysis

We define a function `Analyze` that takes a program c and an abstract value a —representing the relation gathered so far between the input states and the current state—and returns the abstract value $\text{Analyze}(c)(a)$ that over-approximates the effect of running c after a . This section deals with basic constructs, while §5.3 explains how we analyse functions.

Definition 18 (Intra-procedural version of the analysis function).

$$\begin{aligned} \text{Analyze}(\text{assert}(b))(a) &= \text{Cond}^{\text{D}\uparrow\text{S}\uparrow\text{R}(T,T)}(b)(a) \\ \text{Analyze}(x := t)(a) &= \text{Assign}^{\text{D}\uparrow\text{S}\uparrow\text{R}(T,T)}(x := t)(a) \\ \text{Analyze}(c_1 ; c_2)(a) &= \text{Analyze}(c_2) (\text{Analyze}(c_1)(a)) \\ \text{Analyze}(\text{branch } c_1 \text{ or } \dots \text{ or } c_n \text{ end})(a) &= \bigsqcup_{i \in 1, \dots, n}^{\text{D}\uparrow\text{S}\uparrow\text{R}(T,T)} \text{Analyze}(c_i)(a) \\ \text{Analyze}(\text{while } b \text{ do } c \text{ end})(a) &= \text{Cond}^{\text{D}\uparrow\text{S}\uparrow\text{R}(T,T)}(-b) (\lim_{n \rightarrow \infty} a_n) \\ &\text{where } a_0 = a \text{ and } a_{n+1} = a_n \nabla^{\text{D}\uparrow\text{S}\uparrow\text{R}(T,T)} \text{Analyze}(\text{assert}(b); c)(a_n) \end{aligned}$$

Assertion and assignment use the transfer functions we built in previous sections. Sequence and branching follow the structure outlined in Lem. 1.

We analyse loops in a standard way, using a widening-based Kleene iteration, which ensures that we reach a post-fixpoint in a finite number of iterations. In practice, our implementation performs a *loop unrolling* [39, p.131] of the first iteration, so as to obtain better precision.

The `Analyze` function is *sound*, in the sense that it over-approximates the relational collecting semantics.

Theorem 1 (Soundness w.r.t. the collecting semantics). *For any command c and abstract value $a \in \text{D} \uparrow \text{S} \uparrow \text{R}(T, T)$,*

$$\mathbb{P} \llbracket c \rrbracket \left(\gamma^{\text{D}\uparrow\text{S}\uparrow\text{R}(T,T)}(a) \right) \subseteq \gamma^{\text{D}\uparrow\text{S}\uparrow\text{R}(T,T)} \left(\text{Analyze}(c)(a) \right)$$

By instantiating Thm. 1 with the abstraction of the identity relation, we get a soundness result with respect to the relational semantics of commands:

Corollary 1 (Soundness w.r.t. the relational semantics). *For any command c , $\mathbb{S} \llbracket c \rrbracket \subseteq \gamma^{\text{D}\uparrow\text{S}\uparrow\text{R}(T,T)} \left(\text{Analyze}(c) \left(\text{Id}^{\text{D}\uparrow\text{S}\uparrow\text{R}(T,T)} \right) \right)$.*

5.3 Analysis of Function Calls

In this section, we add function definitions and function calls to our language, and extend the intra-procedural analysis of §5.2 into a modular inter-procedural analysis, based on function summaries.

Extended syntax and semantics for functions. We extend our language to support function calls in commands and function declarations:

$$\begin{aligned} c \in \text{Cmd} &::= \dots \mid x := f(x_1, \dots, x_n) \\ d \in \text{Decl} &::= \text{def } f(\tau_1 x_1, \dots, \tau_n x_n) : \tau = \{c; \text{return } x\} \\ P \in \text{Prog} &::= d_1; \dots; d_n \end{aligned}$$

For simplicity, the command for function calls $y := f(x_1, \dots, x_n)$ immediately saves in a variable y the result of calling a function f . This restriction forbids to call functions within expressions, so that the semantics of expressions and the transfer function for assignment remain unchanged.

A program is a sequence of function declarations $\text{def } f(\tau_1 x_1, \dots, \tau_n x_n) : \tau = \{c; \text{return } r\}$, that specify for the function f what are its input and output variables and their types, and defines its body c . A program effectively defines a map Δ , that associates to every declared function f a quadruplet $\Delta(f) = ((x_1, \dots, x_n), c, r, \Gamma)$ that holds the formal parameters x_i of f , its body c , its formal return variable r , and the typing context Γ that specifies the types of its formal and local variables.

The operational semantics is extended in a standard manner to support function calls and returns, by adding a call stack to the states. The new rules are given in appendix §A.2.

Analysing functions. We have chosen to develop a modular analysis, by analysing each function *only once* and computing a *function summary*, that summarises a function's behaviour. This summary is then reused and instantiated each time that function is called. Such a modular analysis allows to better scale to large code bases, as highlighted by [10].

Definition 19 (Function summaries). For a function f defined by $\Delta(f) = ((x_1, \dots, x_n), c_f, y_f, \Gamma_f)$, we call *summary of f* the quadruplet given by:

$$\left((x_1, \dots, x_n), \text{Analyze}(c_f) \left(\text{Id}^{\text{D}\uparrow\text{S}\uparrow\text{R}}(\Gamma_f) \right), y_f, \Gamma_f \right)$$

The second component of the summary of a function f is an abstract value summarising f 's behaviour by over-approximating the input-output relation between its formal arguments and its formal return variable. Thus, this abstract value deals with the variables that are *local* to the execution of f : no information about the caller's environment is recorded in the summary. This abstract value is obtained by analysing the body of f , starting with the identity relation. This means that we make no assumption on the actual arguments that will be given to f , hence we can reuse the *same* summary in *every* calling context.

To use a function summary at some call site, we *instantiate* the summary on the actual arguments and output variable used at the call site. Our method to

instantiate summaries is based on an abstraction of relational composition, that sequentially chains together two abstract values that represent binary relations.

Definition 20 (Abstract composition). *Let Γ_1, Γ_2 and Γ_3 be typing contexts. Let $a_1 \in A \uparrow R(\Gamma_1, \Gamma_2)$ and $a_2 \in A \uparrow R(\Gamma_2, \Gamma_3)$ be two abstract values. The abstract composition $a_1 ;^A a_2$ of the abstract values a_1 and a_2 is defined by:*

$$a_1 ;^A a_2 = \text{Remove}_{\Gamma_2''} \left(\text{Add}_{\Gamma_3'} c_1 \sqcap^{A(\Gamma_1 \cup \Gamma_2'' \cup \Gamma_3')} \text{Add}_{\Gamma_1} c_2 \right)$$

where $c_1 = \text{Rename}_{\text{second} \circ \text{prime}^{-1}} a_1$ and $c_2 = \text{Rename}_{\text{second}} a_2$.

Abstract composition chains the effects of a_1 and a_2 by introducing auxiliary names—*i.e.*, variables of the form y'' —for the states that are in the output of a_1 and the input of a_2 , before taking the intersection, and then removing the temporarily introduced variables. The calls to Add are necessary name management steps, that ensure that the abstract values deal with the same sets of variables. Abstract composition is a sound approximation of relational composition, as stated by the following lemma:

Lemma 2 (Soundness of composition). *Let $a_1 \in A \uparrow R(\Gamma_1, \Gamma_2)$ and $a_2 \in A \uparrow R(\Gamma_2, \Gamma_3)$ be two abstract values. We have:*

$$\gamma^{A \uparrow R(\Gamma_1, \Gamma_2)}(a_1); \gamma^{A \uparrow R(\Gamma_2, \Gamma_3)}(a_2) \subseteq \gamma^{A \uparrow R(\Gamma_1, \Gamma_3)}(a_1 ;^A a_2)$$

Based on abstract composition, we express summary instantiation as follows:

Definition 21 (Summary instantiation). *The instantiation of the function summary $S_f = ((x_1, \dots, x_n), a_f, y_f, \Gamma_f)$ on the actual parameters (z_1, \dots, z_n) , the actual return variable y and the caller typing context Γ is defined as follows:*

$$\begin{aligned} \text{Inst}(S_f, (z_1, \dots, z_n), y, \Gamma) &= \text{ins} ;^{\text{D}\uparrow\text{S}} a_f ;^{\text{D}\uparrow\text{S}} \text{outs} \\ \text{where } \text{ins} &= \text{Cond}^{\text{D}\uparrow\text{S}\uparrow\text{R}(\Gamma, \Gamma_f)} \left(\bigwedge_{i \in \{1, \dots, n\}} z_i = x'_i \right) \\ \text{and } \text{outs} &= \text{Cond}^{\text{D}\uparrow\text{S}\uparrow\text{R}(\Gamma_f, \Gamma)} (y_f = y') \end{aligned}$$

Summary instantiation simply works by composing three abstract values, using abstract composition. Instantiation first ties each actual parameter to its formal parameter by *pre*-composing the abstract value a_f for f 's body with the *ins* abstract value, and then ties the formal output to the actual output by *post*-composing with the *outs* value. The values *ins* and *outs* are simply expressed as mere conjunctions of equalities. The first composition deals with the *call* of the function, whereas the second composition handles the *return*.

During a function call $y := f(z_1, \dots, z_n)$, the instantiation of f 's summary deals with which variables might have changed and how, but does not deal with the fact that *only* the variable y may have changed: every other variable that is available before the call remains the same after the call. Thus, the transfer function for function call augments the instantiation of the function summary S_f

with equalities for the unaltered variables, before extending the so far gathered relation a with the effect of the call to f :

$$\text{Analyze}(y := f(z_1, \dots, z_n))(a) = a ;^{\text{D}\uparrow\text{S}} \left(\begin{array}{l} \text{Inst}(S_f, (z_1, \dots, z_n), y, \Gamma) \sqcap^{\text{D}\uparrow\text{S}\uparrow\text{R}(\Gamma, \Gamma)} \\ \prod_{x \neq y}^{\text{D}\uparrow\text{S}\uparrow\text{R}(\Gamma, \Gamma)} \text{Cond}^{\text{D}\uparrow\text{S}\uparrow\text{R}(\Gamma, \Gamma)}(x = x') \end{array} \right)$$

The transfer function for function calls is sound:

Lemma 3 (Soundness of function call analysis). *For every function definition $\Delta(f) = ((x_1, \dots, x_n), c_f, y_f, \Gamma_f)$ in a program, and any function summary $S_f = ((x_1, \dots, x_n), a_f, y_f, \Gamma_f)$ such that $\mathbb{S} \llbracket c_f \rrbracket \subseteq \gamma^{\text{D}\uparrow\text{S}\uparrow\text{R}(\Gamma_f, \Gamma_f)}(a_f)$, we have:*

$$\mathbb{P} \llbracket y := f(z_1, \dots, z_n) \rrbracket (\gamma^{\text{D}\uparrow\text{S}\uparrow\text{R}(\Gamma, \Gamma)}(a)) \subseteq \gamma^{\text{D}\uparrow\text{S}\uparrow\text{R}(\Gamma, \Gamma)}(\text{Analyze}(y := f(z_1, \dots, z_n))(a))$$

Lem. 3 ensures that the soundness result for the intra-procedural analysis (Thm. 1) extends to the language with function calls that we have described in this section. We give in appendix §A.5 a proof of Lem. 3.

6 Implementation and Experimental Results

We have implemented our analyser in approximately 5000 lines of OCaml. Our implementation together with instructions on how to add new test cases and run the tests cases is packaged and published as a virtual machine artefact [3]. Similarly to our formal development, our analyser is parametrised by an abstract domain for integers. A command-line option allows to choose among numeric domains provided by Apron [23], such as intervals, octagons or polyhedra.

We have tested our analyser on a total of 42 programs (summarised on Table 1), that comprise some complex examples: some sorting algorithms, the `do_ticks` function from §2.3, and 6 functions inspired from the abstract specification of the seL4 micro-kernel [28]. We now review the results that our analyser computed for these examples, using polyhedra as numeric domain.

Sorting integer arrays. To circumvent the absence of support for arrays in our language and in our abstract domain, we modelled arrays of fixed length using tuples, and we defined `get` and `set` functions. With this encoding, we wrote several sorting algorithms for arrays of integers, for arrays of size 5. The analyser could not infer that the output array was sorted. Still, it was able to infer that the sum of the values of the array was preserved by the sorting function.

The `do_ticks` function. The `do_ticks` function (§2.3) is inspired from a process scheduler from operating system code. As reported in §5.1, the analysis result for `do_ticks` captures all the properties we expected.

seL4-inspired functions. We have extracted from the abstract specification of the seL4 formally verified micro-kernel [28] several functions, that work both on ADTs and on scalar values, and translated them in our `while` language. Specifically, those functions are related to either thread management, capability management or scheduling (`decode_set_priority`, `check_prio`, `mask_cap`, `validate_vm_rights`, `cap_rights_update`, `timer_tick`). Our analyser infers exact abstractions for all of them, but for `timer_tick`. This program is slightly different from `do_ticks`: when a thread’s time budget is over, this budget is reset to its original value, and the thread is then re-scheduled, which might select a new current thread. The case constraints of our abstract domain cannot distinguish whether the current thread remains the same or not, so a join of those two cases is performed. This results in some expected information loss on the thread’s time.

For the `mask_cap` program, we experimented with two encodings of bitmasks, using either integers or ADTs to represent booleans. The integer-based encoding produced a function summary that is compact—only 4 cases—but hard to understand for a human being, whereas the summary produced with the ADT-based encoding was easy to interpret, but large—it involved 326 disjunctions.

We consider that the precision we obtained on the seL4 examples is satisfying. Still, the last example illustrates a limitation of our approach. Indeed the function summaries can significantly grow when the analysed program pattern matches on many distinct variables. Abstract domains that leverage BDDs have been successfully used to reduce analysis costs by sharing common results [11,12,20,40], and could also help in our situation.

7 Related Work

The idea of exploiting a *relational* semantics to verify `while` programs was developed by Kozen [30]. He introduced Kleene Algebra with Tests, an extension of relation algebra [43] with co-reflexive relations named *tests*, that serves as a foundation for the semantics of imperative programs, their verification, and as an effective formal tool for proving the correctness of program transformations.

A number of static analyses for approximating the input-output relation of a program have been proposed. Cousot and Cousot [10] used abstract interpretation for designing modular and relational analyses, and argue that compositionality can improve the scalability of analysers. Compositional Recurrence Analysis (CRA) [13] is a *compositional* static analysis that infers numeric relations between the inputs and the outputs of programs. CRA first builds a regular expression to describe the set of program paths, that is then interpreted as a *relation* in a compositional way, in a second stage. Their approach is context insensitive, and is similar to the relational semantics of Def. 15. Whereas we follow the standard iteration-based analysis of loops, they use a special operator to compute the reflexive transitive closure of a relation, that is specialised on linear recurrence equations. Interestingly, they discuss in their benchmarks a variation of their analysis, named CRA+OCT, that “*uses an intra-procedural octagon analysis to gain some contextual information, but which is otherwise compositional*”, and that

Table 1: List of test cases for our analyser. The different columns indicate whether each test uses **sum** types, **numeric** comparisons or arithmetic operations, while **loops** or function **calls**. The **time** taken by the analysis, and the maximum number of **cases** per function summary in the result are also reported in the table. The * symbol is used to denote multiple tests whose name start identically, and that are either similar or work in pairs. Times larger than 1 second are in bold. The rest are in milliseconds. Measures were done on an Intel[®] Core[™] i7 @2.30GHz16; instructions to reproduce them are included in the artefact [3].

Name	Sums	Numeric	Loops	Calls	Time	Cases
Hand-crafted tests:						
do_ticks	Yes	Yes	Yes	No	148 ms	3
nondeterministic_bubble_sort	Yes	Yes	Yes	Yes	2.0 s	5
selection_sort	Yes	Yes	Yes	Yes	10.4 s	25
Inspired from SeL4:						
decode_set_priority	Yes	Yes	No	Yes	7 ms	2
mask_cap_boolean	Yes	No	No	Yes	7.7 s	324
mask_cap_int	Yes	Yes	No	Yes	1.5 s	4
timer_tick_scheduling	Yes	Yes	Yes	Yes	38.0 s	81
Simple tests:						
assert*	Yes	Yes	No	No	1 ms	1
call_inside_loop_*	No	Yes	Yes	Yes	14 ms	1
drift	Yes	Yes	Yes	Yes	24 ms	2
exchange	No	No	No	No	2 ms	1
facto*	No	Yes	Yes	No	6 ms	1
false_type_collision	No	No	No	Yes	3 ms	1
gauss*	No	Yes	Yes	No	10 ms	1
ghost_equality	No	No	No	No	< 1 ms	1
hidden_incompat	Yes	No	No	No	2 ms	0
id	No	No	No	No	< 1 ms	1
if	No	Yes	No	No	2 ms	1
incompat	Yes	No	No	No	< 1 ms	0
indirect_swap	Yes	No	No	Yes	3 ms	2
long_id	Yes	No	No	Yes	4 ms	2
modulo	No	Yes	Yes	Yes	38 ms	2
multiplication_larger	No	Yes	No	No	1 ms	1
or_constructor	Yes	No	No	No	< 1 ms	0
plus_*	Yes	Yes	No	No	< 1 ms	1
record_assignment*	No	Yes	No	No	2 ms	1
reduction	No	Yes	No	No	2 ms	1
struct_exchange	Yes	No	No	No	< 1 ms	1
swap	Yes	No	No	No	< 1 ms	2
test_loop	No	Yes	Yes	No	3 ms	1
two_by_two	No	Yes	Yes	No	2 ms	1
while_true	No	No	Yes	No	< 1 ms	0
widening_convergence	No	Yes	Yes	No	43 ms	1
xor	Yes	No	No	Yes	8 ms	3

leads to more precise results than pure CRA. Although no precise definition is given for CRA+OCT, we believe that it follows our *relational collecting semantics* of Def. 15, again with the exception of the treatment of loops. As we have also observed, exploiting the information available at loop entries is crucial to obtain sufficiently precise results. ICRA [27] is an inter-procedural extension of CRA, where function summaries are computed once and for all, independently of their calling contexts—an approach we have followed too in §5.3. In contrast to CRA and ICRA, our analysis can deal with programs that are not purely numeric, and that can handle algebraic data types. We have not found any detailed description of *how* the function summaries of CRA and ICRA are instantiated. We are therefore not able to compare the way we instantiate function summaries (§5.3) with CRA or ICRA. In contrast to CRA and ICRA, our analysis does not yet support recursively defined functions.

The same approach of computing context-insensitive function summaries was followed in the context of correlation analysis [1]. This analysis infers binary equalities between the parts of structured inputs and outputs of programs, using the *correlation abstract domain*. We improve on that work because we can also express numeric relations between parts of structured values—in addition to n-ary equalities. Our domain differs significantly from the correlation domain, in the sense that correlations are recursively defined so that parts of abstract values relate parts of structured values, whereas our domain is not a recursive structure, and instead exploits *extended variables* to relate the parts of structured values that are accessible through projection paths. A preliminary version of our approach was published in [2]. It did not include the domain for structural equalities that allows to concisely express n-ary equality relations between parts of structured values, nor an implementation. It was not an input-output relational analysis, and had less precise formulas for the transfer functions. Indeed, the implementation [3] allowed to discover precision issues that were fixed with the reduced product, relational lifting and transfer functions presented in this paper.

Several relational analyses were developed for the inter-procedural analysis of numeric programs [4, 22, 35, 41], and in the context of inter-procedural shape analysis [19, 21, 42]. They all feature a form of function summary, that helps reduce the analysis cost of large programs, by enabling modular analyses. A domain that supports both shape abstraction and numeric constraints was developed by [5]. It is defined in a modular fashion, based on the cofibered abstract domain [44]. As in our construction, theirs also features a disjunctive completion, but leaves open the question of how to keep the number of disjuncts under control.

In the context of the static analysis of languages with algebraic data types, techniques based on tree automata [6] have been developed. Tree automata are well suited to represent regular sets of trees, and several works propose to extend their expressive power further. Lattice tree automata [15, 16] augment tree automata with elements of an arbitrary abstract domain at their leaves, and allow to express *non-relational* integer constraints on the leaves of trees. More recently, [24, 25] use a combination of tree automata and of a relational domain whose keys are regular expressions to express relational constraints between the

numeric leaves of trees. They use regular expressions to denote sets of access paths within those trees, and thus to support structures of unbounded heights.

As a particular case of algebraic values, the analysis of programs with *optional* numerical values was handled in [32] by associating to optional variables two *avatars*, that respectively model lower- and upper-constraints on that variable. When the avatars of some variable x induce a contradictory constraint, this denotes that x is in the `None` case. It is unclear how this approach generalises to deeply nested algebraic values.

Controlling the number of disjuncts in a disjunctive completion is admittedly difficult, as a cost *vs* precision balance must be found. Since we deal with finite types only, our number of disjuncts is bounded by the products of the sizes of types used in a program. Other works have used *silhouettes* [31]—abstractions of the shapes of the abstract values—to control disjunctions. Following [26], our disjunctions, that are guided by paths in values, can be understood as a form of *control sensitivity*. It is worth noticing that our disjuncts do not form a *partition* since some disjuncts may overlap—a degree of freedom that is advocated by [26]. Based on our present work, we will investigate whether we can re-cast our disjuncts as *conjunctions of implications*, which could both improve precision and lead to a more parsimonious representations of abstract values.

8 Conclusion and Future Work

In the context of programs that combine arithmetic operations and algebraic data types, we have shown how to construct an abstract domain that extends *any* abstract domain for numeric relations into an abstract domain for relations between algebraic values. The main idea is to consider extended variables—*i.e.*, a variable, and an access path—as the variables used in the numeric abstract domain. To lower the sizes of abstract values, we add a domain that keeps track of equalities between non-numeric values. The domains are combined using a reduced product that propagates equalities and are rendered more expressive using an adaptation of disjunctive completion for handling the different, incompatible cases that an algebraic value can exhibit. This abstract domain is called `RAND`—the Relational Algebraic Numeric Domain—and can be exploited in a static analyser.

We have given a formal justification to the folklore result of static analysis that “*an analysis can be made relational by duplicating variables*”, by effectively turning a non-relational analysis into a relational one that infers relations between inputs and outputs of programs. One key observation is that the relational analyser and the non-relational one share the same *structure*: only a few transfer functions need to be redefined. The second observation is that any relational domain can easily be used to express relations between different stores: the necessary transfer functions can be redefined once and for all, in a generic manner.

Finally, we have exploited our abstract domain to design and implement [3] a static analyser for a `while` language with algebraic data types and function calls that exploits the relational feature of `RAND` to infer function summaries.

Summaries express the input-output behaviours of functions, and enable a *modular* inter-procedural analysis of programs: every function is analysed *exactly once*.

Further work will address the challenging problem of handling *recursive* algebraic data types and functional arrays. To that end, we will need to adapt our language of *paths*, *e.g.*, by using regular languages, or by extending techniques based on tree automata [25]. Another direction of research is to analyse recursive programs, which will require the computation of a fixpoint at the level of function summaries for groups of mutually defined functions.

Finally, we intend to apply our analyser to help the verification of programs, by mixing automatic techniques based on abstract interpretation with standard deductive verification tools, such as Why3 [14]. Previous work [1] have indeed demonstrated that a large number of proof obligations could be discharged automatically in such a way, and could alleviate the verification of large programs.

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A Appendix

A.1 Intra-procedural Semantics

$$\begin{aligned}
\llbracket x.p \rrbracket_s^{\text{exp}} &= \begin{cases} \{s(x) \Downarrow^{\text{val}} p\} & \text{if } s(x) \text{ is defined and } s(x) \Downarrow^{\text{val}} p \neq \text{Undef} \\ \emptyset & \text{otherwise} \end{cases} \\
\llbracket \{f_i = t_i\}^{i \in I} \rrbracket_s^{\text{exp}} &= \left\{ \overline{\{f_i = v_i\}^{i \in I}} \mid \forall i \in I, v_i \in \llbracket t_i \rrbracket_s^{\text{exp}} \right\} \\
\llbracket A(t) \rrbracket_s^{\text{exp}} &= \{A(v) \mid v \in \llbracket t \rrbracket_s^{\text{exp}}\} \\
\llbracket t_1 \boxplus t_2 \rrbracket_s^{\text{exp}} &= \{v_1 \boxplus v_2 \mid v_1 \in \llbracket t_1 \rrbracket_s^{\text{exp}} \wedge v_2 \in \llbracket t_2 \rrbracket_s^{\text{exp}}\}
\end{aligned}$$

Fig. 5: Denotation of expressions.

$$\begin{aligned}
\llbracket t_1 \bowtie t_2 \rrbracket_s^{\text{bool}} &= \{v_1 \bowtie v_2 \mid v_1 \in \llbracket t_1 \rrbracket_s^{\text{exp}} \wedge v_2 \in \llbracket t_2 \rrbracket_s^{\text{exp}}\} \\
\llbracket b_1 \wedge b_2 \rrbracket_s^{\text{bool}} &= \{\mathbf{b}_1 \wedge \mathbf{b}_2 \mid \mathbf{b}_1 \in \llbracket b_1 \rrbracket_s^{\text{bool}} \wedge \mathbf{b}_2 \in \llbracket b_2 \rrbracket_s^{\text{bool}}\} \\
\llbracket b_1 \vee b_2 \rrbracket_s^{\text{bool}} &= \{\mathbf{b}_1 \vee \mathbf{b}_2 \mid \mathbf{b}_1 \in \llbracket b_1 \rrbracket_s^{\text{bool}} \wedge \mathbf{b}_2 \in \llbracket b_2 \rrbracket_s^{\text{bool}}\} \\
\llbracket \neg b \rrbracket_s^{\text{bool}} &= \{\neg \mathbf{b} \mid \mathbf{b} \in \llbracket b \rrbracket_s^{\text{bool}}\}
\end{aligned}$$

Fig. 6: Denotation of conditions. $\mathbf{B} = \{\mathbf{t}, \mathbf{ff}\}$ is the set of boolean values.

$$\begin{array}{c}
\text{SEQSTEP} \\
\frac{(c_1, s) \rightarrow (c'_1, s')}{(c_1 ; c_2, s) \rightarrow (c'_1 ; c_2, s')} \\
\\
\text{SEQSKIP} \\
\frac{}{(\text{skip} ; c_2, s) \rightarrow (c_2, s)} \\
\\
\text{ASSIGN} \\
\frac{v \in \llbracket t \rrbracket_s^{\text{exp}}}{(x := t, s) \rightarrow (\text{skip}, s(x \mapsto v))} \\
\\
\text{BRANCH} \\
\frac{1 \leq i \leq n}{(\text{branch } c_1 \text{ or } \dots \text{ or } c_n \text{ end}, s) \rightarrow (c_i, s)} \\
\\
\text{ASSERT} \\
\frac{\mathbf{t} \in \llbracket b \rrbracket_s^{\text{bool}}}{(\text{assert}(b), s) \rightarrow (\text{skip}, s)} \\
\\
\text{WHILETRUE} \\
\frac{\mathbf{t} \in \llbracket b \rrbracket_s^{\text{bool}}}{(\text{while } b \text{ do } c \text{ end}, s) \rightarrow (c ; \text{while } b \text{ do } c \text{ end}, s)} \\
\\
\text{WHILEFALSE} \\
\frac{\mathbf{ff} \in \llbracket b \rrbracket_s^{\text{bool}}}{(\text{while } b \text{ do } c \text{ end}, s) \rightarrow (\text{skip}, s)}
\end{array}$$

Fig. 7: Small-step semantics of commands.

A.2 Inter-procedural Operational Semantics

We extend the small-step reduction rules from Fig. 7 as follows. First, we add to our semantic states (c, s) , that are composed of a program and a store, a third

$$\begin{array}{c}
\text{FUNCTIONCALL} \\
\frac{\Delta(f) = ((x_1, \dots, x_n), c_f, r, \Gamma_f) \quad \forall i \in \{1, \dots, n\}, v_i \in \llbracket z_i \rrbracket_s^{\text{exp}}}{(y := f(z_1, \dots, z_n), s, \pi) \rightarrow (c_f ; \text{return } r, [x_1 \mapsto v_1, \dots, x_n \mapsto v_n], (y, s) : \pi)} \\
\\
\text{FUNCTIONRETURN} \\
\frac{v_r \in \llbracket r \rrbracket_s^{\text{exp}}}{(\text{return } r, s, (y_r, s_r) : \pi) \rightarrow (\text{skip}, s_r(y_r \mapsto v_r), \pi)}
\end{array}$$

Fig. 8: Small-step semantics for functions

component π that denotes a *call stack*, and is used to properly handle function *returns*. The reduction rule SEQSTEP that handles sequences of commands simply propagates any changes to the stack, whereas the other rules of Fig. 7 leave the stack unchanged.

Then, we augment the reduction relation with two new rules (Fig. 8). Rule FUNCTIONCALL installs the body of the called function f as the new code to execute, and installs a new store that defines the actual values—read in the caller’s store—for the formal parameters of f . Simultaneously, a new element (y, s) is added at the top of the call stack, so as to remember that the caller’s store s should be restored upon f ’s return, and that the value computed by f must be recorded in the variable y . This very action, that must be performed at function return, is specified by rule FUNCTIONRETURN.

A.3 Compositional Characterisation of the Relational Semantics of Programs

We define the relation $Id = \{(s, s)\}$ as the identity relation on stores, and we write $a_1; a_2 = \{(s_1, s_3) \mid \exists s_2, (s_1, s_2) \in a_1 \wedge (s_2, s_3) \in a_2\}$ to denote the composition of the relations a_1 and a_2 . We also write a^* to denote the reflexive transitive closure of the relation a , *i.e.*, the least fixed point of the functional $\lambda a'. Id \cup (a; a')$. We have $a^* = Id \cup (a; a^*) = Id \cup (a^*; a)$.

Lemma 4. *The relational semantics enjoys the following identities:*

$$\begin{aligned}
\mathbb{S} \llbracket \text{skip} \rrbracket &= Id \\
\mathbb{S} \llbracket c_1 ; c_2 \rrbracket &= \mathbb{S} \llbracket c_1 \rrbracket ; \mathbb{S} \llbracket c_2 \rrbracket \\
\mathbb{S} \llbracket \text{branch } c_1 \text{ or } \dots \text{ or } c_n \text{ end} \rrbracket &= \bigcup_{1 \leq i \leq n} \mathbb{S} \llbracket c_i \rrbracket \\
\mathbb{S} \llbracket \text{while } b \text{ do } c \text{ end} \rrbracket &= (\mathbb{C} \llbracket b \rrbracket ; \mathbb{S} \llbracket c \rrbracket)^* ; \mathbb{C} \llbracket \neg b \rrbracket \\
\mathbb{S} \llbracket \text{assert}(b) \rrbracket &= \mathbb{C} \llbracket b \rrbracket \\
\mathbb{S} \llbracket x := t \rrbracket &= \{(s, s(x \mapsto v)) \mid (s, v) \in \mathbb{E} \llbracket t \rrbracket\}
\end{aligned}$$

where the semantics of numeric and boolean expressions are defined as follows:

$$\mathbb{E} \llbracket t \rrbracket = \{(s, v) \mid v \in \llbracket t \rrbracket_s^{\text{exp}}\} \quad \mathbb{C} \llbracket b \rrbracket = \{(s, s) \mid \mathbf{tt} \in \llbracket b \rrbracket_s^{\text{bool}}\}$$

A.4 Proof of the Inductive Characterisation of the Collecting Semantics

Proof (Lem. 1). The proof of the three first cases rely on the algebraic properties of relational composition, namely: Id is its neutral element, composition is associative, and it distributes over union. To prove the fourth case, we first notice that for any fixed a and b , for the function $g(r) = a \cup (r; b)$, we have $g^i(\perp) = \bigcup_{0 \leq j < i} a; b^j$. Then, it follows that $\text{lfp } g = \bigcup_{i \geq 0} g^i(\perp) = \bigcup_{i \geq 0} \bigcup_{0 \leq j < i} a; b^j = \bigcup_{j \geq 0} \bigcup_{i > j} a; b^j = \bigcup_{j \geq 0} a; b^j = a; b^*$. The proofs of the last two cases proceed by unfolding definitions. \square

A.5 Proof for the Soundness Lemma of Function Call Analysis

Proof (Lem. 3). Let s_0 be a store, and s_2 be the store that results from the call instruction $y := f(z_1, \dots, z_n)$ on the initial store s_0 . We have $(s_0, s_2) \in \mathbb{P} \llbracket y := f(z_1, \dots, z_n) \rrbracket (\gamma^{\text{D}\uparrow\text{S}\uparrow\text{R}(f, \Gamma)}(a))$. Unfolding the definition of $\mathbb{P} \llbracket \cdot \rrbracket$ gives an intermediate store s_1 such that (s_0, s_1) belongs to $\gamma^{\text{D}\uparrow\text{S}\uparrow\text{R}(f, \Gamma)}(a)$ and (s_1, s_2) belongs to $\mathbb{S} \llbracket y := f(z_1, \dots, z_n) \rrbracket$. The function call can be decomposed as a series of reductions $(y := f(z_1, \dots, z_n), s_1, \pi) \rightarrow (c_f; \text{return } y_f, s', (y, s_1) : \pi) \rightarrow^* (\text{return } y_f, s'', (y, s_1) : \pi) \rightarrow (\text{skip}, s_2, \pi)$, where s' is the store that is initialised at the beginning of the execution of f 's body, and s'' is the store the is obtained at the end of the execution f 's body. Let ins and $outs$ be the abstract values used in Def. 21. By the definition of s' in rule `FUNCTIONCALL`, we get $(s_1, s') \in \gamma^{\text{D}\uparrow\text{S}\uparrow\text{R}(f, \Gamma)}(ins)$. Moreover, $(s', s'') \in \mathbb{S} \llbracket c_f \rrbracket$ is obtained by the hypothesis on f 's summary. Finally, we prove $(s'', s_2) \in \gamma^{\text{D}\uparrow\text{S}\uparrow\text{R}(f, \Gamma)}(outs)$ by the definition of s_2 in rule `FUNCTIONRETURN`: s'' and s_2 indeed coincide on every variable other than the actual return variable. We conclude using the soundness of abstract composition, abstract intersection and abstract condition. \square

A.6 Basic Operators for the $\text{seq}(\Gamma)$ Domain

A pre-order, an abstract union and an abstract intersection for this domain can be defined by

$$\begin{aligned}
 C_1 \sqsubseteq^{\text{seq}(\Gamma)} C_2 &\text{ iff } C_1 = \perp^{\text{seq}(\Gamma)} \text{ or } C_2 \subseteq C_1 \\
 \perp^{\text{seq}(\Gamma)} \sqcup^{\text{seq}(\Gamma)} C &= C \sqcup^{\text{seq}(\Gamma)} \perp^{\text{seq}(\Gamma)} = C \\
 C_1 \sqcup^{\text{seq}(\Gamma)} C_2 &= C_1 \cap C_2 \qquad \text{otherwise} \\
 C_1 \sqcap^{\text{seq}(\Gamma)} C_2 &= \begin{cases} \perp^{\text{seq}(\Gamma)} & \text{if } C_1 = \perp^{\text{seq}(\Gamma)} \text{ or } C_2 = \perp^{\text{seq}(\Gamma)} \\ & \text{or } \text{Env}(C_1) \ll \text{Env}(C_2) \\ C_1 \cup C_2 & \text{otherwise} \end{cases} \\
 C_1 \nabla^{\text{seq}(\Gamma)} C_2 &= C_1 \sqcup^{\text{seq}(\Gamma)} C_2
 \end{aligned}$$

If this domain was used on its own, then the union, intersection and widening given here would be unnecessarily imprecise, because of the lack of closure. But since we use it as part of the reduced product of §3.6, the closure operator of Def. 22 (appendix §A.7) is applied to the values of this domain before applying the above-given union, intersection and widening. See appendix §A.7 for more details.

The reason why taking the abstract union (that is, intersection of the underlying sets of equalities) as widening works here, is because every element of this domain is a *finite* set of equalities, and abstract union can only decrease it. A decreasing sequence of finite sets necessarily stabilises in finite time.

A.7 More Details on the Reduced Product Domain

We give a name to the reduced product domain : the SNPR lifting, for *Structural and Numeric Paths Relations*.

$$D \uparrow \text{SNPR}(\Gamma) = \text{cc}(\Gamma) \times \text{seq}(\Gamma) \times D \uparrow \text{NPR}(\Gamma)$$

We define an operator that adds variables and applies reduction at the same time. It will be used for defining union and intersection.

Definition 22 (Closure with variable addition for cases).

$$\text{clos}_{E_2}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(E, C, N) = \rho(E, C \cup \{(x.p, x.p) \mid x.p \in E_2\}, N)$$

In practice, our product is smashed: if any of the three components is detected to be semantically equivalent to bottom, the whole abstract value is replaced with a special value $\perp^{\text{D}\uparrow\text{SNPR}(\Gamma)}$. However, to simplify formulas, we have chosen not to reflect this on the formalisation. This smashing is done after reduction and after intersection. As an example, reducing the abstract value $(\{x, x.f, x.g, y, y.f, y.g\}, \{x = y\}, \{x.f < y.f\})$, that would normally yield $(\{x, x.f, x.g, y, y.f, y.g\}, \{x = y\}, \perp^{\text{D}\uparrow\text{NPR}(\Gamma)})$, yields $\perp^{\text{D}\uparrow\text{SNPR}(\Gamma)}$ instead.

Some care has to be exerted when taking the union of abstract values. For example, for abstract values

$$\left(\{x, y\}, \{(x, y)\}, \top^{\text{D}\uparrow\text{NPR}(\Gamma)} \right) \text{ and } \left(\{x.f, x.g, y.f, y.g\}, \top^{\text{seq}(\Gamma)}, \{x.f = y.f, x.g = y.g\} \right)$$

a component-wise union would yield $(\{x, y\}, \top^{\text{D}\uparrow\text{NPR}(\Gamma)}, \top^{\text{seq}(\Gamma)})$, which is unnecessarily imprecise. Indeed, $(\{x.f, x.g, y.f, y.g\}, \top^{\text{seq}(\Gamma)}, \{x.f = y.f, x.g = y.g\})$ is also a sound result and accomplishes a better precision.

We achieve this improved precision by first closing each argument with the extended variables of the other argument that do not introduce any new constructor constraint. To alleviate notations, we write $\text{NotNew}(a_1, a_2)$ for the set of extended variables appearing in a_1 that do not introduce new constructor constraints with respect to a_2 . That is:

$$\begin{aligned} \text{NotNew}(a_1, a_2) = \\ \{x.p \in \text{Env}(a_1) \mid \forall x.p' @ A \preceq x.p, x.p' @ A \in \text{Env}(\rho(a_2))\} \end{aligned}$$

Over-lined operators correspond to the component-wise version of the operations.

Abstract union is given by

$$a_1 \sqcup^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2 = \text{clos}_{\text{NotNew}(a_2, a_1)}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a_1) \overline{\sqcup^{\text{D}\uparrow\text{SNPR}(\Gamma)}} \text{clos}_{\text{NotNew}(a_1, a_2)}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a_2)$$

The pre-order for the SNPR lifting uses the same kind of closure:

$$a_1 \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2 \text{ iff } \text{clos}_{\text{NotNew}(a_2, a_1)}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a_1) \overline{\sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)}} a_2$$

The same principle is applied in defining widening; but only on one side to not interfere with the convergence of the numeric component:

$$a_1 \nabla^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2 = a_1 \overline{\nabla^{\text{D}\uparrow\text{SNPR}(\Gamma)}} \text{clos}_{\text{NotNew}(a_1, a_2)}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a_2)$$

Intersection is simply done component-wise, then reduced :

$$a_1 \sqcap^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2 = \rho \left(a_1 \overline{\sqcap^{\text{D}\uparrow\text{SNPR}(\Gamma)}} a_2 \right)$$

When performing fixpoint convergence acceleration using widening, to know when an abstract fixpoint has been reached, we need to be able to test equivalence of two abstract values. Component-wise equivalence test may not be precise enough. For example, the abstract values $(\{x, y\}, \{(x, y)\}, \top^{\text{D}\uparrow\text{SNPR}(\Gamma)})$ and $(\{x, y, x.f, y.f\}, \{(x, y)\}, \{x.f = y.f\})$ are semantically equivalent, but are not component-wise equivalent. To circumvent this problem, we make sure that we consider all pertinent extended variables, which are the extended variables appearing in the intersection of the two abstract values. We call $\text{CrossEnv}(a_1, a_2)$ the set of all extended variables that appear in the reduced intersection of a_1 and a_2 . It is formally defined by:

$$\text{CrossEnv}(a_1, a_2) = \text{Env} \left(a_1 \sqcap^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2 \right)$$

We then use this to define equivalence:

$$a_1 \equiv^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2 \text{ iff } a_1 = a_2 = \perp^{\text{D}\uparrow\text{SNPR}(\Gamma)} \vee \begin{cases} a_1 \sqcap^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2 \neq \perp^{\text{D}\uparrow\text{SNPR}(\Gamma)} \wedge \\ \text{clos}_{\text{CrossEnv}(a_1, a_2)}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a_1) \overline{\equiv^{\text{D}\uparrow\text{SNPR}(\Gamma)}} \text{clos}_{\text{CrossEnv}(a_1, a_2)}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a_2) \end{cases}$$

A.8 More Details on the Disjunctive Completion

We recall that $\text{D}\uparrow\text{SNPR}(\Gamma)$ stands for the reduced product domain (see appendix §A.7).

The disjunctive completion is equipped with Hoare ordering:

$$O_1 \sqsubseteq^{\text{D}\uparrow\text{S}(\Gamma)} O_2 \text{ iff } \forall a \in O_1, \exists a' \in O_2, a \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a'$$

Abstract intersection for the disjunctive completion is given by:

$$O_1 \sqcap^{\text{D}\uparrow\text{S}(\Gamma)} O_2 = \left\{ a_1 \sqcap^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2 \mid \begin{array}{l} a_1 \in O_1 \wedge a_2 \in O_2 \wedge \\ a_1 \sqcap^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2 \neq \perp^{\text{D}\uparrow\text{SNPR}(\Gamma)} \end{array} \right\}$$

We take a disjunction because for certain programs we want to be able to extract information for different cases, in which incompatible constructor names are involved. However, when two abstract values are equivalent with respect to the constructors that they mention, they should be merged, to limit the size of the disjunction. The following definition formalises what it means, for two sets of extended variables, to be equivalent. For convenience, we give the definition for sets of extended variables closed by prefix (see §3.4).

Definition 23 (Equivalence of constructor constraints). *Two sets of constructor constraints E_1 and E_2 are said to be equivalent, written $E_1 \equiv E_2$, if any path ending with a constructor present in one of them is also present in the other. Formally:*

$$E_1 \equiv E_2 \text{ iff } \forall x.(p\@A) \in E_1, x.(p\@A) \in E_2 \wedge \forall y.(q\@B) \in E_2, y.(q\@B) \in E_1$$

\equiv is an equivalence relation for constructor constraints.

We can extend this into an equivalence relation for elements of the SNPR lifting, by:

$$(E_1, C_1, N_1) \equiv^E (E_2, C_2, N_2) \text{ iff } E_1 \equiv E_2$$

This equivalence only talks about the extended variables, and does not correspond to an equivalence with respect to $\sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)}$ nor to a semantic equivalence. The super-script in the notation \equiv^E reminds of that.

Given an element D of $\text{D}\uparrow\text{S}(\Gamma)$, we write D/\equiv^E the set of equivalence classes of D with respect to \equiv^E .

This notion of equivalence allows us to define an operator that collapses together equivalent triplets in an element of $\mathcal{P}(\text{D}\uparrow\text{SNPR}(\Gamma))$.

Definition 24. *We define an operator Collapse that takes a set of elements of $\text{D}\uparrow\text{SNPR}(\Gamma)$ and merges together (by taking the abstract union), the elements that are equivalent with respect to \equiv^E . Formally,*

$$\text{Collapse}(O) = \left\{ \bigsqcup_{a \in \bar{a}}^{\text{D}\uparrow\text{SNPR}(\Gamma)} a \mid \bar{a} \in O/\equiv^E \right\}$$

We use this collapse operator to provide an abstract union for the structural lifting

$$O_1 \sqcup^{\text{D}\uparrow\text{S}(\Gamma)} O_2 = \text{Collapse}(O_1 \cup O_2)$$

A widening for the structural lifting is given by

$$\begin{aligned} O_1 \nabla^{\text{D}\uparrow\text{S}(\Gamma)} O_2 = & \\ & \{ a_1 \nabla^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2 \mid a_1 \in O_1 \wedge a_2 \in O_2 \wedge a_1 \equiv^E a_2 \} \\ & \cup \{ a_2 \in O_2 \mid \nexists a_1 \in O_1, a_1 \equiv^E a_2 \} \cup \{ a_1 \in O_1 \mid \nexists a_2 \in O_2, a_1 \equiv^E a_2 \} \end{aligned}$$

A.9 Additional Hypothesis on the Numeric Domain

A.10 Properties About our Abstract Domains and Liftings

We have proven the soundness of the constructor constraints domain (§3.4).

Theorem 2 (Soundness of the Constructor Constraints domain). *For any typing context Γ , the domain $\text{cc}(\Gamma)$ is sound :*

- $\sqsubseteq^{\text{cc}(\Gamma)}$ is a pre-order
- Concretisation is monotonic:

$$\forall (E_1, E_2) \in \text{cc}(\Gamma)^2, E_1 \sqsubseteq^{\text{cc}(\Gamma)} E_2 \Rightarrow \gamma^{\text{cc}(\Gamma)}(E_1) \subseteq \gamma^{\text{cc}(\Gamma)}(E_2)$$

- Abstract union is sound :

$$\forall (E_1, E_2) \in \text{cc}(\Gamma)^2, \gamma^{\text{cc}(\Gamma)}(E_1) \cup \gamma^{\text{cc}(\Gamma)}(E_2) \subseteq \gamma^{\text{cc}(\Gamma)}(E_1 \sqcup^{\text{cc}(\Gamma)} E_2)$$

- Abstract intersection is sound :

$$\forall (E_1, E_2) \in \text{cc}(\Gamma)^2, \gamma^{\text{cc}(\Gamma)}(E_1) \cap \gamma^{\text{cc}(\Gamma)}(E_2) \subseteq \gamma^{\text{cc}(\Gamma)}(E_1 \sqcap^{\text{cc}(\Gamma)} E_2)$$

- The transfer function for assignment is sound:

$$\forall E \in \text{cc}(\Gamma), \forall x \in \text{Vars}, \forall t \in \text{Exp}, \\ \{s(x \mapsto v) \mid s \in \gamma^{\text{cc}(\Gamma)}(E) \wedge v \in \llbracket t \rrbracket_s^{\text{exp}}\} \subseteq \gamma^{\text{cc}(\Gamma)}(\text{Assign}^{\text{cc}(\Gamma)}(E)(x := t))$$

- The transfer function for assertions is sound:

$$\forall E \in \text{cc}(\Gamma), \forall b \in \text{BExp}, \\ \{s \in \gamma^{\text{cc}(\Gamma)}(E) \mid \mathbf{t} \in \llbracket b \rrbracket_s^{\text{bool}}\} \subseteq \gamma^{\text{cc}(\Gamma)}(\text{Cond}^{\text{cc}(\Gamma)}(E)(b))$$

- $\nabla^{\text{cc}(\Gamma)}$ computes upper-bounds.
- For any sequence $(E_n)_{n \in \mathbf{N}}$ of $\text{cc}(\Gamma)^{\mathbf{N}}$, the sequence $(E'_n)_{n \in \mathbf{N}}$ defined by $E'_0 = E_0 \wedge \forall n \in \mathbf{N}, E'_{n+1} = E'_n \nabla^{\text{cc}(\Gamma)} E_{n+1}$ converges in finite time.

In addition to soundness, we have a result about exactness and a result about inexactness.

Lemma 5 (Exactness of abstract intersection for the Constructor Constraints domain). *For any typing context Γ , the binary operator $\sqcap^{\text{cc}(\Gamma)}$ is exact:*

$$\forall (E_1, E_2) \in \text{cc}(\Gamma)^2, \gamma^{\text{cc}(\Gamma)}(E_1) \cap \gamma^{\text{cc}(\Gamma)}(E_2) = \gamma^{\text{cc}(\Gamma)}(E_1 \sqcap^{\text{cc}(\Gamma)} E_2)$$

Lemma 6 (Inexactness of abstract union for the Constructor Constraints domain). *There exists typing contexts Γ for which the operator $\sqcup^{\text{cc}(\Gamma)}$ is not exact.*

Proof. Let us consider a sum type with three constructors $\tau = [A \rightarrow \mathbb{N}; B \rightarrow \mathbb{N}; C \rightarrow \mathbb{N}]$, a typing context with a single variable $\Gamma = [x \mapsto \tau]$ and the two sets of constructor constraints $E_1 = \{x@a, x\}$ and $E_2 = \{x@B, x\}$. The abstract union of E_1 and E_2 is semantically equivalent to $\top^{cc(\Gamma)}$, since

$$E_1 \sqcup^{cc(\Gamma)} E_2 = E_1 \cap E_2 = \{x\}$$

This entails, in particular, that the state $s = [x \mapsto C(1)]$ where x uses constructor C belongs to the concretisation of $E_1 \sqcup^{cc(\Gamma)} E_2$. Formally, $s \in \gamma^{cc(\Gamma)}(E_1 \sqcup^{cc(\Gamma)} E_2)$. Nevertheless, s does not belong to the concretisation of either E_1 —which requires x to use constructor A —nor E_2 —which requires x to use constructor B . Formally, $s \notin \gamma^{cc(\Gamma)}(E_1) \cup \gamma^{cc(\Gamma)}(E_2)$. Hence, $\gamma^{cc(\Gamma)}(E_1 \sqcup^{cc(\Gamma)} E_2) \neq \gamma^{cc(\Gamma)}(E_1) \cup \gamma^{cc(\Gamma)}(E_2)$ which means that $\sqcup^{cc(\Gamma)}$ is not exact. \square

We also have proven the soundness of the structural equalities domain (§3.5).

Theorem 3 (Soundness of the Structural Equalities domain). *For any typing context Γ , the domain $\text{seq}(\Gamma)$ is sound :*

- $\sqsubseteq^{\text{seq}(\Gamma)}$ is a pre-order and $\gamma^{\text{seq}(\Gamma)}$ is monotonic.
- Abstract union and abstract intersection are sound.
- The transfer functions for assignment and assertions are sound.
- $\nabla^{\text{seq}(\Gamma)}$ computes upper bounds and enforces convergence.

We require three properties about the operator for removing variables of the underlying numeric domain D :

- A property about composition: for any three sets of (extended) variables $E_3 \subseteq E_2 \subseteq E_1$, and any abstract value $d \in D(E_1)$, we require that

$$\text{Rem}_{E_1 \setminus E_3}^{D(E_1)}(d) \sqsubseteq^{D(E_3)} \text{Rem}_{E_2 \setminus E_3}^{D(E_2)} \left(\text{Rem}_{E_1 \setminus E_2}^{D(E_1)}(d) \right)$$

- A property about monotony: for any two sets of (extended) variables E_1 and E_2 and any two abstract values $(d_1, d_2) \in D(E_1)^2$, we require that

$$d_1 \sqsubseteq^{D(E_1)} d_2 \Rightarrow \text{Rem}_{E_2}^{D(E_1)}(d_1) \sqsubseteq^{D(E_1 \setminus E_2)} \text{Rem}_{E_2}^{D(E_1)}(d_2)$$

- A property about removing an empty set of variables: for any set of variables V and any abstract value $d \in D(V)$, we require that

$$\text{Rem}_{\emptyset}^{D(V)}(d) = d$$

The two first properties are used to prove that the pre-order of the NPR lifting is transitive. The last property is used when proving that the widening of the NPR lifting enforces convergence in finite time.

Provided that the three properties mentioned above hold—and we believe that all the numeric domains implemented in Apron verify these hypotheses—, we have proven the soundness of the NPR lifting.

Theorem 4 (Soundness of the NPR lifting). *For any numeric abstract domain D following the properties of §3.1 and any typing context Γ . If the operator $\text{Rem}^{D(\cdot)}$ satisfies the aforementioned properties, then the domain $D \uparrow \text{NPR}(\Gamma)$ is sound:*

- $\sqsubseteq^{D \uparrow \text{NPR}(\Gamma)}$ is a pre-order and $\gamma^{D \uparrow \text{NPR}(\Gamma)}$ is monotonic.
- Abstract union and abstract intersection are sound.
- The transfer functions for assignment and assertions are sound.
- $\nabla^{D \uparrow \text{NPR}(\Gamma)}$ computes upper bounds and enforces convergence.

We require from the abstraction of boolean conditions of the underlying numeric domain

- that it yields an abstract value that is at least as precise as the initial one

$$\text{Cond}^{D(V)}(b)(d) \sqsubseteq^{D(V)} d$$

- that it is monotonic with respect to abstract values

$$d_1 \sqsubseteq^{D(V)} d_2 \Rightarrow \text{Cond}^{D(V)}(b)(d_1) \sqsubseteq^{D(V)} \text{Cond}^{D(V)}(b)(d_2)$$

- and that it yields a result at least as precise for a conjunction of boolean conditions than for a single conjunct

$$\text{Cond}^{D(V)}(b_1 \wedge b_2)(d) \sqsubseteq^{D(V)} \text{Cond}^{D(V)}(b_1)(d)$$

These three hypotheses allow us to prove that the reduction operator yields an abstract value that is component-wise smaller than the initial one, and is component-wise monotonic; which in turn allow us to prove that the widening $\nabla^{D \uparrow \text{SNPR}(\Gamma)}$ computes upper bounds. Given the first hypothesis, the third one could be replaced by

$$\text{Cond}^{D(V)}(b_1 \wedge b_2)(d) \sqsubseteq^{D(V)} \text{Cond}^{D(V)}(b_2) \left(\text{Cond}^{D(V)}(b_1)(d) \right)$$

Theorem 5 (Soundness of the reduced product). *If the three hypothesis mentioned above for the operator $\text{Cond}^{D(\cdot)}$ are satisfied, then the reduced product that we defined is sound*

- The reduction operator does not change the semantics: For any abstract value $a \in D \uparrow \text{SNPR}(\Gamma)$, $\gamma^{D \uparrow \text{SNPR}(\Gamma)}(\rho(a)) = \gamma^{D \uparrow \text{SNPR}(\Gamma)}(a)$.
- The reduction operator yields a result that is component-wise more precise: For any abstract value $a \in D \uparrow \text{SNPR}(\Gamma)$, $\rho(a) \sqsubseteq^{D \uparrow \text{SNPR}(\Gamma)} a$.
- $\sqsubseteq^{D \uparrow \text{SNPR}(\Gamma)}$ is a pre-order and $\gamma^{D \uparrow \text{SNPR}(\Gamma)}$ is monotonic.
- The abstract union and intersection are sound.
- The transfer functions for assignment and boolean conditions are sound.
- $\nabla^{D \uparrow \text{SNPR}(\Gamma)}$ computes upper bounds and enforces convergence.

Proof. The first soundness properties (the reduction operator does not change the semantics) is the object of lemma 35. The other soundness properties will be proven here, one by one.

The reduction operator yields a result that is component-wise more precise. Let $(E, C, N) = a$ be an element of the product domain, and $(E_1, C_1, N_1) = \rho(a)$ be its reduced version. By definition of reduction, $C_1 = \text{clos}^{\text{seq}(\Gamma)}_{\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a))}(C)$. By definition of closure for structural equalities, we know that $C \subseteq C_1$, which implies that $C_1 \sqsubseteq^{\text{seq}(\Gamma)} C$. By definition of reduction, $E_1 = E \sqcap^{\text{cc}(\Gamma)} \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(C_1) \cup \text{Env}(N))$. Since $\sqcap^{\text{cc}(\Gamma)}$ computes lower bounds (Lem. 14) we have, in particular, that $E_1 \sqsubseteq^{\text{cc}(\Gamma)} E$. Given the definition of N_1 and the fact that the abstraction of conditions for the NPR lifting $\text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)}$ yields abstract values that are smaller than the initial ones (γ), we have that $N_1 \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} N$, which allows to conclude.

$\sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)}$ is a pre-order. Reflexivity is proved in Lem. 44, while transitivity is proved in Lem. 46.

$\gamma^{\text{D}\uparrow\text{SNPR}(\Gamma)}$ is monotonic.

The abstract union is sound.

The abstract intersection is sound.

The transfer function for assignment is sound.

The transfer function for boolean conditions is sound.

The widening computes upper bounds.

The widening enforces convergence.

We will not transcribe all the proofs, but here are the main intermediate lemmas that proved useful during the proofs of Thm. 2 and 3 and Lem. 5 and 6:

Lemma 7 (Distributivity of prefix closure over union). *For any typing context Γ and any pair of sets of extended variables $(E_1, E_2) \in (\mathcal{P}(\mathcal{E}(\Gamma)))^2$, we have*

$$\text{clos}^{\text{cc}(\Gamma)}(E_1 \cup E_2) = \text{clos}^{\text{cc}(\Gamma)}(E_1) \cup \text{clos}^{\text{cc}(\Gamma)}(E_2)$$

Proof. By double inclusion. Let $x.p$ be an extended variable in $\text{clos}^{\text{cc}(\Gamma)}(E_1 \cup E_2)$. By definition of prefix closure, there exists a (possibly empty) path p' such that $x.pp' \in E_1 \cup E_2$. By definition of union, there exists $i \in \{1, 2\}$ such that $x.pp' \in E_i$. Hence, $x.p \in \text{clos}^{\text{cc}(\Gamma)}(E_i)$ and therefore $x.p \in \text{clos}^{\text{cc}(\Gamma)}(E_1) \cup \text{clos}^{\text{cc}(\Gamma)}(E_2)$. The other inclusion uses the same arguments : for an extended variable $x.p$ in $\text{clos}^{\text{cc}(\Gamma)}(E_1) \cup \text{clos}^{\text{cc}(\Gamma)}(E_2)$, there exists an index $i \in \{1, 2\}$ such that $x.p \in \text{clos}^{\text{cc}(\Gamma)}(E_i)$; which means that there exists a path p' such that $x.pp' \in E_i$, hence $x.pp' \in E_1 \cup E_2$, which allows to conclude that $x.p \in \text{clos}^{\text{cc}(\Gamma)}(E_1 \cup E_2)$. \square

Lemma 8 (Monotony of prefix closure). *For any typing context Γ and any pair of sets of extended variables $(E_1, E_2) \in (\mathcal{P}(\mathcal{E}(\Gamma)))^2$, we have*

$$E_1 \subseteq E_2 \Rightarrow \text{clos}^{\text{cc}(\Gamma)}(E_1) \subseteq \text{clos}^{\text{cc}(\Gamma)}(E_2)$$

Proof. Let $x.p$ be an extended variable in $\text{clos}^{\text{cc}(\Gamma)}(E_1)$. There exists a (possibly empty) path p' such that $x.pp' \in E_1$. Hence, by hypothesis, $x.pp' \in E_2$, from which we can deduce $x.p \in \text{clos}^{\text{cc}(\Gamma)}(E_2)$. \square

Lemma 9 (Correctness of assignment decomposition). *For any expression $t \in \text{Exp}$, any value $v \in \text{Values}$, any set of variables V and any state $s : V \rightarrow \text{Values}$; if $v \in \llbracket t \rrbracket_s^{\text{exp}}$ then*

$$\forall (x.pp' := t') \in \text{Decomp}(x.p := t), v \Downarrow^{\text{val}} p' \in \llbracket t' \rrbracket_s^{\text{exp}}$$

Proof. Let V and s be fixed. We do an induction on t .

Two first cases (base cases) : t is an extended variable or is a numeric expression i.e., $t \in \mathcal{E}(\Gamma) \vee \Gamma \vdash t : \text{N}$. In that case, by definition of the decomposition of assignment (Def. 6), we have that $\text{Decomp}(x.p := t) = \{x.p := t\}$; which implies that $p' = \varepsilon$ and $t' = t$. By definition of projection (Def. 3), $v \Downarrow^{\text{val}} \varepsilon = v$, which allows to conclude by hypothesis.

Third case (inductive case) : t is a variant i.e., there exists a constructor A and an expression t_2 such that $t = A(t_2)$. In this case, the inductive hypothesis states the following (x_2, p_2 and v_2 are universally quantified):

$$\forall (x_2.p_2p'_2 := t'_2) \in \text{Decomp}(x_2.p_2 := t_2), v_2 \in \llbracket t_2 \rrbracket_s^{\text{exp}} \Rightarrow v_2 \Downarrow^{\text{val}} p'_2 \in \llbracket t'_2 \rrbracket_s^{\text{exp}}$$

Let $(x.pp' := t')$ be an element of $\text{Decomp}(x.p := t)$. By definition of assignment decomposition for variant expression, we know that there exists p'' such that $p' = @Ap''$. By definition of the denotational semantics of expressions, there exists v' such that $v = A(v')$ and $v' \in \llbracket t_2 \rrbracket_s^{\text{exp}}$. By applying the induction hypothesis (with $x_2 = x, p_2 = p @ A, p'_2 = p'', v_2 = v'$ and $t'_2 = t'$), we have $v' \Downarrow^{\text{val}} p'' \in \llbracket t'_2 \rrbracket_s^{\text{exp}}$. Since $v \Downarrow^{\text{val}} p' = v' \Downarrow^{\text{val}} p''$, we have $v \Downarrow^{\text{val}} p' \in \llbracket t' \rrbracket_s^{\text{exp}}$.

Forth case (inductive case) : t is a record i.e., there exists a finite set of indices I , a family of expressions $(t_i)_{i \in I}$ and a family of fields $(f_i)_{i \in I}$ such that $t = \left\{ \overline{f_i = t_i}^{i \in I} \right\}$. By definition of $\llbracket \cdot \rrbracket_s^{\text{exp}}$ we know that there exists a family of values $(v_i)_{i \in I}$ such that $v = \left\{ \overline{f_i = v_i}^{i \in I} \right\}$. Let $(x.pp' := t')$ be an element of $\text{Decomp}(x.p := t)$. By definition of assignment decomposition, we know that there exists an index $i_0 \in I$ and a path p'' such that $p' = .f_{i_0}p''$ and $(x.p.f_{i_0}p'' := t') \in \text{Decomp}(x.p.f_{i_0} := t_{i_0})$. By applying the induction hypothesis to v_{i_0} , we have that $v_{i_0} \Downarrow^{\text{val}} p'' \in \llbracket t' \rrbracket_s^{\text{exp}}$, hence $v \Downarrow^{\text{val}} p' \in \llbracket t' \rrbracket_s^{\text{exp}}$. \square

Lemma 10 (Congruence of the undefinedness of value projection). *For any value $v \in \text{Values}$ and any two paths $p \in \text{Paths}$ and $p' \in \text{Paths}$, if $v \Downarrow^{\text{val}} pp' \neq \text{Undef}$ then $v \Downarrow^{\text{val}} p \neq \text{Undef}$.*

Proof. By induction on the length of p .

Base case: for paths p of length zero. If $p = \varepsilon$ (only path of length zero), then $v \Downarrow^{\text{val}} p = v$ which is a value of the set Values, hence cannot be Undef.

Inductive case. Let's distinguish two cases. If $p = @Ap''$ for some constructor A then, since we know that $v \Downarrow^{\text{val}} @Ap''p' \neq \text{Undef}$, by definition of value projection we know that there exists a value v' such that $v = A(v')$ and $v \Downarrow^{\text{val}} @Ap''p' = v' \Downarrow^{\text{val}} p''p'$. By applying the induction hypothesis to p'' (whose length is one less than the one of p), we know that $v' \Downarrow^{\text{val}} p'' \neq \text{Undef}$. Furthermore, by definition of value projection, $v' \Downarrow^{\text{val}} p'' = v \Downarrow^{\text{val}} p$, hence $v \Downarrow^{\text{val}} p \neq \text{Undef}$. \square

Lemma 11 (Consistency between the semantics of expressions and the extended variables being mentioned). *For any expression $t \in \text{Exp}$, any boolean expression $b \in \text{BExp}$, any set of variables V and any state $s : V \rightarrow \text{Values}$*

- If $\llbracket t \rrbracket_s^{\text{exp}} \neq \emptyset$ then $\forall x.p \in \text{Env}(t), s(x) \Downarrow^{\text{val}} p \neq \text{Undef}$.
- If $\llbracket b \rrbracket_s^{\text{bool}} \neq \emptyset$ then $\forall x.p \in \text{Env}(b), s(x) \Downarrow^{\text{val}} p \neq \text{Undef}$.

Proof. We will first do the proof for expressions, and then for boolean expressions.

For expressions we reason by induction on t .

If $t = x.p$ is an extended variable then $\text{Env}(t) = \{x.p\}$ and by the definition of the semantics of expressions we have that $s(x) \Downarrow^{\text{val}} p \neq \text{Undef}$.

If $t = A(t')$ is a variant expression then $\text{Env}(t) = \text{Env}(t')$ and $\llbracket t \rrbracket_s^{\text{exp}} = \{A(v) \mid v \in \llbracket t' \rrbracket_s^{\text{exp}}\}$. Since $\llbracket t \rrbracket_s^{\text{exp}} \neq \emptyset$, we also have that $\llbracket t' \rrbracket_s^{\text{exp}} \neq \emptyset$, which allows to use the induction hypothesis, which allows to conclude this case.

If $t = \left\{ \overline{f_i = t_i}^{i \in I} \right\}$ is a record expression then $\text{Env}(t) = \bigcup_{i \in I} \text{Env}(t_i)$ and $\llbracket t \rrbracket_s^{\text{exp}} = \left\{ \left\{ \overline{f_i = v_i}^{i \in I} \right\} \mid \forall i \in I, v_i \in \llbracket t_i \rrbracket_s^{\text{exp}} \right\}$. Since $\llbracket t \rrbracket_s^{\text{exp}} \neq \emptyset$, we also know that $\forall i \in I, \llbracket t_i \rrbracket_s^{\text{exp}} \neq \emptyset$. Let $x.p \in \text{Env}(t)$. There exists $i \in I$ such that $x.p \in \text{Env}(t_i)$. By applying the induction hypothesis to t_i , $s(x) \Downarrow^{\text{val}} p \neq \text{Undef}$.

If $t = t_1 \boxplus t_2$ is an arithmetic operation on two sub-expressions then $\text{Env}(t) = \text{Env}(t_1) \cup \text{Env}(t_2)$ and $\llbracket t \rrbracket_s^{\text{exp}} = \{v_1 \boxplus v_2 \mid v_1 \in \llbracket t_1 \rrbracket_s^{\text{exp}} \wedge v_2 \in \llbracket t_2 \rrbracket_s^{\text{exp}}\}$. Let $x.p \in \text{Env}(t)$. There exists $i \in \{1, 2\}$ such that $x.p \in \text{Env}(t_i)$. Since $\llbracket t \rrbracket_s^{\text{exp}}$ is not empty, $\llbracket t_i \rrbracket_s^{\text{exp}}$ is not empty either. By applying the induction hypothesis to t_i , we know that $s(x) \Downarrow^{\text{val}} p \neq \text{Undef}$.

For boolean expressions we reason by induction on b , like we did for expressions. Like the proof for expressions, the non-emptiness of denotations propagates to any sub-expression, and any extended variable mentioned in b is mentioned in one of its sub-expressions, which allows to use the induction hypothesis to conclude. Notice that this proof for boolean expressions uses the proof for expressions; since boolean expressions can be numeric comparisons over numeric expressions. \square

Lemma 12 (Consistency between typing and evaluation). *For any set of variables V , any typing context $\Gamma : V \rightarrow \text{Types}$, any expression $t \in \text{Exp}$, any state $s : V \rightarrow \text{Values}$, and any type $\tau \in \text{Types}$*

$$\left. \begin{array}{l} \Gamma \vdash t : \tau \\ \Gamma \vdash s \\ v \in \llbracket t \rrbracket_s^{\text{exp}} \end{array} \right\} \Rightarrow \Gamma \vdash v : \tau$$

Lemma 13 (Composition of value projections). *For any $v \in \text{Values} \cup \{\text{Undef}\}$, and any two paths $(p, p') \in \text{Paths}^2$, we have that*

$$(v \Downarrow^{\text{val}} p) \Downarrow^{\text{val}} p' = v \Downarrow^{\text{val}} pp'$$

Proof. By induction on the length of p . If $p = \varepsilon$ (only path of length 0), then, by definition of value projection, $v \Downarrow^{\text{val}} p = v$ and, by definition of path concatenation, $pp' = p'$, which allows to conclude this case. For the inductive case, we distinguish two different cases for p :

If $p = @Ap''$. We distinguish two different cases for v : either it is a variant using constructor A , or it is not. If there exists a value v' such that $v = A(v')$, then we have both $v \Downarrow^{\text{val}} p = v' \Downarrow^{\text{val}} p''$ and $v \Downarrow^{\text{val}} pp' = v' \Downarrow^{\text{val}} p''p'$. Using the inductive hypothesis on p'' and v' allows to conclude this case for v . If there is no such v' , then we have both $v \Downarrow^{\text{val}} p = \text{Undef}$ and $v \Downarrow^{\text{val}} pp' = \text{Undef}$. Using the definition of value projection and a simple induction on p' allows to prove that $v \Downarrow^{\text{val}} p = \text{Undef}$ implies $(v \Downarrow^{\text{val}} p) \Downarrow^{\text{val}} p' = \text{Undef}$; which allows to conclude.

If $p = .f_i p'$. There are two cases for v . First case: there exists a set of indexes I and a family of values $(v_j)_{j \in I}$ such that $v = \{\overline{f_j} = \overline{v_j}^{j \in I}\}$ and $i \in I$. In this case, by definition of value projection, we have both $v \Downarrow^{\text{val}} p = v_i \Downarrow^{\text{val}} p''$ and $v \Downarrow^{\text{val}} pp' = v_i \Downarrow^{\text{val}} p''p'$, which allows to conclude, by applying the induction hypothesis to v_i and p'' . The case in which v is not a record with a field f_i is treated in the same way as the case (in previous paragraph) in which v was not a variant using constructor A . \square

Lemma 14 (Abstract intersection of constructor constraints computes lower bounds). *Let Γ be a typing context and $(E_1, E_2) \in \text{cc}(\Gamma)$ be two elements of the constructor constraints domain. We have that:*

$$E_1 \sqcap^{\text{cc}(\Gamma)} E_2 \sqsubseteq^{\text{cc}(\Gamma)} E_1 \quad E_1 \sqcap^{\text{cc}(\Gamma)} E_2 \sqsubseteq^{\text{cc}(\Gamma)} E_2$$

Lemma 15 (Expressiveness of the constructor constraints domain relative to structural equalities and the NPR lifting). *For any typing context Γ , any set of structural equalities $C \subseteq (\mathcal{E}(\Gamma))^2$ and any element of the NPR lifting $N \in \text{D} \uparrow \text{NPR}(\Gamma)$, we have both*

$$\gamma^{\text{seq}(\Gamma)}(C) \subseteq \gamma^{\text{cc}(\Gamma)}(\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(C)))$$

and

$$\gamma^{\text{D} \uparrow \text{NPR}(\Gamma)}(N) \subseteq \gamma^{\text{cc}(\Gamma)}(\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(N)))$$

Proof. We prove the two inclusions separately.

For structural equalities. Let s be an element of $\gamma^{\text{seq}(\Gamma)}(C)$. Let $x.p \in \text{Env}(C)$ be an extended variable mentioned in C . There exists another (possibly equal) extended variable $y.q$ such that either $(x.p, y.q) \in C$ or $(y.q, x.p) \in C$. In either of those two cases, by definition of concretisation for structural equalities, we have that $s(x) \Downarrow^{\text{val}} p = s(y) \Downarrow^{\text{val}} q \neq \text{Undef}$. In particular, we have that $s(x) \Downarrow^{\text{val}} p \neq \text{Undef}$. This being true for all the extended variables in $\text{Env}(C)$, the lemma of congruence of the undefinedness of value projection (Lem. 10) allows to deduce that it is true for all the extended variables in $\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(C))$. Hence, $s \in \gamma^{\text{cc}(\Gamma)}(\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(C)))$.

For the NPR lifting. Let s be an element of $\gamma^{\text{D}\uparrow\text{NPR}(\Gamma)}(N)$. Let (d, E) be the two components of N . We have $\text{Env}(N) = E$ and $s \Downarrow^{\text{sto}} E \in \gamma^{\text{D}(E)}(d)$. Since $\gamma^{\text{D}(E)}(d) \subseteq E \rightarrow \text{Values}$ and $\text{Undef} \notin \text{Values}$, we know that $\forall x.p \in E, (s \Downarrow^{\text{sto}} E)(x.p) \neq \text{Undef}$. Said otherwise, $\forall x.p \in E, s(x) \Downarrow^{\text{val}} p \neq \text{Undef}$. As before, Lem. 10 allows to conclude. \square

Lemma 16 (Monotony of the transfer function for conditions, in the NPR domain). *Let Γ be a typing context, $b \in \text{BExp}$ be a numeric condition and N_1 and N_2 be two abstract values from $\text{D}\uparrow\text{NPR}(\Gamma)$. We have:*

$$N_1 \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} N_2 \Rightarrow \text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)}(b)(N_1) \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} \text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)}(b)(N_2)$$

Lemma 17 (Transitivity of the pre-order for the NPR domain). *The relation $\sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)}$ is transitive.*

Lemma 18 (Abstract conditions and pre-order for the NPR domain).

Adding a condition to an abstract value from the NPR domain yields a result that is at least as precise as the input. Formally, for any typing context Γ , any abstract value N from $\text{D}\uparrow\text{NPR}(\Gamma)$ and any numeric condition $b \in \text{BExp}$, we have:

$$\text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)}(b)(N) \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} N$$

Lemma 19 (Composition of abstract conditions for the NPR domain).

In the NPR domain, adding a conjunction of conditions is at least as precise as adding them one after the other. Formally, for any typing context Γ , any abstract value N from $\text{D}\uparrow\text{NPR}(\Gamma)$ and any two numeric conditions b_1 and b_2 , we have:

$$\begin{aligned} & \text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)}(b_1 \wedge b_2)(N) \\ & \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} \text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)}(b_1) \left(\text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)}(b_2)(N) \right) \end{aligned}$$

Lemma 20 (Irrelevance of the order of conjuncts in boolean conjunction for abstract conditions). *Inverting the order of two conditions in the NPR domain yields abstract values that are equivalent (with respect to pre-order).*

Formally, for any typing context Γ , any abstract value $N \in \text{D}\uparrow\text{NPR}(\Gamma)$, and any pair of numeric conditions $(b_1, b_2) \in \text{BExp}^2$, we have:

$$\text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)}(b_1 \wedge b_2)(N) \equiv^{\text{D}\uparrow\text{NPR}(\Gamma)} \text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)}(b_2 \wedge b_1)(N)$$

where the relation $\equiv^{\text{D}\uparrow\text{NPR}(\Gamma)}$ is defined by $N_1 \equiv^{\text{D}\uparrow\text{NPR}(\Gamma)} N_2$ if and only if we have both $N_1 \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} N_2$ and $N_2 \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} N_1$.

Lemma 21 (Removing variables that are not present, in the NPR domain). *In the NPR domain, trying to remove from an abstract value variables that are not there, does not change the abstract value. Formally, for any typing context Γ , any abstract value $N \in \text{D}\uparrow\text{NPR}(\Gamma)$ and any set of extended variables E , we have :*

$$E \cap \text{Env}(N) = \emptyset \Rightarrow \text{Rem}_E^{\text{D}\uparrow\text{NPR}(\Gamma)}(N) = N$$

Lemma 22 (Monotony of removing variables for the NPR domain).

In the NPR domain, the operator that removes variables is monotonous on abstract values. Formally, for any typing context Γ , any pair of abstract values $(N_1, N_2) \in \text{D}\uparrow\text{NPR}(\Gamma)^2$ and any set of extended variables E , we have :

$$N_1 \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} N_2 \Rightarrow \text{Rem}_E^{\text{D}\uparrow\text{NPR}(\Gamma)}(N_1) \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} \text{Rem}_E^{\text{D}\uparrow\text{NPR}(\Gamma)}(N_2)$$

Lemma 23 (Removing previously unconstrained variables from a conjunction of equality constraints). *For any $N \in \text{D}\uparrow\text{NPR}(\Gamma)$, any set C of structural equalities closed by transitivity and any set E of extended variables such that all the variables in E are unconstrained in N , we have:*

$$\text{Rem}_E^{\text{D}\uparrow\text{NPR}(\Gamma)}(N + \text{Num}(C)) = \text{Rem}_E^{\text{D}\uparrow\text{NPR}(\Gamma)}(N) + \text{Num}(C \cap (\text{Env}(C) \setminus E)^2)$$

In particular, when $E \cap \text{Env}(N) = \emptyset$, the variables in E are necessarily unconstrained in N and we have

$$\text{Rem}_E^{\text{D}\uparrow\text{NPR}(\Gamma)}(N + \text{Num}(C)) = N + \text{Num}(C \cap (\text{Env}(C) \setminus E)^2)$$

Lemma 24 (More conjuncts implies more precision). *For any typing context Γ , any abstract value $N \in \text{D}\uparrow\text{NPR}(\Gamma)$ and any two families of boolean conditions $(b_i)_{i \in I}$ and $(b_j)_{j \in J}$ with $J \subseteq I$, we have*

$$\text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)}\left(\bigwedge_I b_i\right)(N) \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} \text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)}\left(\bigwedge_J b_j\right)(N)$$

When reasoning about closure of structural equalities, we use two custom induction principles, detailed below.

Lemma 25 (Two-variable induction principle for closure of structural equalities). *Let Γ be a typing context, C be a set of structural equalities, E be a set of extended variables and $C' = \text{clos}^{\text{seq}(\Gamma)}_E(C)$ be the closure of C with E . For any property P on two extended variables, the following induction principle*

holds :

$$\left. \begin{array}{l}
 \text{First base case :} \\
 (x.p, y.q) \in C \Rightarrow P(x.p, y.q) \\
 \text{Second base case :} \\
 x.p \in E \Rightarrow P(x.p, x.p) \\
 \text{Symmetry :} \\
 \left. \begin{array}{l} P(x.p, y.q) \\ (x.p, y.q) \in C' \end{array} \right\} \Rightarrow P(y.q, x.p) \\
 \text{Reflexivity :} \\
 \left. \begin{array}{l} P(x.p, y.q) \\ (x.p, y.q) \in C' \end{array} \right\} \Rightarrow P(x.p, x.p) \\
 \text{Transitivity :} \\
 \left. \begin{array}{l} P(x.p, y.q) \\ P(y.q, z.r) \\ (x.p, y.q) \in C' \\ (y.q, z.r) \in C' \end{array} \right\} \Rightarrow P(x.p, z.r) \\
 \text{Congruence :} \\
 \left. \begin{array}{l} P(x.p, y.q) \\ P(x.pr, x.pr) \\ (x.p, y.q) \in C' \\ (x.pr, x.pr) \in C' \end{array} \right\} \Rightarrow P(x.pr, y.qr)
 \end{array} \right\} \Rightarrow \forall (x.p, y.q) \in C', P(x.p, y.q)$$

This induction principle can be simplified when the property to be proved has the special form $P(x.p, y.q) = P'(x.p) \wedge P(y.q)$. That is, when we want to prove a property on the individual variables being mentioned in the closure, rather than on the pairs of variables.

Lemma 26 (Single-variable induction principle for closure of structural equalities). *Let Γ be a typing context, C be a set of structural equalities, E be a set of extended variables and $C' = \text{clos}^{\text{seq}(\Gamma)}_E(C)$ be the closure of C with E . For any property P on extended variables, the following induction principle holds :*

$$\left. \begin{array}{l}
 \text{First base case :} \\
 (x.p, y.q) \in C \Rightarrow P(x.p) \wedge P(y.q) \\
 \text{Second base case :} \\
 x.p \in E \Rightarrow P(x.p) \\
 \text{Congruence :} \\
 \left. \begin{array}{l} P(x.p) \\ P(y.q) \\ P(x.pr) \\ (x.p, y.q) \in C' \\ (x.pr, x.pr) \in C' \end{array} \right\} \Rightarrow P(y.qr)
 \end{array} \right\} \Rightarrow \forall x.p \in \text{Env}(C'), P(x.p)$$

Lemma 27 (Semantics of closure for structural equalities). *For any typing context Γ , any set of structural equalities $C \in \text{seq}(\Gamma)$, and any set of*

extended variables $E \subseteq \mathcal{E}(\Gamma)$

$$\gamma^{\text{seq}(\Gamma)} \left(\text{clos}_E^{\text{seq}(\Gamma)}(C) \right) = \gamma^{\text{seq}(\Gamma)}(C) \cap \gamma^{\text{cc}(\Gamma)} \left(\text{clos}^{\text{cc}(\Gamma)}(E) \right)$$

Proof. This proof proceeds by double inclusion.

First inclusion : let's prove that $\gamma^{\text{seq}(\Gamma)}(\text{clos}_E^{\text{seq}(\Gamma)}(C)) \subseteq \gamma^{\text{seq}(\Gamma)}(C) \cap \gamma^{\text{cc}(\Gamma)}(E)$. Let $s \in \gamma^{\text{seq}(\Gamma)}(\text{clos}_E^{\text{seq}(\Gamma)}(C))$. Let's start by proving that $s \in \gamma^{\text{seq}(\Gamma)}(C)$. Let $(x.p, y.q) \in C$. By definition of closure for structural equalities, we have $(x.p, y.q) \in \text{clos}_E^{\text{seq}(\Gamma)}(C)$ as well. Hence, by the definition of concretisation for structural equalities, $s(x) \Downarrow^{\text{val}} p = s(y) \Downarrow^{\text{val}} q \neq \text{Undef}$. Thus, $s \in \gamma^{\text{seq}(\Gamma)}(C)$. Now, let's prove that $s \in \gamma^{\text{cc}(\Gamma)}(E)$. Let $z.r \in E$. By definition of closure, $(z.r, z.r) \in \text{clos}_E^{\text{seq}(\Gamma)}(C)$. Hence, by definition of concretisation for structural equalities, $s(z) \Downarrow^{\text{val}} r = s(z) \Downarrow^{\text{val}} r \neq \text{Undef}$. Thus, $s \in \gamma^{\text{cc}(\Gamma)}(E)$. By definition of set intersection, $s \in \gamma^{\text{seq}(\Gamma)}(C) \cap \gamma^{\text{cc}(\Gamma)}(E)$.

Second inclusion : let's now prove the converse inclusion $\gamma^{\text{seq}(\Gamma)}(C) \cap \gamma^{\text{cc}(\Gamma)}(E) \subseteq \gamma^{\text{seq}(\Gamma)}(\text{clos}_E^{\text{seq}(\Gamma)}(C))$. Let $s \in \gamma^{\text{seq}(\Gamma)}(C) \cap \gamma^{\text{cc}(\Gamma)}(E)$. By the definition of concretisation for structural equalities, what we need to prove is

$$\forall (x.p, y.q) \in \text{clos}_E^{\text{seq}(\Gamma)}(C), s(x) \Downarrow^{\text{val}} p = s(y) \Downarrow^{\text{val}} q \neq \text{Undef}$$

We reason by induction (Lem. 25) on the elements of $\text{clos}_E^{\text{seq}(\Gamma)}(C)$.

Case 1 (base case) : $(x.p, y.q) \in C$. Since store s belongs to $\gamma^{\text{seq}(\Gamma)}(C)$, we have that $s(x) \Downarrow^{\text{val}} p = s(y) \Downarrow^{\text{val}} q \neq \text{Undef}$, which is what we want.

Case 2 (base case) : $x.p \in E$. Since store s belongs to $\gamma^{\text{cc}(\Gamma)}(E)$ (and since equality is reflexive), we have that $s(x) \Downarrow^{\text{val}} p = s(x) \Downarrow^{\text{val}} p \neq \text{Undef}$.

Case 3 (inductive case) : some pair $(x.p, y.q)$ belongs to $\text{clos}_E^{\text{seq}(\Gamma)}(C)$ because it was deduced by *symmetry* from another pair $(y.q, x.p)$. Applying the induction hypothesis to $(y.q, x.p)$, we have that $s(y) \Downarrow^{\text{val}} q = s(x) \Downarrow^{\text{val}} p \neq \text{Undef}$. Equality being symmetric, we deduce that $s(x) \Downarrow^{\text{val}} p = s(y) \Downarrow^{\text{val}} q \neq \text{Undef}$, which allows to conclude.

Case 4 (inductive case) : for the pairs deduced by *transitivity*, we reason in the same fashion than for symmetry : the inductive hypothesis plus the transitivity of equality allow to conclude.

Case 5 (inductive case) : some pair $(x.p, x.p)$ belongs to $\text{clos}_E^{\text{seq}(\Gamma)}(C)$ because it was deduced by *reflexivity* from a pair $(x.p, y.q)$. The inductive hypothesis gives us that $s(x) \Downarrow^{\text{val}} p = s(y) \Downarrow^{\text{val}} q \neq \text{Undef}$. In particular, we have that $s(x) \Downarrow^{\text{val}} p \neq \text{Undef}$, which allows to conclude.

Case 6 (inductive case) : some pair $(x.pr, y.qr)$ belongs to $\text{clos}^{\text{seq}(\Gamma)}_E(C)$ because it was deduced by *congruence*. That is, the two pairs $(x.p, y.q)$ and $(x.pr, x.pr)$ belong to $\text{clos}^{\text{seq}(\Gamma)}_E(C)$. The induction hypothesis associated with $(x.p, y.q)$ gives us that $s(x) \Downarrow^{\text{val}} p = s(y) \Downarrow^{\text{val}} q \neq \text{Undef}$. By projecting this equality on path r , and using Lem. 13, we deduce that $s(x) \Downarrow^{\text{val}} pr = s(y) \Downarrow^{\text{val}} qr$. Furthermore, the induction hypothesis associated with $(x.pr, x.pr)$ gives us that $s(x) \Downarrow^{\text{val}} pr \neq \text{Undef}$. Hence $s(x) \Downarrow^{\text{val}} pr = s(y) \Downarrow^{\text{val}} qr \neq \text{Undef}$, which allows to conclude. \square

Lemma 28 (Characterisation of closure for structural equalities). *For any set of extended variables E and any set of structural equalities C , we have that*

$$\text{clos}^{\text{seq}(\Gamma)}_E(C) = \text{clos}^{\text{seq}(\Gamma)}_{\emptyset}(C \cup \{(x.p, x.p) \mid x.p \in E\})$$

Proof. We reason by double inclusion.

First inclusion : $\text{clos}^{\text{seq}(\Gamma)}_E(C) \subseteq \text{clos}^{\text{seq}(\Gamma)}_{\emptyset}(C \cup \{(x.p, x.p) \mid x.p \in E\})$. The set $\text{clos}^{\text{seq}(\Gamma)}_{\emptyset}(C \cup \{(x.p, x.p) \mid x.p \in E\})$ contains all the equalities in C , mentions all the variables in E and is closed by reflexivity, transitivity, symmetry and congruence. The set $\text{clos}^{\text{seq}(\Gamma)}_E(C)$ being by definition the smallest set with these characteristics, we have the first inclusion.

Second inclusion : By definition of closure for structural equalities, we have that $C \subseteq \text{clos}^{\text{seq}(\Gamma)}_E(C)$. In addition, from the fact that the set $\text{clos}^{\text{seq}(\Gamma)}_E(C)$ mentions all the extended variables in E and is closed by reflexivity, we deduce that $\{(x.p, x.p) \mid x.p \in E\} \subseteq \text{clos}^{\text{seq}(\Gamma)}_E(C)$. Hence, $C \cup \{(x.p, x.p) \mid x.p \in E\} \subseteq \text{clos}^{\text{seq}(\Gamma)}_E(C)$. In addition, the set $\text{clos}^{\text{seq}(\Gamma)}_E(C)$ is closed by reflexivity, transitivity, symmetry and congruence. The set $\text{clos}^{\text{seq}(\Gamma)}_{\emptyset}(C \cup \{(x.p, x.p) \mid x.p \in E\})$ being by definition the smallest with these characteristics, we have the second inclusion. \square

Lemma 29 (Monotony of closure for structural equalities). *For any typing context Γ , any three sets of extended variables E , E_1 and E_2 and any three sets of structural equalities C , C_1 and C_2 , we have both*

$$C_1 \subseteq C_2 \Rightarrow \text{clos}^{\text{seq}(\Gamma)}_E(C_1) \subseteq \text{clos}^{\text{seq}(\Gamma)}_E(C_2)$$

and

$$E_1 \subseteq E_2 \Rightarrow \text{clos}^{\text{seq}(\Gamma)}_{E_1}(C) \subseteq \text{clos}^{\text{seq}(\Gamma)}_{E_2}(C)$$

Proof. We begin by proving monotony with respect to the sets of structural equalities: $C_1 \subseteq C_2 \Rightarrow \text{clos}^{\text{seq}(\Gamma)}_E(C_1) \subseteq \text{clos}^{\text{seq}(\Gamma)}_E(C_2)$. Since C_1 is included in C_2 , which in turn is included in $\text{clos}^{\text{seq}(\Gamma)}_E(C_2)$, we can deduce that C_1 is included in $\text{clos}^{\text{seq}(\Gamma)}_E(C_2)$. This, together with the properties of $\text{clos}^{\text{seq}(\Gamma)}_E(C_2)$ and the definition of $\text{clos}^{\text{seq}(\Gamma)}_E(C_1)$ allow to prove the inclusion. Using the monotony with respect to sets of structural equalities that we just showed and the characterisation of closure for structural equalities (Lem. 28) allows to prove monotony with respect to sets of extended abstract values. \square

Lemma 30 (Composition of closures for structural equalities). *For any typing context Γ , any three sets of extended variables E_1 , E_2 and E_3 and any set of structural equalities $C \subseteq (\mathcal{E}(\Gamma))^2$*

$$\text{clos}_{E_2}^{\text{seq}(\Gamma)} \left(\text{clos}_{E_1}^{\text{seq}(\Gamma)}(C) \right) = \text{clos}_{E_1 \cup E_2}^{\text{seq}(\Gamma)}(C)$$

Proof. By applying three times the characterisation of closure for structural equalities (Lem. 28), we have that:

$$\begin{aligned} \text{clos}_{E_2}^{\text{seq}(\Gamma)} \left(\text{clos}_{E_1}^{\text{seq}(\Gamma)}(C) \right) &= \text{clos}_{\emptyset}^{\text{seq}(\Gamma)} \left(C \cup \{(x.p, x.p) \mid x.p \in (E_1 \cup E_2)\} \right) \\ &= \text{clos}_{E_1 \cup E_2}^{\text{seq}(\Gamma)}(C) \end{aligned}$$

□

Lemma 31 (Union of closures for structural equalities). *For any finite set of indices I , any sequence of extended variables $(E_i)_{i \in I}$ and any set of structural equalities C*

$$\bigcup_{i \in I} \text{clos}_{E_i}^{\text{seq}(\Gamma)}(C) \subseteq \text{clos}_{\bigcup_{i \in I} E_i}^{\text{seq}(\Gamma)}(C)$$

Proof. Let j be an index of I . We have that $E_j \subseteq \bigcup_{i \in I} E_i$. The closure of structural equalities being monotonous with respect to the extended variables being added (second property of Lem. 29), we can deduce that $\text{clos}_{E_j}^{\text{seq}(\Gamma)}(C) \subseteq \text{clos}_{\bigcup_{i \in I} E_i}^{\text{seq}(\Gamma)}(C)$. This being true for all j of I , we have the desired inclusion. □

Lemma 32 (Self-closure for structural equalities). *For any typing context Γ , any set of structural equalities $C \subseteq (\mathcal{E}(\Gamma))^2$ and any set of extended variables $E \subseteq \mathcal{E}(\Gamma)$, we have that*

$$E \subseteq \text{Env}(C) \Rightarrow \text{clos}_E^{\text{seq}(\Gamma)}(C) = \text{clos}_{\emptyset}^{\text{seq}(\Gamma)}(C)$$

Proof. Both $\text{clos}_E^{\text{seq}(\Gamma)}(C)$ and $\text{clos}_{\emptyset}^{\text{seq}(\Gamma)}(C)$ include C , mention all the variables in E (since they are mentioned in C) and are closed by reflexivity, transitivity, symmetry and congruence. Since they are both the smallest set with these characteristics, they are equal. □

Lemma 33 (Existence of prefixes for non-trivial structural equalities after closure). *For any typing context Γ , any set of structural equalities $C \in \text{seq}(\Gamma)$, any set of extended variables E , and any pair $(x.p, y.q)$ of extended variables, we have that:*

$$\left. \begin{array}{l} (x.p, y.q) \in \text{clos}_E^{\text{seq}(\Gamma)}(C) \\ x.p \neq y.q \end{array} \right\} \Rightarrow \exists (p', q', s) \in \text{Paths}^3, \left\{ \begin{array}{l} x.p = x.p' s \\ y.q = y.q' s \\ (x.p', y.q') \in \text{clos}_{\emptyset}^{\text{seq}(\Gamma)}(C) \end{array} \right.$$

Proof. We reason by induction (Lem. 25) on $\text{clos}_E^{\text{seq}(\Gamma)}(C)$.

First case (Base case) : $(x.p, y.q) \in C$. The set C being included in $\text{clos}^{\text{seq}(\Gamma)}_{\emptyset}(C)$, we can take $r = \varepsilon$, $p' = p$ and $q' = q$ to conclude.

Second case (Base case) : $x.p = y.q \wedge x.p \in E$. The contradiction with the hypothesis $x.p \neq y.q$ allows to conclude this case.

Third case (Inductive case) : $(x.p, y.q)$ is in $\text{clos}^{\text{seq}(\Gamma)}_E(C)$ because of *transitivity*. In this case, there exists an extended variable $z.r$ such that $(x.p, z.r) \in \text{clos}^{\text{seq}(\Gamma)}_E(C) \wedge (z.r, y.q) \in \text{clos}^{\text{seq}(\Gamma)}_E(C)$. The induction hypothesis gives us four paths p' , r_1 , s_1 , r_2 , q' and s_2 such that

$$\begin{cases} x.p = x.p' s_1 \\ z.r = z.r_1 s_1 \\ z.r = z.r_2 s_2 \\ y.q = y.q' s_2 \\ (x.p', z.r_1) \in \text{clos}^{\text{seq}(\Gamma)}_{\emptyset}(C) \\ (z.r_2, y.q') \in \text{clos}^{\text{seq}(\Gamma)}_{\emptyset}(C) \end{cases}$$

If r_2 is longer than r_1 then there exists a path s'_1 such that $r_2 = r_1 s'_1$. The set $\text{clos}^{\text{seq}(\Gamma)}_{\emptyset}(C)$ being closed by reflexivity and symmetry, both $(z.r_2, z.r_2)$ and $(z.r_1, x.p')$ belong to $\text{clos}^{\text{seq}(\Gamma)}_{\emptyset}(C)$. This set also being closed by congruence, $(z.r_2, x.p' s'_1)$ belongs to $\text{clos}^{\text{seq}(\Gamma)}_{\emptyset}(C)$. Hence, by reflexivity and transitivity of $\text{clos}^{\text{seq}(\Gamma)}_{\emptyset}(C)$, we have that $(x.p' s'_1, y.q') \in \text{clos}^{\text{seq}(\Gamma)}_{\emptyset}(C)$. We have both $r = r_2 s_2 = r_1 s_1$ and $r_2 = r_1 s'_1$; from which we can deduce that $s_1 = s'_1 s_2$. This allows to conclude, since we have that:

$$\begin{cases} x.p = x.p' s'_1 s_2 \\ y.q = y.q' s_2 \\ (x.p' s'_1, y.q') \in \text{clos}^{\text{seq}(\Gamma)}_{\emptyset}(C) \end{cases}$$

Fourth case (Inductive case) : $(x.p, y.q)$ is in $\text{clos}^{\text{seq}(\Gamma)}_E(C)$ because of *congruence*. In this case, there exists three paths p_0 , q_0 and r such that $x.p = x.p_0 r \wedge y.q = y.q_0 r$ and both $(x.p, x.p)$ and $(x.p_0, y.q_0)$ belong to $\text{clos}^{\text{seq}(\Gamma)}_E(C)$. The induction hypothesis gives us the existence of three paths p_2 , q_1 and s_2 , such that:

$$\begin{cases} x.p_0 = x.p_2 s_2 \\ y.q_0 = y.q_1 s_2 \\ (x.p_2, y.q_1) \in \text{clos}^{\text{seq}(\Gamma)}_{\emptyset}(C) \end{cases}$$

This allows us to conclude :

$$\begin{cases} x.p = x.p_2 s_2 r \\ y.q = y.q_1 s_2 r \\ (x.p_2, y.q_1) \in \text{clos}^{\text{seq}(\Gamma)}_{\emptyset}(C) \end{cases}$$

Fifth and sixth cases (Inductive cases) : In the case where $(x.p, y.q)$ is in $\text{clos}^{\text{seq}(\Gamma)}_E(C)$ because of *reflexivity*, we have, in particular, that $x.p = y.q$. The contradiction with the hypothesis $x.p \neq y.q$ allows to conclude. For the case involving *symmetry*, the fact that $\text{clos}^{\text{seq}(\Gamma)}_\emptyset(C)$ is closed by symmetry allows to conclude. \square

Lemma 34 (Parsimony of closure for structural equalities). *For any typing context Γ , any set of extended variables $E \subseteq \mathcal{E}(\Gamma)$, any set of structural equalities $C \subseteq (\mathcal{E}(\Gamma))^2$ and any pair of extended variables $(x.p, y.q) \in (\mathcal{E}(\Gamma))^2$*

$$(x.p, y.q) \in \text{clos}^{\text{seq}(\Gamma)}_E(C) \Rightarrow (x.p, y.q) \in \text{clos}^{\text{seq}(\Gamma)}_{\{x.p, y.q\}}(C)$$

Proof. We call C_1 the set $\text{clos}^{\text{seq}(\Gamma)}_{\{x.p, y.q\}}(C)$. We distinguish two cases. **Frist case** : if $x.p = y.q$. Then, by definition of C_1 , we have $(x.p, x.p) \in C_1$. **Second case** : if $x.p \neq y.q$. Then, using Lem. 33, we know that there exists three paths p', q' and r such that $x.p = x.p'r \wedge y.q = y.q'r \wedge (x.p', y.q') \in \text{clos}^{\text{seq}(\Gamma)}_\emptyset(C)$. By monotony (Lem. 29) we have $\text{clos}^{\text{seq}(\Gamma)}_\emptyset(C) \subseteq C_1$, hence $(x.p', y.q') \in C_1$. By definition of closure, $(x.p, x.p) \in C_1$. Hence, C_1 being closed by congruence, $(x.p, y.q) \in C_1$. \square

Lemma 35 (Semantics of the reduction operator). *For any typing context Γ , any abstract value $a \in \text{D} \uparrow \text{SNPR}(\Gamma)$*

$$\gamma^{\text{D} \uparrow \text{SNPR}(\Gamma)}(\rho(a)) = \gamma^{\text{D} \uparrow \text{SNPR}(\Gamma)}(a)$$

Proof. Let $(E, C, N) = a$ be the three components of the abstract value a . By the definition of reduction and semantics for the product, we have that

$$\gamma^{\text{D} \uparrow \text{SNPR}(\Gamma)}(\rho(a)) = \gamma^{\text{cc}(\Gamma)}(E) \cap \gamma^{\text{seq}(\Gamma)} \left(\text{clos}^{\text{seq}(\Gamma)}_{\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a))}(C) \right) \cap \gamma^{\text{D} \uparrow \text{NPR}(\Gamma)}(N)$$

From there, using the lemma about the semantics of closure for the structural equalities domain (Lem. 27), we deduce that $\gamma^{\text{D} \uparrow \text{SNPR}(\Gamma)}(\rho(a)) = \gamma^{\text{cc}(\Gamma)}(E) \cap \gamma^{\text{seq}(\Gamma)}(C) \cap \gamma^{\text{cc}(\Gamma)}(\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a))) \cap \gamma^{\text{D} \uparrow \text{NPR}(\Gamma)}(N)$. Given the distributivity over union of the prefix closure (Lem. 7), the definition of concretisation for constructor constraints and the fact that $\text{Env}(a) = E \cup \text{Env}(C) \cup \text{Env}(N)$, we can deduce that $\gamma^{\text{cc}(\Gamma)}(\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a))) = \gamma^{\text{cc}(\Gamma)}(\text{clos}^{\text{cc}(\Gamma)}(E)) \cap \gamma^{\text{cc}(\Gamma)}(\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(C))) \cap \gamma^{\text{cc}(\Gamma)}(\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(N)))$. The set E being an element of $\text{cc}(\Gamma)$, it is already closed by prefix, so $E = \text{clos}^{\text{cc}(\Gamma)}(E)$. Using this and the expressiveness of the constructor constraints domain relative to the Seq domain and the NPR lifting (Lem. 15), we can conclude :

$$\gamma^{\text{D} \uparrow \text{SNPR}(\Gamma)}(\rho(a)) = \gamma^{\text{cc}(\Gamma)}(E) \cap \gamma^{\text{seq}(\Gamma)}(C) \cap \gamma^{\text{D} \uparrow \text{NPR}(\Gamma)}(N) = \gamma^{\text{D} \uparrow \text{SNPR}(\Gamma)}(a)$$

\square

Lemma 36 (Environment of product reduction). *For any typing context Γ , and any abstract value $a = (E, C, N)$ from the product domain; if we call $(E_1, C_1, N_1) = \rho(a)$ the three components of the reduction of a , we have that:*

$$\text{Env}(\rho(a)) = \text{Env}(C_1)$$

Proof. Let's start by proving that $\text{Env}(C_1)$ is closed by prefix, since it will prove useful later on the proof. We recall that C_1 is defined as follows : $C_1 = \text{clos}^{\text{seq}(\Gamma)}_{\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a))}(C)$. We proceed by induction (Lem. 26) on the elements of C_1 . *First base case* : if $(x.p, y.q) \in C$, then $\{x.p, y.q\} \subseteq \text{Env}(C)$, so for any prefix $x.p' \preceq x.p$, we have $x.p' \in \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(C)) \subseteq \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a))$, hence, by definition of C_1 , $(x.p', x.p') \in C_1$ and $x.p' \in \text{Env}(C_1)$. The proof for $y.q$ is the same than the one for $x.p$. *Second base case* : if $x.p \in \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a))$ then, by definition of $\text{clos}^{\text{cc}(\Gamma)}$, we also have, for any prefix $x.p' \preceq x.p$, that $x.p' \in \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a))$, hence $(x.p', x.p') \in C_1$ and $x.p' \in \text{Env}(C_1)$. *Inductive case* : if a variable $y.qr$ belongs to $\text{Env}(C_1)$ because an equality $(x.pr, y.qr)$ was introduced into C_1 by congruence, then we have both $(x.p, y.q) \in C_1$ and $(x.pr, x.pr) \in C_1$. In that case, for any prefix $y.q' \preceq y.qr$ there are two cases. Either $y.q' \preceq y.q$ and the induction hypothesis allows to conclude. Or there exists r' such that $r' \preceq r$ and $y.q' = y.qr'$; in which case the induction hypothesis applied to $x.pr$ allows to say that $x.pr'$ belongs to $\text{Env}(C_1)$ and $(x.pr', x.pr') \in C_1$, which, by congruence, allows to deduce that $(x.pr', y.qr') \in C_1$ and $y.q' \in \text{Env}(C_1)$. This concludes the intermediary proof that $\text{Env}(C_1)$ is closed by prefix.

By definition of the environment for the product domain, we have that $\text{Env}(\rho(a)) = E_1 \cup \text{Env}(C_1) \cup \text{Env}(N_1)$. Hence, we already have that $\text{Env}(C_1) \subseteq \text{Env}(\rho(a))$, and what we want to show is the reverse inclusion : $\text{Env}(\rho(a)) \subseteq \text{Env}(C_1)$. By definition of reduction, we have that

$$E_1 = E \sqcap^{\text{cc}(\Gamma)} \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(C_1))$$

$$N_1 = \text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)} \left(\bigwedge_{(x.p, y.q) \in C_1 \wedge \Gamma \vdash x.p : N} x.p = y.q \right) (N)$$

By definition of $\sqcap^{\text{cc}(\Gamma)}$, we know that $E_1 = E \cup \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(C_1))$. From this we deduce $E_1 \subseteq \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(C_1))$. In addition, $\text{Env}(C_1)$ being closed by prefix, we have $\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(C_1)) = \text{Env}(C_1)$ and $E_1 \subseteq \text{Env}(C_1)$.

Given the definitions of $\text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)}$ and N_1 , we have that

$$\begin{aligned} \text{Env}(N_1) &= \text{Env}(N) \cup \bigcup_{(x.p, y.q) \in C_1 \wedge \Gamma \vdash x.p : N} \{x.p, y.q\} \\ &\subseteq \text{Env}(N) \cup \text{Env}(C_1) \end{aligned}$$

Furthermore, by definition of C_1 , $\text{Env}(a) \subseteq \text{Env}(C_1)$. Hence $\text{Env}(N) \subseteq \text{Env}(C_1)$ and $\text{Env}(N_1) \subseteq \text{Env}(C_1)$, which allows to conclude. \square

Lemma 37 (Semantics of closure for the product). *For any typing context Γ , any abstract value $a \in \text{D}\uparrow\text{SNPR}(\Gamma)$ and any set of extended variables*

$$E_1 \subseteq \mathcal{E}(\Gamma)$$

$$\gamma^{\text{D}\uparrow\text{SNPR}(\Gamma)} \left(\text{clos}_{E_1}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a) \right) = \gamma^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a) \cap \gamma^{\text{cc}(\Gamma)} \left(\text{clos}^{\text{cc}(\Gamma)}(E_1) \right)$$

Proof. Let $(E, C, N) = a$ be the three components of abstract value a . By the definition of closure for the product and the semantics of reduction (Lem. 35), we have that $\gamma^{\text{D}\uparrow\text{SNPR}(\Gamma)} \left(\text{clos}_{E_1}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a) \right) = \gamma^{\text{D}\uparrow\text{SNPR}(\Gamma)} \left(E, \text{clos}^{\text{seq}(\Gamma)}_{E_1}(C), N \right)$. By the definition of concretisation for the product, this is equal to $\gamma^{\text{cc}(\Gamma)}(E) \cap \gamma^{\text{seq}(\Gamma)} \left(\text{clos}^{\text{seq}(\Gamma)}_{E_1}(C) \right) \cap \gamma^{\text{D}\uparrow\text{SNPR}(\Gamma)}(N)$. Using the semantics of closure for structural equalities (Lem. 27), we know that $\gamma^{\text{seq}(\Gamma)} \left(\text{clos}^{\text{seq}(\Gamma)}_{E_1}(C) \right) = \gamma^{\text{seq}(\Gamma)}(C) \cap \gamma^{\text{cc}(\Gamma)} \left(\text{clos}^{\text{cc}(\Gamma)}(E_1) \right)$. Then, the definition of concretisation for the product allows to conclude. \square

Lemma 38 (Environment of product closure). *For any typing context Γ , any set of extended variables E_0 , and any abstract value $a = (E, C, N)$ from the product domain; if we call $(E', C', N') = \text{clos}_{E_0}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a)$ the three components of the closure of a with variables E , we have that:*

$$\text{Env} \left(\text{clos}_{E_0}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a) \right) = \text{Env}(C')$$

Proof. This is a direct consequence of Lem. 36, since, by definition of closure for the product domain, $\text{clos}_{E_0}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a) = \rho \left(E, \text{clos}^{\text{seq}(\Gamma)}_{E_0}(C), N \right)$. \square

Lemma 39 (Structural equalities after product closure). *For any typing context Γ , any set of extended variables $E_0 \subseteq \mathcal{E}(\Gamma)$ and any abstract value $a = (E, C, N)$ from the product domain, if we call $(E_1, C_1, N_1) = \text{clos}_{E_0}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a)$ the three components of the closure of a with the extended variables E_0 , then we have that*

$$C_1 = \text{clos}_{\text{clos}^{\text{cc}(\Gamma)}(E_0 \cup \text{Env}(a))}^{\text{seq}(\Gamma)}(C)$$

Proof. Let us call E^0 the set $\{(x.p, x.p) \mid x.p \in E_0\}$. By definition of closure for the product, we have $C_1 = \rho(E, C \cup E^0, N) = \text{clos}_{\text{clos}^{\text{cc}(\Gamma)}(E_0 \cup \text{Env}(a))}^{\text{seq}(\Gamma)}(C \cup E^0)$. Using Lem. 28 and 30, we deduce that $C_1 = \text{clos}_{E_0 \cup \text{clos}^{\text{cc}(\Gamma)}(E_0 \cup \text{Env}(a))}^{\text{seq}(\Gamma)}(C)$. By Lem. 7 we have that $\text{clos}^{\text{cc}(\Gamma)}(E_0 \cup \text{Env}(a)) = \text{clos}^{\text{cc}(\Gamma)}(E_0) \cup \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a))$. In addition, by definition of $\text{clos}^{\text{cc}(\Gamma)}$, we have that $E_0 \subseteq \text{clos}^{\text{cc}(\Gamma)}(E_0)$, which allows to conclude. \square

Lemma 40 (Self-closure for the product domain). *For any typing context Γ and any abstract value $a \in \text{D}\uparrow\text{SNPR}(\Gamma)$*

$$\text{clos}_{\text{Env}(a)}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a) = \rho(a)$$

Proof. We start by giving names to the different components of the abstract values being considered.

$$(E, C, N) = a \quad (E_0, C_0, N_0) = \rho(a) \quad (E_1, C_1, N_1) = \text{clos}_{\text{Env}(a)}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a)$$

We first prove $C_0 = C_1$. By definition, $C_0 = \text{clos}_{\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a))}^{\text{seq}(\Gamma)}(C)$. Thanks to Lem. 39, we have $C_1 = \text{clos}_{\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a) \cup \text{Env}(a))}^{\text{seq}(\Gamma)}(C) = C_0$. E_0 (respectively E_1) being defined by a function of E and C_0 (resp. C_1), we deduce that $E_0 = E_1$. Similarly, N_0 (resp. N_1) is defined by a function of Γ , N and C_0 (resp. C_1), so $N_0 = N_1$. This allows to conclude: $\rho(a) = \text{clos}_{\text{Env}(a)}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a)$. \square

Lemma 41 (Environment after closing a product with ‘not new’ variables). *For any typing context Γ and any two abstract values a_1 and a_2 from the product domain, we have that:*

$$\forall \left(x.p \in \text{Env} \left(\text{clos}_{\text{NotNew}(a_2, a_1)}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a_1) \right) \right), \forall (x.p' @ A \preceq x.p), x.p' @ A \in \text{Env}(\rho(a_1))$$

Proof. We start by giving a name to the components of the different abstract values that we are interested in:

$$(E_1, C_1, N_1) = a_1 \quad (E'_1, C'_1, N'_1) = \text{clos}_{\text{NotNew}(a_2, a_1)}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a_1) \quad (E_0, C_0, N_0) = \rho(a_1)$$

Thanks to Lem. 36, 38 and 39 and the definition of reduction for product closure, we know that

$$\begin{aligned} \text{Env}(\rho(a_1)) &= \text{Env}(C_0) & \text{Env} \left(\text{clos}_{\text{NotNew}(a_2, a_1)}^{\text{D}\uparrow\text{SNPR}(\Gamma)}(a_1) \right) &= \text{Env}(C'_1) \\ C_0 &= \text{clos}_{\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a_1))}^{\text{seq}(\Gamma)}(C_1) & C'_1 &= \text{clos}_{\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a_1) \cup \text{NotNew}(a_2, a_1))}^{\text{seq}(\Gamma)}(C_1) \end{aligned}$$

We prove, by induction (Lem. 26) on C'_1 , that $\forall x.p \in \text{Env}(C'_1), P(x.p)$, where property P is given by $P(x.p) : \forall (x.p' @ A \preceq x.p), x.p' @ A \in \text{Env}(C_0)$.

First case (base case) : let $(x.p, y.q) \in C_1$. By definition of reduction for the product domain, any prefix of either $x.p$ or $y.q$ belongs to C_0 . In particular, any prefix ending by a constructor access. Hence, we have both $P(x.p)$ and $P(y.q)$.

Second case (base case) : let $x.p \in \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a_1) \cup \text{NotNew}(a_2, a_1))$. Using Lem. 7, we can distinguish two cases: First sub-case : $x.p \in \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a_1))$. In this sub-case, since $x.p \in \text{Env}(C_0)$ and $\text{Env}(C_0)$ is closed by prefix, any prefix of $x.p$ (and in particular any prefix ending with a constructor access) belongs to $\text{Env}(C_0)$. Second sub-case : $x.p \in \text{clos}^{\text{cc}(\Gamma)}(\text{NotNew}(a_2, a_1))$. Let $x.p' @ A$ be a prefix of $x.p$ ending with a constructor access. By definition of $\text{clos}^{\text{cc}(\Gamma)}$, we know that there exists p_0 such that $x.pp_0 \in \text{NotNew}(a_2, a_1)$. By definition of the NotNew operator, we know that any prefix of $x.pp_0$ ending with a constructor access, belongs to $\text{Env}(\rho(a_1))$. In particular, $x.p' @ A$ is such a prefix, hence $x.p' @ A \in \text{Env}(\rho(a_1))$.

Third case (inductive case) : congruence. In this case, we consider two pairs $(x.p, y.q) \in C'_1$ and $(x.pr, x.pr) \in C'_1$, we get to assume $P(x.p) \wedge P(y.q) \wedge P(x.pr)$ and we have to show $P(y.qr)$. Let $y.s@A$ be a prefix of $y.qr$ ending with a constructor access. We distinguish two sub-cases. First sub-case : $y.s@A$ is a prefix of $y.q$. In this sub-case, using $P(y.q)$ allows to conclude. Second sub-case : $y.q$ is a prefix of $y.s@A$. Said otherwise, there exists a path r_0 such that $y.s@A = y.qr_0@A$ and $r_0@A$ is a prefix of r . If $x.p = y.q$, then $P(x.pr)$ allows to conclude. If $x.p \neq y.q$, then we use Lem. 33, which allows to deduce that there exists paths p_0, q_0, p_1 and q_1 such that:

$$(x.p_0, y.q_0) \in C_1 \wedge p = p_0p_1 \wedge q = q_0q_1$$

By definition of C_0 , we know that $(x.p_0, y.q_0) \in C_0$. Using $P(x.pr)$, we know that $(x.pr_0@A, x.pr_0@A) \in C_0$. The set of structural equalities C_0 being closed by congruence, we have $(x.pr_0@A, y.qr_0@A) \in C_0$. Hence, $y.qr_0@A \in \text{Env}(C_0)$. \square

Lemma 42 (Intersection of environments is included in ‘non-new’ variables). *For any two abstract values a_1 and a_2 , we have*

$$\text{Env}(a_1) \cap \text{Env}(a_2) \subseteq \text{NotNew}(a_1, a_2)$$

Proof. Let $x.p$ be an extended variable in $\text{Env}(a_1) \cap \text{Env}(a_2)$. Let $x.p'@A \preceq x.p$ be a prefix of $x.p$ ending with a constructor access. Using the fact that $x.p \in \text{Env}(a_2)$ and the definition of $\text{clos}^{\text{cc}(\Gamma)}$, we have that $x.p'@a \in \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a_2))$; which allows to conclude. \square

Lemma 43 (Transposition in non-new variables). *For any typing context Γ , any three abstract values a_1, a_2 and a_3 from the product domain and any extended variable $x.p$, we have that:*

$$\left. \begin{array}{l} x.p \in \text{NotNew}(a_2, a_1) \\ x.p \in \text{Env}(a_3) \end{array} \right\} \Rightarrow x.p \in \text{NotNew}(a_3, a_1)$$

Proof. In order to prove that $x.p \in \text{NotNew}(a_3, a_1)$ we need to prove two things:

- that $x.p$ belongs in $\text{Env}(a_3)$
- and that any prefix of $x.p$ ending with a constructor access is contained in the prefix-closure of $\text{Env}(a_1)$.

The first is an hypothesis. The second is a consequence of $x.p \in \text{NotNew}(a_2, a_1)$, by definition of $\text{NotNew}(a_2, a_1)$. \square

Lemma 44 (Reflexivity of $\sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)}$). *For any typing context Γ and any abstract value $a \in \text{D}\uparrow\text{SNPR}(\Gamma)$ of the product domain, we have that*

$$a \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a$$

Proof. Let $(E, C, N) = a$ be the three components of a , and let C_0 be the set of structural equalities such that $\text{clos}^{\text{D}\uparrow\text{SNPR}(\Gamma)}_{\text{Env}(a)}(a) = \rho(E, C_0, N)$ — that is

$C_0 = \text{clos}^{\text{seq}(\Gamma)}_{\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a))}(C)$. By definition of closure for structural equalities, we have $C \subseteq C_0$, hence $C_0 \sqsubseteq^{\text{seq}(\Gamma)} C$ and $(E, C_0, N) \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a$. Using the fact that the reduction operator computes values that are component-wise more precise, we can deduce that $\text{clos}^{\text{D}\uparrow\text{SNPR}(\Gamma)}_{\text{Env}(a)}(a) \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a$. By definition of $\sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)}$, we can conclude : $a \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a$. \square

Lemma 45 (Link between pre-order, environment and reduction for the product domain). *For any typing context Γ , and any two abstract values a_1 and a_2 of $\text{D}\uparrow\text{SNPR}(\Gamma)$, we have that:*

$$a_1 \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2 \Rightarrow \text{Env}(\rho(a_2)) \subseteq \text{Env}\left(\text{clos}^{\text{D}\uparrow\text{SNPR}(\Gamma)}_{\text{NotNew}(a_2, a_1)}(a_1)\right)$$

Proof. We begin by introducing some names. Let a'_1 be the abstract value $\text{clos}^{\text{cc}(\Gamma)}_{\text{NotNew}(a_2, a_1)}(a_1)$ and a'_2 be $\rho(a_2)$. Let C_2 , C'_2 and C'_1 be the structural equalities components of respectively a_2 , a'_2 and a'_1 . Thanks to Lem. 38, we know both that $\text{Env}(a'_1) = \text{Env}(C'_1)$ and $\text{Env}(a'_2) = \text{Env}(C'_2)$. Hence our goal can be reformulated as $\text{Env}(C'_2) \subseteq \text{Env}(C'_1)$. The definition of reduction gives us that $C'_2 = \text{clos}^{\text{seq}(\Gamma)}_{\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a_2))}(C_2)$. Hence by definition, C'_2 is the smallest set containing C_2 , mentioning all the variables in $\text{Env}(a_2)$ and closed by symmetry, reflexivity, transitivity and congruence. By definition, C'_1 is closed by symmetry, reflexivity, transitivity and congruence. From $a_1 \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2$ we can deduce, by definition, that $a'_1 \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2$, hence $\text{Env}(a_2) \subseteq \text{Env}(a'_1) = \text{Env}(C'_1)$. In addition, Lem. 38 also gives us that $\text{Env}(C'_1)$ is closed by prefix, so $\text{clos}^{\text{cc}(\Gamma)}_{\text{Env}(a_2)} \subseteq \text{Env}(C'_1)$. That is, C'_1 mentions all the variables in $\text{Env}(a_2)$. From $a'_1 \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2$ we can also deduce that $C_2 \subseteq C'_1$. Therefore $C'_2 \subseteq C'_1$ and $\text{Env}(a'_2) \subseteq \text{Env}(a'_1)$. \square

Lemma 46 (Transitivity of $\sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)}$). *For any typing context Γ , and any three abstract values a_1 , a_2 and a_3 of $\text{D}\uparrow\text{SNPR}(\Gamma)$, we have that:*

$$\left. \begin{array}{l} a_1 \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2 \\ a_2 \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_3 \end{array} \right\} \Rightarrow a_1 \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_3$$

Proof. Let a_1 , a_2 and a_3 be three abstract values of the product domain such that $a_1 \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2$ and $a_2 \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_3$. By definition of the relation $\sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)}$, we have both $\text{clos}^{\text{D}\uparrow\text{SNPR}(\Gamma)}_{\text{NotNew}(a_2, a_1)}(a_1) \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2$ and $\text{clos}^{\text{D}\uparrow\text{SNPR}(\Gamma)}_{\text{NotNew}(a_3, a_2)}(a_2) \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_3$. To alleviate notations, we introduce some aliases:

$$a'_1 = \text{clos}^{\text{D}\uparrow\text{SNPR}(\Gamma)}_{\text{NotNew}(a_2, a_1)}(a_1) \quad a'_2 = \text{clos}^{\text{D}\uparrow\text{SNPR}(\Gamma)}_{\text{NotNew}(a_3, a_2)}(a_2) \quad a''_1 = \text{clos}^{\text{D}\uparrow\text{SNPR}(\Gamma)}_{\text{NotNew}(a_3, a_1)}(a_1)$$

We also give a name to the three components of the different abstract values being considered:

$$\begin{array}{lll} (E_1, C_1, N_1) = a_1 & (E_2, C_2, N_2) = a_2 & (E_3, C_3, N_3) = a_3 \\ (E'_1, C'_1, N'_1) = a'_1 & (E'_2, C'_2, N'_2) = a'_2 & (E''_1, C''_1, N''_1) = a''_1 \end{array}$$

Thanks to Lem. 39, we know that

$$C'_1 = \frac{\text{clos}^{\text{seq}(\Gamma)}}{\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a_1) \cup \text{NotNew}(a_2, a_1))} (C_1) \quad (C_1)$$

$$C'_2 = \frac{\text{clos}^{\text{seq}(\Gamma)}}{\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a_2) \cup \text{NotNew}(a_3, a_2))} (C_2) \quad (C_2)$$

$$C''_1 = \frac{\text{clos}^{\text{seq}(\Gamma)}}{\text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a_1) \cup \text{NotNew}(a_3, a_1))} (C_1) \quad (C_1)$$

The goal is to prove that $a''_1 \overline{\sqsubseteq}^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_3$. We will start by proving that $C''_1 \sqsubseteq^{\text{seq}(\Gamma)} C_3$, i.e., $C_3 \subseteq C''_1$. Let $(x.p, y.q)$ be an element of C_3 . During this proof, there will be two inductions (Lem. 25), one embedded inside the other:

- An outer induction on the set C'_2
- An inner induction on the set C'_1

For both of these inductions, the goal will be to prove that $(x.p, y.q)$ belongs to C''_1 . The set C''_1 being closed by reflexivity, symmetry, transitivity and congruence, each one of the four inductive cases in the induction is immediate: it suffices to use the induction hypotheses and the fact that C''_1 is closed by the property corresponding to the inductive case. We will hence only consider the base cases. Since $a'_2 \overline{\sqsubseteq}^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_3$, we know that $(x.p, y.q)$ is also an element of C'_2 . We reason by induction on C'_2 :

First case for C'_2 (outer induction) : $(x.p, y.q) \in C_2$. Since $a'_1 \overline{\sqsubseteq}^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2$, we know, in particular, that $C_2 \subseteq C'_1$. Hence, $(x.p, y.q) \in C'_1$. We reason by induction on C'_1 :

First case for C'_1 (inner induction) : $(x.p, y.q) \in C_1$. By definition of closure for structural equalities, we have that $C_1 \subseteq C''_1$. In particular, we have that $(x.p, y.q) \in C''_1$.

Second case for C'_1 (inner induction) : $x.p = y.q \wedge x.p \in \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a_1) \cup \text{NotNew}(a_2, a_1))$. In this case, there exist an extended variable $x.p'$ such that $x.p \preceq x.p' \wedge x.p' \in \text{Env}(a_1) \cup \text{NotNew}(a_2, a_1)$. We distinguish two cases for $x.p'$. If $x.p' \in \text{Env}(a_1)$, then $x.p \in \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a_1) \cup \text{NotNew}(a_3, a_1))$ and hence we can conclude that $(x.p, x.p) \in C''_1$. If $x.p' \in \text{NotNew}(a_2, a_1)$ then we can use the fact that $(x.p, x.p) \in C_3$ —hence $x.p \in \text{Env}(a_3)$ — to apply Lem. 43 and deduce that $x.p \in \text{NotNew}(a_3, a_1)$ and therefore that $x.p \in \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a_1) \cup \text{NotNew}(a_3, a_1))$ and $(x.p, x.p) \in C''_1$.

This concludes the inner induction on C'_1 .

Second case for C'_2 (outer induction) : $x.p = y.q \wedge x.p \in \text{clos}^{\text{cc}(\Gamma)}(\text{Env}(a_2) \cup \text{NotNew}(a_3, a_2))$. There exists $x.p'$ such that $x.p \preceq x.p' \wedge x.p' \in \text{Env}(a_2) \cup \text{NotNew}(a_3, a_2)$. We distinguish two cases for $x.p'$.

First case for $x.p' : x.p' \in \text{Env}(a_2)$. Since $a_1 \sqsubseteq^{\overline{\text{D}\uparrow\text{SNPR}(\Gamma)}} a_2$, we know that $\text{Env}(a_2) \subseteq \text{Env}(a_1)$. According to Lem. 38, we have both $\text{Env}(a_1) = \text{Env}(C'_1)$ and the fact that $\text{Env}(C'_1)$ is closed by prefix. Hence, we can deduce both $x.p' \in \text{Env}(C'_1)$ and $x.p \in \text{Env}(C'_1)$. Re-using the result of the inner induction that we did on C'_1 allows to conclude that $(x.p, x.p) \in C''_1$.

Second case for $x.p' : x.p' \in \text{NotNew}(a_3, a_2)$. We begin by proving that $x.p' \in \text{NotNew}(a_3, a_1)$. Let $x.p''@A \preceq x.p'$ be a prefix of $x.p'$ ending with a constructor access. Since $x.p' \in \text{NotNew}(a_3, a_2)$, we have $x.p''@A \in \text{Env}(\rho(a_2))$ and hence, by Lem. 45, that $x.p''@A \in \text{Env}(a'_1)$. From there, we deduce by Lem. 41 that $x.p''@A \in \text{Env}(\rho(a_1))$. This being true for any prefix $x.p''@A \preceq x.p'$, we can conclude that $x.p' \in \text{NotNew}(a_3, a_1)$ and hence $x.p' \in \text{Env}(C''_1)$. The set $\text{Env}(C''_1)$ being closed by prefix (Lem. 38), we also have $x.p \in \text{Env}(C''_1)$ and hence $(x.p, x.p) \in C''_1$.

This concludes the proof that $C''_1 \sqsubseteq^{\text{seq}(\Gamma)} C_3$. We have yet to prove $E''_1 \sqsubseteq^{\text{cc}(\Gamma)} E_3$ and $N''_1 \sqsubseteq^{\text{cc}(\Gamma)} N_3$.

From $a_2 \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_3$ we can deduce $\text{Env}(a_3) \subseteq \text{Env}(a'_2)$. By reusing what we have proven so far (in particular the first induction, on C'_2) we know that $\text{Env}(a_3) \cap \text{Env}(a'_2) \subseteq \text{Env}(a''_1)$. Hence, $\text{Env}(a_3) \subseteq \text{Env}(a''_1)$. Said otherwise, $E''_1 \sqsubseteq^{\text{cc}(\Gamma)} E_3$.

We will now show that $N''_1 \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} N_3$. In order to alleviate notations, for any abstract value N from the $\text{D}\uparrow\text{NPR}(\Gamma)$ domain and any set of structural equalities C , we will write $N + \text{Num } C$ to stand for the abstract value $\text{Cond}^{\text{D}\uparrow\text{NPR}(\Gamma)} \left(\bigwedge_{(x.p, y.q) \in C \wedge \Gamma \vdash x.p : N} x.p, y.q \right) (N)$, that is N together with the equalities in C that are *numeric*. Given that $a_1 \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2$ and $a_2 \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_3$, we have both $N_1 + \text{Num } C'_1 \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} N_2$ and $N_2 + \text{Num } C'_2 \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} N_3$. Using Lem. 16 and 17, we deduce that $N_1 + \text{Num } C'_1 + \text{Num } C'_2 \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} N_3$. Thanks to Lem. 19, we can rewrite the previous statement into $N_1 + \text{Num}(C'_2 \cup C'_3) \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} N_3$.

Given that $\text{Env}(N_3) \subseteq \text{Env}(C''_1)$, the set $(\text{Env}(C'_1) \cup \text{Env}(C'_2)) \setminus \text{Env}(C''_1)$ is disjoint with the set $\text{Env}(N_3)$. This allows to use Lem. 21 to deduce

$$\text{Rem}_{(\text{Env}(C'_1) \cup \text{Env}(C'_2)) \setminus \text{Env}(C''_1)}^{\text{D}\uparrow\text{NPR}(\Gamma)}(N_3) = N_3$$

this, together with the monotony of variable removal (Lem. 22) yields

$$\text{Rem}_{(\text{Env}(C'_1) \cup \text{Env}(C'_2)) \setminus \text{Env}(C''_1)}^{\text{D}\uparrow\text{NPR}(\Gamma)}(N_1 + \text{Num } C'_1 + \text{Num } C'_2) \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} N_3$$

Similarly, the fact that $\text{Env}(N_1) \subseteq \text{Env}(C''_1)$ implies that the two sets $\text{Env}(N_1)$ and $(\text{Env}(C'_1) \cup \text{Env}(C'_2)) \setminus \text{Env}(C''_1)$ are disjoint.

In order to use Lem. 23, we need first to show that $C'_1 \cup C'_2$ is closed under transitivity. Let $(x.p, y.q)$ and $(y.q, z.r)$ be two elements of $C'_1 \cup C'_2$. There exists two indices i and j in $\{1, 2\}$ such that $(x.p, y.q) \in C'_i$ and $(y.q, z.r) \in C'_j$. If $i = j$, then the fact that C'_i is closed under transitivity allows to conclude. If $i \neq j$ then we can assume without loss of generality that $i = 1$ and $j = 2$ (indeed, both

C'_1 and C'_2 are closed under symmetry, and $x.p$ and $z.r$ play symmetric roles). Applying Lem. 33 to $(y.q, z.r)$, we know that there exists three paths q' , r' and s , such that $q = q's \wedge r = r's \wedge (y.q', z.r') \in C_2$. Since $a_1 \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_2$, we also have $C_2 \subseteq C'_1$, hence $(y.q', z.r') \in C'_1$. The set C'_1 being closed under congruence, we deduce that $(y.q, z.r) \in C'_1$. Knowing that C'_1 is closed by transitivity allows to conclude.

We can hence apply Lem. 23, which gives us

$$N_1 + \text{Num}((C'_1 \cup C'_2) \cap (\text{Env}(C''_1))^2) \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} N_3$$

We will now prove that $(C'_1 \cup C'_2) \cap (\text{Env}(C''_1))^2 \subseteq C''_1$. Let $(x.p, y.q)$ be an element of $(C'_1 \cup C'_2) \cap (\text{Env}(C''_1))^2$. If $(x.p, y.q) \in C'_1$, then Lem. 33, the fact that $x.p \in \text{Env}(C''_1)$ and the fact that C''_1 is closed by congruence allow to conclude. If $(x.p, y.q) \in C'_2$, then using Lem. 33, the fact that $C_2 \subseteq C'_1$ and what we have proven so far allow to conclude.

Since $(C'_1 \cup C'_2) \cap (\text{Env}(C''_1))^2 \subseteq C''_1$, we can use Lem. 24 to deduce that

$$N_1 + \text{Num } C''_1 \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} N_3$$

Said otherwise, $N''_1 \sqsubseteq^{\text{D}\uparrow\text{NPR}(\Gamma)} N_3$. This concludes the proof that $a_1 \sqsubseteq^{\text{D}\uparrow\text{SNPR}(\Gamma)} a_3$. \square

A.11 Analysis of while Loops Using Loop Unrolling

Our analysis of **while** loops is quite standard. We over-approximate reflexive transitive closure by iterating until we find an abstract fixpoint. We use widening to ensure convergence in finite time. We use loop unrolling once to improve precision.

In order to compute $\text{Analyze}(\text{while } b \text{ do } c \text{ end})(a)$, we first over-approximate the effect of executing the loop one or more times, starting from a . Let's call that abstract value $\text{Lalo}(b, c, a)$, for ‘‘Loop at least once’’. $\text{Lalo}(b, c, a)$ is computed as the limit of the series defined by $a_0 = \text{Analyze}(\text{assert}(b); c)(a)$ and for any $n \in \mathbf{N}^*$,

$$a_{n+1} = a_n \nabla^{\text{D}\uparrow\text{S}\uparrow\text{R}(\Gamma, \Gamma)} (\text{Analyze}(\text{assert}(b); c)(a) \sqcup^{\text{D}\uparrow\text{S}\uparrow\text{R}(\Gamma, \Gamma)} \text{Analyze}(\text{assert}(b); c)(a_n))$$

The analysis of the whole loop is then given, from Lalo , by adding the exit condition of the loop, and the possibility that the loop did not execute at all:

$$\begin{aligned} \text{Analyze}(\text{while } b \text{ do } c \text{ end})(a) = \\ \text{Cond}^{\text{D}\uparrow\text{S}\uparrow\text{R}(\Gamma, \Gamma)}(-b)(\text{Lalo}(b, c, a)) \sqcup^{\text{D}\uparrow\text{S}\uparrow\text{R}(\Gamma, \Gamma)} \text{Cond}^{\text{D}\uparrow\text{S}\uparrow\text{R}(\Gamma, \Gamma)}(-b)(a) \end{aligned}$$