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# Optimal Control of the Controlled Lotka-Volterra Equations with Applications-The Permanent Case\*

Bernard Bonnard<sup>†</sup> and Jérémy Rouot<sup>‡</sup>

**Abstract.** In this article motivated by the control of complex microbiota in view to reduce the infection by a pathogenic agent, we introduce the theoretical frame from optimal control to analyze the problem. Two complementary approaches can be applied in the analysis: one is the so-called permanent case, where no digital constraints are concerning the control (taken as a measurable mapping) versus the sampled-data control case taking into account the logistic constraints, e.g. frequency of the medical interventions. The model is the  $n$ -dimensional Lotka–Volterra equation controlled using either probiotics or antibiotic agents or transplantation and bactericides. In the permanent case the Maximum principle is used to parameterize the geodesics and the optimal synthesis boils down to analyze mainly the singular trajectories and their concatenation with bang arcs. This analysis is preliminary and complementary to the study of the problem in the frame of optimal sampled-data control, where convergence of the MPC algorithms is related to singular trajectories and non-uniqueness of the optimal dynamics encountered in the permanent case.

**Key words.** Optimal control in the permanent case, sampled-data control, biomathematics and population dynamics, geometric control theory.

**MSC codes.** 49K15, 92B05, 93C10, 93C15, 92D25

**1. Introduction.** The book by Vito Volterra [26] "Leçons sur la théorie mathématique pour la lutte pour la vie" leads to the Lotka–Volterra model to predict in a general frame the evolution on interacting biological species. The problem was studied independently by Lotka which makes the connection with chemical networks. The relation with control systems was already present in the original predator-prey model with two species set by Umberto d’Ancona to explain the evolution of the species in relation with fishing activity during the first World War. The original memoir starts with an interesting discussion about the evolution of the species in relation with integrability properties of the conservative model. It validates Ancona observations and opens the road to analyze different problems of populations dynamics. This leads to extend the model to the non conservative  $2d$ -case in any dimension.

Recently our attention was attracted by the works of Jones et al. [16] based on the model by Stein et al. [25] in an attempt to model and cure a gut mouse microbiota infected by the *C-difficile* bacteria which leads to a 11-dimensional controlled Lotka–Volterra model, using either fecal transplantation or one antibiotic agent. The problem is analyzed using 2d-reduced (projection) non conservative Lotka–Volterra model. This model validates the effect of a single fecal injection to cure the infection by constructing a separatrix which allows

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<sup>†</sup>Institut Mathématique de Bourgogne, Université de Dijon and Inria Sophia Antipolis Equipe McTao, 9 rue Alain Savary, 21078 Dijon ([bernard.bonnard@u-bourgogne.fr](mailto:bernard.bonnard@u-bourgogne.fr)).

<sup>‡</sup>Laboratoire de Mathématiques de Bretagne Atlantique, UMR 6205, 6 avenue Le Gorgeu, 29238 Brest ([jeremy.rouot@univ-brest.fr](mailto:jeremy.rouot@univ-brest.fr)).

36 to decide about success or failure of the procedure due to delay in the transplantation or  
 37 insufficient dosing.

38 To understand this approach let us introduce the Lotka–Volterra dynamics and analyze  
 39 its limits, already discussed in [26]. The system is given by the equations

$$40 \quad (1.1) \quad \frac{dx}{dt}(t) = (\text{diag}(x(t)))(Ax(t) + r),$$

41 where  $x = (x_1, x_2, \dots, x_n)^\top$  belongs to the positive quadrant  $x_i \geq 0$  and is the vector of popu-  
 42 lations of the interacting species,  $\text{diag}x$  denotes in short the diagonal matrix with coefficients  
 43  $x_i$ ,  $A = (a_{ij})$  is the matrix containing the interacting coefficients between the species and  
 44  $r = (r_1, r_2, \dots, r_n)^\top$  is the vector describing the individual growth rate without interaction.  
 45 Assuming the matrix  $A$  invertible, the system possesses a unique interior equilibrium (called  
 46 persistent) given by  $x_0 = -A^{-1}r$ . The model is only valid provided that each species is  
 47 contained in an interval  $[\varepsilon, M]$ ,  $\varepsilon > 0$  and hence in the model only the so-called persistent  
 48 trajectories defined for positive times and contained in  $[\varepsilon, M]^n$  have physical signification.  
 49 Therefore the differential equation represents an analytic continuation of the dynamics to the  
 50 whole euclidean space. But the model in fine describes the interaction between the persistent  
 51 equilibrium and equilibria of an hierarchy of reduced Lotka–Volterra dynamics associated to  
 52 extinction of the different species, e.g. extinction of  $x_1$  leads to analyze the reduced model  
 53 of  $n - 1$  interacting species where we substitute in the dynamics  $x \rightarrow x = (x_2, \dots, x_n)^\top$  and  
 54 again we can compute the persistent equilibria of the reduced dynamics. Therefore this leads  
 55 to interpret the model as a system of interaction between a given number  $k$  of equilibria and  
 56 from this view point different generalizations of the dynamics can be proposed in the frame  
 57 of dynamical systems.

58 From this point of view the analysis of [16] is precisely to describe a policy in the ex-  
 59 perimental setting where a mouse is treated prior to infection by antibiotic and a final fecal  
 60 injection in order to cure the C-difficile infection. The  $2d$ -reduced model is with a saddle per-  
 61 sistent point with stability and unstability domain delimited by separatrixes and the success  
 62 of the therapy is to reach a stability domain of an healthy (non persistent) equilibrium.

63 At this time the connection with the network of chemical species pointed by Lotka has  
 64 to be made in relation with intense research activities in the seventies. They were motivated  
 65 by the analysis of chemical batch reactors and realized mainly by Feinberg-Horn-Jackson, see  
 66 for instance the recent book [14]. The study is related to the graph of the reactions using  
 67 the concept of deficiency. The simplest situation (zero deficiency) being dynamics with no  
 68 (non trivial) singular point nor periodic trajectory in the irreversible case like the network:  
 69  $A \rightarrow B$ ,  $B \rightarrow C$ , and a single persistent point with asymptotically stable dynamics in the  
 70 reversible case like the network:  $T + M \rightarrow A$ ,  $A \rightarrow B$ ,  $B \rightarrow T + M$ ,  $A \rightarrow T + M$ .

71 Similarly concerning the Lotka–Volterra equations there was an intense research activities  
 72 due in particular to Zeeman, Smale, Hirsch to analyze the dynamics in the frame of the theory  
 73 of dynamical systems where again the concept of reduced dynamics is handled using roughly  
 74 the concept of reduced attractors as the limit behaviors of the trajectories. The mathematical  
 75 limit of the study is well explained in [15] where such reduction is shown to be useful in the 3d-  
 76 case in the special case of competitive Lotka–Volterra system that is when  $A$  is a matrix with  
 77 negative entries. The model reduction is associated to a 2d-system with a simplex separatrix

78 cycle connecting three non persistent (saddle type) equilibria and is surrounding the persistent  
79 equilibrium of the model. Without the competing assumption the road to chaos is open using  
80 for instance the period doubling bifurcation. Note this leads to consider Generalized Lotka–  
81 Volterra dynamics by gluing such simplicia.

82 The objective of this paper is to investigate the problem of reducing the infection of a  
83 complex microbiote and if the study of the dynamics of the associated Lotka–Volterra model  
84 plays a role, the control problem is associated to compute the accessibility set. Indeed from  
85 dynamical point of view denoting  $X$  the vector field  $(\text{diag}x)(Ax + r)$  only this dynamics is  
86 studied. But from a control point of view the problem is to assign to each different control  
87 action, e.g. antibiotic, probiotic, transplantation or bactericide, an additional vector field  $Y$   
88 and to study from geometric point of view the action of the pseudo-semigroup  $S(D)$  generated  
89 by concatenation of the positive orbits of the vectors fields denoted respectively  $\varphi_t = \exp tX$ ,  
90  $\psi_t = \exp tY$ , where  $D$  is the polysystem  $\{X, Y\}$ . The accessibility set at time  $t$  denoted  
91  $A(x_0, t)$  is the orbit when the total time is  $t$  and its boundary contains the extremities of the  
92 time minimal geodesics. It can be evaluated for small time using Lie brackets computations of  
93  $X$  and  $Y$ . This boundary can have a complicated structure but an intense research activities  
94 at the end of the eighties was devoted to stratify this set under some suitable assumptions in  
95 relation with the problem of computing a regular synthesis, see in particular [9], [11]. In this  
96 research program the Maximum Principle [23] is a very powerful tool to analyze the optimal  
97 solutions since it gives a parametrization of the boundary of the accessibility set, where the  
98 geodesics dynamics is the projection of an  $n$ -Hamiltonian dynamics that is a differential system  
99 in dimension  $2n$  and this system is not smooth.

100 Let us formulate this principle to analyze for instance the effect of an antibiotic or probiotic  
101 treatment only, whence the control system takes the form

$$102 \quad \frac{dx}{dt}(t) = X(x(t)) + u(t)Y(x(t)),$$

103 where the control  $u(\cdot)$  in the permanent case belongs to the set  $\mathcal{U}$  of bounded measurable  
104 mappings valued in the convex set  $U = [-1, +1]$ , where  $u = 0$  corresponds to zero dosing that  
105 is  $X - Y = (\text{diag}x)(Ax + r)$  while to maximal dosing corresponds the dynamics  $(\text{diag}x)(Ax +$   
106  $r) + G(x)$ , where  $G(x) = (\text{diag}x)(\varepsilon_1, \dots, \varepsilon_n)^\top$  and  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)^\top$  is the sensitivity vector of  
107 the antibiotic or probiotic agent.

108 Consider the problem of reaching in minimum time a terminal manifold  $N$  related to reduce  
109 the infected agent  $x_1$ , e.g. reach a small population  $x_1$  in minimal time (a dual formulation  
110 of the problem is  $\min x_1$  for fixed final time).

111 **Theorem 1.1.** *The Maximum Principle tells us that if  $(x(\cdot), u(\cdot))$  is a time minimal control-*  
112 *trajectory pair on  $[0, t_f]$ , then there is exists a non vanishing adjoint  $n$ -dimensional vector  $p(\cdot)$*   
113 *such that the triple  $(z(\cdot), u(\cdot))$ ,  $z = (x, p)$  satisfies the equations that we introduce next.*

114 Denoting  $H(z, u) := H_X(z) + uH_Y(z)$  the pseudo-Hamiltonian, where if  $Z$  is a vector field,  
115  $H_Z(z) = p \cdot Z(x)$  denotes the Hamiltonian lift and  $M$  is the maximized or true Hamiltonian  
116  $\max_{u \in [-1, +1]} H(z, u)$ . Then for almost every  $t$  one has

$$117 \quad (1.2) \quad \frac{dx}{dt}(t) = \frac{\partial H}{\partial x}(x(t), p(t), u(t)), \quad \frac{dp}{dt}(t) = -\frac{\partial H}{\partial p}(x(t), p(t), u(t)).$$

118 *Moreover the optimal control satisfies the maximization condition a.e.*

$$119 \quad (1.3) \quad H(z(t), u(t)) = M(z(t)),$$

120 *and  $M$  is a positive constant.*

121 *At the terminal time  $t_f$  the transversality condition holds*

$$122 \quad (1.4) \quad p(t_f) \perp T_{x(t_f)}^* N.$$

123 From the maximization condition one deduces that an optimal control is the concatenation  
124 of:

- 125 • Regular subarcs where  $u(t) = \text{sign} H_Y(z(t))$  a.e.
- 126 • Singular subarcs defined by the implicit relation  $H_Y(z(t)) = 0$ .

127 Such singular arcs define the geodesics solutions when relaxing the control bound to the whole  
128  $u \in \mathbb{R}$ . They form an Hamiltonian flow constrained to the switching set  $\Sigma : H_Y(z) = 0$  in  
129 which they filled in general a subset of codimension one. Hence they play an important role  
130 in our study, see for instance the reference [5].

131 The main objective of this article is to analyze geometrically this dynamics in the case  
132 of the controlled Lotka–Volterra equations and from the control optimal point of view it is  
133 the analogue of the program of classifying the geometric dynamics for the Lotka–Volterra  
134 model. The limit being essentially the same that is to handle the curse of dimension and the  
135 computational complexity. Making the connection of the problem of minimizing the infection  
136 with the problem of maximizing the production of one species for chemical network whose  
137 dynamics is given by Arrhenius law, our objective is to use in this frame a series of articles  
138 started in the eighties, see for instance [7, 20], aiming to optimize the production of batch  
139 chemical reactors and recently pursued in [8]. Besides the study of the geodesics dynamics  
140 parameterized by an Hamiltonian system, thanks to the Maximum Principle, a complementary  
141 point is to characterize the optimality properties of the geodesics in relation with the concept  
142 of conjugate time where the time optimality is lost for the  $C^1$  topology while for a cut point  
143 the optimality is lost globally. In this frame the series of results and techniques presented in  
144 [5, 1] will be used in their totality.

145 Moreover in practice our objective is to compute an approximation of the time minimal  
146 synthesis that is to reach the terminal manifold for every initial condition determining the  
147 closed loop optimal control:  $x \rightarrow u^*(x)$ . Regularity conditions have to be satisfied in order to  
148 define the solutions and they are related to the regularity properties of the time minimal value  
149 function. Relation with convergence of numerical methods in the sampled-data control case  
150 will be pointed in this article but not analyzed due to space restrictions. Also the sampled-data  
151 control frame is different since the dynamics of the evolution is given using a MPC method  
152 and the controlled Lotka–Volterra dynamics is only an intermediate tool to estimate the effect  
153 of the control on a finite horizon on a domain where the Lotka–Volterra model or an extension  
154 are valid.

155 The article is organized as follows. In section 2 the controlled Lotka–Volterra model is  
156 introduced and the optimal control problem presented in the frame of permanent controls or  
157 sampled-data controls. In the permanent case the Maximum Principle leads to the classifica-  
158 tion of the geodesics in the context of geometric optimal control and singularity theory using

159 the seminal earliest contributions of [17, 18, 13]. Some consequences are deduced in the case  
 160 of a problem with a terminal manifold of codimension one. The section 3 is based on the series  
 161 of articles [7, 20, 8], which are rather technical and our contribution being to introduce two  
 162 main concepts. The first one is the notion of Whitney chart to determine in an appropriated  
 163 coordinates systems the time minimal synthesis in a neighborhood of the terminal manifold  
 164 using the construction of semi-normal forms to estimate the switching and cut loci up to all  
 165 cases of codimension two for the  $C^\infty$ -Whitney topology. Secondly the concept of unfolding  
 166 from singularity theory is introduced in our control frame to reduce the classification to 2d  
 167 and 3d cases and using their description (dictionary) of the aforementioned references. The  
 168 final section is devoted to the analysis and the classification of singular arcs in the 2d and 3d  
 169 cases to deduce the time minimal syntheses. In this program the computation can be autom-  
 170 atized in the 2d case using two classical invariants: the collinear set and the singular locus. In  
 171 dimension 3 the problem is intricate due to the complexity of the singular dynamics. But a  
 172 program of computations in the general case can be outlined based on gluing Whitney charts  
 173 as an alternative of programs as for example in [2] to derive patchy feedbacks to approximate  
 174 the time minimal function, our construction differing by the dominant use of the feedback  
 175 singular control.

## 176 2. Model of Controlled Lotka–Volterra Equation and Optimal Control.

177 **2.1. A quick tour in the 2d-Lotka–Volterra predator-prey model.** The original Lotka–  
 178 Volterra model analyzed in [26] describes, in the frame of (conservative) integrable dynamics,  
 179 the interaction between two species. More precisely it was constructed to explain the evo-  
 180 lution of the averaged populations of two fishing species in relation with diminution of the  
 181 fishing activity observed by d’Ancona during the first World War and succeeds to explain the  
 182 observation. From dynamical point of view it concerns the case of a center, the mechanical  
 183 analogue being the oscillating non linear pendulum and it was extended to the case of an  
 184 integrable saddle by Volterra himself. The limit of the model are clearly indicated in the  
 185 memoir and a parallel can be made with the 2d-reduced model of the gut microbiote that we  
 186 are analyzing in this paper.

187 We conserve the same notation than in the memoir. In the oscillatory situation the prey-  
 188 predator population  $N = (N_1, N_2)$  satisfies the dynamics:

$$189 \quad (2.1) \quad \frac{dN_1}{dt} = N_1(\varepsilon_1 - \gamma_1 N_2), \quad \frac{dN_2}{dt} = -N_2(\varepsilon_2 - \gamma_2 N_1)$$

190 where  $(\varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2) > 0$ , which can be written in a general form as

$$191 \quad (2.2) \quad \frac{dN_1}{dt} = N_1(\lambda_1 + \mu_1 N_1) \quad \frac{dN_2}{dt} = N_2(\lambda_2 + \mu_2 N_2)$$

192 for real parameters  $\lambda_i, \mu_i, i = 1, 2$ .

193 This leads to the equation

$$194 \quad (2.3) \quad \mu_2 \frac{dN_1}{dt} + \lambda_2 \frac{1}{N_1} \frac{dN_1}{dt} - \mu_1 \frac{dN_2}{dt} - \lambda_1 \frac{1}{N_2} \frac{dN_2}{dt} = 0$$

195 and integrating one gets

$$196 \quad \mu_2 N_1 + \lambda_2 \ln N_1 - (\mu_1 N_2 + \lambda_1 \ln N_2) = \text{constant.}$$

197 Hence

$$198 \quad N_1^{\lambda_2} e^{\mu_2 N_1} = C N_2^{\lambda_1} e^{\mu_1 N_2},$$

199 where  $C$  is a constant depending upon the initial conditions  $(N_1(0), N_2(0))$ .

200 Volterra describes the solution using the auxiliary curves

$$201 \quad (2.4) \quad Y = N_1^{-\varepsilon_2} e^{\gamma_2 N_1}, \quad X = N_1^{\varepsilon_1} e^{-\gamma_1 N_2}$$

202 so that the solution can be locally either represented as a graph  $Y = CX$  or  $X = CY$ .

203 One denotes

$$204 \quad \Omega := (K_1, K_2) = \left( \frac{\varepsilon_2}{\gamma_2}, \frac{\varepsilon_1}{\gamma_1} \right)$$

205 the persistent equilibrium. One can use polar coordinates  $(\rho, \omega)$  centered at  $\Omega$  and introducing  
206 the speed of the area swept by the vector  $\Omega M$  satisfying the equation

$$207 \quad \frac{1}{2} \rho^2 \frac{d\omega}{dt} = \frac{1}{2} \left[ (N_1 - K_1) \frac{dN_2}{dt} - (N_2 - K_2) \frac{dN_1}{dt} \right] > 0,$$

208 which takes the form

$$209 \quad \rho^2 \frac{d\omega}{dt} = \rho^2 \left[ \gamma_1 \sin^2 \omega (K_1 + \rho \cos \omega) + \gamma_2 \cos^2 \omega (K_2 + \rho \sin \omega) \right].$$

210 Hence:

$$211 \quad (2.5) \quad t - t_0 = \int_{\omega_0}^{\omega} \frac{d\omega}{\sum_{i=1,2} [\gamma_i \sin^2 \omega (K_i + \rho \cos \omega)]}.$$

212 In particular the period is given by the formulae:

$$213 \quad (2.6) \quad T = \int_0^{2\pi} \frac{d\omega}{\rho \sin \omega \cos \omega (\gamma_1 \sin \omega + \gamma_2 \cos \omega) + K_1 \gamma_1 \sin^2 \omega + K_2 \gamma_2 \cos^2 \omega}$$

214 while the limit value at  $\Omega$  is given by

$$215 \quad \int_0^{2\pi} \frac{d\omega}{K_1 \gamma_1 \sin^2 \omega + K_2 \gamma_2 \cos^2 \omega} = \frac{2\pi}{\sqrt{\varepsilon_1 \varepsilon_2}}$$

216 and this gives the analog with the nonlinear pendulum oscillation but the transcendence of  
217 the period is different.

218 Similar formulae gives the time evolution of the population  $t \mapsto N_i(t)$ . But from physical  
219 point of view, what matters is the averaged population  $\langle N_i \rangle$  and a simple computation gives  
220 the following.

221 **Theorem 2.1.** *The averaged populations are given by:*

$$222 \quad \langle N_i \rangle = \frac{1}{T} \int_0^T N_i(t) dt = K_i, i = 1, 2,$$

223 which are not depending upon the initial conditions but only from the equilibrium  $\Omega$ .

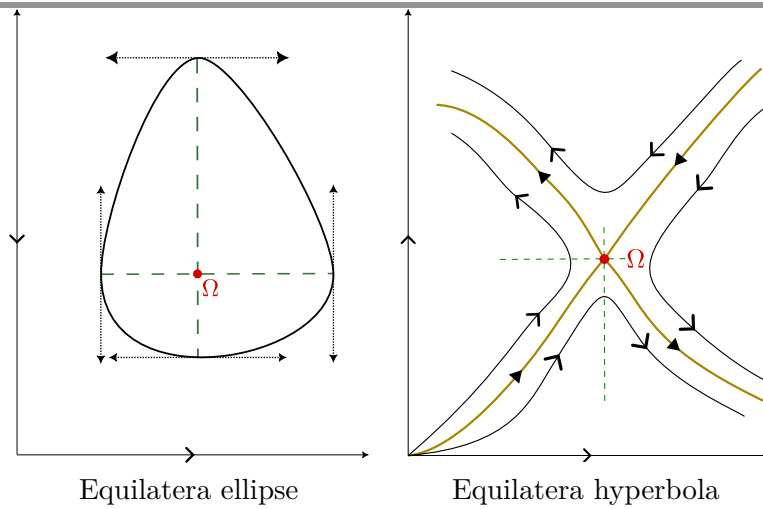


Figure 1. Volterra memoir.

224 The Lotka–Volterra model can be normalized in the same category using the dimensionless  
225 variables:

$$226 \quad n_1 := \frac{N_1}{K_1}, \quad n_2 := \frac{N_2}{K_2}$$

227 and we get

$$228 \quad (2.7) \quad \frac{dn_1}{dt} = \varepsilon_1 n_1 (1 - n_2), \quad \frac{dn_2}{dt} = -\varepsilon_2 n_2 (1 - n_1)$$

229 and the linearized system at  $\Omega$  takes the form

$$230 \quad (2.8) \quad \frac{\delta n_1}{dt} = -\varepsilon_1 \delta n_2, \quad \frac{\delta n_2}{dt} = \varepsilon_2 \delta n_1.$$

231 The geometric construction of the dynamics in the memoir is based on the auxiliary curves  
232 (2.4) to compute  $Y = CX$ .

233 This opens the path to treat in the same frame the case where  $\varepsilon_1 \varepsilon_2 < 0$  so that  $\Omega$  is a  
234 saddle point to get the form  $XY = C$ . Volterra gives a geometric representation of the two  
235 cases in a single figure, see Fig. 1.

236 Let us analyze in the oscillatory case the role of the fishing activity, introducing control  
237 in the model. The system takes the form:

$$238 \quad (2.9) \quad \frac{dN_1}{dt} = (\varepsilon_1 - \alpha\lambda - \gamma_1 N_2)N_1, \quad \frac{dN_2}{dt} = -(\varepsilon_2 + \beta\lambda - \gamma_2 N_1)N_2,$$

239 where  $\alpha, \beta \geq 0$  are the modes of destruction and  $\lambda \geq 0$  is the intensity.

240 Assuming  $\varepsilon_1 - \alpha\lambda > 0$  so that the population is still oscillating, the averaged values of  
241  $N_1, N_2$  become

$$242 \quad \frac{\varepsilon_2 + \beta\lambda}{\gamma_2}, \frac{\varepsilon_1 - \alpha\lambda}{\gamma_1} \quad \text{versus} \quad \frac{\varepsilon_2}{\gamma_2}, \frac{\varepsilon_1}{\gamma_1}$$





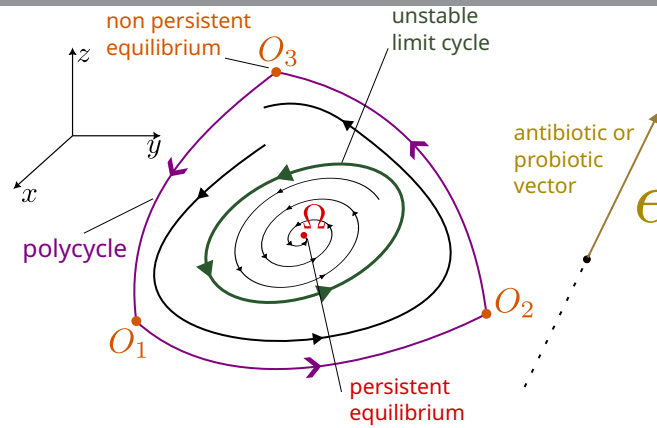


Figure 3. Generalized Controlled Lotka–Volterra model.

268 by the fecal transplantation. The mathematical limit of the model being that each species  
 269 population shall satisfies  $M \geq x_i \geq \varepsilon$ , in particular if the  $x_2$  population becomes too small  
 270 the model is not valid. Also from medical point of view a probiotic injection versus antibiotic  
 271 injection increases the healthy population aiming to struggle against infection.

272 Since Lotka–Volterra model has to be understood as interaction between a set of com-  
 273 putable non persistent singularities with computable domains of attraction with a persistent  
 274 singular point  $\Omega$ , the dynamics being restricted to an open set  $U$  centered at  $\Omega$  representing  
 275 the viability domain of the mathematical model, this opens the road to more complicated  
 276 models.

277 For instance, a normalized dynamics of the form

$$278 \quad (2.11) \quad \frac{dx}{dt} = x(1 - x) - \frac{axy}{d + x}, \quad \frac{dy}{dt} = by(1 - \frac{y}{x}),$$

279 where the persistent equilibrium is given by the intersection of a straight-line and a parabola

$$280 \quad x = y, \quad ay = (1 - x)(d + x)$$

281 allows to construct dynamics with a limit cycle obtained by Hopf bifurcation, which is not  
 282 possible for a  $2d$ –Lotka–Volterra model. The limit of the construction being to analyze sta-  
 283 bility.

284 More complicated constructions lead to consider systems with a single interior equilibria  
 285 with a simplex formed by a polycycle of separatrices connecting three non persistent saddle  
 286 equilibria with one or more interior limit cycles generated by Hopf bifurcation, see Fig. 3.

287 **2.2. Some remarks about constructing general controlled Lotka–Volterra models.** In  
 288 the original memoir starting from the integrable case, one considers the interaction between  
 289 the persistent equilibrium  $\Omega$  and the origin  $O$  which is not persistent. But the distinction  
 290 between the center case of the elliptic situation and the saddle case of the hyperbolic situation  
 291 can be deduced from the nature of  $O$  only which can be either saddle or a node. Then  
 292 conversely this leads to construct a model where the persistent singularity  $\Omega$  can be identified

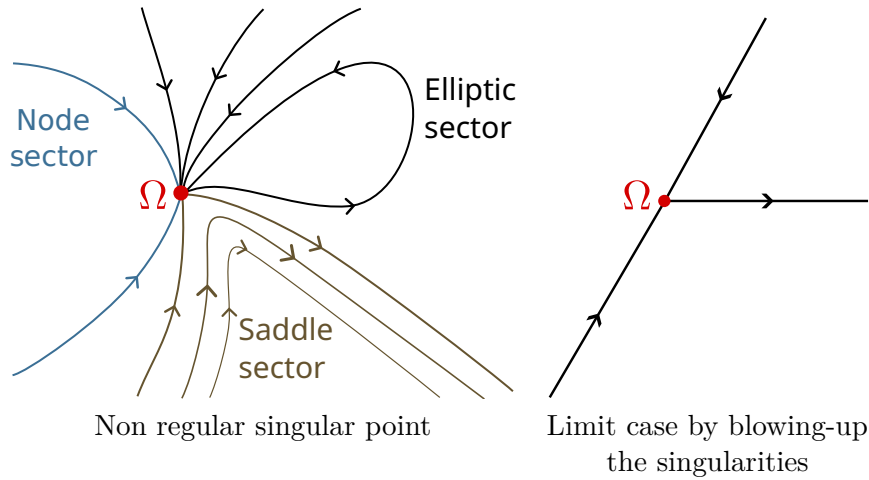


Figure 4. Generalized Controlled Lotka–Volterra model.

293 by confluence to the origin which is the only equilibrium point but with a linear system which  
 294 is degenerated.

295 This construction leads to consider persistent singular point  $\Omega$  with multiple sectors: node  
 296 sector, saddle sector and a new type of sector called an elliptic sector, see Fig. 4.

297 This leads to consider polynomial dynamics:  $\frac{dx}{dt} = H(x)$ , where at the limit,  $H$  is homo-  
 298 geneous with degree  $k$ :  $H(\lambda x) = \lambda^k H(x)$  and the separatrices of the different domains are  
 299 lines.

300 In the polynomial category the system can be compactified using the Poincaré compacti-  
 301 fication, identifying the Euclidean plane to the plane  $(x, z = 1)$  where the system reads

$$302 \quad \frac{dx}{dt} = H(x, z), \quad \frac{dz}{dt} = 0,$$

303 and  $H(x, z)$  is homogeneous with respect to  $(x, z)$ . Such a system gives us a differential  
 304 equation, which can be projected on the two-sphere  $\mathbf{S}^2$ . The construction is similar in the  
 305  $n$ -dimensional case.

306 From control point of view, one considers a pair of vector fields  $(X, Y)$ , where  $X$  is the  
 307 Lotka–Volterra model or a generalization of this model, with a single persistent equilibrium  
 308  $\Omega$ , while  $Y$  corresponds to the controlled vector field. It can be of two types:

- 309 • antibiotic or probiotics which belongs to the same category:  $Y(x) = (\text{diag } x) \in$ .
- 310 • transplantation or bactericide: in this case this leads to a jump in the initial condition:  
 311  $x(0) \rightarrow x(0) + \lambda v$ .

312 The second case has to be interpreted as the response to a Dirac  $t \mapsto \delta(t - t')$  generated as  
 313 the limit of an unbounded control sequence:  $u_n = \lambda n$ ,  $t \in [0, 1/n]$ , while the control vector  
 314 direction is given by the constant vector field  $Y = v$ .

315 In particular this relates our study to the general frame of control systems of the following  
 316 forms.

**Bilinear systems.**

317 
$$\frac{dx}{dt}(t) = Ax(t) + u(t)(Bx(t) + b),$$

318 where  $A, B$  are constant matrices and  $b$  is a constant vector.

**Homogeneous systems.**

319 
$$\frac{dx}{dt}(t) = H(x(t)) + u(t)b,$$

320 where  $H$  is a polynomial homogeneous vector field and  $b$  is a constant vector field.

321 **2.3. Accessibility problem for 2d-systems. The affine case.** In this section, we make a  
322 complete study of the accessibility problem for 2d-affine systems, in relation with the action  
323 of the antibiotic-probiotic agent near the saddle point as in Jones et al. model reduction.

324 **Definitions and Notations.** We work in the  $\mathbb{C}^\omega$  (real analytic) category and  $M$  denotes  
325 a (regular) connected manifold. The set of (local) diffeomorphisms  $\varphi$  denoted by  $\text{Diff}(M)$   
326 and let  $\Xi(M)$  be the set of vector fields on  $M$ . If  $X \in \Xi(M)$ , one denotes by  $\varphi_t = \exp tX$   
327 the one-parameter pseudo-group of local diffeomorphisms:  $\varphi_t(x_0) = x(t, x_0)$ ,  $x(\cdot)$  being the  
328 solution at time  $t$  of the Cauchy problem:  $\frac{dx}{dt}(t) = X(x(t))$ ,  $x(0) = x_0$ . A special case is  
329 the case of linear vector fields  $X(x) = Ax$ ,  $A$  being a constant matrix,  $\varphi_t = \exp tA$  being  
330 the exponential matrix and moreover  $X$  is complete so that  $\exp tA$  is a one-parameter group  
331 acting linearly.

332 **Definition 2.2.** Let  $D = \{X\}$  be a family of  $\Xi(M)$  called a polysystem. We denote by  $S(D)$   
333 the pseudo-semigroup generated by the family  $\exp tX$ ,  $X \in D$ , that is:

334 
$$S(D) = \{\exp t_1 X_1 \circ \cdots \circ \exp t_k X_k, X_i \in D, t_i > 0\},$$

335 and we denote by  $t = t_1 + t_2 + \cdots + t_k$  the total transfer time. The pseudo-group  $S(D)$  can be  
336 extended to the whole group  $G(D)$  by taking  $t_i$  positive or negative.

337 Let  $x, y \in M$ , then  $y$  is called accessible from  $x$  if there exists  $X_i \in D$  and  $t_i > 0$  such that

338 (2.12) 
$$y = \exp t_1 X_1 \circ \cdots \circ \exp t_k X_k(x)$$

339 and  $y$  is said normally accessible to  $x$  if the mapping  $(s_1, \dots, s_k) \mapsto \exp s_1 X_1 \circ \cdots \circ \exp s_k X_k(x)$   
340 is of rank  $n = \dim M$  at  $(t_1, \dots, t_k)$ . Fixing  $x$ , the set of accessible points is denoted  $A^+(x)$   
341 and it is the orbit of  $S(D)$ .

342 If  $y$  is accessible to  $x$  this means that  $x$  can be steered to  $y$  and the set of points which  
343 can be steered to  $y$  is denoted  $A^-(y)$ .

344 Let  $D = \{X\}$  be a polysystem, then  $\text{Sym}(D)$  is the symmetric set  $\{\pm X\}$  and we called  
345 orbit of  $x$  the orbit of the pseudo-group  $G(D)$ .

346 The polysystem  $D$  is called (normally) controllable on a subset  $P \subset M$  if for each pair  
347  $(x, y) \in P$ ,  $y$  is (normally) accessible to  $x$ . If  $P = M$ , the system is said (normally) controllable  
348 on  $M$ .

349 **Definition 2.3.** Let  $X, Y \in \Xi(M)$ , the Lie bracket in local coordinates  $x$  is calculated with  
 350 the convention:  $[X, Y](x) = \frac{\partial X}{\partial x}(x)Y(x) - \frac{\partial Y}{\partial x}(x)X(x)$ . Let  $D = \{X\}$  be a polysystem, we  
 351 denote  $D_{L.A.}$  the Lie algebra generated by  $D$ . It can be computed introducing:

$$352 \quad D^1 = D, \quad D^2 = D \cup [D, D], \dots, \quad D^k = D^{k-1} \cup [D^{k-1}, D]$$

353 and  $D_{L.A.} = \text{span} \cup_{k \geq 1} D^k$ .

354 **Theorem 2.4.** The following assertions are equivalent:

- 355 1. For every  $x, y \in M$ , the Lie algebra  $D_{L.A.}(x)$  is of maximal rank =  $\dim M$ .
- 356 2. For every  $x \in M$  and every neighborhood  $U$  of  $x$ , there exists  $y$  in  $U$  normally accessible  
 357 to  $x$ .
- 358 3. The symmetric system  $\{\pm X\}$  is controllable.

359 The third assertion is Chow-Rashevskii theorem [12].

360 **2.3.1. The case of Lie groups.** Let  $G$  be a Lie group and  $\mathfrak{g}$  its Lie algebra identified to  
 361 the set of right-invariant vector fields on  $G$ . Take  $G = Gl(n, \mathbb{R})$  be the set of  $n \times n$  invertible  
 362 matrices,  $Gl^+(n, \mathbb{R})$  the subset of matrices with positive determinant. The Lie algebra  $\mathfrak{gl}(n, \mathbb{R})$   
 363 is the set of  $n \times n$  matrices and the Lie bracket of two matrices is  $[A, B] = AB - BA$ . If  
 364  $A \in \mathfrak{gl}(n, \mathbb{R})$  the one-parameter group  $\exp tA$  is given by the series  $\exp A = \sum_{n \geq 0} \frac{A^n}{n!}$ .

365 The group  $Gl(n, \mathbb{R})$  acts transitively on  $\mathbb{R}^n \setminus \{0\}$  and the action defines linear vector  
 366 fields  $X^+(x) = Ax$ . The affine group is the subgroup of  $Gl(n, \mathbb{R})$  of matrices  $\begin{pmatrix} 1 & 0 \\ v & S \end{pmatrix}$ ,

367 where  $v \in \mathbb{R}^n$ ,  $S \in Gl(n, \mathbb{R})$  and its Lie algebra is the set of matrices  $\begin{pmatrix} 0 & 0 \\ a & A \end{pmatrix}$ ,  $a \in \mathbb{R}^n$ ,

368  $A \in \mathfrak{gl}(n, \mathbb{R})$ . It acts transitively on  $\mathbb{R}^n$  identified to  $\begin{pmatrix} 1 \\ \mathbb{R}^n \end{pmatrix}$ . This induced action defines  
 369 affine vector fields  $X^+ = Ax + a$ . We denote by  $\mathfrak{gl}(n, \mathbb{R}) \oplus_s \mathbb{R}^n$  the semi-direct Lie algebra  
 370 where the Lie bracket is:  $[(A, a), (B, b)] = ([A, B] = AB - BA, Ab - Ba)$ .

371 **2.3.2. Accessibility results for 2d-affine systems.** The Lie algebra  $\mathfrak{gl}(n, \mathbb{R})$  decomposes  
 372 into the direct sum  $c \oplus \mathfrak{sl}(n, \mathbb{R})$ , where  $c$  is the center formed by diagonal matrices  $cI_n$ ,  $I_n$   
 373 being the  $n \times n$  identity matrix, while  $\mathfrak{sl}(n, \mathbb{R})$  is the set of matrices with zero trace associated  
 374 to linear conservative vector fields. We denote by  $Sl(n, \mathbb{R})$  the special linear group of matrices  
 375 with determinant one. Note that for  $n = 2$ ,  $\mathfrak{sl}(2, \mathbb{R}) = \mathfrak{sp}(1, \mathbb{R})$  the second being associated to  
 376 planar linear Hamiltonian vector fields and with the identification  $Sl(2, \mathbb{R}) = Sp(1, \mathbb{R})$ .

377 We shall restrict our study to the case of conservative vector fields, extension to the non  
 378 conservative case being straightforward.

379 **Definition 2.5.** Let  $A$  be a non zero element of  $\mathfrak{sp}(1, \mathbb{R})$ . Then:

- 380 •  $A$  is elliptic if the eigenvalues are imaginary:  $\{\pm i\sigma\}$ ,  $\sigma \in \mathbb{R}$ ,
- 381 •  $A$  is parabolic if 0 is an eigenvalue,
- 382 •  $A$  is hyperbolic if the eigenvalues are real numbers:  $\{\pm\lambda\}$ ,  $\lambda \in \mathbb{R}$ .

383 **Lemma 2.6.**  $A, A' \in \mathfrak{sp}(1, \mathbb{R})$  are called (symplectically) equivalent:  $A \sim A'$  if there exists  
 384  $P \in Sp(1, \mathbb{R})$  such that  $P^{-1}AP = A'$ . One has:

- 385 •  $A$  is elliptic if and only if  $A \sim \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}$ ,  $\alpha \in \mathbb{R}$ .

386 •  $A$  is parabolic if and only if  $A \sim \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$ ,  $\varepsilon = \pm 1$ .

387 •  $A$  is hyperbolic if and only if  $A \sim \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ .

388 Only in the third case, the equivalent class is given by the spectrum.

389 Next we introduce the normal forms of pairs  $\{A, B\}$  of vector fields in  $\mathfrak{sp}(1, \mathbb{R})$  to study  
390 controllability.

391 **Definition 2.7.** Let  $D = \{A, B\}$  and  $D' = \{A', B'\}$  be two pairs of vector fields in  $\mathfrak{sp}(1, \mathbb{R})$ .  
392 They are called equivalent if there exists  $a, b \in \mathbb{R}, ab > 0$  such that:  $aA \sim A', bB \sim B'$ .

393 We shall analyze controllability properties using normal forms for pairs  $D = \{A, B\}$ .

394 **Lemma 2.8.** Let  $D = \{A, B\}$  be a polysystem on  $\mathfrak{sp}(1, \mathbb{R})$ . One has  $D_{L.A.}(x) = \mathbb{R}^2$  for  
395  $x \neq 0$  if and only if  $A, B$  are non proportional and with distinct eigenvectors.

396 **2.3.3. Criteria of controllability.** From general properties of nonlinear systems we have  
397 the following.

398 **Criteria.** Let  $A, B \in \mathfrak{sp}(1, \mathbb{R})$  non proportional and with distinct eigenspaces. Then the  
399 system is controllable if and only if for every  $x$  non zero there exists an element  $\varphi = \exp t_1 X_1 \circ$   
400  $\dots \circ \exp t_k X_k$  of  $S(D)$ ,  $\varphi(x)$  forming a periodic path surrounding the point 0.

401 **Corollary 2.9.** Assume  $A$  elliptic. Then  $D = \{A, B\}$  is controllable if and only if they are  
402 non proportional.

403 **Corollary 2.10.** Assume  $A$  parabolic and  $B$  hyperbolic so that  $D_{L.A.}(x) = \mathbb{R}^2$  for  $x \neq 0$ .  
404 Then only one polysystem among  $\{Ax, Bx\}$  and  $\{Ax, -Bx\}$  is controllable. Moreover  $D$  is  
405 controllable if and only if  $D \sim \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}$ .

406 **Corollary 2.11.** Take  $D = \{A, B\}$  with  $A$  and  $B$  hyperbolic. Let  $a > 0$  and  $-a$  denotes  
407 the eigenvalues of  $A$  and  $b > 0$  and  $-b$  the eigenvalues of  $B$ . Let  $M_1$  one intersection of the  
408 eigenspace of  $a$  with the unit circle and using the positive orientation starting from  $M_1$  denotes  
409  $M_2, M_3, M_4$  the first intersection with the unit circle of the eigenspaces associated respectively  
410 to  $-a, b, -b$ . Then the only controllable polysystems are associated to  $(M_1, M_2, M_3, M_4)$  or  
411  $(M_1, M_4, M_3, M_2)$ .

412 **2.3.4. Necessary and sufficient conditions results on  $Sp(1, \mathbb{R})$ .** A very interesting result  
413 is that the previous controllability analysis can be lifted on the  $3d$ -manifold  $Sp(1, \mathbb{R})$  in relation  
414 with the action on the Poincaré half-plane. One needs some technical results.

415 **Lemma 2.12.** Let  $SO(2)$  be the group of rotations with determinant one given by

416 
$$\left\{ \exp t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, t \in [0, 2\pi] \right\}.$$

417 Let  $K$  be a maximal compact subgroup of  $Sp(1, \mathbb{R})$ . Then there exists a symplectic matrix  $P$   
418 such that  $K = PSO(2)P^{-1}$ .

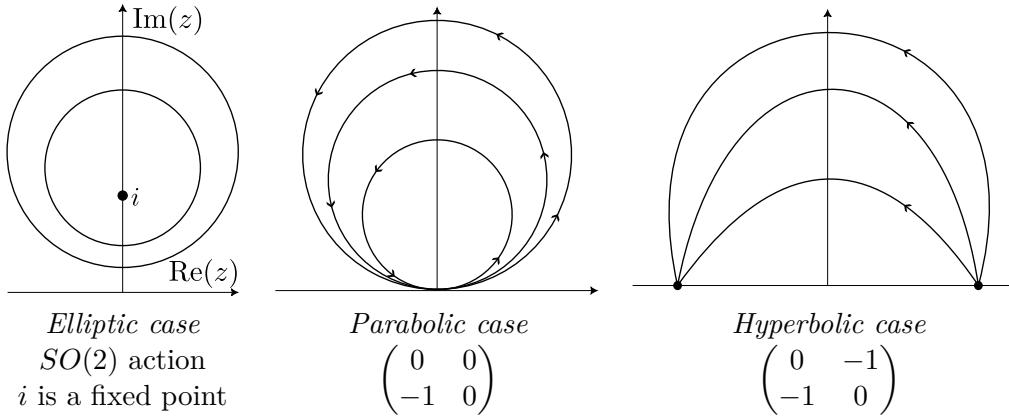


Figure 5. One-parameter dynamics in Poincaré upper plane.

419 **Poincaré half-plane.** The Poincaré half-space  $\mathcal{P}$  is the set of complex numbers  $z$  such that  
 420  $\text{Im}z > 0$ . Then  $Sp(1, \mathbb{R})$  acts transitively on  $\mathcal{P}$  with the action:

$$421 \quad S \cdot z = \frac{az + b}{cz + d}, \quad z \in \mathcal{P}, S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(1, \mathbb{R}).$$

422 Hence for every one-parameter subgroup  $\exp tA$ ,  $A \in Sp(1, \mathbb{R})$  one can associate on  $\mathcal{P}$  the  
 423 vector field  $X^+(z) = \frac{d}{dt}|_{t=0}(\exp tA) \cdot z$ .

424 One has the following characterization of maximal compact subgroup  $K$ :

- 425 •  $SO(2)$  is the set of matrices  $S$  in  $Sp(1, \mathbb{R})$  so that  $S \cdot i = i$ .
- 426 •  $K = PSO(2)P^{-1}$  is the set of matrices so that  $S \cdot z = z$  with  $z = P \cdot i$ .

427 This action gives the key to determine the controllability of pairs  $D = \{A, B\}$  on  $Sp(1, \mathbb{R})$ .  
 428 Indeed one must find elements of  $S(D)$  associated to compact elements of  $K$ . They can be  
 429 seen as periodic paths in the Poincaré half-space leaving fixed a point. See Fig. 5 illustrating  
 430 this action.

431 One deduces the following, see [4] for the details.

432 **Theorem 2.13.** Let  $D = \{A, B\}$  be a pair of right-invariant vectors fields on  $Sp(1, \mathbb{R})$ .  
 433 Then the following conditions are equivalent:

- 434 1. The polystem  $D$  is controllable on  $Sp(1, \mathbb{R})$ .
- 435 2. The polystem  $D = \{Ax, Bx\}$  is controllable on  $\mathbb{R}^2 \setminus \{0\}$ .

436 This is a strong controllability result which can be translated to affine pairs  $\{(A, a), (B, b)\}$   
 437 where  $A, B \in \mathfrak{sp}(1, \mathbb{R})$ . This comes from [3].

438 **Lemma 2.14.** Assume  $\{A, B\}$  controllable on  $\mathbb{R}^2 \setminus \{0\}$ . Then there exists  $R > 0$  large  
 439 enough such that  $\{(A, a), (B, b)\}$  is controllable for  $|x| \geq R$ .

440 **Proposition 2.15.** Assume  $\{A, B\}$  controllable on  $\mathbb{R}^2 \setminus \{0\}$ . Then the polystem  $\{(A, a),$   
 441  $(B, b)\}$  is controllable on  $\mathbb{R}^2$  if and only if  $Ax + a$  and  $Bx + b$  have no common equilibrium  
 442 point.

443 **2.4. Controlled Lotka–Volterra model and optimal control.** Next we introduce the def-  
 444 initions and concepts in a general frame.

445 **Definition 2.16.** Let  $x = (x_1, \dots, x_n)^\top \in \mathbb{R}_+^n$  be the state of interacting species,  $x_1$  being the  
446 infected agent,  $x' = (x_2, \dots, x_n)^\top$  being the state of healthy agents. The dynamics is described  
447 by

$$448 \quad (2.13) \quad \frac{dx}{dt} = (\text{diag}x)(Ax + r)$$

449 the matrix  $A = (a_{ij})$  being the matrix of coefficients of interaction and  $r = (r_1, \dots, r_n)^\top$  is  
450 the vector of individual growth rate. We denote by  $M^+ = \mathbb{R}_+^n$  the invariant domain  $x_i > 0$   
451 and  $M^\sim$  the union of  $M^+$  with its boundary. The dynamics is called regular if  $A$  is invertible.  
452 The interior or persistent equilibrium is the point  $\Omega = (K_1, \dots, K_n)$  given by  $x = -A^{-1}r$ .

453 We note  $n = (n_1, \dots, n_n)^\top$  the dimensionless coordinates so that  $\Omega$  is identified to  $(1, \dots,$   
454  $1)^\top$ .

455 One can associate to (2.13) an hierarchy of dynamics replacing  $x \rightarrow x' = (x_2, \dots, x_n)^\top$   
456 and this leads to up to  $2^n$  equilibria for the dynamics in the physical space, which can be  
457 easily computed by recurrence.

458 The dynamics can be compactified using Poincaré compactification, identifying  $\mathbb{R}^n$  to the  
459 hyperplane  $(x, z = 1)$  in  $\mathbb{R}^{n+1}$  to define the system:

$$460 \quad \frac{dx}{dt} = (\text{diag}x)(Ax + r), \quad \frac{dz}{dt} = 0,$$

461 where the right-member can be homogenized to define an homogeneous vector field which can  
462 be projected on the  $n$ -sphere  $\mathcal{S}^n$ .

463 Each equilibrium of the hierarchy of dynamics can be classified according to the  $L$ -(linear)  
464 stability status associated to the linearized dynamics at equilibrium.

465 Our study is related to the entanglement of  $k$  non persistent equilibria interacting with  
466 the persistent equilibrium and one can introduce a model reduction consisting in a polynomic  
467 dynamics of the form:  $\frac{dx}{dt} = P(x)$ .

468 Introducing the ln-coordinates  $x = e^y$  so that the system takes the form:

$$469 \quad (2.14) \quad \frac{dy}{dt} = (Ae^y + r),$$

470 where  $y \in \ln M^+$  and the non persistent equilibria are at the infinity.

471 We denote by  $X(x)$  the vector field defined by (1.1) where  $x$  can be taken as the normalized  
472 coordinates.

473 **2.4.1. Antibiotic or probiotic agent.** For a single antibiotic or probiotic agent the control  
474 system takes the form

$$475 \quad (2.15) \quad \frac{dx}{dt} = X(x) + uY(x),$$

476 with  $Y(x) = (\text{diag}x)(\varepsilon_1, \dots, \varepsilon_n)^\top$  is the sensitivity vector and the control  $u(t)$  describes the  
477 dosing regimen. Note that  $u$  can be restricted to domain  $[0, 1]$  using normalizations.

478 This notation can be applied to the hierarchy of models so that this leads to an hierarchy  
479 of control problems which can be analyzed independently. For instance, prior to infection one



480 can analyze the effect of probiotics agents where  $\varepsilon_i > 0$ , while posterior to infection one can  
 481 consider the effect of antibiotic treatment.

482 The set of admissible controls fits in the frame of permanent control, where  $u(\cdot)$  is a  
 483 measurable mapping on  $[0, t_f]$  valued in  $[0, 1]$ . Each measurable bounded mapping can be  
 484 approximated by a sequence of piecewise constant mappings in the  $L^\infty$ -topology and acces-  
 485 sibility can be studied restricting to this class. This is the point of view of geometric control  
 486 which leads to introduce the polysystem:  $D = \{X + uY; u \text{ constant in the interval } [0, 1]\}$ .

487 **2.4.2. Fecal transplantation and bactericide.** In the case of transplantation or bacteri-  
 488 cide the control system takes the form:

$$489 \quad \frac{dx}{dt} = X(x) + uY(x),$$

490 where  $Y = (v_1, \dots, v_n)^\top$  is a constant vector field and  $u$  takes its values in the whole  $\mathbb{R}^+$ , where  
 491 the action being to get Dirac pulse  $\delta(t - t')$ , defined as the limit of controls sequence:  $u_n = \lambda n$   
 492 on  $[t', t' + 1/n]$ .

493 **2.4.3. Optimal control problems: the permanent case versus the sampled-data frame.**

494 Resuming the previous discussion, one can write the general control system in the form:

$$495 \quad \frac{dx}{dt}(t) = X(x(t)) + \sum_{\text{ant.,pro.}} u_i(t)Y_i(x(t)) + \sum_{\text{transp.,bac.}} u'_i(t)Y'_i(x(t)),$$

496 where the two sums are related respectively to probiotic, antibiotic agents and transplantations  
 497 and bactericides. Moreover discontinuity in the state  $x = (x_1, \dots, x_n)^\top \rightarrow x' = (x_2, \dots, x_n)^\top$   
 498 can be understood as the jump action  $x \rightarrow x'$ . Hence this leads in the general case to a mixture  
 499 of permanent and sampled-data control systems. The first action is related to permanent  
 500 control, but due to logistic medical constraints it can fit in the sampled-data frame, e.g. a  
 501 finite number of medical interventions at some predefined times:  $0 < t_1 < \dots < t_k < t_f$  so  
 502 modify the treatment. The second action fits in the sampled-data frame, since in particular  
 503 it corresponds to invasive therapies.

504 Hence from a general context, the problem is associated to the following optimal control  
 505 problems related to successive use of the different treatments.

506 **OCP1 in the permanent case.** The problem is either to reduce the  $x_1$ -infection or to  
 507 increase the production or ratio of healthy agents, prior to infection. This leads to consider  
 508 in a dual formulation problems of the form:

- 509 • Reach in minimum time  $t_f$  a given terminal manifold  $N$  of codimension one for the  
 510 control system written as:  $\frac{dx}{dt} = X(x) + uY(x)$ ,  $Y$  being associated to a specific  
 511 treatment.

512 This formulation leads to analyze the candidates as computing minimizers in the permanent  
 513 case using the Maximum Principle, stated in the introduction.

514 **OCP2 in the sampled-data control frame.** In the sampled-data control context, one  
 515 introduces a finite dimensional input-output mapping:  $E : (\lambda \in C \subset \mathbb{R}^m, t) \rightarrow \mathbb{R}$  where the  
 516 output is the time response  $x(t)$  of the dynamics. One denote by  $0 < t_1 < t_2 < \dots < t_p < t_f$   
 517 the sampling times.

518 In this formulation the problem is analyzed using a Model Predictive Control (MPC)  
519 technique:

- 520 • Start at time  $t = 0$  from  $x(0)$  and fix an horizon of length  $k$  in the sampling times.  
521 Compute the optimized (finite dimensional) control on this horizon.
- 522 • Replace  $t = 0$  by  $t = t_1$ , where on  $[0, t_1]$  the control is the restriction on this interval  
523 of the optimized control computed on the whole given horizon. This leads to replace  
524  $x(0)$  by  $x(t_1)$ . Then go to the first step.

525 In this algorithm, the dynamics is used to estimate over an horizon of length  $k$  the effect  
526 of the control action. It can be understood as a non causal approach to the problem, since  
527 the state-evolution is governed by the optimization algorithm, where the dynamics is used to  
528 make the prediction only.

529 In this article we shall concentrate on the permanent case. But convergence of the al-  
530 gorithm in the sampled-data case is related to the properties of the optimal control in the  
531 permanent case.

532 **2.5. Singular trajectories and time optimal control.** In this section, we make a brief  
533 recap of the properties of singular trajectories, crucial in our analysis, for full details see [5].

534 **Definition 2.17.** Let  $(X, Y)$  the pairs of  $C^\omega$  vector fields on  $M$  associated to control system:  
535  $\frac{dx}{dt} = X(x) + uY(x)$ . The feedback group  $G_f$  is the set of triples  $(\varphi, \alpha, \beta)$ , where  $\varphi$  is a local  
536 diffeomorphism and  $u = \alpha(x) + \beta(x)v$ ,  $\beta \neq 0$  is a feedback, the group structure being induced  
537 by the actions:

- 538 • local diffeomorphism  $\varphi : (X, Y) \rightarrow (\varphi * X, \varphi * Y)$ , where  $\varphi * Z = d\varphi(Z \circ \varphi^{-1})$  denotes  
539 the image of a vector field  $Z$ .
- 540 • feedback:  $u = \alpha(x) + \beta(x)v : (X, Y) \rightarrow (X + Y\alpha, Y\beta)$ .

541 The control system can be lifted on the cotangent bundle  $T^*M$  with symplectic structure  
542 defined by  $d\omega$ , where  $\omega = p dx$  is the Liouville form. The Hamiltonians  $H_X(z) = p \cdot X(x)$ ,  
543  $H_Y(z) = p \cdot Y(x)$ , where  $z = (x, p)$  are the symplectic coordinates, are the Hamiltonian lifts  
544 of  $X, Y$ . The system lift takes the form:  $\frac{dz}{dt} = H_X(z) + uH_Y(z)$ , and  $H_X(z) + uH_Y(z)$  is  
545 the pseudo or non maximized Hamiltonian. One can lift every local diffeomorphism  $\varphi$  into a  
546 Mattheiu symplectomorphism  $\varphi$  defined by:  $x = \varphi(y)$ ,  $p = q \frac{\partial \varphi^{-1}}{\partial x}$ , where  $p, q$  are row vectors.  
547 This induces an action of  $G_f$  on the pairs  $(H_X, H_Y)$ .

548 **Computations of singular extremals.** Relaxing the control bound to  $u \in \mathbb{R}$ , from the  
549 Maximum Principle the candidates as time minimizers are the so-called singular extremals  
550 control-trajectory pairs written shortly  $(z, u)$  solutions of the constrained Hamiltonian dynam-  
551 ics:  $\frac{dz}{dt}(t) = \mathbf{H}(z(t))$ ,  $\mathbf{H}$  being the Hamiltonian vector field, the constraints coming from the  
552 maximization condition:  $\frac{\partial H}{\partial u} = H_Y(z) = 0$ .

553 Hence they are solutions contained in the switching set  $\Sigma : H_Y(z(t)) = 0$ . They can  
554 be computed, deriving this equation with respect to  $t$ . Introducing the Poisson bracket of  
555  $H_{Z_1}(z) = p \cdot Z_1(x)$ ,  $H_{Z_2}(z) = p \cdot Z_2(x)$ , by  $\{H_{Z_1}, H_{Z_2}\}(z) = p \cdot [Z_1, Z_2](x)$ .

556 Hence we deduce:

$$557 \quad (2.16) \quad \begin{aligned} H_Y(z(t)) &= \{H_Y, H_X\}(z(t)) = 0, \\ \{\{H_Y, H_X\}, H_X\}(z(t)) + u(t)\{\{H_Y, H_X\}, H_Y\}(z(t)) &= 0. \end{aligned}$$

558 We introduce the following.

559 **Definition 2.18.** *The Generalized Legendre-Clebsch condition (GLC) along a singular ex-*  
560 *tremal  $(z(\cdot), u(\cdot))$  on  $(0, t_f]$  is given by:*

$$561 \quad \frac{\partial}{\partial u} \frac{d^2}{dt^2} \frac{\partial H}{\partial u}(z(t)) = \{\{H_Y, H_X\}, H_Y\}(z(t)) \neq 0,$$

562 *for every  $t$  in  $[0, t_f]$ . The switching surface is  $\Sigma : H_Y(z) = 0$  and we denote  $\Sigma'$  the sub-*  
563 *set:  $H_Y(z) = \{H_Y, H_X\}(z) = 0$ . Then outside the collinear set of  $Y(x)$  and  $[Y, X](x)$ , if*  
564  *$\{\{H_Y, H_X\}, H_Y\} \neq 0$ , the restriction of the symplectic form  $d\omega$  to  $\Sigma'$  defines a symplectic*  
565 *manifold  $(M', \omega|_{M'})$ .*

566 This gives the following.

567 **Proposition 2.19.** *Assume that the GLC-condition holds along  $(z(t), u(t))$  then the extremal*  
568 *is called of minimal order. We have:*

- 569 1. *The singular control  $u_s(\cdot)$  is the dynamic feedback:  $u_s(z) = -\frac{\{\{H_Y, H_X\}, H_X\}(z)}{\{\{H_Y, H_X\}, H_Y\}(z)}$ .*
  - 570 2. *Introduce the true Hamiltonian  $\mathbf{H}_s(z) := H_X(z) + u_s(z) H_Y(z)$ , the singular extremals*  
571 *of minimal order are the solutions of  $\mathbf{H}_s(z)$  contained in the set  $\Sigma'$ .*
- 572 *They are the solutions of  $\mathbf{H}_s(z)$  restricted to the symplectic manifold  $M'$ .*

573 Higher-order singular extremals can be determined at any order using the following algo-  
574 rithm. If

$$575 \quad \{\{H_Y, H_X\}, H_Y\}(z(t)) = 0,$$

576 Then additionally

$$577 \quad \{\{H_Y, H_X\}, H_X\}(z(t)) = 0.$$

578 Deriving both relations one gets:

$$579 \quad \begin{aligned} & \{\{\{H_Y, H_X\}, H_Y\}, H_X\}(z(t)) + u(t) \{\{\{H_Y, H_X\}, H_Y\}, H_Y\}(z(t)) = 0, \\ & \{\{\{H_Y, H_X\}, H_X\}, H_X\}(z(t)) + u(t) \{\{\{H_Y, H_X\}, H_X\}, H_Y\}(z(t)) = 0. \end{aligned}$$

580 If the control cannot be derived from the previous equations, we repeat the derivation  
581 procedure.

### 582 2.5.1. Singular extremals as feedback invariants.

583 **Definition 2.20.** *Let  $E, F$  be two vector spaces and  $G$  a group acting linearly on  $E, F$ . An*  
584 *homomorphism  $\chi: G \rightarrow \mathbb{R} \setminus \{0\}$  is called a character. Let  $\chi$  be a character, a semi-invariant*  
585 *of weight  $\chi$  is a map  $\lambda: E \rightarrow \mathbb{R}$  such that for all  $x \in E, g \in G, \lambda(g \cdot x) = \chi(g)\lambda(x)$ . It is called*  
586 *an invariant if  $\chi = 1$ . A map  $\lambda: E \rightarrow F$  is a semi-covariant of weight  $\chi$  if for all  $x \in E,$   
587  $g \in G, \lambda(g \cdot x) = \chi(g)g \cdot \lambda(x)$ . It is called a covariant if  $\chi = 1$ .*

588 **Theorem 2.21.** *Denote by  $\lambda_s$  the map which associates to the pair  $(X, Y)$  the Hamiltonian*  
589 *vector field  $\mathbf{H}_s$  restricted to  $\Sigma'$ . Then it is a covariant for the respective actions of the feedback*  
590 *group. In particular singular extremals are feedback invariants.*

591 *Proof.* Direct computations give us:

$$592 \quad D^{\phi*X, \phi*Y}(x) = \det\left(\frac{\partial\phi}{\partial x}\right) D^{X,Y}(\phi^{-1}(x)), \quad D^{X+\alpha Y, \beta Y}(x) = \beta^4 D^{X,Y}(x),$$

$$593 \quad D'^{\phi*X, \phi*Y}(x) = \det\left(\frac{\partial\phi}{\partial x}\right) D'^{X,Y}(\phi^{-1}(x)), \quad D'^{X+\alpha Y, \beta Y}(x) = \beta^3 \left( D'^{X,Y}(x) + \alpha D^{X,Y}(x) \right),$$

$$594 \quad D''^{\phi*X, \phi*Y}(x) = \det\left(\frac{\partial\phi}{\partial x}\right) D''^{X,Y}(\phi^{-1}(x)), \quad D''^{X+\alpha Y, \beta Y}(x) = \beta^2 D''^{X,Y}(x). \\ 595$$

596 Hence  $\lambda_s$  is a covariant. ■

### 597 2.5.2. High order Maximum Principle [19].

598 **Proposition 2.22.** *Assume  $p$  is oriented using the convention of the Maximum Principle*  
599 *along the singular extremal  $z(\cdot): H_X(z(t)) \geq 0$ . Then a necessary time optimality condition*  
600 *on  $]0, t_f]$  is given by*

$$601 \quad \frac{\partial}{\partial u} \frac{d^2}{dt^2} \frac{\partial H}{\partial u}(z(t)) = \{\{H_Y, H_X\}, H_Y\}(z(t)) \geq 0.$$

602

603 **Definition 2.23.** *The singular extremal  $z(t) = (x(t), p(t))$ ,  $t \in [0, t_f]$  is called strict if  $p$  is*  
604 *unique up to a scalar.*

605 **Corollary 2.24.** *Assume the strict case. Then the singular trajectories projections of sin-*  
606 *gular extremals of minimal order are stratified according to the following:*

- 607 • *Hyperbolic case:  $H_X \cdot \{\{H_Y, H_X\}, H_Y\}(z) > 0$ ,*
- 608 • *Elliptic case:  $H_X \cdot \{\{H_Y, H_X\}, H_Y\}(z) < 0$ ,*
- 609 • *Abnormal or exceptional case:  $H_X(z) = 0$ .*

### 610 2.5.3. Applications.

611 **2d-case.** Singular extremals satisfy  $H_Y = \{H_Y, H_X\} = 0$  so that singular trajecto-  
612 ries are located on the set  $\mathcal{S}: \det(Y(x), [Y, X](x)) = 0$ . Outside the collinear set  $\mathcal{C}: \det(Y(x), X(x)) = 0$  one can takes  $Y, X$  as a frame and writing  $[[Y, X], Y](x) = \alpha(x)X(x) +$   
613  $\beta(x)Y(x)$ . The singular control is given by:  $[[Y, X], X](x) + u_s(x) [[Y, X], Y](x) = 0$ . Hyper-  
614 bolic case corresponds to  $\alpha(x) > 0$  and elliptic case to  $\alpha(x) < 0$ .  
615

616 **3d-case.** The 3d-case is already a very rich situation to analyze the singular extremals  
617 and the program goes as follows.

618 Introduce the following determinants:

$$619 \quad (2.17) \quad \begin{aligned} D(x) &= \det(Y(x), [Y, X](x), [[Y, X], Y](x)), \\ D'(x) &= \det(Y(x), [Y, X](x), [[Y, X], X](x)), \\ D''(x) &= \det(Y(x), [Y, X](x), X(x)), \end{aligned}$$

620 and using the relations

$$621 \quad (2.18) \quad \begin{aligned} H_Y(z) &= \{H_Y, H_X\}(z) = 0, \\ \{\{H_Y, H_X\}, H_X\}(z) &+ u_s \{\{H_Y, H_X\}, H_Y\}(z) = 0, \end{aligned}$$

622 We can eliminate  $p$  and the singular control is given by the feedback:

$$623 \quad u_s(x) = -\frac{D'(x)}{D(x)}.$$

624 Defining the vector field:

$$625 \quad (2.19) \quad X_s(x) := X(x) + u_s(x)Y(x),$$

626 we have:

627 **Proposition 2.25.** *Singular trajectories of minimal order stratified the dynamics into:*

- 628 • *Hyperbolic arcs in  $DD'' > 0$ ,*
- 629 • *Elliptic arcs in  $DD'' < 0$ ,*
- 630 • *Exceptional or abnormal arcs in  $D'' = 0$ .*

631 **Proposition 2.26.** *For the action of the feedback group  $G_f = \{(\varphi, \alpha, \beta)\}$  reducing to  $\varphi$ -*  
 632 *changes of coordinates on vector fields  $X_s$ , the map  $\lambda_s : (X, Y) \rightarrow X_s$  is a covariant. Hence*  
 633 *this allows to generate feedback invariants using the dynamics (2.19).*

634 Equilibria of this dynamics split into two types:

- 635 • *If  $D \neq 0$ , they are given by the solutions of  $X_s(x) = 0$ .*
- 636 • *If  $D = 0$ , one can reparameterize the dynamics to get the vector field  $D(x)X(x) -$*   
 637  *$D'(x)Y(x)$  and additional (non isolated) singular points are located on  $D(x) = D'(x) =$*   
 638 *0.*

639 Exceptional trajectories are contained in the invariant set  $D''(x) = 0$  for the dynamics.

640 **2.6. Conclusion.** In the case of controlled Lotka–Volterra model, the first step in our study  
 641 is to classify the singular trajectories using the previous stratified dynamics into hyperbolic,  
 642 elliptic and exceptional trajectories.

### 643 3. Extremals classification and local time minimal syntheses near a terminal manifold 644 of codimension one.

645 **3.1. Introduction and definitions.** In this section we consider the time minimal con-  
 646 trol problem for the system:  $\frac{dx}{dt} = X(x) + uY(x)$ ,  $|u| \leq 1$ , with terminal manifold  $N$  of  
 647 codimension one. We denote by  $H(z, u) = H_X(z) + uH_Y(z)$  the pseudo-Hamiltonian and  
 648  $M(z) = \max_{|u| \leq 1} H(z, u)$  the true or maximized Hamiltonian. In this setting the control  
 649 domain is taken as  $U = [-1, +1]$ , where extreme control value is either  $X(x) - Y(x)$  or  
 650  $X(x) + Y(x)$  that is the maximal dosing regimen.

651 **Definition 3.1.** *The extremals are concatenation of regular extremals for which almost eve-*  
 652 *rywhere*

$$653 \quad u(t) = \text{sign}H_Y(z(t))$$

654 *and singular extremals if*

$$655 \quad H_Y(z(t)) = 0$$

656 *holds identically.*

657 *An extremal is called exceptional if the maximized Hamiltonian is such that  $M(z) = 0$ .*  
 658 *A BC-extremal is an extremal satisfying the transversality condition. A switching time is*

659 *an instant such that the extremal control is discontinuous. A bang-bang extremal is a regular*  
660 *extremal with a finite number of switches.*

661 Since the control domain is  $U = [-1, +1]$ , a singular extremal is called strictly feasible  
662 (admissible) if  $|u_s| < 1$  and saturating at time  $t$  if  $|u_s(t)| = 1$ . A regular or singular extremal is  
663 called strict if  $p$  is unique up to a scalar. In the strict case singular extremals can be classified  
664 into hyperbolic, elliptic and exceptional extremals. We denote by  $\sigma_+, \sigma_-$  bang arcs associated  
665 respectively to  $u = +1$  or  $u = -1$ ,  $\sigma_s$  is a singular arc associated to  $u_s$ . We denote by  $\sigma_1\sigma_2$   
666 an arc  $\sigma_1$  followed by  $\sigma_2$ .

667 **Definition 3.2.** *Taking an open set  $V$  of  $M$ , the problem is called geodesically complete on*  
668  *$V$  if for each  $x_0, x_1 \in V$  there exists a time minimizing geodesic in  $V$  joining  $x_0$  to  $x_1$ . Fixing*  
669 *the target to  $N (=N \cap V)$ , a time minimal synthesis is a discontinuous feedback  $x \rightarrow u^*(x)$  so*  
670 *that the solutions of  $\frac{dx}{dt} = X(x) + u^*(x)Y(x)$  are well defined and  $u^*(x)$  is the optimal feedback*  
671 *solution to steer  $x$  to the target in minimum time.*

672 **3.2. Small time classification of regular extremals.** In this section we recall some basic  
673 properties of regular extremals, see [17] but also [18] for the analysis of the Fuller phenomenon  
674 as a recommended reading.

675 The surface  $\Sigma : H_Y(z) = 0$  is called the switching surface and we denote by  $\Sigma'$  the set  
676  $H_Y(z) = \{H_Y, H_X\}(z) = 0$ . Let  $z(\cdot) = (x(\cdot), p(\cdot))$  be a reference extremal on  $[0, t_f]$ . We note  
677  $\Phi(t) := H_Y(z(t))$  the switching function coding the switching times.

678 Deriving twice  $\Phi$  with respect to times, one gets:

$$\begin{aligned} 679 \quad & \frac{d\Phi}{dt}(t) = \{H_Y, H_X\}(z(t)), \\ 680 \quad (3.1) \quad & \frac{d^2\Phi}{dt^2}(t) = \{\{H_Y, H_X\}, H_X\}(z(t)) + u(t)\{\{H_Y, H_X\}, H_Y\}(z(t)). \end{aligned}$$

682 **Definition 3.3.** *The time  $t$  is called an ordinary switching time if  $\Phi(t) = 0$  and  $\frac{d\Phi}{dt}(t) \neq$*   
683 *0.*

684 **Lemma 3.4.** *Assume  $t$  be an ordinary switching time, then near  $z(t)$  every extremal projects*  
685 *onto:*

- 686 •  $\sigma_+\sigma_-$  if  $\frac{d\Phi}{dt}(t) > 0$ ,
- 687 •  $\sigma_-\sigma_+$  if  $\frac{d\Phi}{dt}(t) < 0$ .

688 **Definition 3.5.** *Let  $z(\cdot)$  be a bang extremal on  $[0, t_f]$  with  $u = \varepsilon \in \{-1, +1\}$ . We note*  
689 *by  $\frac{d^2\Phi_\varepsilon}{dt^2}$  the expression (3.1) in which  $u \equiv \varepsilon$ . The point  $z(t)$  is called a fold point if  $\Phi(t) =$*   
690  *$\frac{d\Phi}{dt}(t) = 0$  and  $\frac{d^2\Phi_\varepsilon}{dt^2}(t) \neq 0$ . Assume that  $\Sigma'$  is a regular surface of codimension two. We have*  
691 *three cases:*

- 692 • *Parabolic case:*  $\frac{d^2\Phi_+}{dt^2}(t) \frac{d^2\Phi_-}{dt^2}(t) > 0$ ,
- 693 • *Hyperbolic case:*  $\frac{d^2\Phi_+}{dt^2}(t) > 0$  and  $\frac{d^2\Phi_-}{dt^2}(t) < 0$ ,
- 694 • *Elliptic case:*  $\frac{d^2\Phi_+}{dt^2}(t) < 0$  and  $\frac{d^2\Phi_-}{dt^2}(t) > 0$ .

695 This leads to:

696 **Proposition 3.6.** *In the neighborhood of a fold point every extremal projects onto:*

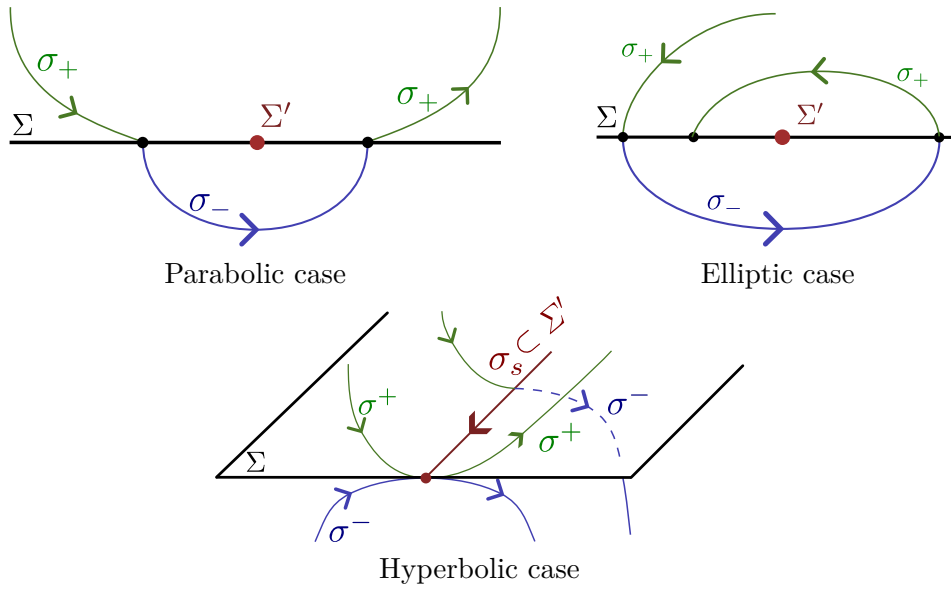


Figure 6. Fold case.

- 697 • In the parabolic case:  $\sigma_+\sigma_-\sigma_+$  or  $\sigma_-\sigma_+\sigma_-$ .  
 698 • In the hyperbolic case  $\sigma_\pm\sigma_s\sigma_\pm$ .  
 699 • In the elliptic case, every extremal is bang-bang but the number of switches is not  
 700 uniformly bounded.

701 This is illustrated in Fig. 6.

702 Note that in the elliptic case, there is a foliation by a cylinder of the regular extremal dynam-  
 703 ics and the number of switches is related to the distance to  $\Sigma'$  versus the Fuller phenomenon,  
 704 where the sequence of regular arcs is not bang-bang and converges to  $\Sigma'$ .

705 **Application to the 3d-case.** Consider the case where  $Y$ ,  $[Y, X]$  and  $X$  form a frame i.e.

$$706 \quad D'' = \det(Y, [Y, X], X)$$

707 is not vanishing. The problem is strict for singular extremals since  $Y$  and  $[Y, X]$  are indepen-  
 708 dent. Hyperbolic trajectories are small time minimizing trajectories, while elliptic trajectories  
 709 are small time maximizing, this up to the first conjugate time  $t_{1c}$  computed in [6] if they are  
 710 strictly admissible (even in the limit case with no constraints on the control). In the parabolic  
 711 case, they can be absent or not feasible. Consider the separating case where the singular arc  
 712 is exceptional (abnormal) and assume that it is strictly admissible (for instance relaxing the  
 713 control bound to the whole  $\mathbb{R}$ ). In this case using again [6] an exceptional arc  $\sigma_s$  is time  
 714 minimizing and time maximizing, up to the first conjugate time  $t_{1c}$ . Such a point is absent in  
 715 the 3d-case. Such an arc can be lifted into two extremals  $(\pm p, \sigma_s)$  and it corresponds either  
 716 to an hyperbolic or elliptic situation in  $\Sigma'$ . One contribution of [20] is to analyze the time  
 717 minimal syntheses near the terminal manifold in this situation.

718 **3.3. General concepts of regular synthesis with a terminal manifold of codimension**  
 719 **one.** Take a triple  $(X, Y, N)$  and let  $x_0 \in N$  which can be identified to 0 while  $N$  is the plane

720  $x_1 = 0$ . Let  $U$  be a neighborhood of 0, which divides the space into neighborhoods  $V$  and  $W$   
721 contained respectively in  $x_1 < 0$  and  $x_1 > 0$  so that  $U = V \cup W$ . The problem is to compute  
722 the time minimal regular synthesis to steer each point of  $U$  to the terminal manifold.

723  $N$  can be taken locally as  $f^{-1}(0)$  where  $f$  is a submersion from  $U$  into a neighborhood of  
724 0 in  $\mathbb{R}$ . The set of triples  $(X, Y, f)$  is endowed with the  $C^\infty$ -Whitney topology and we denote  
725 by  $j^k X$  (resp.  $j^k Y, j^k f$ ) the  $k$ -jet of  $X$  (resp.  $Y, f$ ) obtained by taking the Taylor expansion  
726 at  $x_0 = 0$  up to order  $k$ . We say that the triple  $(X, Y, f)$  has at 0 a singularity of codimension  
727  $i$  if  $(j^k X, j^k Y, j^k f)$  belongs to a semi-algebraic set of codimension  $i$  in the jet space.

728 Our aim is to make a short presentation of the results of [7, 20, 8] to classify local syn-  
729 theses up to an homeomorphisms preserving the target  $N$  for all cases of codimension  $\leq 2$ , by  
730 considering two cases occurring in the application that we introduce next.

731 **Definition 3.7.** *When  $Y$  is everywhere tangent to the target  $N$ , the control  $u$  is indirect and  
732 this case is called the flat case. In the non flat case, the action of the control is direct and the  
733 set of points where  $Y$  is tangent to  $N$  is of codimension  $\geq 1$ .*

734 **Stratified synthesis.** Our aim is to describe the local time minimal synthesis in a neigh-  
735 borhood of  $N$  by estimating at any order the different strata. Actually, the optimal control  
736 feedback  $u^*(x)$  is not always defined on the whole subset  $V$  of  $U$  in the domain  $x_1 < 0$ , since  
737 for some  $x \in V$  the target  $N$  is not accessible.

738 In fact we can reduce our study to two cases:

- 739 • The case when the convex cone  $C$  generated by  $\{X \pm Y\}$  is strict and the set of  
740 admissible directions points towards the space  $x_1 > 0$ .
- 741 • The exceptional case where the set of admissible directions are tangent to the terminal  
742 manifold.

743 This leads to introduce the exceptional locus in the construction of the stratification of  $N$ .

744 **Definition 3.8.** *Let  $n$  be the normal to  $N$  oriented toward  $x_1 > 0$ . The exceptional locus  
745 (restricted to  $N$ )  $\mathcal{E}$  is the set of points of  $N$  such that:  $n(x) \cdot Y(x) = n(x) \cdot X(x) = 0$ .*

746 The second part of the stratification amounts to introduce the singular locus.

747 **Definition 3.9.** *The singular locus (restricted to  $N$ )  $\mathcal{S}$  is the set of points of  $N$  such that:  
748  $n(x) \cdot Y(x) = n(x) \cdot [Y, X](x) = 0$ .*

749 **Definition 3.10.** *A stratified synthesis amounts to find a partition of  $V$  (or a partition of  
750  $U$  in the exceptional case) into  $V^+$  (resp.  $U^+$ ) where  $u^*(x) = +1$  and  $V^-$  (resp.  $U^-$ ) where  
751  $u^*(x) = -1$  and a stratified surface separating  $V^+$  and  $V^-$  (resp.  $U^+, U^-$ ) with three kind of  
752 strata:*

- 753 • *Switching locus.* It is the closure of the set of ordinary switching points and forming  
754 the set  $W_{\#} (\# \in \{-1, +1\})$  where  $W_+$  is associated to  $\sigma_+ \sigma_-$  and  $W_-$  to  $\sigma_- \sigma_+$ .
- 755 • *Cut locus.* Let  $\sigma : [t_f, 0] \rightarrow M$  be a minimizing curve, integrating backwards from  $N$  so  
756 that  $\sigma(0) \in N, t_f < 0$ . The cut-locus is the closure of the set of points where optimality  
757 is lost. It is denoted  $C$  and contains the splitting locus  $L$  where the optimal feedback  
758 is not unique.
- 759 • *Switching singular locus.* It is foliated by optimal singular arcs and is denoted  $\Gamma_s$ .  
760 Recall that if  $u_s \in ]-1, +1[$  the singular arc is strictly feasible but it can be saturated  
761 if  $u_s^*(x) \in \{-1, +1\}$ .



762 To estimate the different strata we use semi-normal forms restricting the action of the feedback  
 763 group to local diffeomorphisms  $\varphi$  preserving 0 and feedbacks  $u \rightarrow -u$  so that  $\sigma_+$  and  $\sigma_-$   
 764 be inverted in the classification.

765 This leads to the following important concept.

766 **Definition 3.11.** Let  $x_0 \in N$  which can be identified to 0. A Whitney chart is a pair  $(U, \varphi)$   
 767 where  $U$  is a neighborhood and  $\varphi$  a system of coordinates  $(x, y, y_1, \dots, y_{n-3}, z)$  such that:

- 768 1.  $Y = \frac{\partial}{\partial z}$ ,
- 769 2.  $N \sim (\frac{1}{2}ks^2, w, w_1, \dots, w_{n-3}, s)$ ,
- 770 3. The time minimal synthesis in  $U$  is  $C^0$  described with foliations by  $2d$  or  $3d$  syntheses  
 771 with triples  $(X^b(x), Y^b(x), N^b(x))$ ,  $x \in \mathbb{R}^2$  or  $\mathbb{R}^3$  where the  $3d$ -cases occur only in the  
 772 exceptional case if  $n \geq 3$ .

773 Moreover in  $U$ , one can construct the stratified optimal synthesis, where each strata can be  
 774 estimated at any order.

775 **Definition 3.12.** In the previous construction the restriction from  $n$  to  $n - 1$  decomposing  
 776  $x = (x', \lambda)$ ,  $N = \cup_\lambda N'(\lambda)$ , where  $\lambda \in \mathbb{R}$ , is a parameter is called a one dimensional unfolding.

777 The previous definitions will be clarified in the examples we present next.

778 **Main points of the geometric and analytic construction.** Let  $n(x)$  be the normal to  $N$ ,  
 779 oriented towards  $x_1 > 0$ . We denote by  $N^\perp$  the symplectic lift of  $N$ :  $N^\perp = \{(x, p); x \in$   
 780  $N, p = n(x)\}$ . The stratification of  $N$  by the conditions  $n(x) \cdot Y(x) = 0$  and  $n(x) \cdot [Y, X](x) = 0$   
 781 selects ordinary switching points or fold points classified in Proposition 3.6.

- 782 • Ordinary switching locus  $W$ : One can restrict for simplicity the analysis to the  $2d$  non  
 783 flat case. Let  $x_0 \in N$  such that  $n \cdot Y(x_0) = 0$  and both  $n \cdot X(x_0)$  and  $n \cdot [Y, X](x_0)$  non  
 784 zero. This leads to compute a switching locus  $K$  terminating at  $x_0$ . Such a switching  
 785 locus is part of  $W$  if the corresponding bang-bang extremal crosses the switching locus  
 786 versus reflects on the switching locus. This leads to estimate the slope of  $W$ .
- 787 • Singular locus  $I_s$ : Consider again the non flat case. Let  $x_0$  such that  $(n(x_0), x_0)$  is a  
 788 fold point hence  $n \cdot Y(x_0) = n \cdot [Y, X](x_0) = 0$ . Moreover assume that  $n \cdot X(x_0) \neq 0$   
 789 (non exceptional case) and that the singular arc  $\sigma_s$  terminating at  $x_0$  is strictly ad-  
 790 missible, such an arc being small time minimizing for the problem with fixed extrem-  
 791 ities (assuming the condition  $n \cdot [[Y, X], Y](x_0) \neq 0$ ). One can choose coordinates  
 792  $(x, y)$  such that  $Y = \frac{\partial}{\partial y}$  and  $N$  is given by  $(\frac{1}{2}ks^2, s)$ , while  $X$  can be normalized to  
 793  $(1 - y^2 X_1(x, y)) \frac{\partial}{\partial x} + (u - u_s(x) + y X_2(x, y)) \frac{\partial}{\partial y}$ , where  $\sigma_s$  is identified to  $t \rightarrow (t, 0)$   
 794 and  $u_s$  is the singular control. Note that such normalizations were introduced in [6]  
 795 in a more general context.

796 In order to decide about optimality one can compare by direct computations in the normalized  
 797 coordinates the curvature of the boundary of the accessibility set along the singular arc with  
 798 the curvature of  $N$  given by  $k$ . This leads to the cases described in Fig. 7. In the first case,  
 799 there exists a cut locus  $C$  with slope  $-u_s(0)$ .

800 The  $3d$ -case with an hyperbolic point terminating at  $x_0 = 0$  can be obtained by construct-  
 801 ing a semi-normal form in coordinates  $(x, y, z)$  by unfolding the case  $k > 0$  and this leads to  
 802 Fig. 8.

803 The central picture corresponds to Fig. 7 with  $k > 0$ . Left we have represented a Whitney

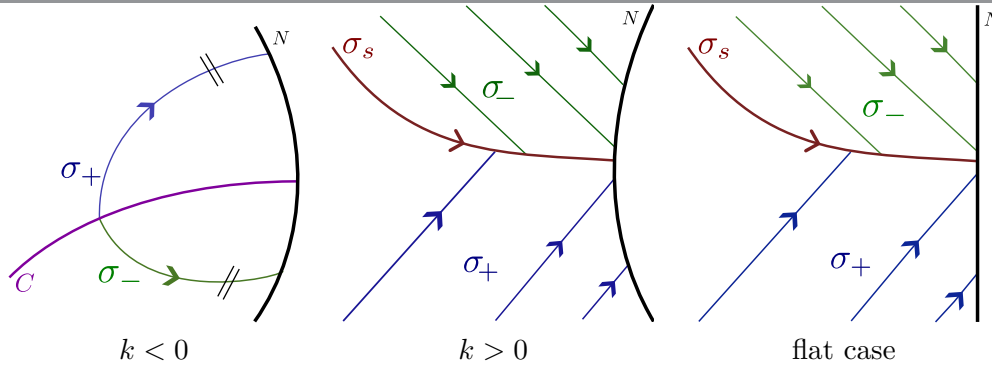


Figure 7. Optimal synthesis in the 2d-hyperbolic case.

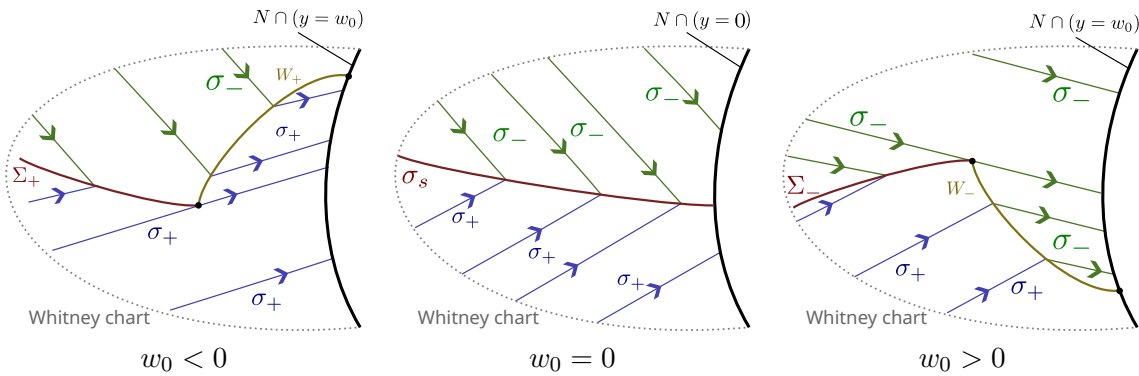


Figure 8. Unfolding in the 2d-hyperbolic case with parameter  $w_0$ .

804 neighborhood containing the optimal switching locus  $W_+$  intersecting the hyperbolic singular  
805 locus. They can be easily computed using the stratification of  $N^\perp$  into sectors with ordinary  
806 switching points and fold points while the singular arc terminating at 0 corresponds to an  
807 hyperbolic fold point.

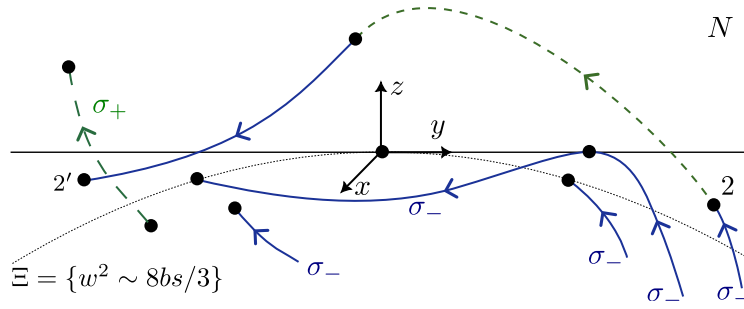
808 This gives the procedure to compute the optimal syntheses using [7] in the non exceptional  
809 case as unfolding of  $2d$ -cases. The procedure fails in the exceptional case where  $n \geq 3$ , where  
810 the following example from [20] shows why it cannot be reduced to  $2d$ -foliations.

811 **An exceptional  $3d$ -case not  $2d$ -reducible.** One take a flat case so that  $N$  can be identified  
812 to  $(0, w, s)$  in  $(x, y, z)$  coordinates and  $Y = \frac{\partial}{\partial z}$ . In the construction the main point is to take  
813 a bang arc  $\sigma_-$  which is optimal in the domain  $x \geq 0$  with a contact of order 3 at 0 with the  
814 surface  $N$ . Hence this gives birth in the domain  $x > 0$  to arcs  $\sigma_-$  intersecting three times the  
815 target  $N$ , thanks to contact analysis.

816 Such a situation occurs for instance for the model which in fine is a  $C^0$ -normal form  
817 describing the situation:

$$818 \quad \frac{dx}{dt} = z, \quad \frac{dy}{dt} = b, \quad \frac{dz}{dt} = 1 + u + y,$$

819 the target  $N$  given by  $(w, s) \rightarrow (0, w, s)$ .



**Figure 9.** Synthesis exceptional case when  $b < 0$ .  $\sigma_-$  has a contact of order 3 with  $N$ .

820 One consider the situation with  $b > 0$  where each point of the neighborhood  $U$  can be  
821 steered to the target.

822 In the domain  $x < 0$ , every time optimal trajectory is of the form  $\sigma_+$  and the contact of  
823  $\sigma_+$  at 0 with  $N$  is of order 2.

824 Optimal arcs  $\sigma_-(t)$  included in  $x \geq 0$  are satisfying:

$$825 \quad (3.2) \quad x_-(t) = t(s + wt/2 + bt^2/6 + \dots),$$

826 where the weight of  $s$  is one and the weight of  $w$  is two, neglected having weights greater than  
827 3.

828 The curve  $\Xi$  is the set of points  $(0, w, s)$   $N$  of such that:  $\frac{x_-(t)}{t} = \frac{d}{dt} \left( \frac{x_-(t)}{t} \right)$  have a common  
829 zero and is given using (3.2) by:  $w^2 \sim \frac{8bs}{3}$ .

830 The optimal synthesis is represented on Fig. 9. Optimal arcs  $\sigma_-$  in the domain  $x \geq 0$  are  
831 cutting twice  $N$  in the subsets of the target denoted 2 and 2' and have two subarcs which are  
832 optimal in  $x > 0$ , but the subarc in  $x < 0$  is not optimal.

833 **4. Computations and preliminary results on the Controlled Lotka–Volterra model.** The  
834 aim of this section is to present the geometric study of the controlled Lotka–Volterra model:

$$835 \quad \frac{dx}{dt}(t) = X(x(t)) + u(t)Y(x(t)),$$

836 with  $X(x) = (\text{diag } x)(Ax + r)$  and  $Y(x) = (\text{diag } x)\epsilon$ , where  $x = (x_1, \dots, x_n)^\top$  is the population  
837 species,  $x_1$  represents the infected species and  $\epsilon = (\epsilon_1, \dots, \epsilon_n)^\top$ . The system can be written  
838 in ln-coordinates:  $y = \ln x$  and it takes the form:

$$839 \quad (4.1) \quad \frac{dy}{dt} = (Ae^y + r) + u\epsilon.$$

840 **4.1. Equilibria and the collinear set.** The collinear set  $\mathcal{C}$  is one of the main feedback in-  
841 variant, related to computations of the free equilibria of the system for the hierarchy of models  
842 with no treatment  $u = 0$  or maximal dosing treatment  $u = 1$ , but also for all intermediate  
843 dosing.

844 This set is a one dimensional algebraic variety and is the projection of the set:

$$845 \quad (4.2) \quad \{(x_e, u_e) \in \mathbf{R}^n \times \mathbb{R}; \exists u_e, X(x_e) + u_e Y(x_e) = 0\}.$$

846 At such a point introduce the Jacobian matrix:

$$847 \quad J(x_e, u_e) = \frac{\partial}{\partial x|_{(x_e, u_e)}} (X(x) + u_e Y(x)).$$

848 The following dimensionless coordinates are useful in the computations. The system can be  
849 written:

$$850 \quad (4.3) \quad \frac{dx_i}{dt} = x_i r_i - x_i \sum_{j=1}^n a_{ij} x_j + u x_i \varepsilon_i$$

851 and denotes by  $x^*$  the free equilibrium given by:

$$852 \quad r_i - \sum_{j=1}^n a_{ij} x_j^* = 0, i = 1, \dots, n.$$

853 One sets  $v_i = \frac{x_i}{x_i^*}$  so that the dynamics takes the form:

$$854 \quad \frac{dv_i}{dt} = v_i r_i - v_i \sum_{j=1}^n a_{ij} x_j^* v_j + u v_i \varepsilon_i,$$

855 and denoting  $a_{ij}^* = a_{ij} x_j^*$ , one has  $r_i = \sum_{j=1}^n a_{ij}^*$  since the persistent equilibrium is normalized  
856 to  $\Omega = (1, \dots, 1)$ .

857 If we set:  $x_i = v_i - 1$ , the dynamics is given by:

$$858 \quad \frac{dx_i}{dt} = (x_i + 1) \sum_{j=1}^n a_{ij}^* - (x_i + 1) \sum_{j=1}^n a_{ij}^* (x_j + 1) + u (x_i + 1) \varepsilon_i.$$

859 Hence we have:

860 **Proposition 4.1.** *In the dimensionless coordinates the controlled Lotka–Volterra model is*  
861 *given:*

$$862 \quad \frac{dx_i}{dt} = -(x_i + 1) \sum_{j=1}^n a_{ij} x_j + u (x_i + 1) \varepsilon_i,$$

863 so that  $\Omega$  is identified to 0 and the Jacobian matrix at  $(0, 0)$  is  $A = (a_{ij})$ .

864 Similarly one can make the same computation for the forced persistent equilibrium  $\Omega'$   
865 with maximal dosing  $u_e = 1$ . Hence one can associate two affine coordinates systems so that

- 866 • at  $u_e = 0$ , the system is given by:  $-(x + 1)A_0x$ ,
- 867 • at  $u_e = 1$ , the system is given by:  $-(x + 1)A_1x$ .

868 This validates the accessibility discussion of section 2.3, where the separatrices are approxi-  
869 mated by lines. In the Volterra model they can be exactly computed thanks to integrability.  
870 The controllability conditions discussed in 2.3 can be generalized to this case.

871 **4.1.1. Computations in the 2d–case.** We consider the regular 2-dimensional dynamics  
872 given in Proposition 4.1.

873 *Collinear set and classification of equilibria.* The collinear set is one of the main feedback  
 874 invariant related to the computations of free equilibria with no treatment  $u = 0$  and forced  
 875 equilibria with maximal dosing  $u = 1$ .

876 The collinear set is given by the determinantal variety:  $\det(X(x), Y(x)) = 0$ :

$$877 \quad (x_1 + 1)(x_2 + 1)(x_1\kappa_2 - x_2\kappa_1) = 0,$$

878 where  $\kappa_1 = \varepsilon_1 a_{22} - \varepsilon_2 a_{12}$  and  $\kappa_2 = \varepsilon_2 a_{11} - \varepsilon_1 a_{21}$ .

879 Alternatively, it can be viewed as the one dimensional algebraic variety projection of the  
 880 set

$$881 \quad \{(x, u) \in \mathbb{R}^2 \times \mathbb{R}, \exists u \text{ such that } X(x) + uY(x) = 0\}.$$

882 The condition  $u_e \in [0, 1]$  selects a segment of persistent equilibria located on the line:

$$883 \quad \mathcal{C} := \left\{ x_2 = u_e(x_1) \frac{\kappa_2}{\det A}, u_e(x_1) = x_1 \frac{\det A}{\kappa_1} \in [0, 1] \right\}.$$

884 In particular, introduce  $x_e = (x_{1e}, x_{2e}) \in \mathcal{C}$ , the origin  $x_e = 0$  is associated to the control  
 885  $u_e = 0$  while  $x_e = (\kappa_1/\det A, \kappa_2/\det A)$  is associated to the control  $u_e = 1$ .

886 For  $u_e \in [0, 1]$ , define the Jacobian matrix

$$887 \quad J(x_e, u_e) = \frac{\partial}{\partial x} (X(x) + u_e Y(x))|_{x=x_e},$$

888 we have:

889 **Lemma 4.2.** *Let  $x_e \in \mathcal{C}$  associated to the control  $u_e$ , the spectrum of  $J$  is:*

$$890 \quad \text{spec}(J(x_e, u_e)) = \left\{ \left( k_1(x_e) \pm \sqrt{k_1(x_e)^2 + k_1'(x_e)} \right) / 2 \right\},$$

891 where  $k_1(x_e) = x_{1e} a_{22} \kappa_2 / \kappa_1 - a_{22} - a_{11}(x_{1e} + 1)$  and  $k_1'(x_e) = -4 \det A (x_{1e} + 1) (1 + x_{1e} \kappa_2 / \kappa_1)$ .

## 892 **4.2. Computation of the collinearity locus and properties-Construction of a normal** 893 **form in $\ln$ -coordinates.**

894 **Computations about  $\mathcal{C}$  in the  $n$ -dimensional case.** We have the following algorithm,  
 895 taking the system represented in the  $x$ -coordinates.

- 896 • Step 1. The collinear set is the projection of the algebraic curve defined by: There  
 897 exists  $u_e$  constant such that  $X(x_e) = -u_e Y(x_e)$ . This gives  $n$ -equations depending  
 898 upon  $(n + 1)$  variables  $(x_e, u_e)$ .
- 899 • Step 2. Take such a pair  $(x_e, u_e)$  so that  $x_e$  is a forced equilibrium for  $u = u_e$  and they  
 900 form a set with extreme points associated to  $u_e = 0$  and  $u_e = 1$ , when restricting to  
 901  $u_e \in [0, 1]$ .

902 The linear dynamics at the point  $x_e$  is characterized by the Jacobian matrix:

$$903 \quad J(x_e, u_e) = \frac{\partial}{\partial x|_{(x_e, u_e)}} (X(x) + u_e Y(x))$$

904 the spectrum being  $\sigma(J) = (\lambda_1, \dots, \lambda_n)$  with associated generalized eigenspaces  $E_{\lambda_i}, i =$   
905  $1, \dots, n$ .

906 The linear stability is determined by this spectrum, thanks to Lyapunov linear stability  
907 theory.

908 • Step 3. From control point of view we have three cases:

- 909 1.  $u_e \notin [0, 1]$ :  $u_e$  is not feasible,
- 910 2.  $u_e \in ]0, 1[$ :  $u_e$  is strictly feasible,
- 911 3.  $u_e = \pm 1$ :  $u_e$  being feasible but saturating.

912 One can discuss the linear controllability of the pair  $(J(x_e, u_e), b)$  where  $b = Y(x_e)$ .

- 913 • Kalman condition:  $\text{rank} [b, Jb, \dots, J^{n-1}b] = n$  and the singular point  $x_e$  is regular. If  
914 the control  $u_e$  is strictly feasible local controllability holds [21].
- 915 • If  $\text{rank} [b, Jb, \dots, J^{n-1}b] = n - k < n$ , then the singular point is a singular trajectory  
916 (reduced to a point) associated to  $u_e$  and  $k$  is the codimension of the singularity.

917 From linear controllability theory, one can construct a normal form, at a given equilibrium  
918 pair  $(x_e, u_e)$ .

919 We take ln-coordinates so that  $X(x)$  takes the form  $X^b(y) = (Ae^y + r)$  and the controlled  
920 vector field  $Y(x)$  becomes the constant vector field  $Y^b = \epsilon$ .

921 Let  $(y_e, u_e)$  be the selected forced equilibrium in the ln-coordinates and let  $z = y - y_e$ ,  
922  $v = (u - u_e)$  so that the system takes the form:

$$923 \quad \frac{dz}{dt} = J(z) + R(z) + v\epsilon,$$

924 where  $J$  is the Jacobian matrix at  $(y_e, u_e)$ .

925 One can find coordinates such that the linear dynamics decomposes into

$$926 \quad \begin{aligned} \frac{dz_1}{dt} &= J_{11}z_1 + J_{12}z_2 + v\epsilon \\ \frac{dz_2}{dt} &= J_{21}z_2 \end{aligned}$$

927 where the restriction to the controllable space  $z_2 = 0$  is given by the dynamics:

$$928 \quad \frac{dz_1}{dt} = J_{11}z_1 + v\epsilon.$$

929 The pair  $(J_{11}, \epsilon)$  can be set in Brunovsky canonical form:

$$930 \quad J_{11} = \left[ \begin{array}{c|cccc} 0 & & & & \\ \hline & \text{Id}_{n-k-1} & & & \\ \hline -a_1 & -a_2 & \dots & -a_{n-k} & \end{array} \right], \quad \epsilon = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

931 where the coefficients  $(a_1, \dots, a_{n-k})$  are the coefficients of the characteristic polynomial of  
932  $J_{11}$ .

933 Finally this leads to the construction of a normal form:

$$934 \quad \frac{dz}{dt} = J(z) + R(z) + v\epsilon,$$

935 where the pair  $(J(x_e), u_e), \epsilon$  is in linear canonical form. Note in particular that the sensitivity  
 936 vector  $\epsilon$  is normalized to  $(0, \dots, 0, 1)^\top$  and  $R(z)$  is the non linear part of the dynamics, related  
 937 to singular trajectories computations.

938 **4.3. Singular extremals.** Lie brackets can be computed in the original coordinates but  
 939 the computations are simpler in ln-coordinates, since the control vector field  $Y^b$  is constant.  
 940 This comes from the following.

941 **Proposition 4.3.**

- 942 1. If  $X = (\text{diag}x)X_1(x)$ ,  $Y = (\text{diag}x)Y_1(x)$ , where  $X_1, Y_1$  are polynomic then the it-  
 943 erated Lie brackets are in the same category of polynomic vector fields of the form  
 944  $(\text{diag}x)P(x)$ .
- 945 2. Let  $\varphi$  be the diffeomorphism:  $x = e^y$ . Denote by  $X^b, Y^b$  the images of  $X, Y$  by this  
 946 diffeomorphism. Then by invariance of the Lie bracket:  $[X^b, Y^b](y) = d\varphi^{-1}[X, Y](e^y)$ .

947 **4.3.1. Computations in the 2d-case.**

- 948 •  $X^b = \left( r_1 + \sum_{j=1,2} a_{1j}e^{x_j}, r_2 + \sum_{j=1,2} a_{2j}e^{x_j} \right)$ ,
- 949 •  $Y^b = (\varepsilon_1, \varepsilon_2)$ ,
- 950 •  $[X^b, Y^b] = \left( \sum_{j=1,2} \varepsilon_j a_{1j}e^{x_j}, \sum_{j=1,2} \varepsilon_j a_{2j}e^{x_j} \right)$ .

951 Hence the singular locus:

$$952 \quad \mathcal{S} : \det([X, Y](x), Y(x)) = 0$$

953 is given by:

$$954 \quad x_1 x_2 [x_1 (a_{11} \varepsilon_1 \varepsilon_2 - \varepsilon_1^2 a_{21}) + x_2 (\varepsilon_2^2 a_{12} - \varepsilon_1 \varepsilon_2 a_{22})] = 0$$

955 which stratifies into  $x_1 = x_2 = 0$  and a permanent straight-line  $L$ .

956 Moreover:

- 957 •  $[[X^b, Y^b], Y^b] = \left( \sum_{j=1,2} \varepsilon_j^2 a_{1j}e^{x_j}, \sum_{j=1,2} \varepsilon_j^2 a_{2j}e^{x_j} \right)$ .

958 Outside the collinear set  $\mathcal{C}$ ,  $X, Y$  form a frame and writing:

$$959 \quad [[Y, X], X](x) = \alpha(x)X(x) + \beta(x)Y(x),$$

960 so that:

- 961 • Hyperbolic (feasible) subarcs are such that  $\alpha(x) > 0$ ,
- 962 • Elliptic subarcs are such that  $\alpha(x) < 0$ .
- 963 • At the persistent point intersection of  $\mathcal{S}$  and  $\mathcal{C}$ , one gets an exceptional point.

964 **4.3.2. Computations in the 3d-case.** The Lie brackets computations are as before, ex-  
 965 cept that the index  $j$  goes from 1 to 3. So that

$$966 \quad D = \det(Y^b, [Y^b, X^b], [[Y^b, X^b], Y^b])$$

967 is homogeneous and quadratic with respect to the variable  $e^{x_i}$ .

968 Moreover the exceptional locus is given by the relation:

$$969 \quad D'' = \det(Y^b, [Y, X]^b, X^b) = 0,$$

970 this set being a quadratic non homogeneous variety with respect to the variables  $e^{x_i}$ .

971 Note that the computation of the Lie bracket:  $[[Y^b, X^b], X^b]$  is more complex and formal  
972 computations are necessary.

973 This fixes the limit of the computational complexity in the  $n$ -dimensional case.

#### 974 4.4. Applications.

975 **4.4.1. A 2d-working case.** In this section, the general techniques from [7, 20] will be  
976 applied to analyze a 2d-case study which occurs in the Lotka–Volterra model, see also [10] as  
977 a complementary study for the fixed end point problem.

978 *Lie brackets and feedback invariants.* The first step is to compute the collinear set  $\mathcal{C}$  defined  
979 by

$$980 \det(X(x), Y(x)) = 0,$$

981 which takes the form:

$$982 (4.4) \quad x_1 x_2 (\varepsilon_2 (x_1 a_{11} + x_2 a_{12} + r_1)) - \varepsilon_1 (x_1 a_{21} + x_2 a_{22} + r_2) = 0,$$

983 while the singular locus, defined by  $\mathcal{S} : \det(Y(x), [Y, X](x)) = 0$ , takes the form:

$$984 (4.5) \quad x_1 x_2 (\varepsilon_2 x_2 (\varepsilon_2 a_{12} - \varepsilon_1 a_{22}) - \varepsilon_1 x_1 (\varepsilon_1 a_{21} - \varepsilon_2 a_{11})) = 0$$

985 and Lie brackets of length 3 are:

$$\begin{aligned} & [[Y, X], Y](x) = -x_1 (\varepsilon_1^2 x_1 a_{11} - \varepsilon_2^2 x_2 a_{12}) \frac{\partial}{\partial x_1} - x_2 (\varepsilon_1^2 x_1 a_{21} + \varepsilon_2^2 x_2 a_{22}) \frac{\partial}{\partial x_2}, \\ 986 & [[Y, X], X](x) = -x_1 \left( \varepsilon_1 x_1 (r_1 a_{11} + x_2 a_{12} (a_{11} - a_{21})) + \varepsilon_2 x_2 a_{12} (x_1 (a_{21} - a_{11}) + r_2) \right) \frac{\partial}{\partial x_1} \\ & \quad - x_2 \left( \varepsilon_2 x_2 (r_2 a_{22} + x_1 a_{21} (a_{22} - a_{12})) + \varepsilon_1 x_1 a_{21} (x_2 (a_{12} - a_{22}) + r_1) \right) \frac{\partial}{\partial x_2}. \end{aligned}$$

987 The geometric situation that we analyze in our working example is the exceptional case  
988 where we consider the intersection of the collinear locus with the singular locus, which corre-  
989 sponds from (4.4) and (4.5) to intersection of two straight-lines. It corresponds to a generic  
990 interaction between an hyperbolic arc and an elliptic arc.

991 *Constructing a semi-normal form.* The second step is to construct a semi-normal form for  
992 the system. The construction is detailed in [7] but computation are simple in the  $2d$ -case.  
993 It consists to choose coordinates such that the intersection is taken as the origin  $(0, 0)$ ,  $Y$  is  
994 identified to the constant vector field  $Y = \frac{\partial}{\partial x_2}$  (this amounts mainly to choose ln-coordinates)  
995 while the reference singular direction is identified to the straight line  $(0x_1)$ .

996 Expanding  $X$  in the jet space at  $(0, 0)$ , this leads to analyze the control system:

$$997 \quad \dot{x}_1 = -\lambda x_1 + \alpha x_2^2, \quad \dot{x}_2 = u - u_e$$

998 with  $u_e \in ]-1, 1[$ ,  $|u| \leq 1$ ,  $\lambda > 0$  and  $\alpha > 0$ .



999 *Properties of the system.* Computing Lie brackets in the new coordinates show relevant  
1000 simplification with respect to the previous formulae:

$$1001 \quad X(x) = (-\lambda x_1 + \alpha x_2^2) \frac{\partial}{\partial x_1} - u_e \frac{\partial}{\partial x_2}, \quad Y(x) = \frac{\partial}{\partial x_2},$$

$$1002 \quad [Y, X](x) = -2\alpha x_2 \frac{\partial}{\partial x_1}, \quad [[Y, X], Y](x) = -2\alpha \frac{\partial}{\partial x_1},$$

1004 and the singular line is given by

$$1005 \quad x_2 = 0.$$

1006 Restricting to  $x_2 = 0$ , one has:

$$1007 \quad X|_{x_2=0} = -\lambda x_1 \frac{\partial}{\partial x_1}, \quad [[Y, X], Y]|_{x_2=0}(x) = -2\alpha \frac{\partial}{\partial x_1}.$$

1009 Hence

$$1010 \quad [[Y, X], Y]|_{x_2=0}(x) = \frac{2\alpha}{\lambda x_1} X|_{x_2=0}(x)$$

1011 for  $x_1 \neq 0$ .

1012 Therefore, we obtain the following lemma.

1013 **Lemma 4.4.** 1. *The origin  $(0, 0)$  is an abnormal singular arc reduced to a point and*  
1014 *the arc  $x_1 > 0$  is hyperbolic and the arc  $x_1 < 0$  is elliptic.*

1015 2. *The singular control along the line  $x_2 = 0$  is given by  $u = u_e$  and is constant and*  
1016 *strictly admissible since  $u_e \in ]-1, +1[$ .*

1017 **Lemma 4.5.** *The collinear set  $\mathcal{C}$  given by  $\det(X, Y) = 0$  is the parabola  $x_1 = \alpha x_2^2 / \lambda$ .*

1018 **Clock form.** To analyze a  $2d$ -time minimal problem with fixed extremities the standard  
1019 technique is to introduce the clock form  $\omega = p dx$  defined outside the collinear set by:

$$1020 \quad p \cdot X(x) = 1, \quad p \cdot Y(x) = 0.$$

1021 Computing one has  $\omega = \frac{1}{-\lambda x_1 + \alpha x_2^2} dx_1$  so that

$$1022 \quad d\omega = \frac{2\alpha x_2}{(-\lambda x_1 + \alpha x_2^2)^2} dx_1 \wedge dx_2.$$

1023 One can decompose  $\mathbb{R}^2 \setminus (\mathcal{C} \cup \mathcal{S})$  in four domains:

- 1024 • domain  $A$ :  $\dot{x}_1 < 0 \cap x_2 > 0$ ,
- 1025 • domain  $B$ :  $\dot{x}_1 > 0 \cap x_2 > 0$ ,
- 1026 • domain  $C$ :  $\dot{x}_1 < 0 \cap x_2 < 0$ ,
- 1027 • domain  $D$ :  $\dot{x}_1 > 0 \cap x_2 < 0$ .

1028 On each domain one can compare the time along arcs  $\gamma_1, \gamma_2$  joining respectively  $x_0$  to  $x_1$   
1029 where  $\gamma_1 = \sigma_+ \sigma_-$ ,  $\gamma_2 = \sigma_- \sigma_+$  using Stokes theorem. One has:

1030 **Lemma 4.6.** *In domain  $A$  and  $D$  for such arcs the time minimal policy is  $\sigma_- \sigma_+$  while in*  
1031 *domain  $B$  and  $C$  the optimal policy is  $\sigma_+ \sigma_-$ .*

1032 *Proof.* Take the case of domain  $A$ , one has:

$$1033 \quad \int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{\gamma_1 \vee (-\gamma_2)} d\omega > 0,$$

1034 hence the time along  $\gamma_1$  is longer than the time along  $\gamma_2$ . The discussion is similar for the  
1035 other cases. ■

1036 *Integrating the extremal curves.* The adjoint system takes the form with  $p = (p_1, p_2)$ :

$$1037 \quad \dot{p}_1 = \lambda p_1, \quad \dot{p}_2 = -2\alpha x_2 p_1.$$

1038 Denoting for  $u = \pm 1$ ,  $\beta = u - u_e$ , one gets:

1039 **Lemma 4.7.** *The extremal system is characterized by:*

- 1040 •  $x_2(t) = x_2(0) + \beta t$ ,
- 1041 •  $x_1(t) = e^{-\lambda t} \left( x_1(0) + \int_0^t e^{\lambda s} (x_2(0) + \beta s)^2 ds \right)$
- 1042 •  $p_1(t) = e^{\lambda t} p_1(0)$ ,
- 1043 •  $p_2(t) = -2\alpha p_1(0) \int_0^t e^{\lambda s} (x_2(0) + \beta s) ds$

1044 and they belong to the polynomial exponential category.

1045 The integrals in the expressions of  $x_1(t)$  and  $p_2(t)$  can be evaluated using:

$$1046 \quad \int_0^t s e^{\lambda s} ds = \frac{t e^{\lambda t}}{\lambda} - \frac{1}{\lambda^2} (e^{\lambda t} - 1), \quad \int_0^t s^2 e^{\lambda s} ds = \frac{t^2 e^{\lambda t}}{\lambda} - \frac{2}{\lambda} \int_0^t s e^{\lambda s} ds.$$

1048 **Lemma 4.8.** *The switching function is  $\Phi(t) = p_2(t)$  so that for  $u = \pm 1$  and one has:*  
1049  $\ddot{p}_2(t) = -2\alpha p_1 (\lambda x_2 + u - u_e)$  where  $p_1(t)$  is of constant sign given by the sign of  $p_1(0)$ .

1050 *Geometric discussion of the synthesis with a terminal manifold  $N$  of codimension one.* Next  
1051 we present the discussion of the time minimal synthesis with a terminal manifold  $N$  of codi-  
1052 mension one by gluing cases discussed in [7].

1053 One takes  $N$  as a circle with radius  $d$  centered at 0 where the synthesis amounts to glue  
1054 the hyperbolic and elliptic situation. The circle intersects the hyperbolic arc  $\sigma_s$  at  $x_1 = d$  and  
1055 the elliptic arc  $\sigma_s$  at  $x_1 = -d$ .

1056 The BC-extremals curves can be parameterized by Lemma 4.7 with  $(p_1(0), p_2(0)) = \pm n(0)$   
1057 where  $n(0)$  is the normal to the circle,  $n(0) = (x(s), y(s))$ . Geodesics curves are integrated  
1058 backwards, geodesics in the interior to the circle are associated to  $n(0)$  and geodesics exterior  
1059 to the circle are associated to  $-n(0)$ .

1060 Since  $Y = \frac{\partial}{\partial x_2}$ , from [7] we can at once deduced the synthesis outside the circle, near  
1061 the hyperbolic point  $(d, 0)$  and the elliptic point  $(-d, 0)$ , using the curvature of  $N$  only at  
1062 such point which are the images of the curve  $(-1/2ks^2, s)$ ,  $k > 0$  at the hyperbolic point and  
1063  $(1/2ks^2, s)$  at the elliptic point.

1064 They are represented on Fig.10.

1065 The main properties are

- 1066 • *hyperbolic case:* the singular arc is optimal and the optimal policy is  $-1$  for  $x_2 > 0$   
1067 and  $+1$  for  $x_2 < 0$ .

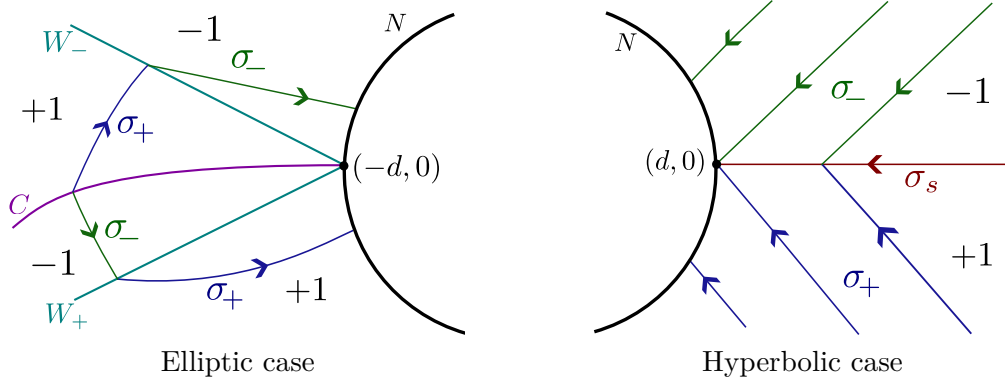


Figure 10. 2d-syntheses.

1068 • *elliptic case*: The time minimal synthesis is defined by the stratification  $W = W_- \cup W_+$   
 1069 of the switching locus and there exists a cut locus  $C$  terminating at  $(-d, 0)$ .

1070 To complete the analysis one must glue the two syntheses along the target  $N$  and consider  
 1071 also geodesics interior to the circle.

1072 To simplify the computations one considers the symmetric case  $u = u_e$  which allows to  
 1073 simplify the estimate of the cut locus prolongating the branch at the elliptic point which  
 1074 coincides with the singular line  $x_2 = 0$  restricted to the domain  $x_1 < -d$ .

The synthesis is represented on Fig.11.

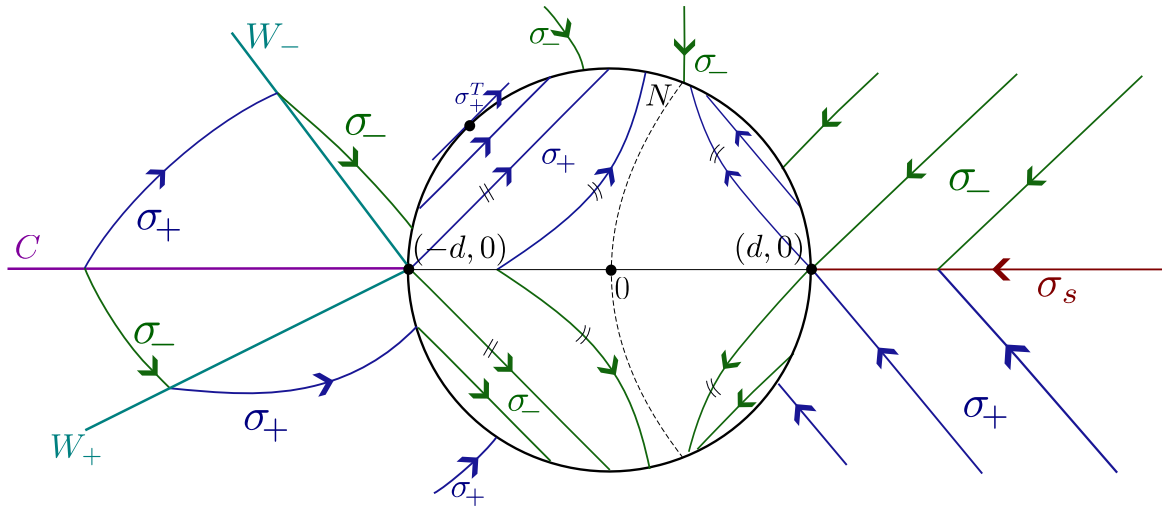


Figure 11. Gluing hyperbolic and elliptic case with  $N$  being a circle; the symmetric case  $u_e = 0$ .

1075 This gives the time minimal synthesis in a neighborhood  $V$  of the origin. In general  
 1076 we combine numerical integration using the parameterization of the BC-extremals and the  
 1077 analysis of [7] to glue different Whitney charts.  
 1078

1079 **4.4.2. Complexity if the singular flow in the 3d-case.** From dynamical system point  
 1080 of view, the complexity of the Lotka–Volterra model is related to the existence of persistent

1081 equilibrium point, which leads to complicated dynamics related to the hierarchy of dynamics  
1082 associated to the hierarchy of at most  $2^n$ -equilibria.

1083 Hence, in this section one analyses the same question regarding to existence of permanent  
1084 equilibria for the singular dynamics associated to the time minimal control problem.

1085 First one must extent the previous  $2d$ -result concerning the existence of (exceptional)  
1086 singular arc reduced to a point, in the context of controlled Lotka–Volterra model.

1087 **Proposition 4.9.** *Consider the controlled Lotka–Volterra model in the  $n$ -dimensional case.  
1088 Then there exists (isolated) exceptional arcs reduced to a point.*

1089 *Proof.* Consider the pair  $(X, Y)$ , and choose ln-coordinates so that  $Y(x) = \frac{\partial}{\partial x_n}$  with  
1090  $X(x) = \sum_{i=1}^n X_i(x) \frac{\partial}{\partial x_i}$ , the collinear set is defined by the  $(n - 1)$ -equations  $X_i(x) = 0$ ,  
1091  $i = 1, \dots, n - 1$  and let  $\lambda = -u_e$  so that  $X_n(x) = -\lambda$ . Let  $\tilde{X}(x) = \sum_{i=1}^{n-1} X_i(x) \frac{\partial}{\partial x_i} +$   
1092  $(X_n(x) + u_e) \frac{\partial}{\partial x_n}$  and by construction there exists  $x_e$  on the collinear set so that  $\tilde{X}(x_e) = 0$ .  
1093 Denote by  $\tilde{J}$  the *Jacobian matrix* of  $\tilde{X}$  at  $x = x_e$  and let  $\sigma(\tilde{J})$  be its spectrum.

1094 Denoting by  $\text{ad}\tilde{X} \cdot Y = [\tilde{X}, Y]$ . Then at  $x_e$  the matrix with columns of iterated Lie brackets

$$1095 \quad K = \left( Y, \text{ad}\tilde{X} \cdot Y, \dots, \text{ad}^{n-1} \tilde{X} \cdot Y \right)$$

1096 coincides with the *Kalman matrix*

$$1097 \quad \left( b, \tilde{J}b, \dots, \tilde{J}^{n-1}b \right)$$

1098 where  $b$  is the constant vector  $Y(x)$ . The singular point is exceptional if and only if  $\text{rank}K \leq$   
1099  $n - 1$ .

1100 At isolated point, the condition  $\text{rank} = n - 1$  can be realized for the controlled Lotka–  
1101 Volterra model (see Example 4.10 for  $n = 3$ ). ■

1102 **Example 4.10.** In the dimensionless coordinates (see Proposition 4.1), take:  $A = \text{diag}(\lambda_1,$   
1103  $\lambda_1, \lambda_3)$ ,  $X(x) = -\text{diag}(x + 1) Ax$ ,  $Y(x) = \text{diag}(x + 1) (\varepsilon_1, \varepsilon_2, \varepsilon_3)^\top$  so that the persistent equi-  
1104 librium is located at  $x_e = 0$ . The columns of the Kalman matrix  $K$  are

$$1105 \quad Y(0) = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^\top,$$

$$1106 \quad [X, Y](0) = (-\varepsilon_1 \lambda_1, -\varepsilon_2 \lambda_1, -\varepsilon_3 \lambda_3)^\top,$$

$$1107 \quad [X, [X, Y]](0) = (\varepsilon_1 \lambda_1^2, +\varepsilon_2 \lambda_1^2, \varepsilon_3 \lambda_3^2)^\top,$$

1109 hence  $\text{rank}K < 3$  and 0 is a singular exceptional point.

1110 Note that the Jacobian matrix of the singular flow, defined for  $\varepsilon_1 \neq \varepsilon_2$ , evaluated at 0 is

$$1111 \quad J = \begin{pmatrix} \frac{\varepsilon_1 \lambda_3}{\varepsilon_1 - \varepsilon_2} - \lambda_1 & \frac{\varepsilon_1 \lambda_3}{\varepsilon_2 - \varepsilon_1} & 0 \\ \frac{\varepsilon_2 \lambda_3}{\varepsilon_1 - \varepsilon_2} & \frac{\varepsilon_1 \lambda_3}{\varepsilon_2 - \varepsilon_1} - \lambda_1 + \lambda_3 & 0 \\ \frac{\varepsilon_3 \lambda_3}{\varepsilon_1 - \varepsilon_2} & -\frac{\varepsilon_3 \lambda_3}{\varepsilon_1 - \varepsilon_2} & -\lambda_3 \end{pmatrix}$$

1112 and its spectrum  $\{\lambda_3 - \lambda_1, -\lambda_3, -\lambda_1\}$  is resonant.

1113 **3d-case.** In the 3d-case, the singular trajectories are solutions of the vector field:  $\dot{x} =$   
 1114  $X_s(x) = X(x) - u_s Y(x)$ , where the singular control feedback is  $u_s = -D'(x)/D(x)$  with

$$1115 \quad D(x) = \det(Y(x), [Y, X](x), [[Y, X], Y](x)), \quad D'(x) = \det(Y(x), [Y, X](x), [[Y, X], X](x)).$$

1116 Moreover exceptional trajectories are located on the exceptional locus  $D''(x) = 0$  with

$$1117 \quad D''(x) = \det(Y(x), [Y, X](x), X(x)).$$

1118 Computing in the original coordinates leads to complicated expressions:

1119

$$1120 \quad D(x)/x_1 x_2 x_3 = (\varepsilon_1^2 x_1 a_{21} + \varepsilon_1 (\varepsilon_2 (x_2 a_{22} - x_1 a_{11}) + \varepsilon_3 x_3 a_{23}) - \varepsilon_2 (\varepsilon_2 x_2 a_{12} + \varepsilon_3 x_3 a_{13})) \\
 1121 \quad (\varepsilon_1^2 x_1 a_{31} + \varepsilon_2^2 x_2 a_{32} + \varepsilon_3^2 x_3 a_{33}) + (\varepsilon_1^2 x_1 a_{11} + \varepsilon_2^2 x_2 a_{12} + \varepsilon_3^2 x_3 a_{13}) (\varepsilon_2^2 x_2 a_{32} + \varepsilon_3 \varepsilon_2 (x_3 a_{33} - x_2 a_{22}) \\
 1122 \quad - \varepsilon_3^2 x_3 a_{23} + \varepsilon_1 x_1 (\varepsilon_2 a_{31} - \varepsilon_3 a_{21})) - (\varepsilon_1^2 x_1 a_{21} + \varepsilon_2^2 x_2 a_{22} + \varepsilon_3^2 x_3 a_{23}) \\
 1123 \quad (\varepsilon_1^2 x_1 a_{31} + \varepsilon_1 (\varepsilon_2 x_2 a_{32} + \varepsilon_3 (x_3 a_{33} - x_1 a_{11})) - \varepsilon_3 (\varepsilon_2 x_2 a_{12} + \varepsilon_3 x_3 a_{13})),$$

1124

1125

1126

$$1127 \quad D'(x)/x_1 x_2 x_3 = (-\varepsilon_1^2 x_1 a_{21} + \varepsilon_1 (\varepsilon_2 (x_1 a_{11} - x_2 a_{22}) - \varepsilon_3 x_3 a_{23}) + \varepsilon_2 (\varepsilon_2 x_2 a_{12} + \varepsilon_3 x_3 a_{13})) \\
 1128 \quad (\varepsilon_2 x_2 (x_1 a_{12} a_{31} - a_{32} (x_1 a_{21} + x_3 (a_{23} - a_{33}) + r_2)) - \varepsilon_1 x_1 (r_1 a_{31} + x_3 (a_{13} - a_{33}) a_{31} \\
 1129 \quad + x_2 (a_{12} a_{31} - a_{21} a_{32})) + \varepsilon_3 x_3 (-r_3 a_{33} + x_1 a_{31} (a_{13} - a_{33}) + x_2 a_{32} (a_{23} - a_{33}))) \\
 1130 \quad + (\varepsilon_2^2 (-x_2) a_{32} + \varepsilon_3 \varepsilon_2 (x_2 a_{22} - x_3 a_{33}) + \varepsilon_3^2 x_3 a_{23} + \varepsilon_1 x_1 (\varepsilon_3 a_{21} - \varepsilon_2 a_{31})) \\
 1131 \quad (-\varepsilon_1 x_1 (r_1 a_{11} + x_2 a_{12} (a_{11} - a_{21}) + x_3 a_{13} (a_{11} - a_{31})) + \varepsilon_2 x_2 (x_3 a_{13} a_{32} - a_{12} (x_1 (a_{21} - a_{11}) \\
 1132 \quad + x_3 a_{23} + r_2)) - \varepsilon_3 x_3 (a_{13} (x_1 (a_{31} - a_{11}) + x_2 a_{32} + r_3) - x_2 a_{12} a_{23})) \\
 1133 \quad - (\varepsilon_1^2 (-x_1) a_{31} + \varepsilon_1 (\varepsilon_3 (x_1 a_{11} - x_3 a_{33}) - \varepsilon_2 x_2 a_{32}) + \varepsilon_3 (\varepsilon_2 x_2 a_{12} + \varepsilon_3 x_3 a_{13})) \\
 1134 \quad (\varepsilon_1 x_1 (x_3 a_{23} a_{31} - a_{21} (x_3 a_{13} + x_2 (a_{12} - a_{22}) + r_1)) + \varepsilon_2 x_2 (-r_2 a_{22} + x_1 a_{21} (a_{12} - a_{22}) \\
 1135 \quad + x_3 a_{23} (a_{32} - a_{22})) + \varepsilon_3 x_3 (x_1 a_{13} a_{21} - a_{23} (x_1 a_{31} + x_2 (a_{32} - a_{22}) + r_3))),$$

1136

1137

1138

$$1139 \quad D''(x)/x_1 x_2 x_3 = (-\varepsilon_1^2 x_1 a_{21} + \varepsilon_1 (\varepsilon_2 (x_1 a_{11} - x_2 a_{22}) - \varepsilon_3 x_3 a_{23}) + \varepsilon_2 (\varepsilon_2 x_2 a_{12} + \varepsilon_3 x_3 a_{13})) \\
 1140 \quad (x_1 a_{31} + x_2 a_{32} + x_3 a_{33} + r_3) + (-\varepsilon_2^2 x_2 a_{32} + \varepsilon_3 \varepsilon_2 (x_2 a_{22} - x_3 a_{33}) + \varepsilon_3^2 x_3 a_{23} + \varepsilon_1 x_1 (\varepsilon_3 a_{21} \\
 1141 \quad - \varepsilon_2 a_{31})) (x_1 a_{11} + x_2 a_{12} + x_3 a_{13} + r_1) + (\varepsilon_1^2 x_1 a_{31} + \varepsilon_1 (\varepsilon_2 x_2 a_{32} + \varepsilon_3 (x_3 a_{33} - x_1 a_{11})) \\
 1142 \quad - \varepsilon_3 (\varepsilon_2 x_2 a_{12} + \varepsilon_3 x_3 a_{13})) (x_1 a_{21} + x_2 a_{22} + x_3 a_{23} + r_2).$$

1144 **Proposition 4.11.** *If  $D(x) \neq 0$ , the equilibria of the singular dynamics  $\dot{x} = X_s(x)$  are*  
 1145 *exceptional trajectories reduced to a point. At such a point  $x_e$ , the spectrum of  $J = \frac{\partial X_s}{\partial x}(x_e)$*   
 1146 *is a feedback invariant. Moreover the dynamics is foliated by the invariant set  $D''(x) = 0$  and*  
 1147  *$D(x)D''(x) > 0$  or  $< 0$  associated respectively to hyperbolic and elliptic arcs. The singular*  
 1148 *feedback  $u_s$  acts as a geometric pole placement of the dynamics on the collinear set.*

1149 **Proof.** The proof is clear following the construction detailed in the proof of Proposition  
 1150 4.9. ■

1151 *The 4d-case.* This gives the road to the  $n$ -dimensional case.

1152 The singular exceptional control can be expressed as a feedback using the relation

$$1153 \quad H_X(z) = H_Y(z) = \{H_X, H_Y\}(z) = 0$$

$$1154 \quad \{\{H_X, H_Y\}, H_X\}(z) + u_{se} \{\{H_X, H_Y\}, H_Y\}(z) = 0$$

1156 and this leads to

$$1157 \quad u_{se}(x) = -\frac{D'(x)}{D(x)},$$

1158 where

$$1159 \quad D(x) = \det(X(x), Y(x), [Y, X](x), [[Y, X], Y](x)),$$

$$1160 \quad D'(x) = \det(X(x), Y(x), [Y, X](x), [[Y, X], X](x)).$$

1162 Similarly to the  $3d$ -case, the singular exceptional dynamics:  $\dot{x} = X(x) + u_{se}Y(x)$  can be used  
1163 to generate feedback invariants.

1164 Remaining singular dynamics are parameterized by a dynamic feedback  $u(x, \lambda)$  depending  
1165 upon a one dimensional coefficient.

1166 **5. Conclusion.** In this article we have presented the general techniques from geometric  
1167 control to analyze in the permanent case, the optimal control problem related to vermin  
1168 reduction in a complex microbiote modelled by the Lotka–Volterra equations.

1169 Our analysis is based on a series of articles classifying the time minimal syntheses for a  
1170 single-input affine system with terminal manifold of codimension one developed for chemical  
1171 networks [7, 8, 20]. Using the concepts of Whitney chart and unfolding the explicit computa-  
1172 tions in a neighbourhood of the terminal manifold can be reduced to problems in dimension 2  
1173 or 3 and a dictionary of the time minimal syntheses is described in [7, 20], up to codimension  
1174 2 cases. Global syntheses can be described by gluing different Whitney charts.

1175 It can be applied to the controlled Lotka–Volterra model to analyze either the problem  
1176 of reducing the infection using an antibiotic agent or to reinforce the body prior to infection  
1177 using a probiotic agent. A case study is given to construct a global optimal synthesis by gluing  
1178 distinct Whitney charts.

1179 Our article shows the parallel between the analysis of the free dynamics in the frame of dy-  
1180 namical systems and the dynamics of the Hamiltonian dynamics deduced from the Maximum  
1181 Principle which parameterizes the extremals candidates as minimizers.

1182 The complexity of the Hamiltonian dynamics for nonlinear control systems is related to  
1183 the existence and their complexity of the singular extremals dynamics. Our contribution is to  
1184 present preliminary analysis of this complexity, in the frame of the controlled Lotka–Volterra  
1185 model. It is shown to be related to the collinear set on which are located the equilibrium  
1186 of the free dynamics where no treatment is applied and the forced equilibrium associated to  
1187 maximal treatment.

1188 An additional step in our analysis will be to analyze the problem in the sampled–data  
1189 control frame in relation with the permanent case.

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Barnesiella (Bar.)						0.3680	Akkermansia (Akk.)					0.2297
undefined genus of Lachnospiraceae (Und. Lac.)						0.3102	Coprobacillus (Cop.)					0.8300
undefined genus of unclassified Mollicutes (Und. Mol.)						0.4706	Clostridium difficile (C. diff.)					0.3918
unclassified Lachnospiraceae (Uncl. La.)						0.3561	Enterococcus (Ent.)					0.2907
Blautia (Bla.)						0.7089	undefined genus of Enterobacteriaceae (Und. En.)					0.3236
Other						0.5400						
	Bar.	Und. Lac.	Uncl. Lac.	Other	Bla.	Und. Mol.	Akk.	Cop.	Und. En.	Ent.	C. diff.	
Bar.	-0.205	0.098	0.167	-0.164	-0.143	0.019	-0.515	-0.391	-0.268	0.008	0.346	
Und. Lac.	0.062	-0.104	-0.043	-0.154	-0.187	0.027	-0.459	-0.413	-0.196	0.022	0.301	
Uncl. Lac.	0.143	-0.192	-0.101	-0.139	-0.165	0.013	-0.504	-0.772	-0.206	-0.006	0.292	
Other	0.224	0.138	0.000	-0.831	-0.223	0.220	-0.205	-1.009	-0.400	-0.039	0.666	
Bla.	-0.180	-0.051	0.000	-0.054	-0.708	0.016	-0.507	0.553	0.106	0.224	0.157	
Und. Mol.	-0.111	-0.037	-0.042	0.041	0.261	-0.422	-0.185	-0.432	-0.264	-0.061	0.164	
Akk.	-0.126	-0.185	-0.122	0.380	0.400	-0.160	-1.212	1.389	-0.096	0.191	-0.379	
Cop.	-0.071	0.000	0.080	-0.454	-0.503	0.169	-0.562	-4.350	-0.207	-0.223	0.443	
Und. Ent.	-0.374	0.278	0.248	-0.168	0.084	0.033	-0.232	-0.395	-0.384	-0.038	0.314	
Ent.	-0.042	-0.013	0.024	-0.117	-0.328	0.020	0.054	-2.096	0.023	-0.192	0.111	
C. diff.	-0.037	-0.033	-0.049	-0.090	-0.102	0.032	-0.181	-0.303	-0.007	0.014	-0.055	

Table 1

(top) Growth rates  $a_{ij}$  of each microbial population  $i$  of the CDI model. (bottom) Interactions between pairwise microbial populations of the CDI model. Both tables are excerpted from [25].

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