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## - To cite this version:

Rangaswami Balakrishnan, Jean-Claude Bermond, Paul Paulraja, Min Li Yu. On Hamilton cycle decompositions of the tensor product of complete graphs. Discrete Mathematics, 2003, 268 (1-3), pp.49-58. 10.1016/S0012-365X(02)00680-5 . hal-03764835

HAL Id: hal-03764835
https://inria.hal.science/hal-03764835
Submitted on 31 Aug 2022

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# On Hamilton Cycle Decompositions of the Tensor Product of Complete Graphs 

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July 19, 2002
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#### Abstract

In this paper, we show that the tensor product of complete graphs is hamilton cycle decomposable.


## 1 Introduction

Let $G$ and $H$ be two simple graphs. The tensor product (also called direct product ) of the graphs $G$ and $H, G \otimes H$, is the graph with the vertex set $V(G \otimes H)=V(G) \times V(H)$ and with the edge set $E(G \otimes H)=$ $\{(u, x)(v, y), u v \in E(G)$ and $x y \in E(H)\}$. A k-regular multigraph $G$ has a hamilton cycle decomposition if its edge set can be partitioned into $k / 2$
hamilton cycles when $k$ is even, or into $(k-1) / 2$ hamilton cycles plus a one factor (or perfect matching), when $k$ is odd. In this paper, we study the hamilton cycle decomposition of $K_{r} \otimes K_{s}$.

The problem of finding hamilton cycle decompositions of product graphs is not new. Hamilton cycle decompositions of various product graphs have been studied by many people (see the survey papers [1] and [5] or the book [6]). One interesting problem is to investigate whether the product graph of two hamilton cycle decomposable graphs is also hamilton cycle decomposable. Many results for various products have been obtained in the last few years $[2,4,7,9,10,11]$. Like other products, the tensor product graph has some interesting properties. For example, the tensor product of two hamilton cycle decomposable graphs may not be connected: consider the product of two even cycles. Jha [8] conjectured that if both $G$ and $H$ are hamilton cycle decomposable and $G \otimes H$ is connected, then the tensor product graph is also hamilton cycle decomposable. But this conjecture was disproved in [3]. It would be interesting to know what extra condition(s) should be added to $G$ and $H$ to ensure a hamilton cycle decomposition of $G \otimes H$.

The following result concerning the tensor product has been known for a long time.

Theorem 1.1. Let $G$ and $H$ be two even regular simple graphs. If both $G$ and $H$ are hamilton cycle decomposable and at least one of them has odd order, then $G \otimes H$ is hamilton cycle decomposable.

The above result can be obtained from the fact that $C_{r} \otimes C_{s}$ has a hamilton cycle decomposition if at least one of $r$ and $s$ is odd and that the tensor product is distributive over edge disjoint union of graphs. The next result follows immediately from Theorem 1.1.

Corollary 1.2. $K_{r} \otimes K_{s}$ has a hamilton cycle decomposition if both $r$ and $s$ are odd and $r, s \geq 3$.

The main result in this paper is Theorem 1.3.
Theorem 1.3. If $r, s \geq 3$, then $K_{r} \otimes K_{s}$ has a hamilton cycle decomposition.
Note that the case when at least one of $r$ and $s$ is less than three is trivial.

## 2 Proof of the main result

The proof of Theorem 1.3 depends on several relatively simple lemmas.
We will use $\lambda G$ (resp. $\lambda D$ ) to denote the graph (digraph) obtained by replacing each edge (resp. arc) of $G$ (resp. $D$ ) with $\lambda$ edges (resp. arcs). Let $a b$ denote the edge between the vertices $a$ and $b$, and ( $a, b$ ) denote the arc from $a$ to $b$. A $k$-cycle is denoted by either $\left(v_{1}, v_{2}, \ldots, v_{k}, v_{1}\right)$ or $e_{1} e_{2} \ldots e_{k}$ where $e_{i}=v_{i} v_{i+1}, i=1,2, \ldots, k-1$, and $e_{k}=v_{k} v_{1}$ and a $k$-path is denoted by $\left[v_{1}, v_{2}, \ldots, v_{k}\right]$.

Let $V\left(K_{2 q}\right)=\{\infty, 1,2, \ldots, 2 q-1\}$, and

$$
H^{i}=(\infty, 1+i, 2+i, 2 q-1+i, 3+i, 2 q-2+i, 4+i, \ldots, q+1+i, \infty)
$$

where $0 \leq i \leq 2 q-2$. (The arithmetic calculations are modulo $2 q-1$ on the residues $1,2, \ldots, 2 q-1$.) Clearly, the $H^{i}$,s are hamilton cycles of $K_{2 q}$.

We will also denote by $e_{j}^{i}, 1 \leq j \leq 2 q$, the $j$ th edge of $H^{i}$. We have

$$
e_{j}^{i}= \begin{cases}\infty(1+i), & \text { if } j=1 \\ (2+i-j / 2)(1+i+j / 2), & \text { if } j \text { is even and } j \neq 2 q \\ (i+(j+1) / 2)(2+i-(j+1) / 2), & \text { if } j \text { is odd and } j \neq 1 \\ (q+1+i) \infty, & \text { if } j=2 q\end{cases}
$$

Example: $q=4$ and 5

$$
\begin{array}{ll}
H^{0}: \infty 1273645 \infty & H^{0}: \infty 129384756 \infty \\
H^{1}: \infty 2314756 \infty & H^{1}: \infty 231495867 \infty \\
H^{2}: \infty 3425167 \infty & H^{2}: \infty 342516978 \infty \\
H^{3}: \infty 4536271 \infty & H^{3}: \infty 453627189 \infty \\
H^{4}: \infty 5647312 \infty & H^{4}: \infty 564738291 \infty \\
H^{5}: \infty 6751423 \infty & H^{5}: \infty 675849312 \infty \\
H^{6}: \infty 7162534 \infty & H^{6}: \infty 786951423 \infty \\
& H^{7}: \infty 897162534 \infty \\
& H^{8}: \infty 918273645 \infty
\end{array}
$$

The following results are some simple observations and they will be used extensively.

Lemma 2.1. (a) $\bigcup_{i=0}^{2 q-2}\left\{e_{j}^{i}\right\}=\bigcup_{i=0}^{2 q-2}\left\{e_{2 q+1-j}^{i}\right\}$
(b) The graphs induced by the edge sets $\bigcup_{i=0}^{2 q-2} \bigcup_{j=1}^{q}\left\{e_{j}^{i}\right\}$ and $\bigcup_{i=0}^{2 q-2} \bigcup_{j=q+1}^{2 q}\left\{e_{j}^{i}\right\}$ are $K_{2 q}$.
(c) The $H^{i}, 0 \leq i \leq 2 q-2$, form a hamilton cycle decomposition of $2 K_{2 q}$.
(d) The $H^{i}, 0 \leq i \leq q-2$, form a hamilton cycle decomposition of $K_{2 q} \backslash F$, where $F$ is the one factor $\bigcup_{j=1}^{q-1}\{(q-j)(q+j)\} \cup\{\infty, q\}$.

We remark that $K_{r} \otimes K_{s}$ can be obtained from $K_{s}$ by replacing each vertex $x$ of $K_{s}$ by a set of $r$ vertices $V_{x}$ and each edge $x y$ by a set of $r-1$ one factors between $V_{x}$ and $V_{y}$. More precisely, we let $V_{x}=\left\{x_{0}, x_{1}, \ldots, x_{r-1}\right\}$ and denote the one factor of distance $k(k=1,2, \ldots, r-1)$ from $V_{x}$ to $V_{y}$, the set of edges $F_{x, y}^{k}=\left\{x_{i} y_{i+k}, i=0,1, \ldots, r-1\right\}$ (the addition on the subscripts is modulo $r$ ). Note that the order between $x$ and $y$ is important as a one factor of distance $k$ from $V_{x}$ to $V_{y}$ is a one factor of distance $r-k(\equiv-k(\bmod r))$ from $V_{y}$ to $V_{x}$. If we fix an orientation for the edges of $K_{s}$ and denote the resulting tournament by $T_{s}$, then we have

$$
\begin{aligned}
& V\left(K_{r} \otimes K_{s}\right)=\bigcup_{x \in V\left(T_{s}\right)} V_{x} \text { and } \\
& E\left(K_{r} \otimes K_{s}\right)=\bigcup_{(x, y) \in A\left(T_{s}\right)} \bigcup_{k=1}^{r-1} F_{x, y}^{k}, \text { where } A\left(T_{s}\right) \text { is the arc set of } T_{s} .
\end{aligned}
$$

Assume that in $(r-1) T_{s}$, the $r-1$ arcs between any pair of vertices are labelled $1,2, \ldots, r-1$. Then the arc $(x, y)$ of $(r-1) T_{s}$ with label $k$ can be associated to the set of edges $F_{x, y}^{k}$ in $K_{r} \otimes K_{s}$ and in fact this is a one to one correspondance relationship. We will construct a hamilton cycle decomposition of $K_{r} \otimes K_{s}$ from an oriented hamilton cycle decomposition of the labelled $(r-1) T_{s}$ by associating a suitable oriented (not directed) hamilton cycle of $(r-1) T_{s}$ with a hamilton cycle of $K_{r} \otimes K_{s}$ (see Lemma 2.3). For that we need the following definition.

Definition 2.2. Let $H=\left(v_{1}, v_{2}, \ldots, v_{s}, v_{1}\right)$ be an oriented (not necessary directed) hamilton cycle of $(r-1) T_{s}$. We can also write $H=a_{1} a_{2} \ldots a_{s}$, where $a_{i}$ is the arc of $(r-1) T_{s}$ between $v_{i}$ and $v_{i+1}$. We define the label of $H, \ell(H)=\sum_{i=1}^{s} \varepsilon_{i} \ell\left(a_{i}\right)$, where $\ell\left(a_{i}\right)$ is the label of the arc $a_{i}$ and $\varepsilon_{i}=1$ if $a_{i}=\left(v_{i}, v_{i+1}\right), \varepsilon_{i}=-1$ otherwise.

Lemma 2.3. Let $H=\left(v_{1}, v_{2}, \ldots, v_{s}, v_{1}\right)=a_{1} a_{2} \ldots a_{s}$ be an oriented hamilton cycle of $(r-1) T_{s}$. If $\ell(H)$ and $r$ are relatively prime, then the edge set in $K_{r} \otimes K_{s}$ corresponding to $H$ forms a hamilton cycle of $K_{r} \otimes K_{s}$.

Proof. The edge set $\bigcup_{i} F_{v_{i}, v_{i+1}}^{\varepsilon_{i} l\left(a_{i}\right)}$ forms a 2-factor of $K_{r} \otimes K_{s}$. Consider the subgraph of $K_{r} \otimes K_{s}$ induced by the edge set $\bigcup_{i \neq 1} F_{v_{i}, v_{i+1}}^{\varepsilon_{i} \ell\left(a_{i}\right)}$, it is a union of $r$ paths of length $s-1$. When the edge set $F_{v_{1}, v_{2}}^{\varepsilon_{1} l\left(a_{1}\right)}$ is added, the condition that $\ell(H)$ and $r$ are relatively prime guarantees that the resulting 2 -factor is a hamilton cycle of $K_{r} \otimes K_{s}$.

Corollary 2.4. If there exists a decomposition of $(r-1) T_{s}$ into oriented hamilton cycles, such that the label of each hamilton cycle is relatively prime to $r$, then there exists a hamilton cycle decomposition of $K_{r} \otimes K_{s}$.

## Proof of Theorem 1.3:

The case where both $r$ and $s$ are odd was already covered by Corollary 1.2. So we can assume that at least one of $r$ and $s$ is even. As the tensor product is commutative, we will assume that $s$ is even. Let $s=2 q$.

In what follows, we give a precise description of $T_{2 q}$ (or $T_{s}$ ) by giving a special orientation to $K_{2 q}$. First we take the hamilton cycle decomposition of $2 K_{2 q}, H^{i}, i=0,1, \ldots, 2 q-2$ as described in the beginning of this section. For $0 \leq i \leq 2 q-2$, we orient the hamilton path $H^{i} \backslash e_{2 q}^{i}=[\infty, 1+i, 2+i, 2 q-1+$ $i, 3+i, 2 q-2+i, \ldots, q+1+i]$ into a directed path from $\infty$ to $q+1+i$. Now we justify that this orientation is well defined. By Lemma 2.1 (c), each edge except those in the form of $\infty i$ has been oriented twice. From the definition of $H^{i}$, each of these edges appears in two different hamilton paths and the end vertices are in the same order. Let $T_{2 q}$ be the resulting tournament; then we have $A\left(T_{2 q}\right)=\bigcup_{i=1}^{2 q-1}\{(\infty, i),(i, i+\alpha)$ for $\alpha$ odd, $\alpha \leq q-1,(i, i-\alpha)$ for $\alpha$ even, $\alpha \leq q-1\}$.

In the remainder of the paper, we will attach the above orientation to $H^{i}$ defined in the beginning of the section, and use $H^{i}$ to denote the resulting oriented hamilton cycles. Hence $H^{i}$ 's form an oriented hamilton cycle decomposition of $2 T_{2 q}$. (Notice that $H^{i}$ 's are not directed hamilton cycles.) $H^{i}$ can also be expressed in terms of the arcs in the form $a_{1}^{i} a_{2}^{i} \ldots a_{2 q}^{i}$, where $a_{j}^{i}$ is $e_{j}^{i}$ with the above orientation. We have the following lemma.

Lemma 2.5. $\ell\left(H^{i}\right)=\sum_{j=1}^{2 q-1} \ell\left(a_{j}^{i}\right)-\ell\left(a_{2 q}^{i}\right)$.
Proof. Notice that in $H^{i}, a_{j}^{i}$ 's, $1 \leq j \leq 2 q-1$, are in the same direction, but $a_{2 q}^{i}=(\infty, q+1+i)$ is not. Therefore $\varepsilon_{i}=1,1 \leq i \leq 2 q-1$ and $\varepsilon_{2 q}=-1$.

We now divide the proof of Theorem 1.3 into two cases.

## Case 1. $r$ is odd.

Let $r=2 p+1$. To exhibit a decomposition of $K_{2 p+1} \otimes K_{2 q}$ into $p(2 q-1)$ hamilton cycles, we will decompose the labelled $2 p T_{2 q}$ into $p(2 q-1)$ oriented hamilton cycles, so that the label of each cycle is relatively prime to $2 p+1$.

Case 1.1. $p$ is even. We decompose $2 p T_{2 q}$ into the $p(2 q-1)$ oriented hamilton cycles, $H_{x}^{i}$ and $H_{x+1}^{i}$ for $0 \leq i \leq 2 q-2$ and $x=0,2,4, \ldots, p-2$. (here $H_{x}^{i}$ and $H_{x+1}^{i}$ are the same as $H^{i}$ defined before). We arrange the labels on the arcs as follows:
in $H_{x}^{i}$, let $\ell\left(a_{1}^{i}\right)=p-x$;

$$
\begin{gathered}
\ell\left(a_{j}^{i}\right)=p+2+x, 2 \leq j \leq q ; \\
\ell\left(a_{j}^{i}\right)=p-1-x, q+1 \leq j \leq 2 q, \text { and } \\
\text { in } H_{x+1}^{i}, \text { let } \ell\left(a_{1}^{i}\right)=p+2+x ; \\
\ell\left(a_{j}^{i}\right)=p-x, 2 \leq j \leq q \\
\ell\left(a_{j}^{i}\right)=p+1+x, q+1 \leq j \leq 2 q
\end{gathered}
$$

Notice that $\{p-1-x, p-x, p+1+x, p+2+x\}, x=0,2, \ldots, p-2$, form a partition of $\{1,2, \ldots, 2 p\}$. By Lemma 2.1(c), $\cup_{i=0}^{2 q-2} H_{x}^{i} \cup H_{x+1}^{i}$ is the union of four copies of $T_{2 q}$. By Lemma 2.1(b) the subgraphs induced by $\bigcup_{i=0}^{2 q-2} \bigcup_{j=1}^{q}\left\{a_{j}^{i}\right\}$ and $\bigcup_{i=0}^{2 q-2} \bigcup_{j=q+1}^{2 q}\left\{a_{j}^{i}\right\}$ are $T_{2 q}$. For example, the label $p-x$ is assigned to $a_{1}^{i}$ in $H_{x}^{i}$ and to $a_{j}^{i}$ for $2 \leq j \leq q$ in $H_{x+1}^{i}$. So we can conclude that one copy of $T_{2 q}$ is labeled $p-x$. Similarly, one copy of each is labelled $p-1-x, p+1+x$ and $p+2+x$.

Finally it remains to prove that the label of each oriented hamilton cycle is relatively prime to $2 p+1$. By Lemma 2.5 ,

$$
\begin{aligned}
\ell\left(H_{x}^{i}\right) & =(p-x)+(q-1)(p+2+x)+(q-1)(p-1-x)-(p-1-x) \\
& =q(2 p+1)-2 p \equiv 1(\bmod 2 p+1), \text { and } \\
\ell\left(H_{x+1}^{i}\right) & =(p+2+x)+(q-1)(p-x)+(q-1)(p+1+x)-(p+1+x) \\
& =q(2 p+1)-2 p \equiv 1(\bmod 2 p+1) .
\end{aligned}
$$

In all the cases the labels of the hamilton cycles are relatively prime to $2 p+1$. By Corollary 2.4, the above decomposition of $2 p T_{2 q}$ will produce a hamilton cycle decomposition of $K_{2 p+1} \otimes K_{2 q}$.

Case 1.2. p is odd. As in Case 1.1 we use the $(2 q-1)(p-1)$ hamilton cycles $H_{x}^{i}$ and $H_{x+1}^{i}$ for $0 \leq i \leq 2 q-2$ and $x=0,2, \ldots, p-3$. Now the 4 -subsets $\{p-1-x, p-x, p+1+x, p+2+x\}, x=0,2, \ldots, p-3$, form a partition of $\{2,3, \ldots, 2 p-1\}$. Hence we decompose the arcs with labels in $\{2,3, \ldots, 2 p-1\}$ into $(2 q-1)(p-1)$ hamilton cycles whose labels are relatively prime to $2 p+1$.

It remains to decompose the arcs of $2 T_{2 q}$ with labels 1 and $2 p$ into the $2 q-1$ hamilton cycles, $H_{p-1}^{i}$ for $0 \leq i \leq 2 q-2$ :
in $H_{p-1}^{i}, \ell\left(a_{j}^{i}\right)=1$ for $1 \leq j \leq q$, and

$$
\ell\left(a_{j}^{i}\right)=2 p \text { for } q+1 \leq j \leq 2 q .
$$

$\ell\left(H_{p-1}^{i}\right)=q+(q-1) 2 p-2 p=(2 p+1)(q-2)-2 \equiv-2(\bmod 2 p+1)$. $\ell\left(H_{p-1}^{i}\right)$ and $2 p+1$ are relatively prime as $2 p+1$ is odd.

So far we have shown that $K_{r} \otimes K_{s}$ has a hamilton cycle decomposition if one of $r$ and $s$ is odd.

## Case 2. r is even.

Let $r=2 p$. We will show that $K_{2 p} \otimes K_{2 q}$ can be decomposed into ( $p-$ 1) $(2 q-1)+q-1$ hamilton cycles and a one factor. Again we will exhibit a decomposition of $(2 p-1) T_{2 q}$ into oriented hamilton cycles with labels relatively prime to $r=2 p$.

Case 2.1. $\mathbf{p}$ is even. Note that the set of labels $\{1,2, \ldots, 2 p-1\}$ can be partitioned into $(p-2) / 24$-subsets,
$\{p-2-x, p-1-x, p+1+x, p+2+x\}, x=1,3, \ldots, p-3$ and a 3 -subset
$\{p-1, p, p+1\}$.
We first assign the labels to the arcs of the following $(p-2)(2 q-1)$ hamilton cycles:
for $0 \leq i \leq 2 q-2$ and $x=1,3, \ldots, p-3$,
in $H_{x}^{i}$, let $\ell\left(a_{1}^{i}\right)=p-1-x$;

$$
\ell\left(a_{j}^{i}\right)=p+2+x, 2 \leq j \leq q
$$

$$
\ell\left(a_{j}^{i}\right)=p-2-x, q+1 \leq j \leq 2 q, \text { and }
$$

in $H_{x+1}^{i}$, let $\ell\left(a_{1}^{i}\right)=p+2+x$;

$$
\begin{gathered}
\ell\left(a_{j}^{i}\right)=p-1-x, 2 \leq j \leq q ; \\
\ell\left(a_{j}^{i}\right)=p+1+x, q+1 \leq j \leq 2 q . \\
\ell\left(H_{x}^{i}\right)=p-1-x+(q-1)(p+2+x)+(q-1)(p-2-x)-(p-2-x) \\
=2 p(q-1)+1 \equiv 1(\bmod 2 p) \\
\ell\left(H_{x+1}^{i}\right)=p+2+x+(q-1)(p-1-x)+(q-1)(p+1+x)-(p+1+x) \\
=2 p(q-1)+1 \equiv 1(\bmod 2 p) .
\end{gathered}
$$

Hence the labels of the above hamilton cycles are relatively prime to $2 p-1$. So far we have decomposed all the arcs of $(2 p-1) T_{2 q}$ with labels in $\{1,2, \ldots, 2 p-1\} \backslash\{p-1, p, p+1\}$. It remains to partition the arcs of the three copies of $T_{2 q}$ with labels $p-1, p, p+1$ into $(2 q-1)+(q-1)$ hamilton cycles and a one factor. We first assign the labels to the arcs of the $2 q-1$ hamilton cycles, $H_{p-1}^{i}, 0 \leq i \leq 2 q-2$ :
in $H_{p-1}^{i}$, for $1 \leq j \leq q$ and $j \neq j_{0}$,

$$
\begin{aligned}
& \ell\left(a_{j}^{i}\right)=p+1, \text { and } \\
& \text { for } j=j_{0}, \\
& \quad \ell\left(a_{j_{0}}^{i}\right)=p \text {, where } \\
& \quad j_{0}=1 \text {, for } 0 \leq i \leq q-2 \text { and } \\
& \quad i=q, q+2, \ldots, 2 q-2 \text { if } q \text { is even, and } \\
& \quad i=q, q+2, \ldots, 2 q-3 \text { if } q \text { is odd; } \\
& \quad j_{0}=2 \text {, for } i=q-1, q+1, \ldots, 2 q-3 \text { if } q \text { is even and } \\
& \quad i=q-1, q+1, \ldots, 2 q-2 \text { if } q \text { is odd; }
\end{aligned}
$$

$$
\text { for } q+1 \leq j \leq 2 q \text {, }
$$

$$
\ell\left(a_{j}^{i}\right)=\bar{p}-\overline{1}
$$

Note that the label of each of these hamilton cycles is

$$
\ell\left(H_{p-1}^{i}\right)=p+(q-1)(p+1)+(q-1)(p-1)-(p-1) \equiv 1(\bmod 2 p)
$$

Example: From the decompositions exhibited in the beginning of the section, it is easy to check that the arcs with label $p$ are the following;
when $2 q=8, a_{1}^{0}=(\infty, 1), a_{1}^{1}=(\infty, 2), a_{1}^{2}=(\infty, 3), a_{1}^{4}=(\infty, 5)$,

$$
a_{1}^{6}=(\infty, 7), a_{2}^{3}=(4,5), a_{2}^{5}=(6,7)
$$

when $2 q=10, a_{1}^{0}=(\infty, 1), a_{1}^{1}=(\infty, 2), a_{1}^{2}=(\infty, 3), a_{1}^{3}=(\infty, 4)$,

$$
a_{1}^{5}=(\infty, 6), a_{1}^{7}=(\infty, 8), a_{2}^{4}=(5,6), a_{2}^{6}=(7,8)
$$

$$
a_{2}^{8}=(9,1)
$$

Note that the arcs not used by the hamilton cycles above form a copy of $T_{2 q}$ and these arcs have the label $p$ except for the $\operatorname{arcs} a_{j_{0}}^{i}$ which have
the label $p+1$. Therefore, the last step is to obtain $q-1$ hamilton cycles by decomposing the arcs of this $T_{2 q}$. Using Lemma 2.1 (d) we take the decomposition of $T_{2 q}$ into the $q-1$ hamilton cycles $H_{p}^{i}$, for $0 \leq i \leq q-2$. The labels of each hamilton cycle are fixed and can be described precisely:
for $0 \leq i \leq q-2$, in $H_{p}^{i}$,
when $i$ is even,

$$
\begin{gathered}
\ell\left(a_{j}^{i}\right)=p \text { if } 1 \leq j \leq 2 q \text { and } j \neq 1,2 q-1 \text { and } 2 q, \\
\ell\left(a_{1}^{i}\right)=\ell\left(a_{2 q-1}^{i}\right)=\ell\left(a_{2 q}^{i}\right)=p+1 ; \\
\left(\ell\left(H_{p}^{i}\right)=p+1+(2 q-3) p+p+1-(p+1) \equiv 1(\bmod 2 p)\right)
\end{gathered}
$$

when $i$ is odd,

$$
\begin{gathered}
\ell\left(a_{1}^{i}\right)=p+1 \text { and } \ell\left(a_{j}^{i}\right)=p, 2 \leq j \leq 2 q . \\
\left(\ell\left(H_{p}^{i}\right)=p+1+(2 q-2) p-p \equiv 1(\bmod 2 p)\right)
\end{gathered}
$$

Hence the decomposition satisfies the conditions.

## Example:

For $2 q=8$, the arcs with labels $p+1$ are
$a_{1}^{0}=(\infty, 1), a_{7}^{0}=(4,5), a_{8}^{0}=(\infty, 5)$ in $H_{p}^{0}$,
$a_{1}^{2}=(\infty, 3), a_{7}^{2}=(6,7), a_{8}^{2}=(\infty, 7)$ in $H_{p}^{2}$, and $a_{1}^{1}=(\infty, 2)$ in $H_{p}^{1}$.
For $2 q=10$, the arcs with labels $p+1$ are
$a_{1}^{0}=(\infty, 1), a_{9}^{0}=(5,6), a_{10}^{0}=(\infty, 6)$ in $H_{p}^{0}$,
$a_{1}^{2}=(\infty, 3), a_{9}^{2}=(7,8), a_{10}^{2}=(\infty, 8)$ in $H_{p}^{2}$,
$a_{1}^{1}=(\infty, 2)$ in $H_{p}^{1}$, and $a_{1}^{3}=(\infty, 4)$ in $H_{p}^{3}$.
Notice that when $q$ is odd, the last arc labelled $p+1$, namely $(9,1)$ (in general $(2 q-1,1))$, appears in the remaining one factor.

The remaining edges form a one factor of $T_{2 q}$. It is clear that this one factor can be used to construct a one factor in $K_{2 p} \otimes K_{2 q}$.

Case 2.2. $\mathbf{p}$ is odd. We can assume that $q$ is odd $(q \geq 3)$ (the case when $q$ is even has been dealt with as in Case 2.1).

We partition the set of labels $\{1,2, \ldots 2 p-1\}$ into $(p-3) / 24$-subsets $\{p-2-x, p-1-x, p+1+x, p+2+x\}$ for $x=2,4, \ldots, p-3$ and a 5 -subset $\{p-2, p-1, p, p+1, p+2\}$. We deal with the arcs of $(2 p-1) T_{2 q}$ with labels other than $\{p-2, p-1, p, p+1, p+2\}$ in the same way as in Case 2.1.

The remaining edges are five copies of $T_{2 q}$ whose arcs are labelled $\{p-$ $2, p-1, p, p+1, p+2\}$ and they are partitioned into $2(2 q-1)+(q-1)$ oriented hamilton cycles as follows:

For $0 \leq i \leq 2 q-2$, in $H_{p-1}^{i}$,
$\ell\left(a_{j}^{i}\right)=p-2$, for $1 \leq j \leq q$ and $j \neq 3 ;$
$\ell\left(a_{3}^{i}\right)=p+1$;
$\ell\left(a_{j}^{i}\right)=p+2$, for $q+1 \leq j \leq 2 q$.
So $\ell\left(H_{p-1}^{i}\right)=p+1+(q-1)(p-2)+(q-1)(p+2)-(p+2) \equiv-1(\bmod 2 p)$.
For the other $2 q-1$ hamilton cycles, we do the following.
For $0 \leq i \leq 2 q-2$, in $H_{p}^{i}$,

$$
\text { for } 1 \leq j \leq q \text { and } j \neq 3, j_{0}
$$

$$
\ell\left(a_{j}^{i}\right)=p+1
$$

$$
\ell\left(a_{3}^{i}\right)=p-2, \text { and }
$$

$$
\ell\left(a_{j_{0}}^{i}\right)=p, \text { where }
$$

$$
j_{0}=1, \text { for } 0 \leq i \leq q-2 \text { and } i=q, q+2, \ldots, 2 q-3, \text { and }
$$

$$
j_{0}=2, \text { for } i=q-1, q+1, \ldots, 2 q-2
$$

for $q+1 \leq j \leq 2 q$ and $j \neq 2 q-2$,

$$
\ell\left(a_{j}^{i}\right)=p-1, \text { and }
$$

$$
\ell\left(a_{2 q-2}^{i}\right)=p
$$

$$
\ell\left(H_{p}^{i}\right)=p+p-2+(q-2)(p+1)+(q-2)(p-1)+p-(p-1) \equiv-1(\bmod 2 p)
$$

Example: When $2 q=10$, from the listed decomposition we can check that the arcs with label $p$ are the following;

$$
\begin{aligned}
& a_{1}^{0}=(\infty, 1), a_{1}^{1}=(\infty, 2), a_{1}^{2}=(\infty, 3), a_{1}^{3}=(\infty, 4), \\
& a_{1}^{5}=(\infty, 6), a_{1}^{7}=(\infty, 8), a_{2}^{4}=(5,6), a_{2}^{6}=(7,8), a_{2}^{8}=(9,1)
\end{aligned}
$$

The arcs not used by the hamilton cycles form a copy of $T_{2 q}$ and they have label $p$ except the $\operatorname{arcs} a_{j_{0}}^{i}\left(\ell\left(a_{j_{0}}^{i}\right)=p+1\right)$ and $a_{2 q-2}^{i}\left(\ell\left(a_{2 q-2}^{i}\right)=p-1\right)$. Like in the Case 2.1, using Lemma 2.1 (d), we partition these arcs into $q-1$ hamilton cycles $H_{p+1}^{i}, 0 \leq i \leq q-2$ and a one factor. The labels on the arcs are fixed as follows:

For $0 \leq i \leq q-2$, in $H_{p+1}^{i}$, when $i$ is even,

$$
\ell\left(a_{1}^{i}\right)=\ell\left(a_{2 q}^{i}\right)=\ell\left(a_{2 q-1}^{i}\right)=p+1,
$$

$$
\ell\left(a_{3}^{i}\right)=\ell\left(a_{2 q-2}^{i}\right)=p-1, \text { and otherwise }
$$

$$
\ell\left(a_{j}^{i}\right)=p
$$

when $i$ is odd,

$$
\begin{aligned}
& \ell\left(a_{1}^{i}\right)=p+1, \\
& \ell\left(a_{3}^{i}\right)=\ell\left(a_{2 q-2}^{i}\right)=p-1, \text { and otherwise } \\
& \ell\left(a_{j}^{i}\right)=p
\end{aligned}
$$

Example: For $2 q=10$, the arcs with labels $p+1$ are
$a_{1}^{0}=(\infty, 1), a_{9}^{0}=(5,6), a_{10}^{0}=(\infty, 6)$ in $H_{p+1}^{0}$,
$a_{1}^{2}=(\infty, 3), a_{9}^{2}=(7,8), a_{10}^{2}=(\infty, 8)$ in $H_{p+1}^{2}$,
$a_{1}^{1}=(\infty, 2)$ in $H_{p+1}^{1}$, and $a_{1}^{3}=(\infty, 4)$ in $\stackrel{p+1}{H_{p+1}^{3}}$.
As in the previous case, the last arc labelled $p+1$, namely, $(9,1)$ (in general $(2 q-1,1))$ appears in the remaining one factor.

When $i$ is even,

$$
\begin{aligned}
\ell\left(H_{p+1}^{i}\right) & =(p+1)+p+(p-1)+(2 q-6) p+(p-1)+(p+1)-(p+1) \\
& \equiv-1(\bmod 2 p)
\end{aligned}
$$

When $i$ is odd,

$$
\ell\left(H_{p+1}^{i}\right)=(p+1)+p+(p-1)+(2 q-6) p+(p-1)+p-p \equiv-1(\bmod 2 p)
$$

So in all cases $\ell\left(H_{p+1}^{i}\right)$ and $2 p$ are relatively prime and we have the required hamilton cycle decomposition of $(r-1) T_{2 q}$.

Aknowledgement. The first three authors thank the Indo-French Centre for the Promotion of Advanced Research (Centre Franco-Indien Pour la Promotion de la Recherche Avancee, New Delhi) for the support through Project 401-1. Part of the research was done while the last author was visiting SLOOP (now MASCOTTE), joint project I3S-CNRS/UNSA/INRIA and he was supported by the Natural Science and Engineering Research Council of Canada.

We would also like to thank the referees for some useful remarks.

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