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## Second-order cone programming for rolling friction contact mechanics

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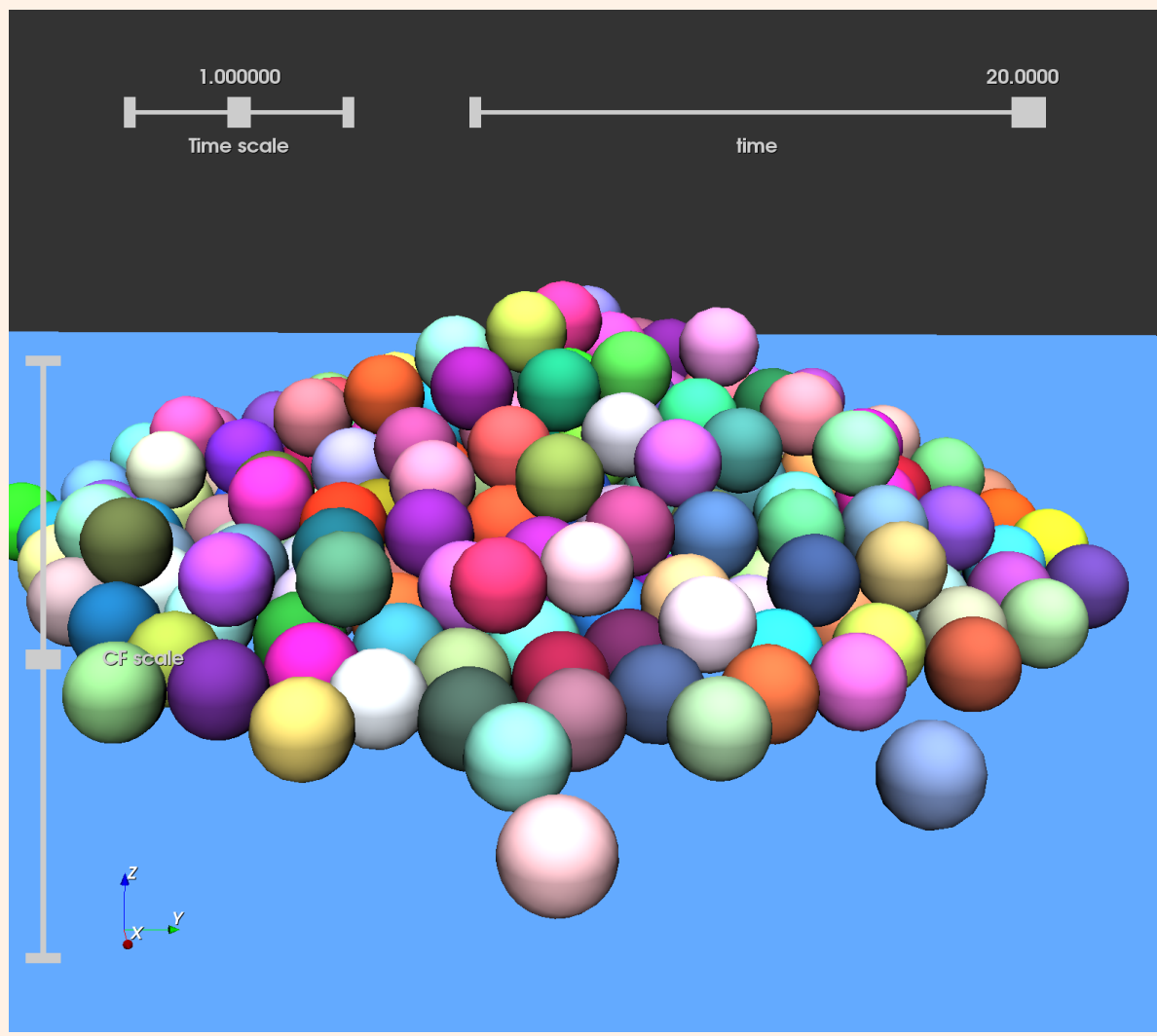
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## GENERAL FRICTION CONTACT PROBLEM



$$\begin{aligned} Mv + f - H^T r &= 0 \\ Hv + w &= u \\ \mathcal{F}^* \ni u + \Phi(u) \perp r &\in \mathcal{F} \end{aligned}$$

where

- **Parameters:**  $n$  No. of contact points,  $m$  No. of degrees of freedom.
- **Given data:**  $0 \prec M = M^T \in \mathbb{R}^{m \times m}$ ,  $H \in \mathbb{R}^{5n \times m}$ ,  $f \in \mathbb{R}^m$ ,  $w \in \mathbb{R}^{5n}$ .
- **Convex problem:**  $\Phi(u)$  is fixed.
- **Variables:** global velocity  $v \in \mathbb{R}^m$ , velocity  $u \in \mathbb{R}^{5n}$ , reaction  $r \in \mathbb{R}^{5n}$ .

## ORIGINAL ROLLING FRICTION CONE

$$\begin{aligned} \mathcal{F} &= \prod_{i=1}^n \mathcal{R}_i := \{r_i = (r_{i0}, \tilde{r}_i, \tilde{r}_i) \in \mathbb{R}^5 : \max\{\|\tilde{r}_i\|_2, \|\tilde{r}_i\|_2\} \leq r_{i0}\} \\ \mathcal{F}^* &= \prod_{i=1}^n \mathcal{R}_i^* := \{u_i = (u_{i0}, \tilde{u}_i, \tilde{u}_i) \in \mathbb{R}^5 : \|\tilde{u}_i\|_2 + \|\tilde{u}_i\|_2 \leq u_{i0}\} \end{aligned}$$

$$\begin{aligned} \min_{v,u} \frac{1}{2} v^T M v + f^T v & \quad \max_{v,r} -\frac{1}{2} v^T M v - w^T r \\ \text{(P)} \quad \text{s.t. } H v + w = u & \quad \text{(D)} \quad \text{s.t. } M v + f - H^T r = 0 \\ u \in \mathcal{F}^* & \quad r \in \mathcal{F} \end{aligned}$$

Since  $\mathcal{F}$  is not self-dual, the Jordan algebra cannot be used as it is.

## NESTEROV & TODD (NT) SCALING

To get a symmetric and non-singular system, we need a change of variables at each iteration of the **Interior Point algorithm**. Define a vector  $p$  such that

$$\hat{z} := Q_p z = Q_{p-1} y =: \tilde{y}$$

where  $Q_p$  is quadratic representation of  $p$ .

$$\begin{aligned} \min_{v,\hat{z}} \frac{1}{2} v^T M v + f^T v & \quad \max_{v,\tilde{y}} -\frac{1}{2} v^T M v - w^T (K Q_p \tilde{y}) \\ \text{(P}_{NT}) \quad \text{s.t. } H v + w = J Q_{p-1} \hat{z} & \quad \text{(D}_{NT}) \quad \text{s.t. } M v + f - H^T K Q_p \tilde{y} = 0 \\ \hat{z} \in \mathcal{L}^{2n} & \quad \tilde{y} \in \mathcal{L}^{2n} \end{aligned}$$

## REFORMULATION WITH PRODUCT OF LORENTZ CONES

To simplify, set  $n = 1$ . By letting  $u_0 = t + t'$ , we obtain

$$\begin{aligned} \min_{v,z} \frac{1}{2} v^T M v + f^T v & \quad \max_{v,y} -\frac{1}{2} v^T M v - w^T K y \\ \text{(P')} \quad \text{s.t. } H v + w = J z & \quad \text{(D')} \quad \text{s.t. } M v + f - H^T K y = 0 \\ z \in \mathcal{L}^2 & \quad y \in \mathcal{L}^2 \end{aligned}$$

where

$$J = \begin{pmatrix} 1 & 1 \\ I_2 & I_2 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \\ I_2 & I_2 \end{pmatrix}, \quad z = \begin{pmatrix} t \\ \tilde{u} \\ t' \\ \tilde{u} \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} r_0 \\ \tilde{r} \\ r_0 \\ \tilde{r} \end{pmatrix}.$$

Note that

- $Jz = u$ ,  $Ky = r$  and  $KJ^T = I$ ,
- $\mathcal{L}^2$  is self-dual, then Jordan algebra can be used.

## SOLVER

The **perturbed KKT system** is

$$\begin{aligned} M v + f - H^T K Q_p \tilde{y} &= 0, \\ J Q_{p-1} \hat{z} - H v - w &= 0, \\ \hat{z} \circ \tilde{y} &= 2\mu e, \\ (\hat{z}, \tilde{y}) &\in \text{int}(\mathcal{L}^4). \end{aligned}$$

The **linear system** to solve at each iteration is of the form

$$\begin{bmatrix} M & -H^T & 0 \\ -H & 0 & J \\ 0 & J^T & Q_{p^2} \end{bmatrix} \begin{bmatrix} d^v \\ d^r \\ d^z \end{bmatrix} = - \begin{bmatrix} M v + f - H^T r \\ J z - H v - w \\ J^T r - 2\mu z^{-1} \end{bmatrix}.$$

**One difficulty** is that  $Q_{p^2}$  is badly conditioned near the optimal solution.

## EQUAL STEPLENGTHS

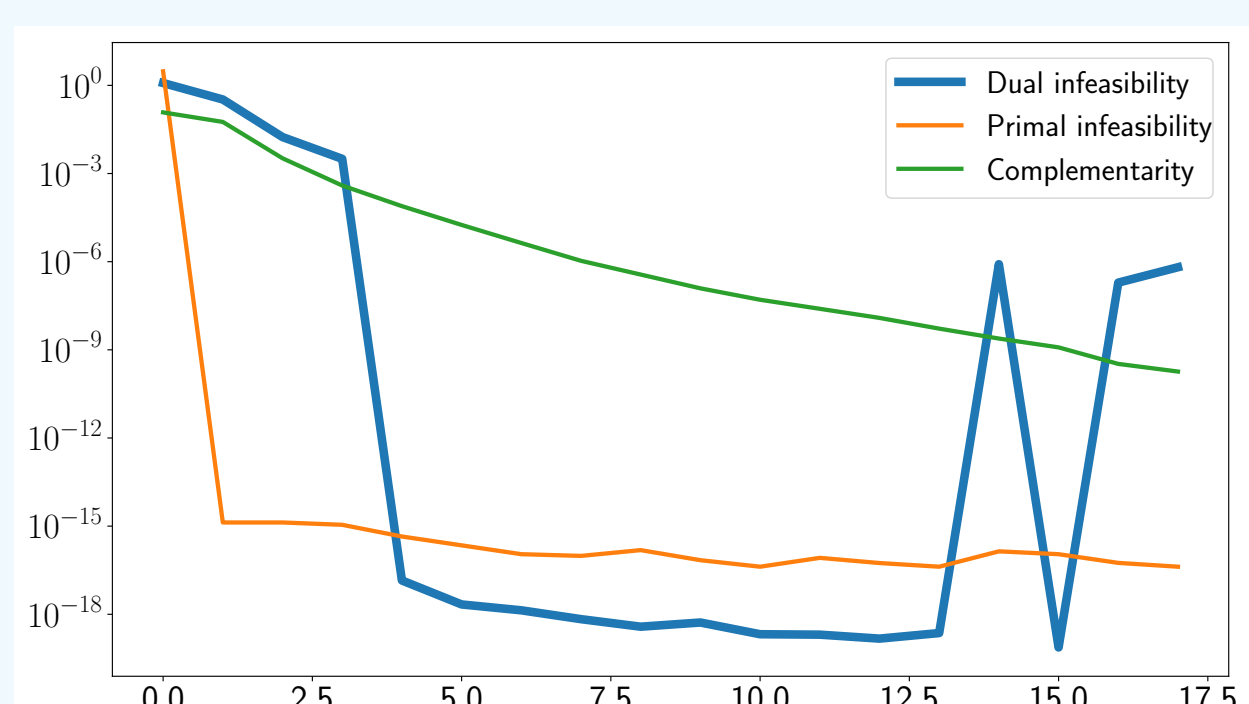


Figure 1.  $n = 70$ , steplengths  $\alpha_p \neq \alpha_b$

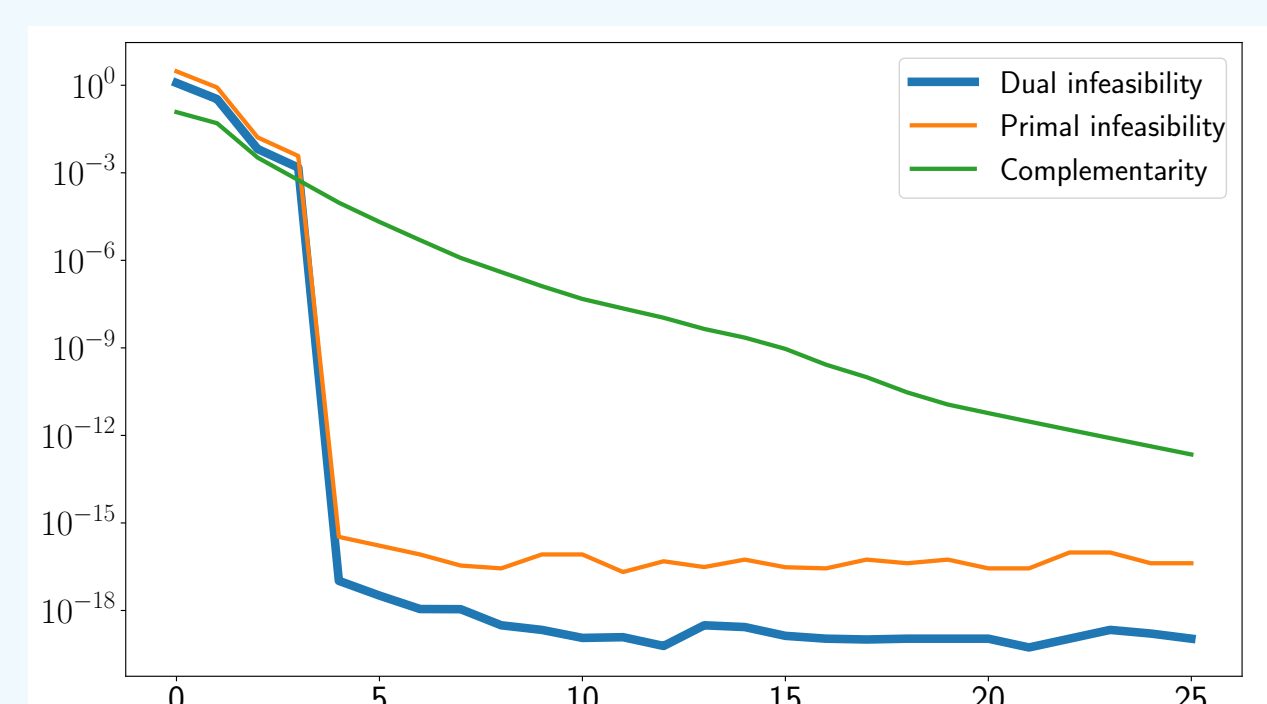


Figure 2.  $n = 70$ , steplengths  $\alpha_p = \alpha_b$

## Primal-dual infeasibilities

$$\begin{aligned} \text{pinfeas}^+ &= (1 - \alpha_p) \text{pinfeas} + \alpha_p \xi_p & \text{where} \\ \text{dinfeas}^+ &= (1 - \alpha_b) \text{dinfeas} + \alpha_b \xi_b + (\alpha_b - \alpha_p) M d^v & \xi_p, \xi_b \text{ are residual vectors.} \end{aligned}$$

## EFFICIENCY & ROBUSTNESS

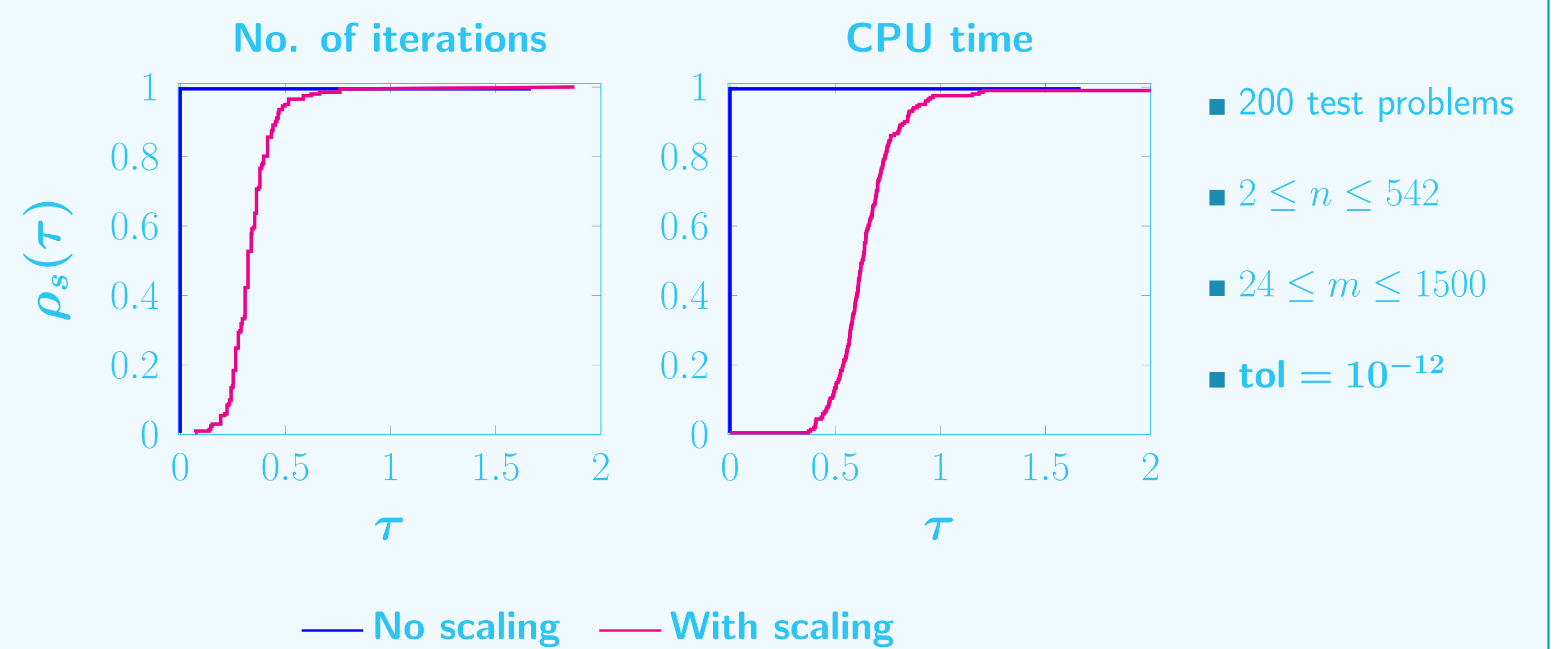


Figure 3. Performance profile

## THEORETICAL RESULT FOR FRICTION CONTACT MECHANICS WITHOUT ROLLING FRICTION

- For the case without rolling friction, the original friction cone  $\mathcal{F}$  is the Lorentz cone  $\mathcal{L}$ . If (P) is feasible, then it has a unique optimal solution  $(v^*, u^*)$ . It is not the case for (D), because  $H$  is rank deficient.
- If the **Slater's conditions** and the **strict complementarity** hold, there are 3 possible cases:

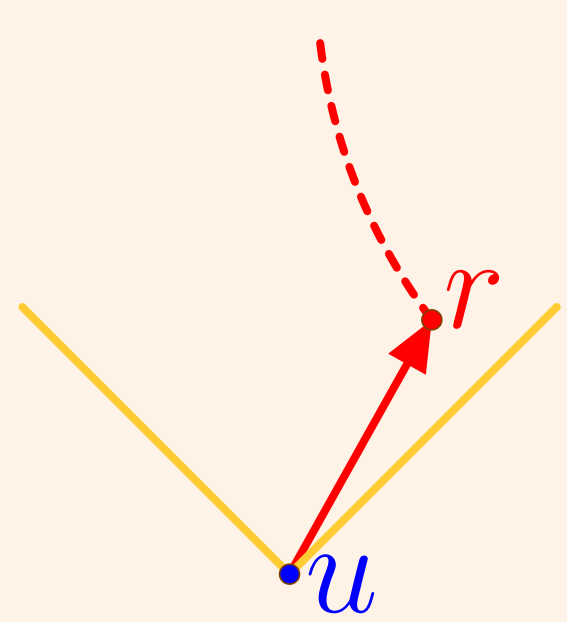


Figure 4.  $u = 0$ ,  $r \in \text{int}(\mathcal{L})$

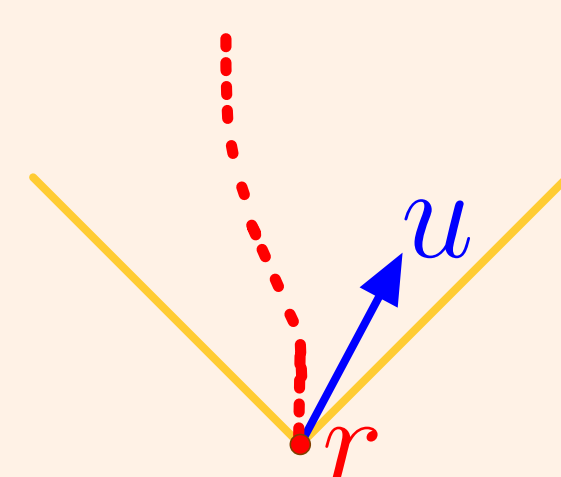


Figure 5.  $u \in \text{int}(\mathcal{L})$ ,  $r = 0$

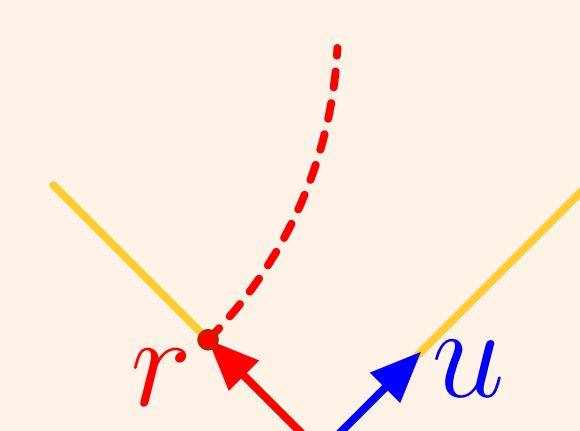


Figure 6.  $u \in \text{bd}(\mathcal{L})$ ,  $r \in \text{bd}(\mathcal{L})$

Then, the optimal solution  $r$  will converge to the **analytic center** of the optimal set  $\hat{R}$  which is the unique optimal solution of the problem

$$\min_{r \in \text{ri}(\hat{R})} \psi(r) := - \sum_{i \in B} \log \det r_i - \sum_{i \in R} \log r_{i,0},$$

where  $(B, N, R)$  is a partition of  $\{1, \dots, n\}$ .

- This result can be extended to the case of rolling friction.