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A *POSTERIORI* ERROR ESTIMATES FOR DISCONTINUOUS GALERKIN METHODS ON POLYGONAL AND POLYHEDRAL MESHES

ANDREA CANGIANI*, ZHAONAN DONG†, AND EMMANUIL H. GEORGIOULIS‡

Abstract. We present a new residual-type energy-norm *a posteriori* error analysis for interior penalty discontinuous Galerkin (dG) methods for linear elliptic problems. The new error bounds are also applicable to dG methods on meshes consisting of elements with very general polygonal/polyhedral shapes. The case of simplicial and/or box-type elements is included in the analysis as a special case. In particular, for the upper bounds, arbitrary number of very small faces are allowed on each polygonal/polyhedral element, as long as certain mild shape regularity assumptions are satisfied. As a corollary, the present analysis generalizes known *a posteriori* error bounds for dG methods, allowing in particular for meshes with arbitrary number of irregular hanging nodes per element. The proof hinges on a new conforming recovery strategy in conjunction with a Helmholtz decomposition formula. The resulting *a posteriori* error bound involves jumps on the tangential derivatives along elemental faces. Local lower bounds are also proven for a number of practical cases. Numerical experiments are also presented, highlighting the practical value of the derived *a posteriori* error bounds as error estimators.

Key words. discontinuous Galerkin; *a posteriori* error bound; polygonal/polyhedral meshes; polytopic elements; irregular hanging nodes.

AMS subject classifications. 65N30, 65M60, 65J10

1. Introduction. Recent years have witnessed an extensive activity in the development of various Galerkin methods posed on meshes consisting of general polygonal/polyhedral (henceforth, collectively referred to as *polytopic*) elements. A central question arising is the derivation of computable error bounds for such discretisations, so that the extreme geometric flexibility of such meshes can be harnessed.

Residual-type *a posteriori* error bounds for interior penalty dG methods on composite/polytopic meshes appeared in [26, 18]. Also, in the context of virtual element methods, corresponding bounds are proven in [12, 16], while for the weak Galerkin approach, an *a posteriori* error analysis can be found in [34]. In addition, corresponding results for the hybrid high-order method can be found in [21]. All aforementioned results are proven for shape-regular polytopic meshes, under the additional assumption that the diameters of elemental faces are of comparable size to the element diameter. Although the latter may be a reasonable assumption in the context of standard, simplicial meshes, it can be rather restrictive for general polytopic elements. This is because general polygons/polyhedra with more than d faces can be simultaneously shape-regular yet containing *small* faces, i.e., faces whose diameter is arbitrarily small compared to the element diameter.

This work aims exactly at rectifying this restrictive state of affairs. We prove new energy-norm *a posteriori* upper error bounds for interior penalty discontinuous Galerkin (dG) methods posed on meshes containing polytopic elements, including with arbitrary number of small faces, as long as certain mild shape-regularity assumptions are satisfied. The case of simplicial and/or box-type elements is included in the analysis as a special case. For accessibility, we restrict the discussion to a model elliptic problem, noting, nevertheless, that various generalizations are possible with minor modifications.

As a general principle, residual-based *a posteriori* error analysis of non-conforming and, in particular, dG methods requires a recovery of the numerical solution into a related conforming function. The pioneering work of Karakashian & Pascal [30] (see also [29]) proposed the recovery

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of the dG solution by a nodal averaging operator for which a crucial stability result was proven [30, Theorem 2.2]; cf., also [29, Theorem 2.1] for an extension. This construction allowed for the first rigorous *a posteriori* error analysis of a dG method for elliptic problems. A number of related results followed, improving various aspects of the theory; for instance, see [3, 28, 15, 40, 1, 6, 27, 20, 32]. A key reason for the aforementioned restrictive assumption that all elemental faces are of comparable size to the element diameter in existing *a posteriori* error analysis for polytopic dG methods [26, 18] is exactly the lack of availability of a stability result corresponding to [30, Theorem 2.2] for polytopic element meshes containing elements with small faces.

In this work, we crucially avoid the use of averaging operators. Instead, the proof of the upper error bound hinges on a new recovery into H^1 -conforming functions, in conjunction with a Helmholtz decomposition formula. To complete the analysis, we also require the existence of appropriate auxiliary simplicial meshes on which quasi-interpolants are defined. This can be verified in practice using simple and efficient algorithms. We provide two such algorithms, one based on a sub-mesh while the other employs tools from computational geometry and, in particular, constrained Delaunay triangulations [17, 37]. The resulting *a posteriori* error bound involves also jumps on the tangential derivatives along elemental faces.

Local lower bounds are also proven for a number of practical cases, indicating the optimality of the new estimators. The key challenge in the proof of the latter is, again, the treatment of small/degenerating element faces and the construction of respective bubble functions. In particular, local lower bounds for the element residuals are proven allowing for arbitrarily small faces. Lower bounds for the flux residuals are proven under more restrictive assumptions (see Assumption 4.6 below), allowing, nevertheless, for arbitrarily small faces.

We note that the case of meshes consisting of simplicial and/or box-type elements is included in the analysis as a special case. For such ‘classical’ mesh concepts, the developments presented below provide a new class of *a posteriori* error bounds applicable even to *hp*-version dG methods, allowing in particular for meshes with arbitrary number of irregular hanging nodes per element.

The remainder of this work is structured as follows. In Section 2, we define the elliptic model problem, the admissible meshes, and finite element spaces and the interior penalty dG method on polytopic meshes. We also prove some important technical results regarding the construction of auxiliary meshes which will be instrumental in the proof of *a posteriori* error bounds. In Section 3, we prove the *a posteriori* upper error bound, using the aforementioned technical developments, while in Section 4 we provide respective lower bounds for the energy-norm error for a number of practical cases. Finally, in Section 5, we present some numerical experiments confirming the robustness and efficiency of the derived *a posteriori* error bound and highlighting its practical value as error estimator.

2. Model problem and numerical method. For a Lipschitz domain $\omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, we denote by $H^s(\omega)$ the Hilbertian Sobolev space of index $s \geq 0$ of real-valued functions defined on ω , endowed with the seminorm $|\cdot|_{H^s(\omega)}$ and norm $\|\cdot\|_{H^s(\omega)}$. Furthermore, we let $L_p(\omega)$, $p \in [1, \infty]$, be the standard Lebesgue space on ω , equipped with the norm $\|\cdot\|_{L_p(\omega)}$. In the case $p = 2$, we shall simply write $\|\cdot\|_\omega$ to denote the L_2 -norm over ω and simplify this further to $\|\cdot\|$ when $\omega = \Omega$, the physical domain. Finally, $|\omega|$ denotes the $(d - 1)$ -dimensional Hausdorff measure of ω .

2.1. Model problem. Let Ω be a simply connected, bounded, and open polygonal/polyhedral domain in \mathbb{R}^d , $d = 2, 3$. The boundary $\partial\Omega$ of Ω is split into two disjoint parts Γ_D and Γ_N with $|\Gamma_D| \neq 0$. For technical reasons, when $d = 3$ and $|\Gamma_N| \neq 0$, the interface between Γ_D and Γ_N is assumed that is made up of straight planar segments. We consider the linear elliptic problem: find $u \in H^1(\Omega)$, such that

$$(2.1) \quad \begin{aligned} -\nabla \cdot (a\nabla u) &= f && \text{in } \Omega, \\ u &= g_D && \text{on } \Gamma_D, \\ a\nabla u \cdot \mathbf{n} &= g_N && \text{on } \Gamma_N, \end{aligned}$$

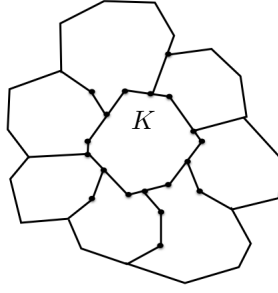


FIG. 2.1. Element $K \in \mathcal{T}$ with its face-wise neighbours; hanging nodes are highlighted by a bullet.

with known $f \in L_2(\Omega)$, $g_D \in H^{1/2}(\Gamma_D)$ and $g_N \in L_2(\Gamma_N)$ and symmetric diffusion tensor $a \in [L_\infty(\Omega)]^{d \times d}$ such that

$$(2.2) \quad \alpha^* |\xi|^2 \geq \xi^\top a(x) \xi \geq \alpha_* |\xi|^2 > 0 \quad \forall \xi \in \mathbb{R}^d, \quad \text{a.e. } x \in \Omega,$$

for some constants $\alpha^*, \alpha_* > 0$. For simplicity of the presentation, we assume that a is piecewise constant, although this is not an essential restriction for the validity of the developments below.

Setting $H_D^1 := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$, the weak formulation of (2.1) is: find $u \in H^1(\Omega)$, $u = g_D$ on Γ_D such that

$$(2.3) \quad \int_{\Omega} a \nabla u \cdot \nabla v \, d\mathbf{x} = \int_{\Omega} f v \, d\mathbf{x} + \int_{\Gamma_N} g_N v \, ds,$$

for all $v \in H_D^1(\Omega)$. The well-posedness is guaranteed by the Lax-Milgram Lemma.

2.2. Finite element spaces and trace operators. We consider meshes \mathcal{T} consisting of general polygonal (for $d = 2$) or polyhedral (for $d = 3$) mutually disjoint open elements $K \in \mathcal{T}$, henceforth termed collectively as *polytopic*, with $\cup_{K \in \mathcal{T}} \bar{K} = \bar{\Omega}$. Given $h_K := \text{diam}(K)$, the diameter of $K \in \mathcal{T}$, we define the mesh-function $\mathbf{h} : \cup_{K \in \mathcal{T}} K \rightarrow \mathbb{R}_+$ by $\mathbf{h}|_K = h_K$, $K \in \mathcal{T}$. Further, we let $\Gamma := \cup_{K \in \mathcal{T}} \partial K$ denote the mesh skeleton and set $\Gamma_{\text{int}} := \Gamma \setminus \partial\Omega$. The mesh skeleton Γ is decomposed into $(d - 1)$ -dimensional simplices F denoting the mesh *faces*, shared by at most two elements. These are distinct from elemental *interfaces*, which are defined as the simply-connected components of the intersection between the boundary of an element and either a neighbouring element or $\partial\Omega$. As such, an interface between two elements may consist of more than one face, separated by hanging nodes/edges shared by those two elements only. This includes both ‘classical’ hanging nodes, typically created by local mesh refinement, and non-standard ones separating non-co-planar faces. The latter may be created, for instance, by a mesh agglomeration procedure; we refer to Fig. 2.1 for an illustration for $d = 2$.

The *finite element space* $S_{\mathcal{T}}$ with respect to \mathcal{T} is defined by

$$S_{\mathcal{T}} \equiv S_{\mathcal{T}}^p := \{u \in L_2(\Omega) : u|_K \in \mathcal{P}_p(K), K \in \mathcal{T}\},$$

for some $p \in \mathbb{N}$ with $\mathcal{P}_p(K)$ denoting the space of d -variate polynomials of total degree up to p on K . We stress that the local elemental polynomial spaces employed within $S_{\mathcal{T}}$ are defined in the *physical coordinate system*, i.e., without mapping from a given reference or canonical frame. This approach allows to retain the full local approximation properties of the underlying finite element space. We refer to [11, 10] for a detailed discussion on the benefits and implementation issues resulting from this choice.

Let K_i and K_j be two adjacent elements of \mathcal{T} sharing a face $F \subset \partial K_i \cap \partial K_j \subset \Gamma_{\text{int}}$. For v and \mathbf{q} element-wise continuous scalar- and vector-valued functions, respectively, we define the *average*

across F by $\{\{v\}\}_F := \frac{1}{2}(v|_{F \cap K_i} + v|_{F \cap K_j})$, $\{\{\mathbf{q}\}\}_F := \frac{1}{2}(\mathbf{q}|_{F \cap K_i} + \mathbf{q}|_{F \cap K_j})$, respectively, and the *jump* across F by $[[v]] := v|_{F \cap K_i} - v|_{F \cap K_j}$, $[[\mathbf{q}]] := \mathbf{q}|_{F \cap K_i} - \mathbf{q}|_{F \cap K_j}$, using the convention $i > j$ in the element numbering to determine the sign. On a boundary face $F \subset \Gamma_D$, with $F \subset \partial K_i$, $K_i \in \mathcal{T}$, we set $\{\{v\}\} := v_i$, $\{\{\mathbf{q}\}\} := \mathbf{q}_i$, $[[v]] := v_i$, and $[[\mathbf{q}]] := \mathbf{q}_i$, respectively.

For $v \in S_{\mathcal{T}}$ we denote by $\nabla_h v$ the element-wise gradient; namely, $(\nabla_h v)|_K := \nabla(v|_K)$ for all $K \in \mathcal{T}$. Also, we denote by $\nabla_T v$ the face-wise tangential gradient operator acting on the traces of v on Γ , noting that $\nabla_T v$ is double-valued on Γ_{int} . With a slight abuse of notation, we use the same symbol to denote the tangential gradient of boundary functions such as the Dirichlet datum g_D .

2.3. Mesh assumptions, inverse inequalities, and approximation results. Each mesh \mathcal{T} is required to conform to the problem data in the following basic way. First, \mathcal{T} must represent exactly the domain, namely $\cup_{K \in \mathcal{T}} K = \Omega$, and be consistent with the subdivision of $\partial\Omega$ into Γ_D and Γ_N . Moreover, we require resolution of multiscale features of the domain, such as complex boundaries and bottlenecks. Note that, in the context of polytopic meshes, such resolution is not intrinsic in that multiscale geometrical features can be represented by relatively ‘large’ elements with ‘small’ faces. Hence we assume that the local mesh size of each mesh \mathcal{T} is comparable to the local finest scale of Ω . It is clear that such saturation-type assumption can always be satisfied, possibly after a finite number of refinements of an original coarse mesh. Further, we require the following general polytopic mesh regularity assumption.

ASSUMPTION 2.1 (Mesh regularity). *We assume that each mesh \mathcal{T} satisfies the following mesh regularity conditions. For each $K \in \mathcal{T}$ it holds:*

- (a) K is star-shaped with respect to an inscribed ball of radius $r_K \geq \tau^{-1}h_K$, centred at some point $\mathbf{x}^0 \equiv \mathbf{x}_K^0 \in K$; see Figure 2.2 (right panel) for an illustration when $d = 2$.
- (b) Each face $F \subset \partial K \cap \partial\Omega$ is star-shaped with respect to a $(d - 1)$ -dimensional ball of radius $r_F \geq \tau^{-1}h_K$.

Here, $\tau > 1$ is a constant independent of the discretization parameters. In what follows, we assume that the centres of the inscribed balls are selected to be chosen so that τ is as minimal.

Remark 2.2. All results below generalize immediately to meshes containing polytopic elements that are finite unions of star-shaped polytopes; see Figure 2.2 (left panel) for an example. The minimal modifications required in the proofs are detailed in Remark 2.9 below.

Assumption 2.1 allows for very general element shapes, including non-convex polytopes with arbitrary number of degenerating faces, i.e., element faces $F \subset \partial K$ with $|F| \ll h_K^{d-1}$. Notable examples of acceptable subdivisions comprise elements with a bounded number of possibly degenerate hanging nodes and elements with ‘many’ faces obtained by agglomeration of very fine triangulations of the problem domain. Note, however, that Assumption 2.1(b) forbids degenerating/shape-irregular elemental faces on the boundary. This restriction can be relaxed to some extent at the expense of introducing unknown/hard-to-estimate constants, as discussed in Remark 3.6 below.

To the best of our knowledge, Assumption 2.1 allows for the most general polytopic meshes for which a posteriori error bounds are proven, for any Galerkin discretization and for any PDE problem. Nevertheless, it is, perhaps inevitably, more restrictive compared to the respective ones required for stability and a priori error analysis of dG methods; see [9], [10, Section 4] and [8], for details. The key advantage of the present setting is that it allows us to be as explicit as possible in the constants involved in the a posteriori error bounds.

We also require the following, standard, local quasi-uniformity assumption.

ASSUMPTION 2.3 (local quasi-uniformity). *Each mesh \mathcal{T} is locally quasiuniform, i.e., there exists a constant $\rho > 0$ such that $\rho^{-1} \leq h_{K_i}/h_{K_j} \leq \rho$, whenever $K_i, K_j \in \mathcal{T}$ share a common face.*

We now state and prove a few results stemming from the above shape-regularity and local quasiuniformity assumptions. A first, geometrical, consequence is that the number of face neighbours of each $K \in \mathcal{T}$ is, in fact, bounded.

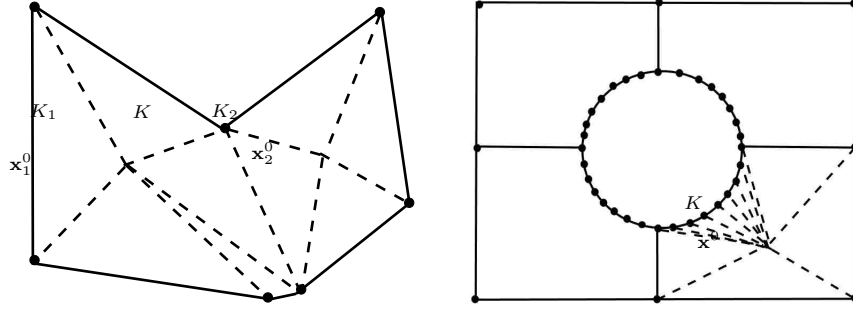


FIG. 2.2. Left panel: A polygonal element K with 7 nodes is subdivided into two polygons K_1 and K_2 star-shaped with respect to the points \mathbf{x}_1^0 and \mathbf{x}_2^0 , respectively. Right panel: a patch of polygonal elements. The element K has 12 nodes and is star-shaped with respect to a ball centred in \mathbf{x}^0 .

LEMMA 2.4. Under Assumptions 2.1 and 2.3, the number of interface neighbours of each element of \mathcal{T} is uniformly bounded.

Proof. Denote by ω_K the set of elements in \mathcal{T} which are neighbours of K . We derive a rough upper-bound on the cardinality n of ω_K as follows. If $K' \in \omega_K$, then $K' \subset B_{h_K+h_{K'}}(\mathbf{x}_K^0) \subseteq B_{(1+\rho)h_K}(\mathbf{x}_K^0)$ thanks to Assumption 2.3. On the other hand, Assumption 2.1 implies that K' contains the ball $B_{r_{K'}}(\mathbf{x}_{K'}^0)$ with $r_{K'} \geq \tau^{-1}h_{K'}$. Letting $C_2 := \pi$ and $C_3 := 4\pi/3$, we thus have $|K'| \geq C_d r_{K'}^d \geq C_d \tau^{-d} h_{K'}^d \geq C_d \tau^{-d} \rho^{-d} h_K^d$. Therefore, $n(C_d \tau^{-d} \rho^{-d} h_K^d) \leq C_d (1+\rho)^d h_K^d$, or $n \leq \tau^d \rho^d (1+\rho)^d$, thereby showing that the number of neighbours of K is uniformly bounded as required. \square

In particular, the presence of ‘many, small’ faces is allowed if these are grouped into a few interfaces only; see Figure 2.2 (right panel) for such a case.

LEMMA 2.5. Let $K \in \mathcal{T}$ satisfying Assumption 2.1. Then, for all $v \in H^1(\Omega)$, we have the trace estimate

$$(2.4) \quad \|v\|_{\partial K}^2 \leq C_{\text{tr}} \left(\frac{\zeta}{h_K} \|v\|_K^2 + \frac{h_K}{\zeta} \|\nabla v\|_K^2 \right),$$

for any $\zeta > 0$ and C_{tr} a positive constant only depending on τ and on d .

Also, for each $v \in \mathcal{P}_p(K)$, the inverse estimate

$$(2.5) \quad \|v\|_{\partial K}^2 \leq \frac{C_{\text{inv}}}{h_K} \|v\|_K^2,$$

holds with $C_{\text{inv}} := \tau(p+1)(p+d)$.

Proof. These estimates are special cases of the corresponding ones presented in [8]. The trace estimate (2.4) follows from [8, Lemma 4.7] in conjunction with Assumption 2.1, while (2.5) follows from [8, Lemma 4.4] along with Assumption 2.1. \square

LEMMA 2.6. Given $K \in \mathcal{T}$ satisfying Assumption 2.1, for each $v \in H^1(K)$, $K \in \mathcal{T}$, we have the bounds

$$(2.6) \quad \|v - \Pi_0 v\|_K \leq C_{PF} h_K \|\nabla v\|_K$$

and

$$(2.7) \quad \|v - \Pi_0 v\|_{\partial K} \leq \tilde{C}_{PF} \sqrt{h_K} \|\nabla v\|_K,$$

with $\Pi_0 : L_2(\Omega) \rightarrow S_{\mathcal{T}}^0$, the orthogonal L_2 -projection onto $S_{\mathcal{T}}^0$, the space of element-wise constants; here $C_{PF}, \tilde{C}_{PF} > 0$ depend on d and on τ only.

Proof. The key technical difficulty is to show that C_{PF} and \tilde{C}_{PF} are independent of the shape of K . Under Assumption 2.1, for $K \in \mathcal{T}$ we can apply the Poincaré-Friedrichs inequalities proven in [44, Theorem 3.5] and [41, Proposition 2.10], with explicit dependence on the shape-regularity constant τ and dimension d , yielding (2.6). Then, (2.7) follows using the trace inequality (2.4). \square

A crucial technical aspect of the analysis below is the availability of a shape-regular, auxiliary triangulation defined as follows.

DEFINITION 2.7 (Auxiliary mesh). *Given the sequence of meshes \mathcal{T} , we name auxiliary mesh a corresponding sequence of conforming simplicial meshes $\hat{\mathcal{T}}$ satisfying: for each $\hat{T} \in \hat{\mathcal{T}}$*

- (a) (shape-regularity) *the radius r_T of the largest circle inscribed in T is such that $r_T \geq \hat{\tau}^{-1}h_T$, where h_T denotes the diameter of T ;*
- (b) (local mesh-size compatibility) *if $K \in \mathcal{T}$ is such that $K \cap T \neq \emptyset$, it holds $\hat{\rho}^{-1} \leq h_T/h_K \leq \hat{\rho}$, with $\hat{\tau}, \hat{\rho} > 1$ constants independent of the discretization parameters.*

An immediate consequence of the above definition is that, if $\hat{\mathcal{T}}$ is an auxiliary mesh sequence, then the number of intersections of each $T \in \hat{\mathcal{T}}$ with the elements of the corresponding polytopic mesh \mathcal{T} is uniformly bounded in function of the shape-regularity and local quasi-uniformity constants of both the polytopic and auxiliary mesh. The proof of this fact follows along the lines of that of Lemma 2.4.

We note that the evaluation of the error estimator presented below does *not* require the construction of auxiliary meshes in practice. As long as their existence can be assumed, the a posteriori error bound holds. Moreover, we do not expect such assumption to limit in any possible way the configurations of very general polytopic mesh sequences allowed by Assumptions 2.1 and 2.3. Rather, the issue is to show that auxiliary meshes tightly close to the polytopic meshes can be constructed in principle. To this end, we present two possible algorithms for the construction of auxiliary meshes which apply to progressively complicated primal mesh configurations. Both algorithms are easily and cheaply implementable. As such, if desired, the corresponding auxiliary mesh quality parameters may be computed in practice, thus permitting the explicit evaluation of their impact on the a posteriori error bound.

2.3.1. Auxiliary sub-mesh. Assume that the mesh \mathcal{T} is *fully shape-regular* in the sense that, for each $K \in \mathcal{T}$, each face $F \in \partial K$ satisfies the shape-regularity property which is stated in Assumption 2.1(b) for boundary faces. Then, an auxiliary mesh can be simply constructed by joining F to \mathbf{x}_K^0 , the centre of star-shapedness of $K \in \mathcal{T}$, for each $K \in \mathcal{T}$ and $F \in \partial K$.

2.3.2. Constrained Delaunay auxiliary mesh. Employing a sub-mesh is not helpful if the element faces are shape-irregular and/or of arbitrarily small size with respect to the elemental size. In this case, it is necessary to consider auxiliary meshes which are not sub-meshes of the corresponding polytopic meshes. One possibility, designed to maximise shape-regularity while maintaining the auxiliary mesh as close as possible to the polytopic mesh in terms of local mesh-size, is to exploit the concept of *constrained Delaunay triangulations*, introduced in [17] for $d = 2$ and generalized to any d in [37]; see also [23, 38]. We recall their definition, limiting ourselves to the case of interest, namely when the constraints are given by the mesh faces laying on the boundary of Ω .

DEFINITION 2.8 (Constrained Delaunay triangulation). *Let $\mathcal{X} = \{\mathcal{P}, \mathcal{F}\}$ with \mathcal{P} a set of points in $\Omega \subset \mathbb{R}^d$ and \mathcal{F} the set of boundary faces of the mesh \mathcal{T} . A constrained Delaunay triangulation (CDT) associated to \mathcal{X} is a triangulation of Ω , which conforms to \mathcal{F} , has \mathcal{P} as its set of internal vertices, and satisfies the following constrained Delaunay property: for every $k \in \{1, \dots, d\}$ and every k -dimensional simplex S in the triangulation which is not on $\partial\Omega$, there exists a circle \mathcal{C} such that:*

- (1) *the vertices of S are on the boundary of \mathcal{C} ;*
- (2) *if a vertex P of the CDT is in the interior of \mathcal{C} , then the straight line connecting P to at least one of the vertices of S intersects $\partial\Omega$. (Then, we say that P cannot be seen from one of the vertices of S .)*

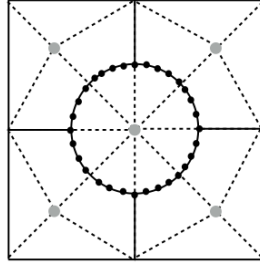


FIG. 2.3. A mesh with 5 elements (solid lines) and its constrained Delaunay auxiliary mesh with 16 elements (dash lines). The dual nodes belonging to the set \mathcal{P} of Definition 2.7 are marked with a grey bullet.

This definition generalizes the concept of Delaunay triangulations in that, if no constraints are given, it would coincide with the definition of Delaunay triangulations. Moreover, as Delaunay triangulations, CDTs maximizes the minimum angle among all triangulations generated by the cloud of points \mathcal{P} and constrained by \mathcal{F} . The existence of CDTs is analyzed in [17, 37, 38]: constrained Delaunay triangulations always exists for $d = 2$ while for $d = 3$ they exist if any ridge formed by \mathcal{F} is *strongly Delaunay*. A simplex is strongly Delaunay if the circle \mathcal{C} of Definition 2.8 does not *enclose* any other point in \mathcal{X} . As shown in [37, 38], this condition can always be satisfied, possibly after the insertion of a finite number of regular nodes on non-strongly Delaunay ridges in the skeleton of \mathcal{F} . Moreover, once every boundary edge is strongly Delaunay, the restriction of the CDT on each interface is Delaunay.

Given the polytopic mesh \mathcal{T} , here we consider the constrained Delaunay triangulation of Ω with seeds $\mathcal{P} = \{\mathbf{x}_K^0\}_{K \in \mathcal{T}}$, possibly after the modification of \mathcal{F} discussed above. We refer to Figure 2.3 for an illustration. If desired, its qualitative parameters may be easily evaluated by constructing the auxiliary mesh using well established and efficient algorithms, cf. [17, 37, 38].

Remark 2.9. In the case of meshes with elements made of finite unions of star-shaped sub-polytopes, auxiliary meshes should be constructed starting from the centres of the sub-polytopes instead. The results below still hold, as long as local quasi-uniformity is assumed for the sub-mesh comprising the star-shaped sub-polytopes.

2.3.3. Auxiliary mesh interpolation and inverse estimates. We recall the following Scott-Zhang-type quasi-interpolation result [36, 35, 42, 14].

LEMMA 2.10 (Quasi-interpolant). *Let $\widehat{\mathcal{T}}$ be a shape-regular simplicial subdivision of Ω not containing any hanging nodes. Then, there exists a quasi-interpolation operator $I_h : H^1(\Omega) \rightarrow H^1(\Omega) \cap S_{\widehat{\mathcal{T}}}^1$, such that*

$$(2.8) \quad \|v - I_h v\|_T + h_T \|\nabla(v - I_h v)\|_T \leq C_{sz} h_T \|\nabla v\|_{\omega_T},$$

where ω_T denotes the patch of elements in $\widehat{\mathcal{T}}$ with non-empty intersection with \overline{T} . The constant $C_{sz} > 0$ depends only on the shape-regularity constant $\widehat{\tau}$ of the auxiliary mesh $\widehat{\mathcal{T}}$. If the function v has nonhomogeneous piecewise linear trace on $\partial\Omega$, we have $I_h v|_F = v|_F$, for all $F \subset \partial T \cap \partial\Omega$, $T \in \widehat{\mathcal{T}}$.

Moreover, we have

$$(2.9) \quad \|v - I_h v\|_{\partial K}^2 \leq C_{in} h_K \|\nabla v\|_{\widehat{\omega}_K}^2,$$

with $C_{in} > 0$ depending only on the shape-regularity of \mathcal{T} and of $\widehat{\mathcal{T}}$. Here, $\widehat{\omega}_K = \cup\{\omega_T : T \cap K \neq \emptyset, T \in \widehat{\mathcal{T}}\}$.

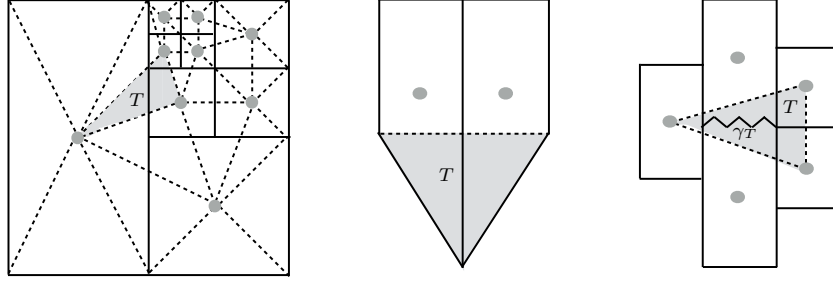


FIG. 2.4. *Constrained Delaunay auxiliary mesh examples. Different configurations for the set Γ_T of (2.10) for the auxiliary element T marked in grey. Left: $\Gamma_T = \Gamma_T^c$. Centre: $\Gamma_T = \Gamma_T^o$. Right: Γ_T^o is made of a single interface γ_T while the rest of Γ_T belongs to Γ_T^c . The simplices required in the proof of Lemma 2.12 to estimate the contributions from each edge in γ_T partially overlap (not shown).*

Proof. The proof of (2.8) can be found in [36, 35, 42, 14] for various levels of generality. Noting that (2.9) refers to the trace on the skeleton of the *original* polytopic mesh, we apply the trace inequality (2.4) with $\zeta = 1$,

$$\begin{aligned} \|v - I_h v\|_{\partial K}^2 &\leq C_{\text{tr}} (h_K^{-1} \|v - I_h v\|_K^2 + h_K \|\nabla(v - I_h v)\|_K^2) \\ &\leq C_{\text{tr}} \sum_{T \in \widehat{\mathcal{T}}: T \cap K \neq \emptyset} (h_K^{-1} \|v - I_h v\|_T^2 + h_K \|\nabla(v - I_h v)\|_T^2), \end{aligned}$$

and (2.9) follows from (2.8), depending on the shape-regularity of \mathcal{T} through C_{tr} and on the shape-regularity of $\widehat{\mathcal{T}}$ through (2.8). \square

The next polynomial inverse estimation result, relating L_2 -norms on subsets of the mesh skeleton Γ of \mathcal{T} with L_2 -norms over elements of the auxiliary mesh $\widehat{\mathcal{T}}$ will be important for the analysis below. In this context, for each $T \in \widehat{\mathcal{T}}$, we consider the set of *cut interfaces* obtained by the intersection of Γ with the simplex T , which we may characterise as follows:

$$(2.10) \quad \Gamma_T := \Gamma \cap T = \Gamma_T^c \cup \Gamma_T^o,$$

with Γ_T^c the set of interfaces $\gamma_T \in \Gamma_T \cap \Omega$ such that $\gamma_T \subset \partial K$ with K an element of \mathcal{T} whose centre of star-shapedness is a vertex of T , and $\Gamma_T^o := \Gamma_T \setminus \Gamma_T^c$. Note that the number of interfaces in Γ_T is bounded since the number of intersections of T with the elements $K \in \mathcal{T}$ is bounded; however, each such interface may be made of unbounded cut faces, depending on the complexity of the intersecting polytopic elements.

Remark 2.11. In the case of sub-mesh auxiliary meshes, we always have $\Gamma_T^o = \emptyset$. Instead, for constrained Delaunay auxiliary meshes, $\Gamma_T^o \neq \emptyset$ in general; we refer to Figure 2.4 for some examples.

LEMMA 2.12. *Let Assumptions 2.1 and 2.3 hold and let $\widehat{\mathcal{T}}$ be an auxiliary mesh, related to \mathcal{T} . Given $q \in \mathbb{N}$, for all $T \in \widehat{\mathcal{T}}$ and $v \in \mathcal{P}_q(T)$, we have*

$$(2.11) \quad \|v\|_{\Gamma_T}^2 \leq \frac{\widehat{C}_{\text{inv}}}{h_T} \|v\|_T^2.$$

The constant \widehat{C}_{inv} depends on d , q , and on shape-regularity and local quasi-uniformity of both \mathcal{T} and $\widehat{\mathcal{T}}$. If $\Gamma_T \equiv \Gamma_T^c$ then $\widehat{C}_{\text{inv}} = \tau(q+1)(q+d)$; otherwise, when $\Gamma_T^o \neq \emptyset$, the constant depends also on the number of overlappings n_o required to cover the elements of Γ_T^c , cf. (2.13). In particular, if $\Gamma_T^o \subset \partial\Omega$, then $n_o \leq d$.

Proof. We consider the two subset defined in (2.10) separately, starting with Γ_T^c . Exploiting the star-shapedness property of Γ_T^c with respect to the vertices of T , which is inherited from Assumption 2.1, the inverse inequality

$$\|v\|_{\Gamma_T^c}^2 \leq \frac{\tau(q+1)(q+d)}{h_T} \|v\|_T^2,$$

follows in the same way as (2.5).

Considering now the set Γ_T^o , we observe that each cut interface belonging to Γ_T^o may be partitioned into a set of $(d-1)$ -dimensional simplices. Indeed, each interface inherits a set of, possibly cut, simplicial faces from Γ . And, if a face $F \in \Gamma$ is only partially contained in Γ_T^o , then $F \cap T$ is still an interval if $d=2$ while it can always be subdivided into four triangles if $d=3$. Let now \widehat{F} be one such $(d-1)$ -dimensional simplex within Γ_T^o . We note that, if a simplex T has inradius r_T , then, for any given intersecting hyperplane Z , there exists a vertex V of T such that $d(V, Z) \geq r$. Otherwise, S must be contained in the region $\{\mathbf{x} \in \mathbb{R}^d : d(\mathbf{x}, Z) < r\}$, in contradiction with the fact that T contains a (closed) ball of radius r . It follows that we can always construct a non degenerate simplex $T_{\widehat{F}}$ by joining \widehat{F} with a vertex V of T such that $d(V, \widehat{F}) \geq r$. We thus have, cf. (2.5), the inverse estimate:

$$(2.12) \quad \|v\|_{\widehat{F}}^2 \leq \widehat{\tau} \frac{(q+1)(q+d)}{h_T} \|v\|_{T_{\widehat{F}}}^2 \quad \forall v \in \mathcal{P}_q(T).$$

Then, summing up over all $\widehat{F} \in \Gamma_T^o$ we conclude

$$(2.13) \quad \|v\|_{\Gamma_T^o}^2 \leq \frac{\widehat{\tau}(q+1)(q+d)}{h_T} \sum_{\widehat{F} \in \Gamma_T^o} \|v\|_{T_{\widehat{F}}}^2 \leq n_o \frac{\widehat{\tau}(q+1)(q+d)}{h_T} \|v\|_T^2,$$

with n_o the number of overlaps of the simplices $T_{\widehat{F}}$, $\widehat{F} \in \Gamma_T^o$. In particular, if Γ_T^o is only made of boundary interfaces, then $n_o \leq d$, as there may be at most d such interfaces, each made of a single $(d-1)$ -dimensional simplex. The required estimate now follows by summing up the contributions from Γ_T^c and Γ_T^o . \square

Remark 2.13. The constant \widehat{C}_{inv} appearing in (2.11) accounts for the complexity of the mesh in terms of topology and shape, quantified by the number n_o of overlap required to cover the mesh skeleton, see the mesh shown in Figure 2.4 (right) for an illustrative example. In typical practical cases, e.g., meshes stemming from standard algorithms such as Voronoi tessellations, as well as shape-regular adaptively generated meshes, we expect to have $\Gamma_T \equiv \Gamma_T^c$ for the vast majority of auxiliary elements. For instance, for the adaptively refined mesh with multiple hanging-nodes shown Figure 2.4 (left), Γ_T^o is either empty or it contains a single boundary edge, i.e., no overlaps are required, resulting in the ‘ideal’ constant $\widehat{C}_{\text{inv}} = \tau(q+1)(q+d)$ for each $T \in \widehat{\mathcal{T}}$.

2.4. Discontinuous Galerkin method. Let $\mathcal{V} := S_{\mathcal{T}} + H^1(\Omega)$. The symmetric interior penalty discontinuous Galerkin method reads: find $u_h \in S_{\mathcal{T}}$ such that

$$(2.14) \quad B(u_h, v_h) = \ell(v_h) \quad \text{for all } v_h \in S_{\mathcal{T}},$$

whereby $B(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ is defined by

$$(2.15) \quad \begin{aligned} B(w, v) := & \int_{\Omega} a \nabla_h w \cdot \nabla_h v \, d\mathbf{x} + \int_{\Gamma \setminus \Gamma_N} \sigma[[w]][[v]] \, ds, \\ & - \int_{\Gamma \setminus \Gamma_N} (\{a(\mathbf{\Pi} \nabla w) \cdot \mathbf{n}\} [[v]] + \{a(\mathbf{\Pi} \nabla v) \cdot \mathbf{n}\} [[w]]) \, ds, \end{aligned}$$

for $w, v \in \mathcal{V}$, and $\ell(\cdot) : \mathcal{V} \rightarrow \mathbb{R}$ by

$$\ell(v) := \int_{\Omega} f v \, d\mathbf{x} - \int_{\Gamma_D} g_D ((a(\mathbf{\Pi}\nabla v)) \cdot \mathbf{n} - \sigma v) \, ds + \int_{\Gamma_N} g_N v \, ds,$$

with $\mathbf{\Pi} : [L_2(\Omega)]^d \rightarrow [S_{\mathcal{T}}]^d$ denoting the orthogonal L_2 -projection operator onto the (vectorial) finite element space, and $\sigma \in L_{\infty}(\Gamma \setminus \Gamma_N)$ being the, so-called, *discontinuity-penalization function* given by

$$(2.16) \quad \sigma(\mathbf{x}) := \begin{cases} C_{\sigma} \max_{K \in \{K_i, K_j\}} \left\{ \frac{\bar{a}_K C_{\text{inv}}}{h_K} \right\}, & \mathbf{x} \in F \in \Gamma_{\text{int}}, \quad F \subseteq \partial K_i \cap \partial K_j, \\ C_{\sigma} \frac{\bar{a}_K C_{\text{inv}}}{h_K}, & \mathbf{x} \in F \in \Gamma_D, \quad F \subset \partial K, \end{cases}$$

with C_{σ} a positive constant and $\bar{a}_K := |\sqrt{a}|_2|_K$, $K \in \mathcal{T}$; here $|\cdot|_2$ denotes the natural matrix- l_2 -norm. The known dependence of the penalty on the local polynomial degree is included in C_{inv} for brevity; see [10, 8] for details. Note that, using (2.3) and that $[[v]] = 0$ on $\Gamma \setminus \Gamma_N$ for all $v \in H_D^1$, we have $B(u, v) = \ell(v)$ for all $v \in H_D^1$, with $u \in H^1(\Omega)$ the solution to (2.3).

Remark 2.14. To avoid further notational overhead, we opted in exposing the main results for element-wise constant diffusion tensors, i.e., $a \in (S_{\mathcal{T}}^0)^{d \times d}$, and for the classical interior penalty dG method. With minor modifications, the results below can also be extended to more general coefficients. Moreover, we expect that a corresponding analysis to what is presented below holds also for the interior penalty dG variants from [25, 22].

Upon defining the dG-norm by $\|v\| := (\|\sqrt{a}\nabla_h v\|^2 + \|\sqrt{\sigma}[[v]]\|_{\Gamma \setminus \Gamma_N}^2)^{1/2}$, we have the following result.

LEMMA 2.15. *Under Assumption 2.1, there exists $C_{\sigma} > 0$, such that*

$$(2.17) \quad B(v, v) \geq C_{\text{coer}} \|v\|^2 \quad \text{and} \quad B(w, v) \leq C_{\text{cont}} \|w\| \|v\| \quad \text{for all } v \in \mathcal{V},$$

respectively, with $C_{\text{coer}}, C_{\text{cont}} > 0$, independent of h and of p_K , $K \in \mathcal{T}$.

We refer to [10, 8] for the proof, whereby the respective trace inverse estimates are shown for this level of mesh generality. *A priori* error bounds are also available [10, 8].

3. A *posteriori* error analysis. The following analysis requires Assumptions 2.1 and 2.3 and that an auxiliary mesh according to Definition 2.8 is given.

We decompose the error into two components:

$$e = u - u_h = (u - u_c) + (u_c - u_h) =: e_c + e_d,$$

whereby $u_c \in H^1(\Omega)$ is the recovery of the discrete solution $u_h \in S_{\mathcal{T}}$, defined by

$$(3.1) \quad B(u_c, v) = B(u_h, v) \quad \forall v \in H_D^1(\Omega),$$

and $u_c = g_D$ on Γ_D . The existence and uniqueness of u_c is guaranteed by the Lax-Milgram Lemma.

Remark 3.1. The construction of u_c is known in the theory of finite element methods and has been used in various contexts, e.g., in [24] for the design of equilibrated flux *a posteriori* error estimators and in [39] for the analysis of domain decomposition preconditioners. A crucial reason of using this recovery instead of the averaging operator as in [30], is that it is essentially independent of the mesh geometry and topology; this is clearly helpful in the present context of very general polytopic meshes.

3.1. Bounding the non-conforming error e_d . Inspired by [19, 13], cf. also [5, 7], we decompose the nonconforming error $e_d = u_c - u_h$ further via a Helmholtz decomposition.

LEMMA 3.2. *For any $\mathbf{w} \in (L_2(\Omega))^d$, there exists $\xi \in H_D^1(\Omega)$ and $\phi \in [H^1(\Omega)]^{2d-3}$, $d = 2, 3$, such that*

$$(3.2) \quad a\mathbf{w} = a\nabla\xi + \text{curl}\phi \quad \text{in } \Omega,$$

and ϕ can be chosen so that

$$(3.3) \quad \text{curl}\phi \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N.$$

Moreover, the following relations hold

$$(3.4) \quad \|\sqrt{a}\mathbf{w}\|^2 = \|\sqrt{a}\nabla\xi\|^2 + \|a^{-1/2}\text{curl}\phi\|^2,$$

and

$$(3.5) \quad \|\nabla\phi\| \leq C_\Omega \|\text{curl}\phi\|,$$

with a constant $C_\Omega > 0$ only depending on Ω .

Proof. The proof of (3.2) is given in [19, Theorem 3.1] for the $d = 2$ case and is extended to $d = 3$ in [13]. Since $\nabla\xi$ is orthogonal to $\text{curl}\phi$, from the symmetry of the diffusion tensor a , the orthogonality (3.4) follows immediately. Finally, the proof of (3.5) can be found in [13]; for $d = 2$, we have $C_\Omega = 1$. \square

Condition (3.3) imposes a constraint on Γ_N for $\phi \in [H^1(\Omega)]^{2d-3}$. Namely, $\text{curl}\phi \cdot \mathbf{n}_F = 0$ on $F \subset \Gamma_N$, implying that $\phi \in [H^1(\Omega)]^3$ has constant components on each $(d-1)$ -dimensional planar subset of Γ_N .

We apply the Helmholtz decomposition with $\mathbf{w} = \nabla_h e_d$. Hence $\xi \in H_D^1(\Omega)$ and $\phi \in [H^1(\Omega)]^{2d-3}$ are such that $a\nabla_h e_d = a\nabla\xi + \text{curl}\phi$, and we have

$$(3.6) \quad \|\sqrt{a}\nabla_h e_d\|^2 = \int_\Omega a\nabla_h e_d \cdot \nabla\xi \, d\mathbf{x} + \int_\Omega \nabla_h e_d \cdot \text{curl}\phi \, d\mathbf{x}.$$

Since $u_c \in H^1(\Omega)$ with $u_c|_{\Gamma_D} = g_D$ and $\xi \in H_D^1(\Omega)$, (3.1) implies

$$\int_\Omega a\nabla_h e_d \cdot \nabla\xi \, d\mathbf{x} = - \int_{\Gamma_{\text{int}}} \{a(\mathbf{\Pi}\nabla\xi) \cdot \mathbf{n}\} [u_h] \, ds - \int_{\Gamma_D} (a(\mathbf{\Pi}\nabla\xi) \cdot \mathbf{n})(u_h - g_D) \, ds.$$

Hence, using the Cauchy-Schwarz inequality, the trace inverse estimate (2.5), the definition of σ , and the orthogonality (3.4), we have, respectively,

$$(3.7) \quad \begin{aligned} & \int_\Omega a\nabla_h e_d \cdot \nabla\xi \, d\mathbf{x} \\ & \leq \|\sigma^{-1/2} \{a(\mathbf{\Pi}\nabla\xi)\}\|_{\Gamma \setminus \Gamma_N} \left(\|\sqrt{\sigma}[u_h]\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{\sigma}(u_h - g_D)\|_{\Gamma_D}^2 \right)^{1/2} \\ & \leq \left(C_\sigma^{-1} \sum_{K \in \mathcal{T}} \|\mathbf{\Pi}(\sqrt{a}\nabla\xi)\|_K^2 \right)^{1/2} \left(\|\sqrt{\sigma}[u_h]\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{\sigma}(u_h - g_D)\|_{\Gamma_D}^2 \right)^{1/2} \\ & \leq (C_\sigma)^{-1/2} \|\sqrt{a}\nabla\xi\| \left(\|\sqrt{\sigma}[u_h]\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{\sigma}(u_h - g_D)\|_{\Gamma_D}^2 \right)^{1/2} \\ & \leq \|\sqrt{a}\nabla_h e_d\| \left(\|\sqrt{\sigma C_\sigma^{-1}}[u_h]\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{\sigma C_\sigma^{-1}}(u_h - g_D)\|_{\Gamma_D}^2 \right)^{1/2}. \end{aligned}$$

To bound the second term on the right-hand side of (3.6), we first decompose

$$(3.8) \quad \int_{\Omega} \nabla_h e_d \cdot \operatorname{curl} \phi \, d\mathbf{x} = \int_{\Omega} \nabla_h e_d \cdot \operatorname{curl}(\phi - I_h \phi) \, d\mathbf{x} + \int_{\Omega} \nabla_h e_d \cdot \operatorname{curl} I_h \phi \, d\mathbf{x},$$

with I_h the (component-wise if $d = 3$) quasi-interpolation operator of Lemma 2.10.

Starting with the first term, observing that $\operatorname{curl} \nabla u_c = \mathbf{0}$ and using the fact that $\phi \in [H^1(\Omega)]^{2d-3}$ satisfying (3.3), implying that ϕ is a constant function on each planar section of Γ_N , and choosing $\phi = I_h \phi$ on Γ_N . Then we have

$$\int_{\Omega} \nabla u_c \cdot \operatorname{curl}(\phi - I_h \phi) \, d\mathbf{x} = \int_{\Gamma_D} (\phi - I_h \phi) \nabla_T g_D \, ds.$$

Applying integration by parts, observing that $\operatorname{curl} \nabla e_d|_K = \mathbf{0}$, and using $(\phi - I_h \phi)$ is single valued on each face, and $(\phi - I_h \phi)_{\Gamma_N} = 0$, yields

$$(3.9) \quad \begin{aligned} & \sum_{K \in \mathcal{T}} \int_K \nabla e_d \cdot \operatorname{curl}(\phi - I_h \phi) \, d\mathbf{x} \\ &= - \int_{\Gamma} (\phi - I_h \phi) \llbracket \nabla_T u_h \rrbracket \, ds + \int_{\Gamma_D} (\phi - I_h \phi) \nabla_T g_D \, ds \\ &= - \int_{\Gamma_{\text{int}}} (\phi - I_h \phi) \llbracket \nabla_T u_h \rrbracket \, ds - \int_{\Gamma_D} (\phi - I_h \phi) \nabla_T (u_h - g_D) \, ds. \end{aligned}$$

Further, using (2.9) and, finally, (3.5) and (3.4), the right-hand side of (3.9) can be further estimated from above by

$$(3.10) \quad \begin{aligned} & \sum_{K \in \mathcal{T}} \|h_K^{-1/2}(\phi - I_h \phi)\|_{\partial K \setminus \Gamma_N} \left(\|\sqrt{h_K} \llbracket \nabla_T u_h \rrbracket\|_{\partial K \cap \Gamma_{\text{int}}} + \|\sqrt{h_K} \nabla_T (u_h - g_D)\|_{\partial K \cap \Gamma_D} \right) \\ & \leq \left(\sum_{K \in \mathcal{T}} C_{\text{tr}} C_I \|\nabla \phi\|_{\tilde{\omega}_K}^2 \right)^{1/2} \left(\|\sqrt{h} \llbracket \nabla_T u_h \rrbracket\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{h} \nabla_T (u_h - g_D)\|_{\Gamma_D}^2 \right)^{1/2} \\ & \leq C_1 \|\nabla \phi\|_{\Omega} \left(\|\sqrt{h} \llbracket \nabla_T u_h \rrbracket\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{h} \nabla_T (u_h - g_D)\|_{\Gamma_D}^2 \right)^{1/2} \\ & \leq C_1 C_{\Omega} \sqrt{\alpha^*} \|\sqrt{a} \nabla_h e_d\| \left(\|\sqrt{h} \llbracket \nabla_T u_h \rrbracket\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{h} \nabla_T (u_h - g_D)\|_{\Gamma_D}^2 \right)^{1/2}, \end{aligned}$$

for C_1 a constant depending only on the shape-regularity constant $\hat{\tau}$ of the auxiliary mesh $\hat{\mathcal{T}}$, and on the local quasi-uniformity constants ρ and $\hat{\rho}$.

We now consider the second term in (3.8). Since $I_h \phi \in [H^1(\Omega)]^{2d-3}$, we have $\nabla \cdot \operatorname{curl} I_h \phi = 0$ on Ω and, hence, $\llbracket \operatorname{curl} I_h \phi \cdot \mathbf{n} \rrbracket = 0$ on Γ_{int} . Moreover, given that $I_h \phi$ is constant on each component of Γ_N , we also have $\operatorname{curl} I_h \phi \cdot \mathbf{n} = 0$ on Γ_N . Then, integration by parts and working as above gives

$$\begin{aligned} \int_{\Omega} \nabla_h e_d \cdot \operatorname{curl} I_h \phi \, d\mathbf{x} &= \sum_{K \in \mathcal{T}} \int_{\partial K} e_d \mathbf{n}_K \cdot \operatorname{curl} I_h \phi \, ds \\ &= - \int_{\Gamma_{\text{int}}} \llbracket u_h \rrbracket (\mathbf{n} \cdot \operatorname{curl} I_h \phi) \, ds - \int_{\Gamma_D} (u_h - g_D) (\mathbf{n} \cdot \operatorname{curl} I_h \phi) \, ds. \end{aligned}$$

Next, applying to $I_h \phi$ the trace inverse inequality with respect to the auxiliary mesh $\hat{\mathcal{T}}$ given

in (2.11), we obtain

$$\begin{aligned}
& \int_{\Omega} \nabla_h e_d \cdot \operatorname{curl} I_h \phi \, d\mathbf{x} \\
& \leq (\|\sqrt{\sigma} \llbracket u_h \rrbracket\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{\sigma}(u_h - g_D)\|_{\Gamma_D}^2)^{1/2} \left(\sum_{T \in \widehat{\mathcal{T}}} \|\sigma^{-1/2} \operatorname{curl} I_h \phi\|_{T \cap (\Gamma \setminus \Gamma_N)}^2 \right)^{1/2} \\
& \leq (\|\sqrt{\sigma} \llbracket u_h \rrbracket\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{\sigma}(u_h - g_D)\|_{\Gamma_D}^2)^{1/2} \left(\sum_{T \in \widehat{\mathcal{T}}} \widehat{C}_{\text{inv}} h_T^{-1} \|\sigma^{-1/2} \operatorname{curl} I_h \phi\|_T^2 \right)^{1/2} \\
& \leq C_2 (\|\sqrt{\sigma C_{\sigma}^{-1}} \llbracket u_h \rrbracket\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{\sigma C_{\sigma}^{-1}}(u_h - g_D)\|_{\Gamma_D}^2)^{1/2} \|\operatorname{curl} I_h \phi\|,
\end{aligned}$$

for $C_2 > 0$ constant depending on C_{inv} , \widehat{C}_{inv} , $\widehat{\rho}$, and on α^* . Next, we use the stability of I_h from Lemma 2.10, together with (3.5) to deduce

$$(3.11) \quad \|\operatorname{curl} I_h \phi\|_{\Omega} \leq C_c \|\nabla \phi\|_{\Omega} \leq C_c C_{\Omega} \sqrt{\alpha_*} \|a^{-1/2} \nabla \phi\| \leq C_c C_{\Omega} \sqrt{\alpha_*} \|\sqrt{a} \nabla_h e_d\|.$$

Hence, combing (3.7), (3.9), (3.10), and (3.11), we arrive at the bound

$$\begin{aligned}
(3.12) \quad \|\sqrt{a} \nabla_h e_d\|^2 & \leq C_{\text{nc}} \left(\|\sqrt{\sigma C_{\sigma}^{-1}} \llbracket u_h \rrbracket\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{\sigma C_{\sigma}^{-1}}(u_h - g_D)\|_{\Gamma_D}^2 \right. \\
& \quad \left. + \|\sqrt{h} \llbracket \nabla_T u_h \rrbracket\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{h} \nabla_T (u_h - g_D)\|_{\Gamma_D}^2 \right);
\end{aligned}$$

the constant $C_{\text{nc}} > 0$ depends on C_{inv} , \widehat{C}_{inv} , $\widehat{\rho}$, C_{Ω} , and on α^* , α_* , but is independent from h and the number and measure of the mesh faces.

Remark 3.3. We stress that the mesh-size h in (3.12) is the local element diameter for \mathcal{T} , i.e., *independent* of the measure of the faces and the number of faces per element. This new bound refines the, now classical, results in [30], by showing that the dG error has, in fact, two sources: the normal flux and the tangential gradient. By applying the inverse inequality on each face F , the L_2 -norm of the tangential jump can be bounded from above by the L_2 -norm of the jump term itself, thus recovering the bound in [30]. However, such bound would be proportional to $\operatorname{diam}(K)/\operatorname{diam}(F)$, for each face $F \in \Gamma$, which may be severely pessimistic for increasingly small faces F .

3.2. Bounding the conforming error e_c . For $v \in H_D^1$, we have

$$(3.13) \quad B(e, v) = \ell(v) - B(u_h, v) = \ell(\eta) - B(u_h, \eta),$$

with $\eta := v - v_h$, for any $v_h \in S_{\mathcal{T}}$. Recalling that $e = e_c + e_d$, since $e_c \in H_D^1$ we can fix $v = e_c$ in (3.13) to further deduce

$$(3.14) \quad \|\sqrt{a} \nabla e_c\|^2 = B(e_c, e_c) = (\ell(\eta) - B(u_h, \eta)) - B(e_d, e_c) = \ell(\eta) - B(u_h, \eta),$$

from (3.1). The right-hand side of (3.14) can now be bounded via standard arguments [30]: integration by parts, application of [4, Eq. (3.3)], the observation that $\mathbf{\Pi} \nabla u_h = \nabla u_h$, and elementary manipulations yield

$$\begin{aligned}
(3.15) \quad \ell(\eta) - B(u_h, \eta) & = \int_{\Omega} (f + \nabla_h \cdot (a \nabla_h u_h)) \eta \, d\mathbf{x} - \int_{\Gamma_{\text{int}}} \llbracket a \nabla u_h \cdot \mathbf{n} \rrbracket \{\eta\} \, ds \\
& \quad - \int_{\Gamma_N} (a \nabla u_h \cdot \mathbf{n} - g_N) \eta \, ds - \int_{\Gamma_{\text{int}}} \sigma \llbracket u_h \rrbracket \llbracket \eta \rrbracket \, ds - \int_{\Gamma_D} \sigma (u_h - g_D) \eta \, ds \\
& \quad + \int_{\Gamma_{\text{int}}} \llbracket a(\mathbf{\Pi} \nabla \eta) \cdot \mathbf{n} \rrbracket \llbracket u_h \rrbracket \, ds + \int_{\Gamma_D} (a(\mathbf{\Pi} \nabla \eta) \cdot \mathbf{n})(u_h - g_D) \, ds =: \sum_{i=1}^7 T_i.
\end{aligned}$$

Setting $\eta = e_c - \Pi_0 e_c$ and using (2.6), we have

$$(3.16) \quad T_1 \leq C_{PF} \alpha_*^{-1/2} \left(\sum_{K \in \mathcal{T}} \|h_K (f + \nabla \cdot (a \nabla u_h))\|_K^2 \right)^{1/2} \|\sqrt{a} \nabla e_c\|.$$

Employing (2.7), along with standard manipulations, we also have

$$(3.17) \quad \begin{aligned} T_2 + T_3 &\leq \sum_{K \in \mathcal{T}} \sum_{F \subset \partial K \cap \Gamma_{\text{int}}} \|h_K^{-1/2} \eta\|_F \|\sqrt{h_K} [a \nabla u_h \cdot \mathbf{n}]\|_F \\ &+ \sum_{K \in \mathcal{T}} \sum_{F \subset \partial K \cap \Gamma_N} \|h_K^{-1/2} \eta\|_F \|\sqrt{h_K} (a \nabla u_h \cdot \mathbf{n} - g_N)\|_F \\ &\leq \tilde{C}_{PF} \alpha_*^{-1/2} \|\sqrt{a} \nabla e_c\| \left(\|\sqrt{h} [a \nabla u_h \cdot \mathbf{n}]\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{h} (a \nabla u_h \cdot \mathbf{n} - g_N)\|_{\Gamma_N}^2 \right)^{1/2}. \end{aligned}$$

Similarly, using the definition of σ from (2.16), we have

$$(3.18) \quad \begin{aligned} T_4 + T_5 &\leq \left(\|\sqrt{\sigma} [u_h]\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{\sigma} (u_h - g_D)\|_{\Gamma_D}^2 \right)^{1/2} \|\sigma^{1/2} [\eta]\|_{\Gamma \setminus \Gamma_N} \\ &\leq \left(\sum_{K \in \mathcal{T}} \max_{F \in \partial K \setminus \Gamma_N} \sigma \|\eta\|_{\partial K \setminus \Gamma_N}^2 \right)^{1/2} \left(\|\sqrt{\sigma} [u_h]\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{\sigma} (u_h - g_D)\|_{\Gamma_D}^2 \right)^{1/2} \\ &\leq \sqrt{C_\sigma \alpha^* / \alpha_*} C_{\text{inv}} \tilde{C}_{PF} \rho \|\sqrt{a} \nabla e_c\| \left(\|\sigma^{1/2} [u_h]\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{\sigma} (u_h - g_D)\|_{\Gamma_D}^2 \right)^{1/2}. \end{aligned}$$

Now, using the trace inverse estimate (2.5), the stability of the L_2 -projection operator and that $\nabla \eta|_K = \nabla e_c|_K$, we deduce

$$(3.19) \quad \begin{aligned} T_6 + T_7 &\leq \left(\|\sqrt{\sigma} [u_h]\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{\sigma} (u_h - g_D)\|_{\Gamma_D}^2 \right)^{1/2} \|\sigma^{-1/2} \{a(\mathbf{\Pi} \nabla e_c) \cdot \mathbf{n}\}\|_{\Gamma \setminus \Gamma_N} \\ &\leq \left(\sum_{K \in \mathcal{T}} \max_{F \in \partial K \setminus \Gamma_N} \sigma^{-1} \|\sqrt{a}(\mathbf{\Pi}(\sqrt{a} \nabla e_c)) \cdot \mathbf{n}\|_F^2 \right)^{1/2} \\ &\quad \times \left(\|\sqrt{\sigma} [u_h]\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{\sigma} (u_h - g_D)\|_{\Gamma_D}^2 \right)^{1/2} \\ &\leq C_\sigma^{-1/2} \rho \|\sqrt{a} \nabla e_c\| \left(\|\sqrt{\sigma} [u_h]\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{\sigma} (u_h - g_D)\|_{\Gamma_D}^2 \right)^{1/2}. \end{aligned}$$

Hence, by collecting above bounds (3.16), (3.17), (3.18), (3.19) and (3.15), we arrive to the following bound on the conforming error:

$$(3.20) \quad \begin{aligned} \|\sqrt{a} \nabla e_c\| &\leq C_{\text{co}} \left(\|h(f + \nabla_h \cdot (a \nabla_h u_h))\|^2 + \|\sqrt{\sigma} [u_h]\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{\sigma} (u_h - g_D)\|_{\Gamma_D}^2 \right. \\ &\quad \left. + \|\sqrt{h} [a \nabla u_h \cdot \mathbf{n}]\|_{\Gamma_{\text{int}}}^2 + \|\sqrt{h} (a \nabla u_h \cdot \mathbf{n} - g_N)\|_{\Gamma_N}^2 \right)^{1/2}, \end{aligned}$$

with C_{co} depending on C_σ , ρ , τ , the polynomial degree p , C_{PF} , and \tilde{C}_{PF} , but independent of h and the number and measure of the elemental faces.

We are now ready to present the *a posteriori* error upper bound.

THEOREM 3.4 (upper bound). *Let u be the solution of (2.1) and let $u_h \in S_{\mathcal{T}}$ be its dG approximation on a polytopic mesh satisfying Assumptions 2.1 and 2.3. Also let an auxiliary mesh according to Definition 2.8 is given. Then, we have the following *a posteriori* error bound*

$$(3.21) \quad \| \|u - u_h\| \|^2 \leq C_{\text{up}} \sum_{K \in \mathcal{T}} (R_K^2 + O_K^2),$$

with the local estimator $R_K^2 = R_{K,E}^2 + R_{K,N}^2 + R_{K,J}^2 + R_{K,T}^2$, and the data oscillation $O_K^2 = O_{K,E}^2 + O_{K,N}^2 + O_{K,J}^2 + O_{K,T}^2$, given by

$$\begin{aligned} R_{K,E} &:= \|h(\Pi f + \nabla \cdot (a \nabla u_h))\|_K, \\ R_{K,N} &:= (\|\sqrt{h} [a \nabla u_h \cdot \mathbf{n}] \|_{\partial K \cap \Gamma_{\text{int}}}^2 + \|\sqrt{h} (a \nabla u_h \cdot \mathbf{n} - \bar{g}_N) \|_{\partial K \cap \Gamma_N}^2)^{1/2}, \\ R_{K,J} &:= (\|\sqrt{\sigma} [u_h] \|_{\partial K \cap \Gamma_{\text{int}}}^2 + \|\sqrt{\sigma} (u_h - \bar{g}_D) \|_{\partial K \cap \Gamma_D}^2)^{1/2}, \\ R_{K,T} &:= (\|\sqrt{h} [\nabla_T u_h] \|_{\Gamma_{\text{int}}}^2 + \|\sqrt{h} \nabla_T (u_h - \bar{g}_D) \|_{\Gamma_D}^2)^{1/2}, \\ O_{K,E} &:= \|h(f - \Pi f)\|_K, \quad O_{K,N} := \|\sqrt{h} (g_N - \bar{g}_N) \|_{\partial K \cap \Gamma_N}, \\ O_{K,J} &:= \|\sqrt{\sigma} (g_D - \bar{g}_D) \|_{\partial K \cap \Gamma_D}, \quad O_{K,T} := \|\sqrt{h} \nabla_T (g_D - \bar{g}_D) \|_{\Gamma_D}, \end{aligned}$$

with C_{up} depending on C_{co} and C_{nc} only, but is independent of h and of the number and measure of the elemental faces; here, for any for $K \in \mathcal{T}$, such that $\partial K \cap \Gamma_S \neq \emptyset$ with $S \in \{D, N\}$, we set $\bar{g}_S|_{\partial K \cap \Gamma_S} \in \mathcal{P}_{p_k}(\partial K \cap \Gamma_S)$ with g_S denoting an approximation of the Dirichlet and Neumann data, respectively.

Proof. The proof follows immediately from the bounds (3.12) and (3.20), together with the triangle inequality $\|u - u_h\| \leq \|e_c\| + \|e_d\|$. \square

Remark 3.5. In the above, we followed a known approach in splitting the estimator into a ‘residual part’ and a ‘data oscillation part’, assuming that $f \in L_2(\Omega)$ and for sufficiently smooth boundary data. In this setting the data oscillation error is typically dominated by the residual estimators. However, if the forcing data $f \in H^{-1}(\Omega)$, then data oscillation may dominate the error [33]. It would be an interesting future development to investigate the approach from [33] in the context of discontinuous Galerkin methods.

Remark 3.6. Theorem 3.4 has been proven under Assumption 2.1(b) which disallows boundary faces with arbitrarily small size relative to the local mesh size. This assumption is reasonable in as much resolution of the problem domain is required in order for the numerical solution to incorporate the boundary conditions. Nevertheless, this assumption can be relaxed in the case of Dirichlet boundary conditions as follows. Noting that the latter is only required to construct the interpolant of the divergence-free component ϕ of the non-conforming error, which is not constrained on the Dirichlet boundary. Thus, the interpolant may be constructed for an extension $\tilde{\phi}$, e.g., as a Stein-type extension operator) defined on an extended domain $\tilde{\Omega} \supset \Omega$ whose respective mesh $\tilde{\mathcal{T}}$ would correspond ‘closely’ to the primal mesh \mathcal{T} and is constructed so that it may contain *no* small boundary faces. The resulting bounds, however, would depend on the, typically unknown, boundedness constant of the extension operator.

4. Lower bounds. We now derive lower bounds for the a posteriori error estimator of Theorem 3.4. Of particular interest is the extend to which the efficiency of the estimator can be shown to be independent of the number and of the relative sizes of $(d-1)$ -dimensional faces in the mesh. The situation differs for the elemental residual and face jump residuals; for clarity, we deal with them separately.

4.1. Elemental residual. Lower bounds for the elemental residual can be derived under no further assumptions on the mesh. The analysis is based on a new element bubble function and some auxiliary results.

LEMMA 4.1 ([8, Corollary 4.24]). *Let \mathcal{T} satisfy the Assumption 2.1. Then, for each $K \in \mathcal{T}$, $p \in \mathbb{N}$ and $v \in \mathcal{P}_p(K)$, the following inverse inequality holds*

$$(4.1) \quad \|\nabla v\|_K^2 \leq C_{\text{inv},K} h_K^{-2} \|v\|_K^2.$$

with $C_{\text{inv},K}$ a positive constant depending only on d , p and τ . Note also the trivial inequality $\|\text{curl } v\|_K^2 \leq (d-1) \|\nabla v\|_K^2$.

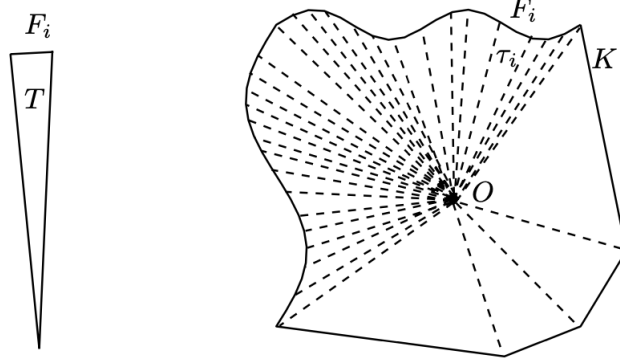


FIG. 4.1. Left: a triangle with one ‘small’ face. Right: a polytopic element with many ‘small’ faces.

Next, for a generic d -dimensional simplex T , we denote its barycentric co-ordinates by λ_T^i , $d = 0, \dots, d$, and denote by F_i , $i = 0, \dots, d$ the corresponding $(d-1)$ -dimensional simplicial face of T such that $\lambda_T^i|_{F_i} = 0$. Note that $\|\nabla \lambda_T^i\|_{L^\infty(T)} = d|F_i|/|T|$, since $\nabla \lambda_T^i$ is constant. Importantly, the maximum norm is determined by the distance of the i -th vertex from the face F_i , but it is *independent of the measure of face F_i* , see Figure 4.1 (left) for an illustration.

Let $K \in \mathcal{T}$ and let m_K be the number of its faces. Given that K is star-shaped by Assumption 2.1, we can construct a non-overlapping subdivision of K into m_K simplicial sub-elements τ_j by joining the face F_j , $j = 1, \dots, m_K$, of K with the centre of the largest ball inscribed in K ; see Figure 4.1 (right) for an illustration. Note that $h_K \geq \text{diam}(\tau_j) \geq r_K \geq \tau^{-1}h_K$. Moreover, letting $\lambda_j := \lambda_{\tau_j}^i$ with i such that $\lambda_{\tau_j}^i$ is the barycentric coordinate of τ_j corresponding to the vertex of τ_j which is internal to K , it follows that

$$(4.2) \quad h_K^{-1} \leq \|\nabla \lambda_j\|_{L^\infty(\tau_j)} = |\partial \tau_j \cap \partial K|/|\tau_j| \leq \tau h_K^{-1}.$$

DEFINITION 4.2 (Element bubble). *Let $K \in \mathcal{T}$ and let m_K be the number of its faces. With the above notation, the element bubble function b_K is defined as*

$$(4.3) \quad b_K|_{\tau_j} = \lambda_j,$$

for $j = 1, \dots, m_k$.

By construction, b_K is a continuous piecewise polynomial function with zero trace and with values in $[0, 1]$ on K . Next, we will derive some important properties of the new bubble function (4.3).

LEMMA 4.3. *For each $K \in \mathcal{T}$ satisfying the Assumption 2.1 and for each $v \in \mathcal{P}_p(K)$, we have*

$$(4.4) \quad \|\nabla(b_K v)\|_K^2 \leq 2((d\tau)^2 + C_{\text{inv},K})h_K^{-2}\|v\|_K^2,$$

with τ as in Assumption 2.1 and $C_{\text{inv},K}$ is in (4.1), and

$$(4.5) \quad \|v\|_K^2 \leq C_{b,K}\|b_K^{1/2}v\|_K^2,$$

with $C_{b,K} := [2(p+2)]^{2d} \frac{d^{d+1}}{(d-1)!}$.

Proof. By using the triangle inequality, the bound (4.2), and the inverse inequality (4.1), we have

$$\begin{aligned}
\|\nabla(b_K v)\|_K^2 &\leq 2\|(\nabla b_K)v\|_K^2 + 2\|b_K(\nabla v)\|_K^2 \\
&\leq 2\sum_{j=1}^{m_K}\|\nabla\lambda_j\|_{L^\infty(\tau_j)}^2\|v\|_{\tau_j}^2 + 2\|b_K\|_{L^\infty(K)}^2\|\nabla v\|_K^2 \\
(4.6) \quad &\leq 2\tau^2 h_K^{-2}\sum_{i=1}^{m_K}\|v\|_{\tau_i}^2 + 2C_{\text{inv},K}h_K^{-2}\|v\|_K^2 = 2((d\tau)^2 + C_{\text{inv},K})h_K^{-2}\|v\|_K^2,
\end{aligned}$$

which is the bound required in (4.4). We now prove the norm equivalence relation (4.5). Recalling the norm equivalence relation for each $v \in \mathcal{P}_p(T)$ on a simplex T from [43, Section 3.6], we have

$$(4.7) \quad \|v\|_T^2 \leq [2(p+2)]^{2d} \left(\frac{d}{d+1}\right)^{d+1} \frac{1}{(d-1)!} \|(\psi_T)^{1/2}v\|_T^2,$$

with $\psi_T := (d+1)^{d+1}(\prod_{i=0}^d \lambda_T^i)$. Then, by using $\|\lambda_T^i\|_{L^\infty(T)} = 1$, $i = 0, \dots, d$, we deduce

$$(4.8) \quad \|v\|_T^2 \leq C_{b,K} \|(\lambda_T^i)^{1/2}v\|_T^2,$$

for each $i \in \{0, \dots, d\}$. Hence, the bound (4.5) is proven using the definition of $b_K(x)$ in (4.3):

$$(4.9) \quad \|v\|_K^2 = \sum_{j=1}^{m_K} \|v\|_{\tau_j}^2 \leq \sum_{j=1}^{m_K} C_{b,K} \|(\lambda_j)^{1/2}v\|_{\tau_j}^2 = C_{b,K} \|b_K^{1/2}v\|_K^2. \quad \square$$

Remark 4.4. The new element bubble function on polytopic meshes in the above Lemma is different from the classical element bubble function on simplices. In particular, we note that the important relations (4.4) and (4.5) are independent of the number and measure of the faces of the element K .

THEOREM 4.5 (Elemental residual lower bound). *Let u be the solution of (2.1) and let $u_h \in S_{\mathcal{T}}$ be its dG approximation under Assumption 2.1 and 2.3. Then, for each $K \in \mathcal{T}$, we have*

$$(4.10) \quad \|h(\Pi f + \nabla(a\nabla u_h))\|_K^2 \leq 2C_{b,K}(2(\tau^2 + C_{\text{inv},K})(\alpha^*)^2\|\sqrt{a}\nabla e\|_K^2 + O_{K,E}^2).$$

Proof. We fix $v \in H_D^1(\Omega)$ as $v|_K = b_K(\Pi f + \nabla \cdot (a\nabla u_h))$, where b_K is the element bubble function in (4.3), and extended to zero outside K . Using relations (3.13), (3.15), (4.4), and $b_K \leq 1$, we obtain

$$\begin{aligned}
\int_K b_K(\Pi f + \nabla \cdot (a\nabla u_h))^2 \, dx &= \int_K (\Pi f - f)v \, dx + \int_K a\nabla e \cdot \nabla v \, dx \\
&\leq \|\Pi f - f\|_K \|v\|_K + \|\sqrt{a}\nabla e\|_K \|\sqrt{a}\nabla v\|_K \\
(4.11) \quad &\leq (\|\Pi f - f\|_K + \sqrt{2(\tau^2 + C_{\text{inv},K})\alpha^* h_K^{-1}} \|\sqrt{a}\nabla e\|_K) \|(\Pi f + \nabla \cdot (a\nabla u_h))\|_K,
\end{aligned}$$

Recalling (4.5), we have $\|\Pi f + \nabla \cdot (a\nabla u_h)\|_K^2 \leq C_{b,K} \|b_K^{1/2}(\Pi f + \nabla \cdot (a\nabla u_h))\|_K^2$ from which the result (4.10) already follows. \square

4.2. Flux residuals. In view of proving the lower bound for the flux residuals, we require the number of faces of each element to be uniformly bounded. Furthermore, in the case $d = 3$ we shall assume that each face F is shape-regular. Note that such assumptions still allows for arbitrarily small faces.

ASSUMPTION 4.6. *The number of faces of every element $K \in \mathcal{T}$ is uniformly bounded. For $d = 3$ only, for every $F \in \Gamma_{\text{int}}$, the radius r_F of the largest $(d-1)$ -dimensional ball inscribed in F satisfies $r_F \geq \tau^{-1}h_F$, with τ as in Assumption 2.1.*

Next, we construct the face bubble function. Here we note that the size of a mesh face can be arbitrarily smaller than that of the elements it belongs to. Taking this into account, we consider the standard face bubble functions supported in a pair of simplices contained in the neighbouring elements, instead than the elements itself.

DEFINITION 4.7 (Face bubble). *Let $K \in \mathcal{T}$ and $F \subset \partial K$ a mesh face satisfying Assumption 4.6. Define $T_F^K \subset K$ as the simplex having F as a face and opposite vertex the point at distance h_F from F along the segment joining the barycentre of F with the centre of star-shapedness of K . The face bubble function b_F is defined on K as the standard edge bubble function of T_F^K , cf. [2, 30], extended by zero to the rest of K .*

LEMMA 4.8. *Let $K \in \mathcal{T}$ and $F \subset \partial K$ a mesh face under Assumption 2.1. Let $v \in \mathcal{P}_p(F)$ and, with slight abuse of notation, denote by the symbol v also the constant extension in T_F^K of v in the direction normal to F . We have*

$$(4.12) \quad \|v\|_F^2 \leq C_{b,F} \|b_F^{1/2} v\|_F^2,$$

with $C_{b,F} := [2(p+2)]^{2(d-1)} \frac{(d-1)^d}{(d-2)!}$, and

$$(4.13) \quad \|\nabla(b_F v)\|_{T_F^K}^2 \leq C_{\text{inv},T} h_F^{-1} \|v\|_F^2.$$

with $C_{\text{inv},T}$ the constant of the inverse inequality (4.1) in the case of $d-1$ -dimensional simplices. Moreover, we have

$$(4.14) \quad \|\text{curl}(b_T v)\|_{T_F^K}^2 \leq (d-1) C_{\text{inv},T} h_F^{-1} \|v\|_F^2.$$

Proof. The bound (4.12) is given in [30] and (4.13) follows immediately from (4.1), the fact that $b_F \leq 1$ and the fact that $\|v\|_{T_F^K}^2 \leq h_F \|[a \nabla u_h \cdot \mathbf{n}]\|_F^2$. Finally, the bound (4.14) is a trivial consequence of (4.13) observing once again that $\|\text{curl} v\|_K^2 \leq (d-1) \|\nabla v\|_K^2$. \square

THEOREM 4.9 (flux residuals lower bound). *Let u be the solution of (2.1) and let $u_h \in S_{\mathcal{T}}$ be its dG approximation under Assumption 2.1, 2.3 and 4.6. Then, for each $K \in \mathcal{T}$, we have*

$$(4.15) \quad \begin{aligned} & \|h^{1/2} \llbracket a \nabla u_h \cdot \mathbf{n} \rrbracket\|_{\partial K \cap \Gamma_{\text{int}}}^2 \\ & \leq 6C_{b,F} \left(2C_{b,K} (\tau^2 + C_{\text{inv},K}) (\alpha^*)^2 \|\sqrt{a} \nabla e\|_{\omega_K}^2 + \sum_{K' \in \omega_K} (1 + 2C_{b,K}) O_{K',E}^2 \right. \\ & \quad \left. + \alpha^* C_{\text{inv},T} \sum_{F \in \partial K \cap \Gamma_{\text{int}}} h_F^{-1} h_{F^\perp} \|\sqrt{a} \nabla e\|_{\omega_F}^2 \right), \end{aligned}$$

and

$$(4.16) \quad \|h^{1/2} \llbracket \nabla_T u_h \rrbracket\|_{\Gamma_{\text{int}}}^2 \leq 2C_{b,F} (d-1) \alpha_* C_{\text{inv},T} \sum_{F \in \partial K \cap \Gamma_{\text{int}}} h_F^{-1} h_{F^\perp} \|\sqrt{a} \nabla e\|_{\omega_F}^2.$$

Here, ω_K is the patch of elements neighbouring K , $\omega_F = T_F^K \cup T_F^{K'}$ with K' the element neighbouring K across F and $h_{F^\perp} := \max\{h_K, h_{K'}\}$.

Proof. In view of proving (4.15), we first consider any $F \in \partial K$ with $F \in \Gamma_{\text{int}}$. Further, we fix $v \in H^1(\omega_F)$ as the constant extension of $\llbracket a \nabla u_h \cdot \mathbf{n} \rrbracket_F$ in the direction normal to F , so that

$b_F v \in H_0^1(\omega_F)$, cf. Lemma 4.8. Then, testing the error equation (3.13) with $b_F v$ prolonged to zero on the whole of Ω , we get

$$\begin{aligned} \int_{\omega_F} a \nabla_h e \cdot \nabla (b_F v) \, d\mathbf{x} &= \int_{\omega_F} (f - \Pi f)(b_F v) \, d\mathbf{x} + \int_{\omega_F} (\Pi f + \nabla \cdot (a \nabla u_h))(b_F v) \\ &\quad - \int_F [[a \nabla u_h \cdot \mathbf{n}]]^2 b_F \, ds. \end{aligned}$$

From this, using (4.13) and the fact that, $\|b_F v\|_K^2 = \|b_F v\|_{T_F^K}^2 \leq h_F \|[[a \nabla u_h \cdot \mathbf{n}]]\|_F^2$, the same bound being true on K' , we obtain

$$\begin{aligned} \|[[a \nabla u_h \cdot \mathbf{n}]] b_F^{1/2}\|_F^2 &\leq \sum_{\mathcal{K}=K, K'} \left[(\|\Pi f - f\|_{T_F^K} + \|(\Pi f + \nabla \cdot (a \nabla u_h))\|_{T_F^K}) \|b_F v\|_{T_F^K} \right. \\ &\quad \left. + \|\sqrt{a} \nabla e\|_{T_F^K} \|\sqrt{a} \nabla (b_F v)\|_{T_F^K} \right] \\ &\leq \sum_{\mathcal{K}=K, K'} \left[h_F^{1/2} (\|\Pi f - f\|_{T_F^K} + \|(\Pi f + \nabla \cdot (a \nabla u_h))\|_{T_F^K}) \right. \\ &\quad \left. + (\alpha^* C_{\text{inv}, T})^{1/2} h_F^{-1/2} \|\sqrt{a} \nabla e\|_{T_F^K} \right] \|[[a \nabla u_h \cdot \mathbf{n}]]\|_F. \end{aligned}$$

This, together with (4.12), gives

$$\begin{aligned} \|h^{1/2} [[a \nabla u_h \cdot \mathbf{n}]]\|_F^2 &\leq 2C_{b, F} \sum_{\mathcal{K}=K, K'} \left[h_F^{1/2} h_{\mathcal{K}}^{-1/2} (\|h(\Pi f - f)\|_{T_F^K} + \|h(\Pi f + \nabla \cdot (a \nabla u_h))\|_{T_F^K}) \right. \\ &\quad \left. + (\alpha^* C_{\text{inv}, T})^{1/2} h_F^{-1/2} h_{\mathcal{K}}^{1/2} \|\sqrt{a} \nabla e\|_{T_F^K} \right]^2. \end{aligned}$$

As $h_F \leq h_{F^\perp} := \max\{h_K, h_{K'}\}$, summing over all internal faces of K noting carefully that the involved domains do not overlap, we finally obtain

$$(4.17) \quad \begin{aligned} \|h^{1/2} [[a \nabla u_h \cdot \mathbf{n}]]\|_{\partial K \cap \Gamma_{\text{int}}}^2 &\leq 6C_{b, F} \left(\|h(\Pi f - f)\|_{\omega_K}^2 + \|h(\Pi f + \nabla \cdot (a \nabla u_h))\|_{\omega_K}^2 \right. \\ &\quad \left. + \alpha^* C_{\text{inv}, T} \sum_{F \in \partial K \cap \Gamma_{\text{int}}} h_F^{-1} h_{F^\perp} \|\sqrt{a} \nabla e\|_{\omega_F}^2 \right), \end{aligned}$$

from which (4.15) now follows by inserting (4.15).

The proof concerning the tangential jump residual is similar. Given $F \in \Gamma_{\text{int}}$, we fix $v \in H^1(\omega_F)$ as the constant extension of $([[\nabla_T u_h]])|_F$ in the direction normal to F , so that $b_F v \in H_0^1(\omega_F)$. Using the fact that $\text{curl } \nabla u = \mathbf{0}$, we have the key observation

$$(4.18) \quad \int_{\omega_F} \text{curl}(b_F v) \cdot \nabla u \, d\mathbf{x} = 0.$$

Then, using integration by parts and then (4.14), we have

$$\begin{aligned} \|[[\nabla_T u_h]] b_F^{1/2}\|_F^2 &= \int_{\omega_F} \text{curl}(b_F v) \cdot \nabla u_h \, d\mathbf{x} = \int_{\omega_F} \text{curl}(b_F v) \cdot \nabla(u_h - u) \, d\mathbf{x} \\ &\leq \alpha_* \sum_{\mathcal{K}=K, K'} \|\sqrt{a} \nabla e\|_{T_F^K} \|\text{curl}(b_F v)\|_{T_F^K} \\ &\leq \alpha_* (d-1)^{1/2} C_{\text{inv}, T}^{1/2} h_F^{-1/2} \|[[\nabla_T u_h]]\|_F \sum_{\mathcal{K}=K, K'} \|\sqrt{a} \nabla e\|_{T_F^K}. \end{aligned}$$

Hence, by using (4.12), we obtain

$$\|h^{1/2}[\nabla_{\mathbf{T}}u_h]\|_F^2 \leq 2\alpha_*(d-1)C_{\text{inv},T}C_{b,F}h_F^{-1}h_{F\perp}\|\sqrt{a}\nabla e\|_{\omega_F}^2.$$

The required lower bound on the jump of the tangential gradient now follows by summing over all internal faces belonging to K . \square

The terms involving norms over the patches ω_F appearing in the lower bounds of Theorem 4.9 reflect, and account for, the presence of relatively small faces. Indeed, linking the size of ω_F to that of F , instead of that of the element K , allows for the ratio h_K/h_F to be unbounded.

If, on the other hand, the size of each of the element's face is comparable to that of the element itself, then we may modify the construction of the face bubble of Definition 4.7 by moving the opposite vertex all the way to the centre of star-shapedness of K . In such a case, $h_F^{-1}h_{F\perp}\|\sqrt{a}\nabla e\|_{\omega_F}^2 \lesssim \|\sqrt{a}\nabla e\|_{K\cup K'}^2$. Thus, for meshes with potentially many but regular hanging nodes, the new flux residuals lower bounds revert to the classical ones, as encapsulated in the following corollary.

COROLLARY 4.10. *Under the assumption of Theorem 4.9, if, moreover, for every $K \in \mathcal{T}$ and $F \in \partial K$ it holds $h_F \geq \tau^{-1}h_K$, then*

$$(4.19) \quad \begin{aligned} & \|h^{1/2}[a\nabla u_h \cdot \mathbf{n}]\|_{\partial K \cap \Gamma_{\text{int}}}^2 \\ & \leq 6C_{b,F} \left((2C_{b,K}(\tau^2 + C_{\text{inv},K})(\alpha^*)^2 + \tau\alpha^*C_{\text{inv},K}) \|\sqrt{a}\nabla e\|_{\omega_K}^2 \right. \\ & \quad \left. + \sum_{K' \in \omega_K} (1 + 2C_{b,K})O_{K',E}^2 \right), \end{aligned}$$

and

$$(4.20) \quad \|h^{1/2}[\nabla_{\mathbf{T}}u_h]\|_{\Gamma_{\text{int}}}^2 \leq 2C_{b,F}(d-1)\tau\alpha_*C_{\text{inv},K}\|\sqrt{a}\nabla e\|_{\omega_K}^2.$$

Remark 4.11. Corollary 4.10 holds in the setting of fully shape-regular meshes allowing for the sub-mesh auxiliary mesh construction, cf. Section 2.3.1. Hence, Corollary 4.10 together with Theorem 4.5 and Theorem 3.4 establishes the reliability and efficiency of the classical residual error estimator for general fully shape-regular polytopic meshes with multiple hanging nodes. The analysis, also in this case, differs from the classical one in that the finite element space used to control the non-conforming error, being based on the auxiliary mesh, is *not* a subspace of the discrete solution space $S_{\mathcal{T}}$. Moreover, the resulting error bound is as explicit as the classical bound because, in this case, the auxiliary mesh quality is fully controlled by that of the polytopic mesh. The element bubble construction is also new and accounts for the polytopic nature of the mesh.

On the other hand, controlling the flux residuals in the extreme case of possibly unbounded non shape-regular interfaces requires further new ideas. Whenever it is possible to construct a face bubble function b_F such that the following bound

$$\|\nabla(b_F v)\|_{T_F^K}^2 \leq C_{\text{inv},T}h_K^{-1}\|v\|_F^2,$$

holds true, the lower bounds of the flux residuals (4.15) and (4.16) will be independent of $h_F^{-1}h_{F\perp}$. An alternative approach could be to consider bubble functions constructed on a neighbouring set of structured elements. Then, the bubble functions b_F will be independent of the individual face size and the number of elements. For instance, this is the approach used in [31] to derive a lower bound of the flux residual of the FEM employing structured anisotropic triangular meshes. However, the construction of such face bubble functions for the general-shaped polytopic meshes considered in this work is highly non-trivial.

5. Numerical experiments. We present two numerical examples testing the new estimators. With the first example we test the impact of polygonal elements with a large number of ‘small’ faces on the effectivity index. With the second, we test the performance of the *a posteriori* error estimator within a non-standard adaptive algorithm. In all cases, we set $C_\sigma = 10$.

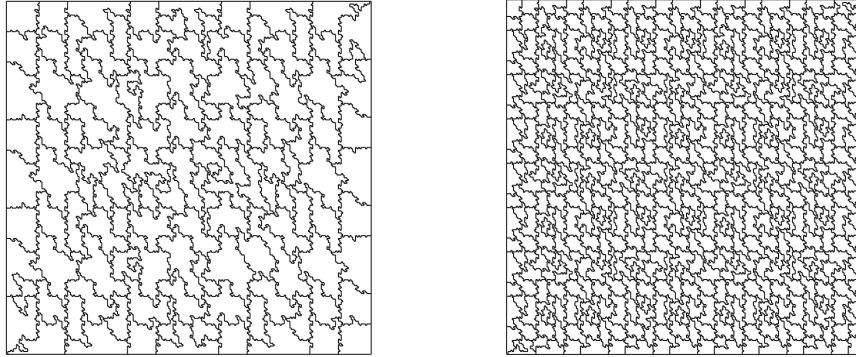


FIG. 5.1. *Example 1. Sample meshes with 114 (left) and 498 (right) polygonal elements obtained by agglomeration of a fine triangular mesh made of one million elements.*

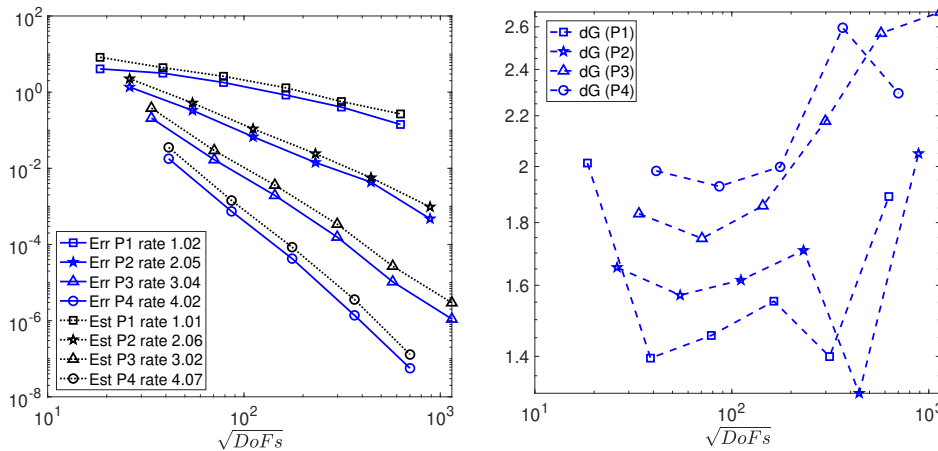


FIG. 5.2. *Example 1. Convergence history of the error and error estimator of Equation 3.21 for $p = 1, 2, 3, 4$ (left) and respective effectivity measured as the ratio estimator over error (right).*

5.1. Example 1. We construct a sequence of polygonal meshes containing 114, 498, 2063, 8912, and 32768 elements obtained by successive agglomeration of a very fine triangular background mesh made of 10^6 elements. Each of the polygonal elements contains at least 50 edges, see Figure 5.1 for an illustration.

We consider the problem (2.1), with $a = I_{2 \times 2}$ and $\Omega := (-1, 1)^2$. The Dirichlet boundary conditions and the source term f are determined by the exact solution $u = \sin(\pi x) \sin(\pi y)$. Numerical results for $p = 1, 2, 3, 4$ are displayed in Figure 5.2. The observed convergence rate of both the error and the estimator is $\mathcal{O}(DoFs^{-\frac{p}{2}})$, i.e., optimal in terms of the total number of degrees of freedom $DoFs$. Moreover, the effectivity index is bounded between 1.2 and 2.6, hence showing its efficiency in practice is not affected by the complexity of the element shapes.

5.2. Example 2. We test a new adaptive algorithm driven by the error estimator from Section 3. Starting from a relatively coarse simplicial mesh, we use the estimator (3.21) to mark simplicial elements for refinement through a bulk-chasing criterion (also known as Dörfler marking), and also mark *pairs* of elements for agglomeration based on the size of the jump residual terms on elemental interfaces. Refinement of simplicial elements is performed via a newest vertex bisection

algorithm. In the agglomeration step, general, polygonal meshes will be generated. In successive iterations, polygonal meshes which are marked for refinement are subdivided into either a finer polygonal mesh or simplicial meshes, depending on their level of agglomeration. For simplicity, we do not consider the data oscillation terms. The adaptive algorithm can thus be described as

Solve \longrightarrow Estimate \longrightarrow Mark \longrightarrow Refine/Agglomerate

We consider the problem (2.1) with $a = I_{2 \times 2}$ on $\Omega := (-1, 1)^2 \setminus (0, 1) \times (-1, 0)$. The Dirichlet boundary conditions and the source term f are determined by the exact solution

$$u = r^{2/3} \sin(2\psi/3) + \exp(-1000((x - 0.5)^2 + (y - 0.25)^2)) \\ + \exp(-1000((x - 0.5)^2 + (y - 0.75)^2)),$$

which has a point singularity at the origin. We test the adaptive dG algorithm described above with $p = 1, 2, 3$, with Dörfler's marking strategy 25% for refinement and the maximum marking strategy 5% for agglomeration. We point out that the agglomeration step is driven by the jump terms $R_{F,N}$, $R_{F,J}$, and $R_{F,T}$ for all faces $F \subseteq \partial K_i \cap \partial K_j$ on the meshes interface between element K_i and K_j .

The performance of the proposed adaptive algorithm is showcased in Figure 5.3. The convergence rates of both error and estimator are optimal in terms of the total number of degrees of freedom ($DoFs$), which is $\mathcal{O}(DoFs^{-\frac{p}{2}})$. Further, in coarse mesh levels, the mesh agglomeration dominates the mesh refinement. Consequently, the number of $DoFs$ is initially reduced by the adaptive algorithm while the error is also reduced. Another important observation is that the coarse mesh level's effectivity index still seems quite reasonable, namely 2 to 3 times greater than the asymptotic value. This is in spite of the presence of some very large polygonal elements next to small shape-regular triangles. These can be seen, for example, in the sequence of meshes produced by the adaptive algorithm with $p = 3$ and shown in Figure 5.4.

REFERENCES

- [1] M. AINSWORTH, *A posteriori error estimation for discontinuous Galerkin finite element approximation*, SIAM J. Numer. Anal., 45 (2007), pp. 1777–1798.
- [2] M. AINSWORTH AND J. T. ODEN, *A posteriori error estimation in finite element analysis*, Pure and Applied Mathematics (New York), Wiley-Interscience [John Wiley & Sons], New York, 2000.
- [3] M. AINSWORTH AND R. RANKIN, *Fully computable error bounds for discontinuous Galerkin finite element approximations on meshes with an arbitrary number of levels of hanging nodes*, SIAM J. Numer. Anal., 47 (2010), pp. 4112–4141.
- [4] D. N. ARNOLD, F. BREZZI, B. COCKBURN, AND L. D. MARINI, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal., pp. 1749–1779.
- [5] R. BECKER, P. HANSBO, AND M. G. LARSON, *Energy norm a posteriori error estimation for discontinuous Galerkin methods*, Comput. Methods Appl. Mech. Engrg., 192 (2003), pp. 723–733.
- [6] A. BONITO AND R. H. NOCHETTO, *Quasi-optimal convergence rate of an adaptive discontinuous Galerkin method*, SIAM J. Numer. Anal., 48 (2010), pp. 734–771.
- [7] Z. CAI, X. YE, AND S. ZHANG, *Discontinuous Galerkin finite element methods for interface problems: a priori and a posteriori error estimations*, SIAM J. Numer. Anal., 49 (2011), pp. 1761–1787.
- [8] A. CANGIANI, Z. DONG, AND E. GEORGOULIS, *hp-version discontinuous Galerkin methods on essentially arbitrarily-shaped elements*, Math. Comp., 91 (2022), pp. 1–35.
- [9] A. CANGIANI, Z. DONG, AND E. H. GEORGOULIS, *hp-version space-time discontinuous galerkin methods for parabolic problems on prismatic meshes*, SIAM J. Sci. Comput., 39 (2017), pp. A1251–A1279.
- [10] A. CANGIANI, Z. DONG, E. H. GEORGOULIS, AND P. HOUSTON, *hp-Version Discontinuous Galerkin Methods on Polygonal and Polyhedral Meshes*, Springer, 2017.
- [11] A. CANGIANI, E. H. GEORGOULIS, AND P. HOUSTON, *hp-version discontinuous Galerkin methods on polygonal and polyhedral meshes*, Math. Models Methods Appl. Sci., pp. 2009–2041.
- [12] A. CANGIANI, E. H. GEORGOULIS, T. PRYER, AND O. J. SUTTON, *A posteriori error estimates for the virtual element method*, Numer. Math., 137 (2017), pp. 857–893.
- [13] C. CARSTENSEN, S. BARTELS, AND S. JANSCHKE, *A posteriori error estimates for nonconforming finite element methods*, Numer. Math., 92 (2002), pp. 233–256.
- [14] C. CARSTENSEN AND S. A. FUNKEN, *Constants in Clément-interpolation error and residual based a posteriori error estimates in finite element methods*, East-West J. Numer. Math., 8 (2000), pp. 153–175.

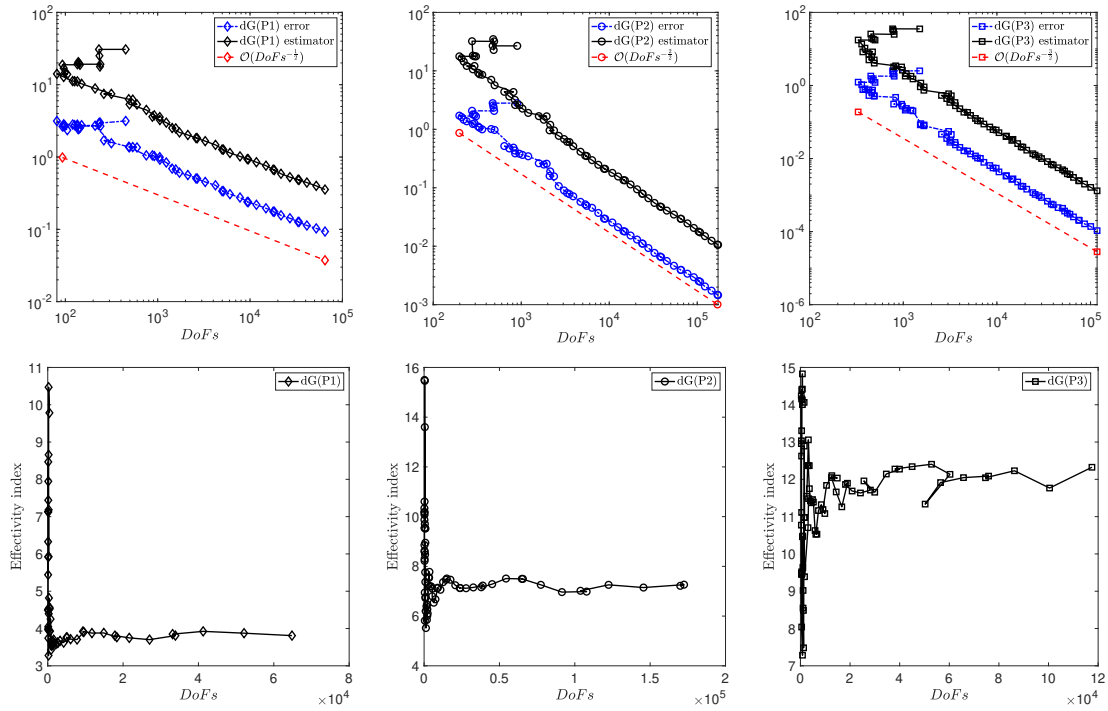


FIG. 5.3. Error, estimator (top) and effectivity index (bottom) of the new adaptive polytopic dG method with $p = 1, 2, 3$.

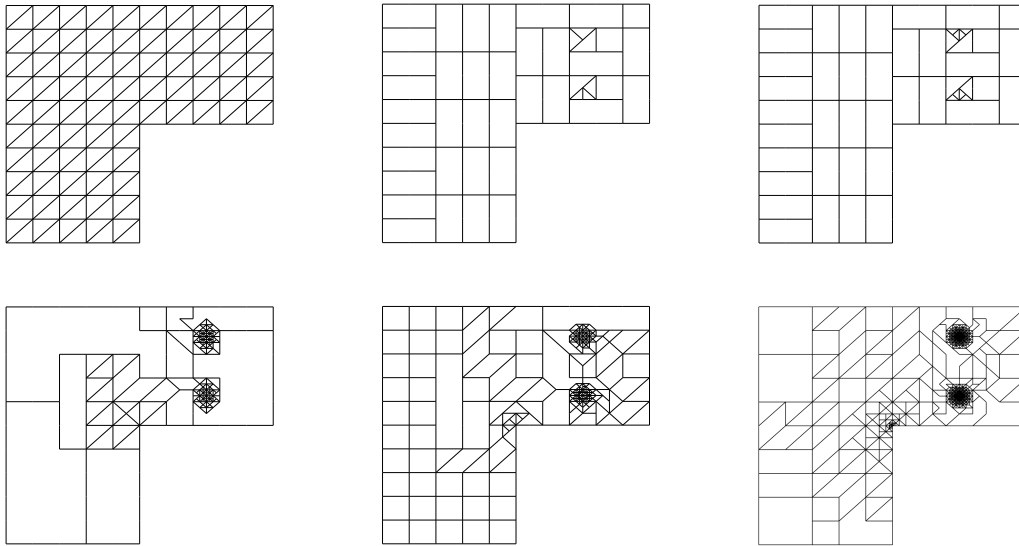


FIG. 5.4. The sequence of meshes generated by the adaptive polytopic dG algorithm with $p = 3$.

[15] C. CARSTENSEN, T. GUDI, AND M. JENSEN, *A unifying theory of a posteriori error control for discontinuous Galerkin FEM*, Numer. Math., 112 (2009), pp. 363–379.
 [16] C. CARSTENSEN, R. KHOT, AND A. K. PANI, *A priori and a posteriori error analysis of the lowest-order ncvm for second-order linear indefinite elliptic problems*, Numer. Math., 151 (2022), pp. 551–600.

- [17] L. P. CHEW, *Constrained Delaunay triangulations*, vol. 4, 1989, pp. 97–108. Computational geometry (Waterloo, ON, 1987).
- [18] J. CUI, F. GAO, Z. SUN, AND P. ZHU, *A posteriori error estimate for discontinuous Galerkin finite element method on polytopal mesh*, Numer. Methods Partial Differential Equations, 36 (2020), pp. 601–616.
- [19] E. DARI, R. DURAN, C. PADRA, AND V. VAMPA, *A posteriori error estimators for nonconforming finite element methods*, ESAIM Math. Model. Numer. Anal., 30 (1996), pp. 385–400.
- [20] A. DEMLOW AND E. H. GEORGOULIS, *Pointwise a posteriori error control for discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal., 50 (2012), pp. 2159–2181.
- [21] D. A. DI PIETRO AND R. SPECOGNA, *An a posteriori-driven adaptive mixed high-order method with application to electrostatics*, J. Comput. Phys., 326 (2016), pp. 35–55.
- [22] Z. DONG AND E. H. GEORGOULIS, *Robust interior penalty discontinuous Galerkin methods*, J. Sci. Comput. Published online. arXiv:2108.02642.
- [23] H. EDELSBRUNNER, *Geometry and topology for mesh generation*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, 1 ed., 2001.
- [24] A. ERN AND M. VOHRÁLIK, *Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous Galerkin, and mixed discretizations*, SIAM J. Numer. Anal., 53 (2015), pp. 1058–1081.
- [25] E. H. GEORGOULIS AND A. LASIS, *A note on the design of hp-version interior penalty discontinuous Galerkin finite element methods for degenerate problems*, IMA J. Numer. Anal., 26 (2006), pp. 381–390.
- [26] S. GIANI AND P. HOUSTON, *hp-adaptive composite discontinuous Galerkin methods for elliptic problems on complicated domains*, Numer. Methods Partial Differential Equations, 30 (2014), pp. 1342–1367.
- [27] R. H. W. HOPPE, G. KANSCHAT, AND T. WARBURTON, *Convergence analysis of an adaptive interior penalty discontinuous Galerkin method*, SIAM J. Numer. Anal., 47 (2008/09), pp. 534–550.
- [28] P. HOUSTON, D. SCHÖTZAU, AND T. P. WIHLE, *Energy norm a posteriori error estimation of hp-adaptive discontinuous Galerkin methods for elliptic problems*, Math. Models Methods Appl. Sci., 17 (2007), pp. 33–62.
- [29] O. A. KARAKASHIAN AND F. PASCAL, *Convergence of adaptive discontinuous Galerkin approximations of second-order elliptic problems*, SIAM J. Numer. Anal., pp. 641–665 (electronic).
- [30] ———, *A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems*, SIAM J. Numer. Anal., 41 (2003), pp. 2374–2399.
- [31] N. KOPTEVA, *Lower a posteriori error estimates on anisotropic meshes*, Numer. Math., 146 (2020), pp. 159–179.
- [32] C. KREUZER AND E. H. GEORGOULIS, *Convergence of adaptive discontinuous Galerkin methods*, Math. Comp., 87 (2018), pp. 2611–2640.
- [33] C. KREUZER AND A. VEESER, *Oscillation in a posteriori error estimation*, Numer. Math., 148 (2021), pp. 43–78.
- [34] H. LI, L. MU, AND X. YE, *A posteriori error estimates for the weak Galerkin finite element methods on polytopal meshes*, Commun. Comput. Phys., 26 (2019), pp. 558–578.
- [35] J. M. MELENK, *hp-interpolation of nonsmooth functions and an application to hp-a posteriori error estimation*, SIAM J. Numer. Anal., 43 (2005), pp. 127–155.
- [36] L. R. SCOTT AND S. ZHANG, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., pp. 483–493.
- [37] J. R. SHEWCHUK, *A condition guaranteeing the existence of higher-dimensional constrained delaunay triangulations.*, in Proceedings of the Fourteenth Annual Symposium on Computational Geometry., Association for Computing Machinery, New York, 1998, pp. 76–85.
- [38] ———, *General-dimensional constrained Delaunay and constrained regular triangulations. I. Combinatorial properties*, Discrete Comput. Geom., 39 (2008), pp. 580–637.
- [39] I. SMEARS, *Nonoverlapping domain decomposition preconditioners for discontinuous Galerkin approximations of Hamilton-Jacobi-Bellman equations*, J. Sci. Comput., 74 (2018), pp. 145–174.
- [40] S. K. TOMAR AND S. I. REPIN, *Efficient computable error bounds for discontinuous Galerkin approximations of elliptic problems*, J. Comput. Appl. Math., 226 (2009), pp. 358–369.
- [41] A. VEESER AND R. VERFÜRTH, *Poincaré constants for finite element stars*, IMA J. Numer. Anal., 32 (2012), pp. 30–47.
- [42] R. VERFÜRTH, *Error estimates for some quasi-interpolation operators*, M2AN Math. Model. Numer. Anal., 33 (1999), pp. 695–713.
- [43] ———, *A posteriori error estimation techniques for finite element methods*, Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, 2013.
- [44] W. ZHENG AND H. QI, *On Friedrichs-Poincaré-type inequalities*, J. Math. Anal. Appl., 304 (2005), pp. 542–551.