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# Globally unsolvability of fractional evolution equations with exponential nonlinearity

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## Abstract

In this paper the Cauchy problem for a time-space fractional evolution equation and a system with a time nonlocal nonlinearity of exponential growth are considered. The results on existence and uniqueness of local mild solution and nonexistence of global weak solution are presented, and under some suitable conditions, an estimate of the life span of blowing-up solutions is established.

## Keywords

blow-up ; fractional integrals and derivatives ; life span ; local existence ; nonlinear fractional evolution equations

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## I INTRODUCTION

In this paper, we consider the following Cauchy problem:

$$\begin{cases} \mathbf{D}_{0|t}^\alpha u + (-\Delta)^{\beta/2} u = J_{0|t}^{1-\alpha} (e^u), & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.1)$$

where  $N \geq 1, 0 < \alpha < 1, 0 < \beta \leq 2$ ,  $\mathbf{D}_{0|t}^\alpha$  the Caputo fractional derivative operator of order  $\alpha$ ,  $J_{0|t}^{1-\alpha} (e^u)$  is the Riemann-Liouville fractional integral of order  $1 - \alpha$  for  $e^u$ , defined by

$$J_{0|t}^{1-\alpha} (e^u) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{u(s)} ds,$$

where  $\Gamma$  is the gamma function,  $(-\Delta)^{\beta/2}$  is the fractional Laplacian operator defined by

$$(-\Delta)^{\beta/2} u(x) = \mathcal{F}^{-1} \left( |\xi|^\beta \mathcal{F}(u)(\xi) \right) (x),$$

for  $u \in D \left( (-\Delta)^{\beta/2} \right) = H^\beta (\mathbb{R}^N)$ , where  $H^\beta (\mathbb{R}^N)$  is the homogeneous Sobolev space defined by

$$H^\beta (\mathbb{R}^N) = \left\{ u \in \mathcal{S}' ; (-\Delta)^{\beta/2} u \in L^2 (\mathbb{R}^N) \right\}, \text{ if } \beta \notin \mathbb{N},$$

$$H^\beta (\mathbb{R}^N) = \left\{ u \in L^2 (\mathbb{R}^N) ; (-\Delta)^{\beta/2} u \in L^2 (\mathbb{R}^N) \right\}, \text{ if } \beta \in \mathbb{N}$$

where  $\mathcal{S}'$  is the Schwartz space,  $\mathcal{F}$  is the Fourier transform, and  $\mathcal{F}^{-1}$  is its inverse, and  $u_0 \in C_0(\mathbb{R}^N)$ , where  $C_0(\mathbb{R}^N)$  denotes the space of all continuous functions decaying to 0 at infinity.

We also consider local existence and global non-existence of the solution for the system:

$$\begin{cases} \mathbf{D}_{0|t}^{\alpha_1} u + (-\Delta)^{\beta/2} u = J_{0|t}^{1-\alpha_1}(e^v), & x \in \mathbb{R}^N, t > 0, \\ \mathbf{D}_{0|t}^{\alpha_2} v + (-\Delta)^{\beta/2} v = J_{0|t}^{1-\alpha_2}(e^u), & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \mathbb{R}^N. \end{cases} \quad (1.2)$$

Before we present and prove our results, let us dwell on existing literature related to our problems. The Cauchy problem for the semi-linear parabolic equation with a nonlinear memory

$$\begin{cases} u_t + (-\Delta)^{\beta/2} u = J_{0|t}^{1-\alpha}(|u|^{p-1}u), & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

and system

$$\begin{cases} u_t + (-\Delta)^{\beta/2} u = J_{0|t}^{1-\alpha_1}(|v|^{p-1}v), & x \in \mathbb{R}^N, t > 0, \\ v_t + (-\Delta)^{\beta/2} v = J_{0|t}^{1-\alpha_2}(|u|^{q-1}u), & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \mathbb{R}^N, \end{cases}$$

has been considered by Fino and Kirane in [3] and [4].

In addition, Ahmad et al [2] considered the semi-linear nonlocal in time and space reaction diffusion equation:

$$\begin{cases} u_t + (-\Delta)^{\beta/2} u = J_{0|t}^{1-\alpha}(e^u), & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \quad (1.3)$$

Moreover, the system of (1.3) was investigated by Ahmad et al [1] in the following form:

$$\begin{cases} u_t + (-\Delta)^{\beta/2} u = J_{0|t}^{1-\alpha_1}(e^v), & x \in \mathbb{R}^N, t > 0, \\ v_t + (-\Delta)^{\beta/2} v = J_{0|t}^{1-\alpha_2}(e^u), & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \mathbb{R}^N, \end{cases}$$

The here above cited papers treated the local existence, the blowing-up solutions and the blow-up profil of exploding solutions.

## II FRACTIONAL DIFFUSION EQUATION WITH EXPONENTIAL NONLINEARITY

### 2.1 Local existence

In this section, we give the local existence and uniqueness of mild solution of the problem (1.1). First, we give the definition of a mild solution of (1.1).

*Definition 1: Mild solution*

Let  $u_0 \in C_0(\mathbb{R}^N)$  and  $T > 0$ . We say that  $u \in C([0, T]; C_0(\mathbb{R}^N))$  is a mild solution of (1.1) if

$$u(t) = P_{\alpha, \beta}(t) u_0 + \int_0^t (t-s)^{\alpha-1} S_{\alpha, \beta}(t-s) J_{0|s}^{1-\alpha}(e^{u(\tau)}) ds,$$

for  $t \in [0, T]$ .

*Theorem 1: Local existence*

Let  $u_0 \in C_0(\mathbb{R}^N)$ . Then, there exists a maximal time  $T_{max} > 0$  such that problem (1.1) has a unique mild solution  $u \in C([0, T_{max}); C_0(\mathbb{R}^N))$ . Furthermore, either  $T_{max} = +\infty$  or  $T_{max} < +\infty$  and  $\lim_{t \rightarrow T_{max}} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = +\infty$ . If, in addition,  $u_0 \geq 0$ ,  $u_0 \not\equiv 0$ , then  $u(t) > 0$  for all  $0 < t < T_{max}$ . Moreover, if  $u_0 \in L^r(\mathbb{R}^N)$ , for  $1 \leq r < \infty$ , then  $u \in C([0, T_{max}); L^r(\mathbb{R}^N))$ .

## 2.2 Blowing up solutions

*Definition 2: Weak solution*

Let  $u_0 \in L_{loc}^\infty(\mathbb{R}^N)$  and  $T > 0$ . We say that  $u$  is a weak solution of (1.1) if  $u \in L^p((0, T); L_{loc}^\infty(\mathbb{R}^N))$  and verify the equation

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} u_0 \mathbf{D}_{t|T}^\alpha \psi(x, t) + \int_0^T \int_{\mathbb{R}^N} J_{0|t}^{1-\alpha}(e^u) \psi(x, t) \\ &= \int_0^T \int_{\mathbb{R}^N} u(x, t) (-\Delta)^{\beta/2} \psi(x, t) + \int_0^T \int_{\mathbb{R}^N} u(x, t) \mathbf{D}_{t|T}^\alpha \psi(x, t), \end{aligned}$$

for all test function  $\psi \in C^1([0, T]; H^\beta(\mathbb{R}^N))$  such that  $\psi(x, T) = 0$ .

*Lemma 2:*

Consider  $u_0 \in C_0(\mathbb{R}^N)$  and let  $u \in C([0, T]; C_0(\mathbb{R}^N))$  be a mild solution of (1.1), then  $u$  is also a weak solution of (1.1).

*Theorem 3:*

Let  $u_0 \in C_0(\mathbb{R}^N)$  with  $u_0 \geq 0$  and  $u_0 \not\equiv 0$ . Then, the solution of (1.1) blows up in a finite time.

## 2.3 Life span of blowing up solutions

In this section, we give an upper bound estimate of the life span of the blowing up solutions with some special initial datum. To this aim, we consider the following problem:

$$\begin{cases} \mathbf{D}_{0|t}^\alpha u_\epsilon + (-\Delta)^{\beta/2} u_\epsilon = J_{0|t}^{1-\alpha}(e^{u_\epsilon}), & x \in \mathbb{R}^N, t > 0, \\ u_\epsilon(x, 0) = \epsilon u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.4)$$

where  $\epsilon > 0$ ,  $0 < \alpha < 1$ ,  $0 < \beta \leq 2$  and  $u_0 \in C_0(\mathbb{R}^N)$  satisfies

$$u_0(x) \geq C_0 |x|^{-\delta}, \quad |x| \geq \epsilon_0, \quad N < \delta < \frac{\beta}{\alpha}, \quad (1.5)$$

for some positive constants  $C_0$  and  $\epsilon_0$ .

*Theorem 4:*

Suppose that (1.5) holds. Let  $[0, T_\epsilon]$  be the life span of the solution  $u_\epsilon$  of the problem (1.4). Then, there exists a positive constant  $C$  such that

$$T_\epsilon \leq C\epsilon^{\frac{1}{\theta}}, \quad \theta = \frac{\alpha\delta}{\beta} - 1 < 0.$$

### III FRACTIONAL DIFFUSION SYSTEM WITH NONLINEARITIES OF EXPONENTIAL GROWTH

In this section, we study the system (1.2). We start with the definition of the mild solution of (1.2).

#### 3.1 Local existence

*Definition 3: Mild solution*

Let  $u_0, v_0 \in C_0(\mathbb{R}^N)$ . We say that  $(u, v) \in C([0, T], C_0(\mathbb{R}^N)) \times C([0, T], C_0(\mathbb{R}^N))$  is a mild solution of (1.2) if  $(u, v)$  satisfies, for  $t \in [0, T]$ , the following equations

$$\begin{aligned} u(t) &= P_{\alpha_1, \beta}(t) u_0 + \int_0^t (t-s)^{\alpha_1-1} S_{\alpha_1, \beta}(t-s) J_{0|s}^{1-\alpha_1}(e^{v(\tau)}), \\ v(t) &= P_{\alpha_2, \beta}(t) v_0 + \int_0^t (t-s)^{\alpha_2-1} S_{\alpha_2, \beta}(t-s) J_{0|s}^{1-\alpha_2}(e^{u(\tau)}). \end{aligned}$$

*Theorem 5: Local existence*

Let  $u_0, v_0 \in C_0(\mathbb{R}^N)$ . Then, there exists a maximal time  $T_{max} > 0$  such that the problem (1.2) has a unique mild solution  $(u, v) \in C([0, T_{max}], C_0(\mathbb{R}^N)) \times C([0, T_{max}], C_0(\mathbb{R}^N))$ . Furthermore, we have the alternative: - either  $T_{max} = +\infty$ ; - or  $T_{max} < +\infty$  and  $\lim_{t \rightarrow T_{max}} (\|u(t)\|_{L^\infty(\mathbb{R}^N)} + \|v(t)\|_{L^\infty(\mathbb{R}^N)}) = +\infty$ . If, in addition,  $u_0, v_0 \geq 0, u_0 \not\equiv 0, v_0 \not\equiv 0$ , then  $u(t), v(t) > 0$  for all  $0 < t < T_{max}$ . Moreover, if  $u_0, v_0 \in L^r(\mathbb{R}^N)$ , for  $1 \leq r < \infty$ , then  $u, v \in C([0, T_{max}]; L^r(\mathbb{R}^N))$ .

#### 3.2 Blowing up solutions

*Definition 4: Weak solution*

Let  $u_0, v_0 \in L_{loc}^\infty(\mathbb{R}^N)$  and  $T > 0$ . We say that  $(u, v)$  is a weak solution of (1.2) if  $(u, v) \in L^p((0, T); L_{loc}^\infty(\mathbb{R}^N)) \times L^p((0, T); L_{loc}^\infty(\mathbb{R}^N))$  and satisfies

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} u_0 \mathbf{D}_{t|T}^{\alpha_1} \psi_1(x, t) dx dt + \int_0^T \int_{\mathbb{R}^N} J_{0|t}^{1-\alpha_1}(e^v) \psi_1(x, t) dx dt \\ &= \int_0^T \int_{\mathbb{R}^N} u(x, t) (-\Delta)^{\beta/2} \psi_1(x, t) + \int_0^T \int_{\mathbb{R}^N} u(x, t) \mathbf{D}_{t|T}^{\alpha_1} \psi_1(x, t), \\ & \int_0^T \int_{\mathbb{R}^N} v_0 \mathbf{D}_{t|T}^{\alpha_2} \psi_2(x, t) dx dt + \int_0^T \int_{\mathbb{R}^N} J_{0|t}^{1-\alpha_2}(e^u) \psi_2(x, t) dx dt \\ &= \int_0^T \int_{\mathbb{R}^N} v(x, t) (-\Delta)^{\beta/2} \psi_2(x, t) + \int_0^T \int_{\mathbb{R}^N} v(x, t) \mathbf{D}_{t|T}^{\alpha_2} \psi_2(x, t), \end{aligned}$$

for all test functions  $\psi_1, \psi_2 \in C^1([0, T]; H^\beta(\mathbb{R}^N))$  such that  $\psi_1(x, T) = \psi_2(x, T) = 0$ .

*Theorem 6:*

Let  $u_0, v_0 \in C_0(\mathbb{R}^N)$  with  $u_0 \geq 0, u_0 \not\equiv 0, v_0 \geq 0$  and  $v_0 \not\equiv 0$ . blows up in a finite time.

### 3.3 Life span of blowing up solutions

We consider the problem:

$$\begin{cases} \mathbf{D}_{0|t}^{\alpha_1} u_\epsilon + (-\Delta)^{\beta/2} u_\epsilon = J_{0|t}^{1-\alpha_1} (e^{v_\epsilon}), & x \in \mathbb{R}^N, t > 0, \\ \mathbf{D}_{0|t}^{\alpha_2} v_\epsilon + (-\Delta)^{\beta/2} v_\epsilon = J_{0|t}^{1-\alpha_2} (e^{u_\epsilon}), & x \in \mathbb{R}^N, t > 0, \\ u_\epsilon(x, 0) = \epsilon u_0(x), \quad v_\epsilon(x, 0) = \epsilon v_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.6)$$

where  $\epsilon > 0, 0 < \alpha_1, \alpha_2 < 1, 0 < \beta \leq 2$  with  $u_0, v_0 \in C_0(\mathbb{R}^N)$  satisfying

$$u_0(x) \geq m_0 |x|^{-\frac{\delta}{\alpha_1}}, \quad v_0(x) \geq n_0 |x|^{-\frac{\delta}{\alpha_2}}, \quad |x| \geq \epsilon_0, \quad \max\{\alpha_1, \alpha_2\} N < \delta < \beta, \quad (1.7)$$

for some positive constants  $m_0, n_0$  and  $\epsilon_0$ .

*Theorem 7:*

Suppose that (1.7) holds. Let  $[0, T_\epsilon)$  be the maximal time of existence of the solution  $(u_\epsilon, v_\epsilon)$  of problem (1.6). Then there exists a constant  $C > 0$  such that

$$T_\epsilon \leq C \epsilon^{\frac{1}{\eta}}, \quad \eta = \frac{\delta}{\beta} - 1 < 0.$$

## IV REFERENCES

### REFERENCES

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