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Second-order cone programming for rolling friction contact mechanics

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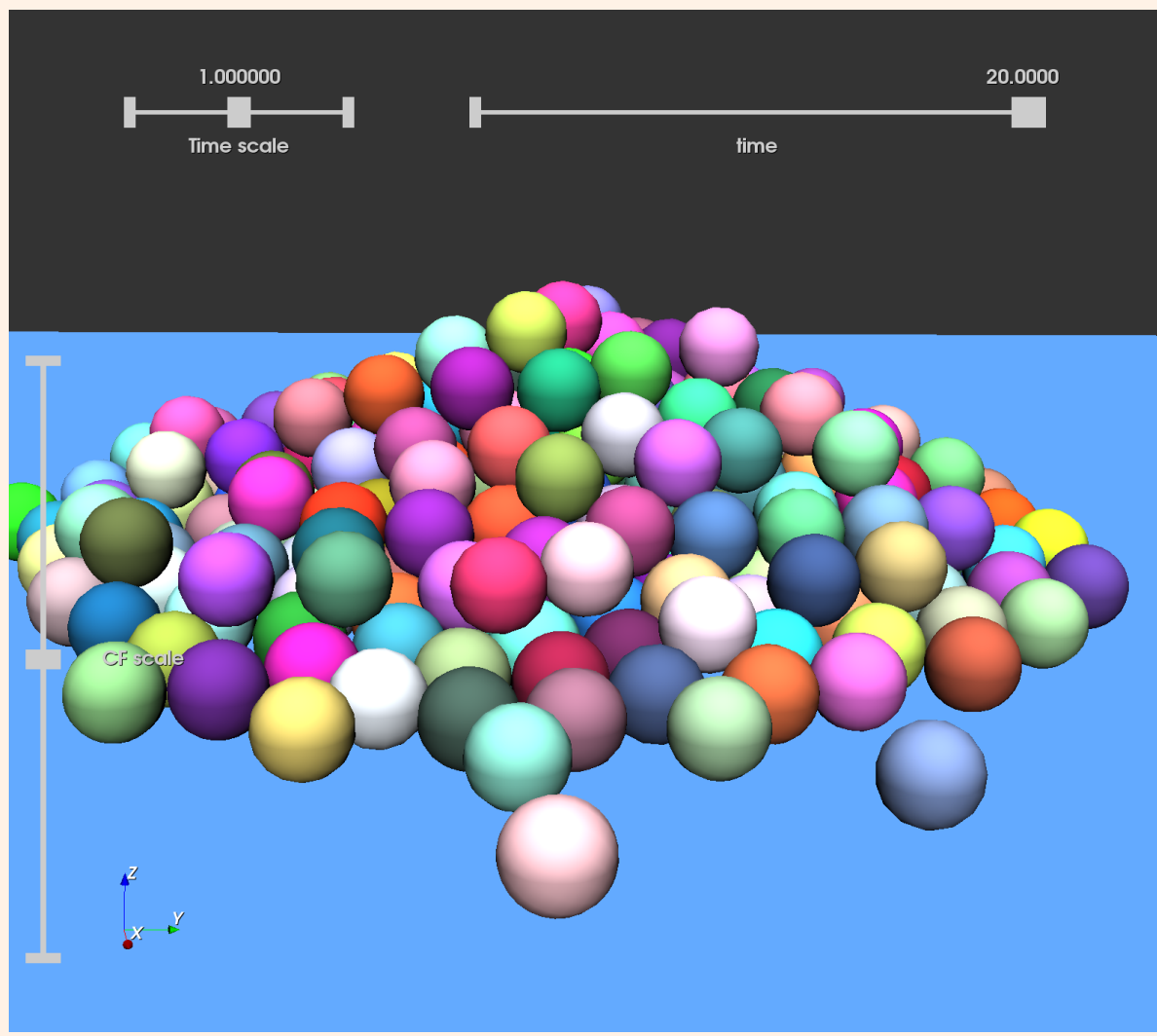
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GENERAL FRICTION CONTACT PROBLEM



$$\begin{aligned} Mv + f - H^T r &= 0 \\ Hv + w &= u \\ \mathcal{F}^* \ni u + \Phi(u) \perp r &\in \mathcal{F} \end{aligned}$$

where

- **Parameters:** n No. of contact points, m No. of degrees of freedom.
- **Given data:** $0 \prec M = M^T \in \mathbb{R}^{m \times m}$, $H \in \mathbb{R}^{5n \times m}$, $f \in \mathbb{R}^m$, $w \in \mathbb{R}^{5n}$.
- **Convex problem:** $\Phi(u)$ is fixed.
- **Variables:** global velocity $v \in \mathbb{R}^m$, velocity $u \in \mathbb{R}^{5n}$, reaction $r \in \mathbb{R}^{5n}$.

ORIGINAL ROLLING FRICTION CONE

$$\begin{aligned} \mathcal{F} &= \prod_{i=1}^n \mathcal{R}_i := \{r_i = (r_{i0}, \tilde{r}_i, \hat{r}_i) \in \mathbb{R}^5 : \max\{\|\tilde{r}_i\|_2, \|\hat{r}_i\|_2\} \leq r_{i0}\} \\ \mathcal{F}^* &= \prod_{i=1}^n \mathcal{R}_i^* := \{u_i = (u_{i0}, \tilde{u}_i, \hat{u}_i) \in \mathbb{R}^5 : \|\tilde{u}_i\|_2 + \|\hat{u}_i\|_2 \leq u_{i0}\} \end{aligned}$$

$$\begin{aligned} \min_{v,u} \frac{1}{2} v^T M v + f^T v & \quad \max_{v,r} -\frac{1}{2} v^T M v - w^T r \\ \text{(P)} \quad \text{s.t. } H v + w = u & \quad \text{(D)} \quad \text{s.t. } M v + f - H^T r = 0 \\ & \quad u \in \mathcal{F}^* & \quad r \in \mathcal{F} \end{aligned}$$

Since \mathcal{F} is not self-dual, the Jordan algebra cannot be used as it is.

NESTEROV & TODD (NT) SCALING

To get a symmetric and non-singular system, we need a change of variables at each iteration of the **Interior Point algorithm**. Define a vector p such that

$$\hat{z} := Q_p z = Q_{p^{-1}} y =: \tilde{y}$$

where Q_p is quadratic representation of p .

$$\begin{aligned} \min_{v,\hat{z}} \frac{1}{2} v^T M v + f^T v & \quad \max_{v,\tilde{y}} -\frac{1}{2} v^T M v - w^T (K Q_p \tilde{y}) \\ \text{(P}_{NT}) \quad \text{s.t. } H v + w = J Q_{p^{-1}} \hat{z} & \quad \text{(D}_{NT}) \quad \text{s.t. } M v + f - H^T K Q_p \tilde{y} = 0 \\ & \quad \hat{z} \in \mathcal{L}^{2n} & \quad \tilde{y} \in \mathcal{L}^{2n} \end{aligned}$$

REFORMULATION WITH PRODUCT OF LORENTZ CONES

To simplify, set $n = 1$. By letting $u_0 = t + t'$, we obtain

$$\begin{aligned} \min_{v,z} \frac{1}{2} v^T M v + f^T v & \quad \max_{v,y} -\frac{1}{2} v^T M v - w^T K y \\ \text{(P')} \quad \text{s.t. } H v + w = J z & \quad \text{(D')} \quad \text{s.t. } M v + f - H^T K y = 0 \\ & \quad z \in \mathcal{L}^2 & \quad y \in \mathcal{L}^2 \end{aligned}$$

where

$$J = \begin{pmatrix} 1 & 1 \\ I_2 & I_2 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \\ I_2 & I_2 \end{pmatrix}, \quad z = \begin{pmatrix} t \\ \tilde{u} \\ t' \\ \tilde{u} \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} r_0 \\ \tilde{r} \\ r_0 \\ \tilde{r} \end{pmatrix}.$$

Note that

- $Jz = u$, $Ky = r$ and $KJ^T = I$,
- \mathcal{L}^2 is self-dual, then Jordan algebra can be used.

SOLVER

The **perturbed KKT system** is

$$\begin{aligned} M v + f - H^T K Q_p \tilde{y} &= 0, \\ J Q_{p^{-1}} \hat{z} - H v - w &= 0, \\ \hat{z} \circ \tilde{y} &= 2\mu e, \\ (\hat{z}, \tilde{y}) &\in \text{int}(\mathcal{L}^4). \end{aligned}$$

The **linear system** to solve at each iteration is of the form

$$\begin{bmatrix} M & -H^T & 0 \\ -H & 0 & J \\ 0 & J^T & Q_{p^2} \end{bmatrix} \begin{bmatrix} d^v \\ d^r \\ d^z \end{bmatrix} = - \begin{bmatrix} M v + f - H^T r \\ J z - H v - w \\ J^T r - 2\mu z^{-1} \end{bmatrix}.$$

One difficulty is that Q_{p^2} is badly conditioned near the optimal solution.

EQUAL STEPLENGTHS

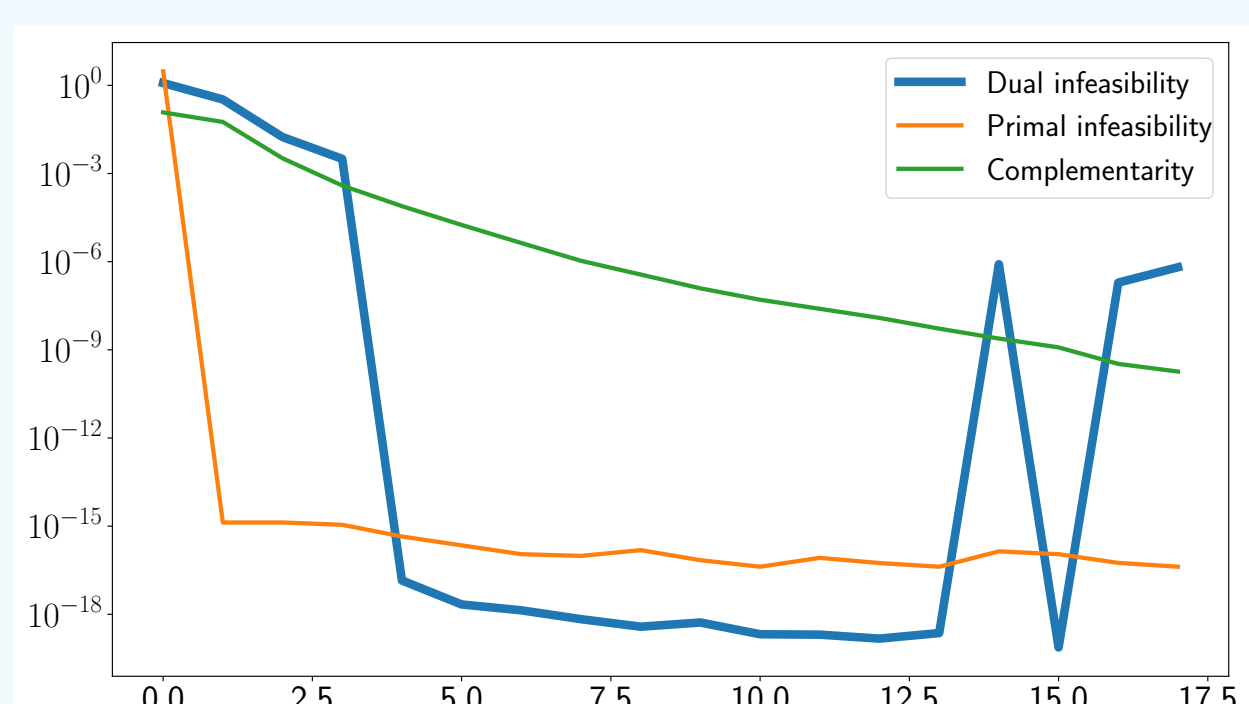


Figure 1. $n = 70$, steplengths $\alpha_p \neq \alpha_d$

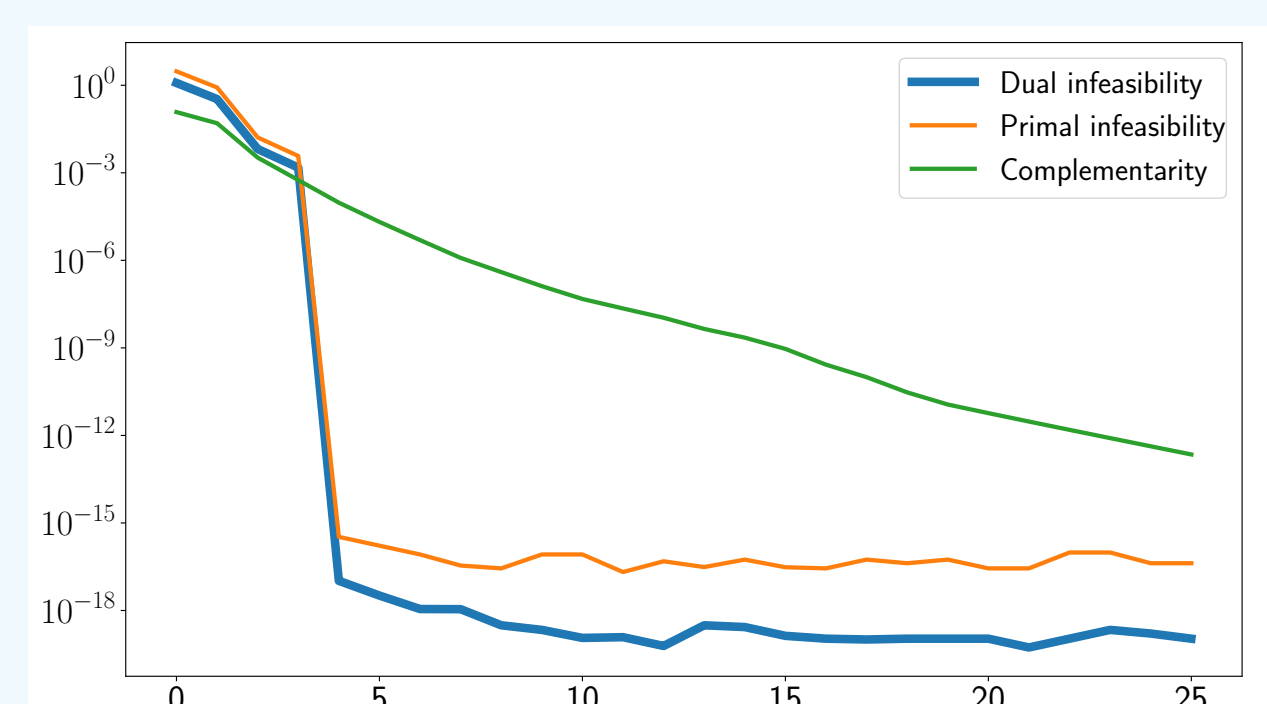
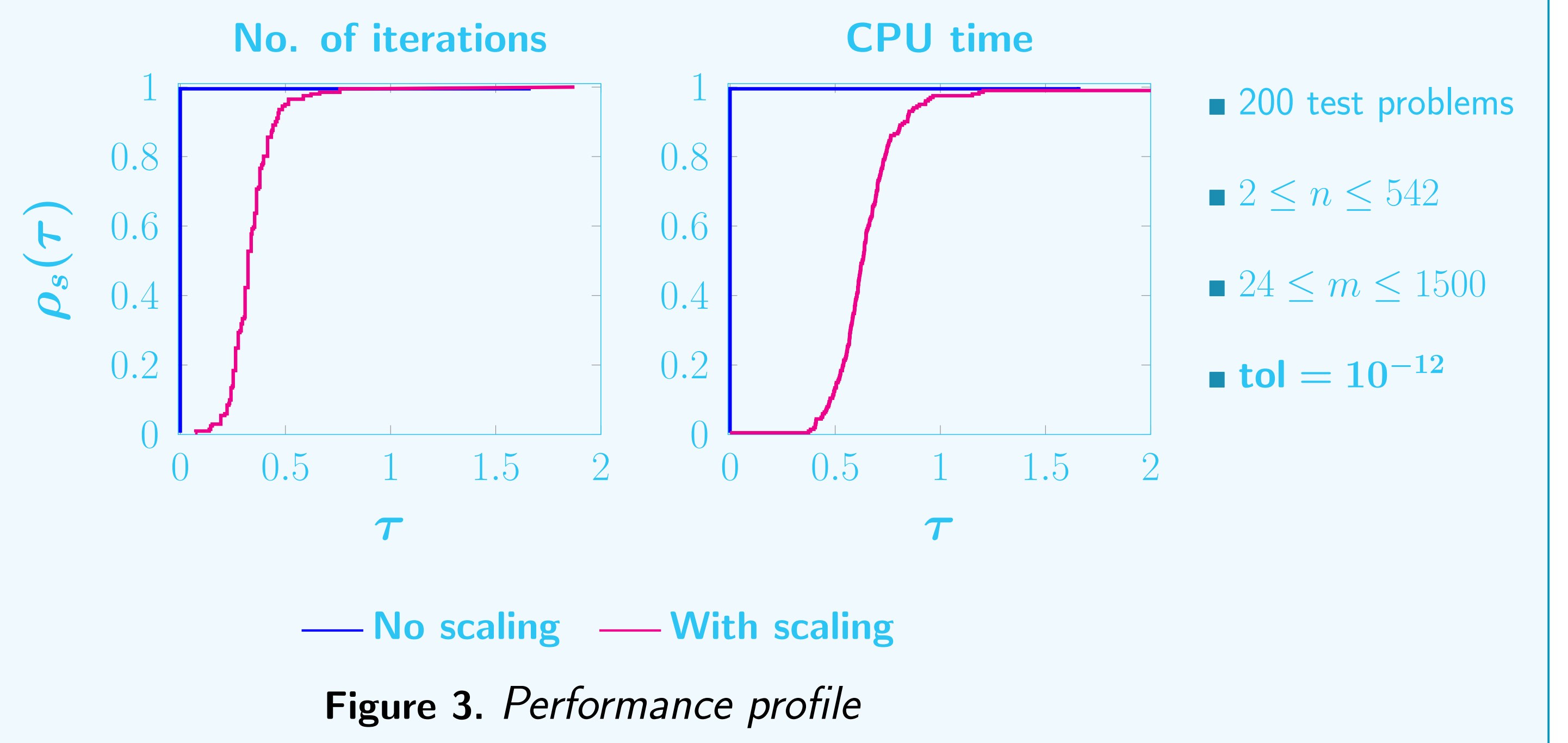


Figure 2. $n = 70$, steplengths $\alpha_p = \alpha_d$

Primal-dual infeasibilities

$$\begin{aligned} \text{pinfeas}^+ &= (1 - \alpha_p) \text{pinfeas} + \alpha_p \xi_p & \text{where} \\ \text{dinfeas}^+ &= (1 - \alpha_d) \text{dinfeas} + \alpha_d \xi_d + (\alpha_d - \alpha_p) M d^v & \xi_p, \xi_d \text{ are residual vectors.} \end{aligned}$$

EFFICIENCY & ROBUSTNESS



THEORETICAL RESULT FOR FRICTION CONTACT MECHANICS WITHOUT ROLLING FRICTION

- For the case without rolling friction, the original friction cone \mathcal{F} is the Lorentz cone \mathcal{L} . If (P) is feasible, then it has a unique optimal solution (v^*, u^*) . It is not the case for (D), because H is rank deficient.
- If the **Slater's conditions** and the **strict complementarity** hold, there are 3 possible cases:

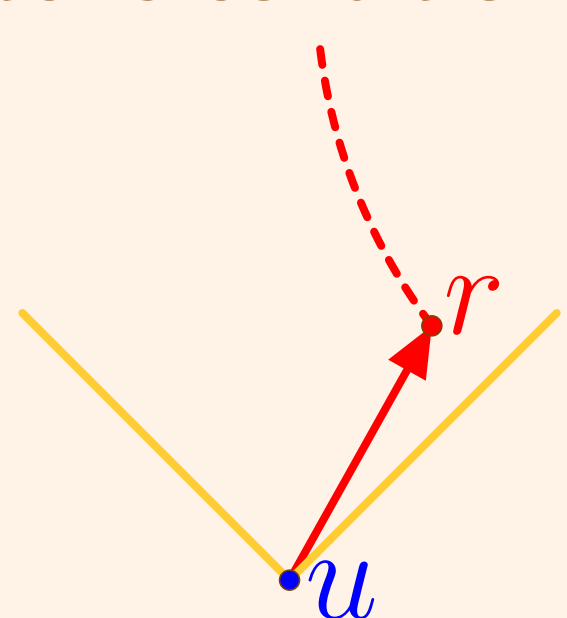


Figure 4. $u = 0$, $r \in \text{int}(\mathcal{L})$

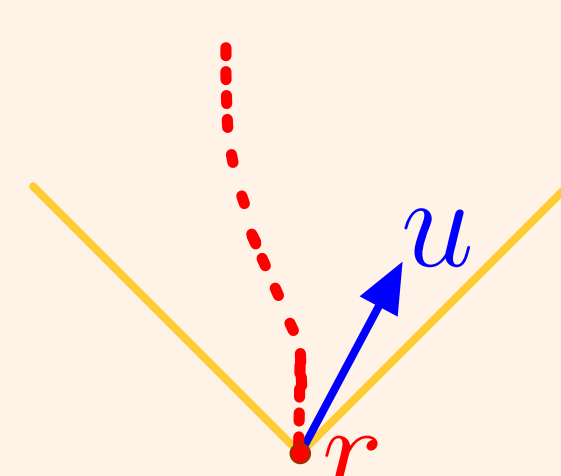


Figure 5. $u \in \text{int}(\mathcal{L})$, $r = 0$

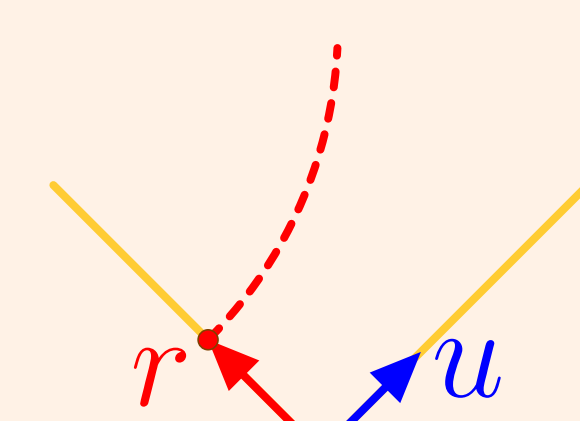


Figure 6. $u \in \text{bd}(\mathcal{L})$, $r \in \text{bd}(\mathcal{L})$

Then, the optimal solution r will converge to the **analytic center** of the optimal set \hat{R} which is the unique optimal solution of the problem

$$\min_{r \in \text{ri}(\hat{R})} \psi(r) := - \sum_{i \in B} \log \det r_i - \sum_{i \in R} \log r_{i,0},$$

where (B, N, R) is a partition of $\{1, \dots, n\}$.

- This result can be extended to the case of rolling friction.