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DECAY OF COEFFICIENTS AND APPROXIMATION RATES IN GABOR GAUSSIAN FRAMES

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ABSTRACT. The aim of this note is to present a self-contained proof of the fact that a function can be approximated using a linear combination of Gaussian coherent states, with a number of terms controlled in terms of the smoothness and of the decay at infinity of the function. This result, which is essential in [3], can easily be obtained using advanced results on modulation spaces, but the proof presented here is completely elementary and self-contained.

KEYWORDS: Approximation theory, Gabor frames, Time-frequency analysis

1. INTRODUCTION

In [3], the authors proposed a novel approach to solve numerically a family of PDEs with a small parameter, including the Helmholtz equation at large frequencies. This approach relied on some ideas coming from semiclassical analysis, as well as on results concerning *Gabor frames*.

Gabor frames are a standard tool to decompose functions into a discrete sum of “coherent states”, which are localised both in position and Fourier spaces. Such expansions are somehow similar to Fourier expansions, but are more subtle, as Gabor frames do not form orthonormal bases. The study of Gabor frames is a field of research which has been very active for the past decades; we refer the reader to the monographs [7, 4, 2] for an account of recent advances.

The main result of time-frequency analysis which is used in [3] is a decay property for the coefficients of functions when decomposed in a Gabor frame, in terms of the regularity of the functions and of their decay at infinity. These results are analogous to the standard decay properties of Fourier coefficients, and permit to show that a finite number of coherent states provide a good approximation to any smooth rapidly decaying function.

Such a result can be easily deduced from advanced results of time-frequency analysis concerning *modulation spaces*., as we will explain in Appendix D.

The aim of this note is to give a completely self-contained proof of all the results concerning Gabor frames which are used in [3], and in particular, of the fact that a smooth rapidly decaying function can be approximated using few coherent states. This note is thus intended mainly for readers who want to understand where the results of [3] come from, without learning the recent developments in time-frequency analysis.

We insist on the fact that the results presented here are not new, and are not as general as they could be; for instance, we will consider only Gabor frames built from Gaussian windows (which simplifies the proofs), while more general window functions could be considered using modulation spaces.

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The remainder of the document is organized as follows. In Section 2, we introduce the main notations we will use in this work. In Section 3, we recall standard properties of Gabor frames and state the main results that are used in [3]. In Section 4, we recall the proof of the frame property for the Gabor system. In Section 5, we prove properties of decay of the coefficients, which will be essential in the proof of our main result, given in Section 6. Finally, Appendices A, B and C collect necessary technical results, while Appendix D explains how the main result of this note can be recovered using results concerning modulation spaces.

2. SETTINGS

2.1. Scaling parameter. In the sequel, all our quantities will depend on a parameter $k \geq 1$, which is an arbitrary but fixed real number. This is more convenient for applications to PDEs, where it is useful to choose k related to the wavelength of the solution.

Although we may not indicate it explicitly, all our results hold true for any value of k , with the estimates holding uniformly. Besides, $d \in \mathbb{N}^*$ denotes the number of space dimensions.

2.2. Multi-indices. For a multi-index $\mathbf{a} \in \mathbb{N}^d$, $[\mathbf{a}] := a_1 + \dots + a_d$ denotes its usual ℓ^1 norm. If $v : \mathbb{R}^d \rightarrow \mathbb{C}$, the notation

$$\partial^{\mathbf{a}} v := \frac{\partial^{a_1}}{\partial x_1} \dots \frac{\partial^{a_d}}{\partial x_d} v$$

is employed for the partial derivatives in the sense of distributions, whereas $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} \dots x_d^{a_d}$.

2.3. Functional spaces. $L^2(\mathbb{R}^d)$ is the usual space of complex-valued square integrable functions over \mathbb{R}^d . We respectively denote by $\|\cdot\|_{L^2(\mathbb{R}^d)}$ and (\cdot, \cdot) its norm and inner-product. If $\omega \subset \mathbb{R}^d$ is a measurable subset and $v \in L^2(\mathbb{R}^d)$, we set $\|v\|_{L^2(\omega)} = \|\chi_\omega v\|_{L^2(\mathbb{R}^d)}$, where χ_ω is the set function of ω . We also denote by $H^p(\mathbb{R}^d)$ the usual Sobolev space of order $p \in \mathbb{N}$, that we equip with the norm

$$\|v\|_{H_k^p(\mathbb{R}^d)}^2 := \sum_{[\mathbf{a}] \leq p} k^{-2[\mathbf{a}]} \|\partial^{\mathbf{a}} v\|_{L^2(\mathbb{R}^d)}^2 \quad \forall v \in H^p(\mathbb{R}^d).$$

Notice that the above norm is equivalent to the usual Sobolev norm, but with equivalence constant depending on k . We refer the reader to, e.g., [1] for an in-depth presentation of the above spaces.

We will also need a family of weighted Sobolev spaces. We first introduce the norms

$$\|v\|_{\widehat{H}_k^p(\mathbb{R}^d)}^2 := \sum_{[\mathbf{a}] \leq p} \sum_{q=0}^{p-[\mathbf{a}]} k^{-2[\mathbf{a}]} \|\mathbf{x}^q \partial^{\mathbf{a}} v\|_{L^2(\mathbb{R}^d)}^2 \quad \forall v \in \mathcal{S}(\mathbb{R}^d)$$

for all $p \in \mathbb{N}$, and we define $\widehat{H}^p(\mathbb{R}^d)$ as the closure of $\mathcal{S}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$ with respect to $\|\cdot\|_{\widehat{H}_k^p(\mathbb{R}^d)}$.

For future references, we note that if $u \in \widehat{H}^{p+1}(\mathbb{R}^d)$, we have $x_j u \in \widehat{H}^p(\mathbb{R}^d)$ with

$$(2.1) \quad \|x_j u\|_{\widehat{H}_k^p(\mathbb{R}^d)} \leq C_p \|u\|_{\widehat{H}_k^{p+1}(\mathbb{R}^d)}$$

for $1 \leq j \leq d$.

2.4. Fourier transform. For $v \in L^2(\mathbb{R}^d)$, we define a weighted Fourier transform by

$$\mathcal{F}_k(v)(\boldsymbol{\xi}) := \left(\frac{k}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} v(x) e^{-ik\mathbf{x}\cdot\boldsymbol{\xi}} d\mathbf{x}$$

for a.e. $\boldsymbol{\xi} \in \mathbb{R}^d$. For all $v \in L^2(\mathbb{R}^d)$, we then have

$$\|\mathcal{F}_k(v)\|_{L^2(\mathbb{R}^d)} = \|v\|_{L^2(\mathbb{R}^d)}.$$

In addition, if $v \in H^1(\mathbb{R}^d)$ or respectively if $|\mathbf{x}|v \in L^2(\mathbb{R}^d)$, then

$$\mathcal{F}_k(\partial_j v) = ik\xi_j \mathcal{F}_k(v), \quad ik\mathcal{F}_k(x_j v) = \partial_j \mathcal{F}_k(v)$$

for $j \in \{1, \dots, d\}$. In particular, for any $p \in \mathbb{N}$, we have for every $v \in H^p(\mathbb{R}^d)$

$$(2.2) \quad \|v\|_{H_k^p(\mathbb{R}^d)}^2 = \sum_{|\mathbf{a}| \leq p} \|\boldsymbol{\xi}^{\mathbf{a}} \mathcal{F}_k v\|_{L^2(\mathbb{R}^d)}^2.$$

2.5. Generic constants. Throughout the manuscript, C denotes a constant that may vary from one occurrence to the other and only depends on the space dimension d . If p, q, \dots are previously introduced symbols, then the notation $C_{p,q,\dots}$ is similarly used to indicate a constant that solely depends on d and p, q, \dots .

3. MAIN RESULTS CONCERNING GABOR FRAMES OF GAUSSIAN STATES

3.1. The Gabor frame. For $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$, we set

$$(3.1) \quad \mathbf{x}^{k,\mathbf{m}} := \sqrt{\frac{\pi}{k}} \mathbf{m}, \quad \boldsymbol{\xi}^{k,\mathbf{n}} := \sqrt{\frac{\pi}{k}} \mathbf{n},$$

so that, the couples $[\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}}]$ form a lattice of the phase space \mathbb{Z}^{2d} . We associate with each point $[\mathbf{m}, \mathbf{n}]$ in the lattice the Gaussian state

$$\Psi_{k,\mathbf{m},\mathbf{n}}(\mathbf{x}) := \left(\frac{k}{\pi}\right)^{d/4} e^{-\frac{k}{2}|\mathbf{x}-\mathbf{x}^{k,\mathbf{m}}|^2} e^{ik(\mathbf{x}-\mathbf{x}^{k,\mathbf{m}})\cdot\boldsymbol{\xi}^{k,\mathbf{n}}}.$$

The set $(\Psi_{k,\mathbf{m},\mathbf{n}})_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}}$ forms a *frame*, i.e., there exist two constants $\alpha, \beta > 0$ that only depends on d , such that

$$(3.2) \quad \alpha^2 \|u\|_{L^2(\mathbb{R}^d)}^2 \leq \sum_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}} |(u, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 \leq \beta^2 \|u\|_{L^2(\mathbb{R}^d)}^2 \quad \forall u \in L^2(\mathbb{R}^d).$$

This fact was first shown in [5] when $d = 1$ and $k = 1$ (from which the case $d > 1$ and $k \neq 1$ follow easily), and we will recall their proof in Section 4. We refer the reader to [4] for generalisations of this result.

3.2. The dual frame. Given $u \in L^2(\mathbb{R}^d)$, we call the sequence $((u, \Psi_{k,\mathbf{m},\mathbf{n}}))_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}}$ the coefficients of u in the frame, and we introduce the operator $T_k : L^2(\mathbb{R}^d) \rightarrow \ell^2(\mathbb{Z}^{2d})$ mapping a function u to its coefficients:

$$(3.3) \quad (T_k u)_{\mathbf{m},\mathbf{n}} := (u, \Psi_{k,\mathbf{m},\mathbf{n}}) \quad [\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}.$$

The fact that T_k maps into $\ell^2(\mathbb{Z}^{2d})$ is a direct consequence of (3.2). Straightforward computations show that the adjoint $T_k^* : \ell^2(\mathbb{Z}^{2d}) \rightarrow L^2(\mathbb{R}^d)$ of T_k is given by

$$T_k^* U = \sum_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}} U_{\mathbf{m},\mathbf{n}} \Psi_{k,\mathbf{m},\mathbf{n}},$$

for all $U \in \ell^2(\mathbb{Z}^{2d})$ and, in particular, we have

$$(3.4) \quad (T_k^* T_k)u = \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} (u, \Psi_{k, \mathbf{m}, \mathbf{n}}) \Psi_{k, \mathbf{m}, \mathbf{n}} \quad \forall u \in L^2(\mathbb{R}^d).$$

The operator $(T_k^* T_k)$ is self-adjoint, and the estimates in (3.2) imply that it is bounded from above and from below. As a result, $(T_k^* T_k)$ is invertible, and we will denote by S_k its inverse.

We call set of functions $(\Psi_{k, \mathbf{m}, \mathbf{n}}^*)_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d}$ defined by

$$\Psi_{k, \mathbf{m}, \mathbf{n}}^* := S_k \Psi_{k, \mathbf{m}, \mathbf{n}} \quad \forall [\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d},$$

the ‘‘dual frame’’ of $(\Psi_{k, \mathbf{m}, \mathbf{n}})_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}}$. It is indeed a frame, since, for any $u \in L^2(\mathbb{R}^d)$, we have

$$\sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} |(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)|^2 = \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} |(u, S_k \Psi_{k, \mathbf{m}, \mathbf{n}})|^2 = \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} |(S_k u, \Psi_{k, \mathbf{m}, \mathbf{n}})|^2 = \|T_k S_k u\|^2,$$

and it suffices to use (3.2) and to note that S_k is continuous with a continuous inverse.

Importantly, recalling (3.4), we see that for all $u \in L^2(\mathbb{R}^d)$, we have

$$(3.5) \quad u = \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} (u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*) \Psi_{k, \mathbf{m}, \mathbf{n}}$$

and

$$(3.6) \quad u = \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} (u, \Psi_{k, \mathbf{m}, \mathbf{n}}) \Psi_{k, \mathbf{m}, \mathbf{n}}^*.$$

3.3. Approximation in Gabor frames. The main result of this note is a generalization of (3.5) when the series is truncated to a finite number of terms. Specifically, considering $D > 0$ such that $k^{1/2}D > \sqrt{\pi}$, we define an ‘‘interpolation operator’’ $\Pi_D : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ by

$$(3.7) \quad \Pi_D u := \sum_{|(\mathbf{x}^{k, \mathbf{m}}, \boldsymbol{\xi}^{k, \mathbf{n}})| \leq D} (u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*) \Psi_{k, \mathbf{m}, \mathbf{n}}.$$

Notice that, recalling the definitions of $\mathbf{x}^{k, \mathbf{m}}$ and $\boldsymbol{\xi}^{k, \mathbf{n}}$ in (3.1), the requirement that $k^{1/2}D > \sqrt{\pi}$ simply means that there is at least one point in the summation set, so that it is not a restrictive assumption.

Theorem 3.1 (Approximation in the Gabor frame). *Consider two non-negative integers p and r . For all $u \in \widehat{H}^{p+r}(\mathbb{R}^d)$, we have $\Pi_D u \in \widehat{H}^p(\mathbb{R}^d)$. In addition, the following estimate holds true:*

$$(3.8) \quad \|u - \Pi_D u\|_{\widehat{H}_k^p(\mathbb{R}^d)} \leq C_{p,r} D^{-r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}.$$

The estimate in Theorem 3.1 is sharp. Specifically, we have the following result.

Proposition 3.2 (Sharp estimate). *For all $p, r \in \mathbb{N}$, there exists a constant $D_\star > 0$ depending on p, r and d , such that for all $D \geq D_\star$, there exists $u \in \widehat{H}^{p+r}(\mathbb{R}^d)$ with*

$$(3.9) \quad \|u - \Pi_D u\|_{\widehat{H}_k^p(\mathbb{R}^d)} \geq C_{p,r} D^{-r} \|u\|_{\widehat{H}^{p+r}(\mathbb{R}^d)}.$$

Theorem 3.1 will be proved in Sections 5 and 6, and we will give an alternative proof of it in Appendix D. Proposition 3.2 will be proven in Appendix C.

4. PROOF OF THE FRAME PROPERTY

The aim of this section is to recall the proof of (3.2), following closely [5].

First of all, note that for $f \in L^2(\mathbb{R}^d)$, if we set

$$(4.1) \quad (\delta_k f)(x) := k^{-d/4} f(k^{-1/2}x),$$

we have $\|\delta_k f\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$ and

$$(4.2) \quad \delta_k \Psi_{k,m,n} = \Psi_{1,m,n}.$$

Therefore, it suffices to prove (3.2) for $k = 1$, and the general case will follow. To lighten notations, we shall write $\Psi_{m,n}$ instead of $\Psi_{1,m,n}$ in this section.

Letting $\square := [-\sqrt{\pi}, \sqrt{\pi}]^d \times [-\sqrt{\pi}/2, \sqrt{\pi}/2]^d$, we define the Zak transform $Z : L^2(\mathbb{R}^d) \rightarrow L^2(\square)$ as follows. If $f \in C_c(\mathbb{R}^d)$, we set

$$(Zf)(\mathbf{v}, \mathbf{q}) := \frac{1}{(2\sqrt{\pi})^{d/2}} \sum_{\boldsymbol{\ell} \in \mathbb{Z}^d} e^{i\sqrt{\pi}\mathbf{v}\cdot\boldsymbol{\ell}} f(\mathbf{q} - \sqrt{\pi}\boldsymbol{\ell}) \quad \forall (\mathbf{v}, \mathbf{q}) \in \square.$$

One readily checks that $\|Zf\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\square)}$, so that Z extends to a unitary transform.

Let $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$. We have

$$\begin{aligned} (Z\Psi_{\mathbf{m},\mathbf{n}})(\mathbf{v}, \mathbf{q}) &= \frac{1}{2^{d/2}\pi^{3d/4}} \sum_{\boldsymbol{\ell} \in \mathbb{Z}^d} e^{i\sqrt{\pi}\mathbf{v}\cdot\boldsymbol{\ell}} e^{i(\mathbf{q}-\sqrt{\pi}\boldsymbol{\ell}-\sqrt{\pi}\mathbf{m})\cdot\sqrt{\pi}\mathbf{n}} e^{-\frac{|\mathbf{q}-\sqrt{\pi}\boldsymbol{\ell}-\sqrt{\pi}\mathbf{m}|^2}{2}} \\ &= \frac{1}{2^{d/2}\pi^{3d/4}} e^{i\sqrt{\pi}\mathbf{v}\cdot\mathbf{m}} e^{i\mathbf{q}\cdot\sqrt{\pi}\mathbf{n}} \sum_{\boldsymbol{\ell}' \in \mathbb{Z}^d} e^{i\sqrt{\pi}\mathbf{v}\cdot\boldsymbol{\ell}'} e^{i\pi\boldsymbol{\ell}'\cdot\mathbf{n}} e^{-\frac{|\mathbf{q}-\sqrt{\pi}\boldsymbol{\ell}'|^2}{2}} \\ &= e^{i\sqrt{\pi}\mathbf{v}\cdot\mathbf{m}} e^{i\mathbf{q}\cdot\sqrt{\pi}\mathbf{n}} (Z\Psi_{0,\mathbf{n}'})(\mathbf{v}, \mathbf{q}), \end{aligned}$$

The, given $f \in L^2(\mathbb{R}^d)$, we can write

$$\begin{aligned} \|T_1 f\|_{\ell^2}^2 &= \sum_{\mathbf{n}' \in \{0,1\}^d} \sum_{\mathbf{m}, \mathbf{p} \in \mathbb{Z}^d} |\langle Z\Psi_{\mathbf{m}, 2\mathbf{p}+\mathbf{n}'}, Zf \rangle|^2 \\ &= \sum_{\mathbf{n}' \in \{0,1\}^d} \sum_{\mathbf{m}, \mathbf{p} \in \mathbb{Z}^d} \left| \int_{\square} (Zf)(\mathbf{v}, \mathbf{p}) e^{-i\sqrt{\pi}\mathbf{v}\cdot\mathbf{m}} e^{-i\mathbf{q}\cdot\sqrt{\pi}(2\mathbf{p}+\mathbf{n}')} (Z\Psi_{0,\mathbf{n}'})(\mathbf{v}, \mathbf{q}) d\mathbf{v} d\mathbf{q} \right|^2 \\ &= (2\pi)^d \int_{\square} |(Zf)(\mathbf{v}, \mathbf{q})|^2 \sum_{\mathbf{n}' \in \{0,1\}^d} |Z\Psi_{0,\mathbf{n}'}(\mathbf{v}, \mathbf{q})|^2 d\mathbf{v} d\mathbf{q}, \end{aligned}$$

by Plancherel's formula. Therefore, if we introduce the function

$$(4.3) \quad \Theta(\mathbf{v}, \mathbf{q}) := \sum_{\mathbf{n}' \in \{0,1\}^d} |Z\Psi_{0,\mathbf{n}'}(\mathbf{v}, \mathbf{q})|^2 \quad \forall (\mathbf{v}, \mathbf{q}) \in \square,$$

we have

$$(2\pi)^d \min_{\mathbf{v}, \mathbf{q}} \Theta(\mathbf{v}, \mathbf{q}) \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \|T_1 f\|_{\ell^2}^2 \leq (2\pi)^d \max_{\mathbf{v}, \mathbf{q}} \Theta(\mathbf{v}, \mathbf{q}) \|f\|_{L^2(\mathbb{R}^d)}^2.$$

We define θ to be Θ when $d = 1$. That is, for $(v, q) \in [-\sqrt{\pi}, \sqrt{\pi}] \times [-\sqrt{\pi}/2, \sqrt{\pi}/2]$, we set

$$\theta(v, q) := \frac{1}{2\pi} \left| \sum_{\ell \in \mathbb{Z}} e^{i\sqrt{\pi}v\ell} e^{-(q-\sqrt{\pi}\ell)^2/2} \right|^2 + \frac{1}{2\pi} \left| \sum_{\ell \in \mathbb{Z}} e^{i\sqrt{\pi}v\ell} e^{i\sqrt{\pi}(q-\sqrt{\pi}\ell)} e^{-(q-\sqrt{\pi}\ell)^2/2} \right|^2.$$

We then have, for $\mathbf{v} = (v_1, \dots, v_d)$, $\mathbf{q} = (q_1, \dots, q_d)$

$$\Theta(\mathbf{v}, \mathbf{q}) = \prod_{j=1}^d \theta(v_j, q_j),$$

and therefore

$$\min_{\mathbf{v}, \mathbf{q}} \Theta(\mathbf{v}, \mathbf{q}) = \left(\min_{v, q} \theta(v, q) \right)^d \quad \max_{\mathbf{v}, \mathbf{q}} \Theta(\mathbf{v}, \mathbf{q}) = \left(\max_{v, q} \theta(v, q) \right)^d.$$

Recalling that the Jacobi Theta functions are given, for any $z, \tau \in \mathbb{C}$ with $\text{Im } \tau > 0$ by

$$\vartheta(z, \tau) := \sum_{n \in \mathbb{Z}} \exp(\pi i n^2 \tau + 2i\pi n z),$$

we see that

$$\begin{aligned} \theta(v, q) &= \frac{e^{-q^2}}{2\pi} \left| \sum_{\ell \in \mathbb{Z}} \exp\left(-\frac{\pi \ell^2}{2} + \ell(i v \sqrt{\pi} + q \sqrt{\pi})\right) \right|^2 + \frac{e^{-q^2}}{2\pi} \left| \sum_{\ell \in \mathbb{Z}} \exp\left(-\frac{\pi \ell^2}{2} + \ell(i v \sqrt{\pi} + q \sqrt{\pi}) - i\pi\right) \right|^2 \\ &= \frac{e^{-q^2}}{2\pi} \left(\left| \vartheta\left(\frac{v - iq}{2\sqrt{\pi}}, \frac{i}{2}\right) \right|^2 + \left| \vartheta\left(\frac{v - iq}{2\sqrt{\pi}} - \frac{1}{2}, \frac{i}{2}\right) \right|^2 \right). \end{aligned}$$

This gives us the upper bound in (3.2), by continuity of ϑ . To obtain the lower bound, we note that the quantity $\vartheta(z, \tau)$ vanishes if and only if we have

$$z = \left(n + \frac{1}{2}\right) + \tau \left(m + \frac{1}{2}\right)$$

for $m, n \in \mathbb{Z}$. Hence, the term $\left| \vartheta\left(\frac{v - iq}{2\sqrt{\pi}}, \frac{i}{2}\right) \right|^2$ vanishes if and only if $v = -\sqrt{\pi}$ and $q = -\frac{\sqrt{\pi}}{2}$. On the other hand, the term $\left| \vartheta\left(\frac{v - iq}{2\sqrt{\pi}} - \frac{1}{2}, \frac{i}{2}\right) \right|^2$ vanishes if and only if $v = 0$ and $q = -\frac{\sqrt{\pi}}{2}$. Therefore, $\min_{v, q} \theta(v, q) > 0$, which proves the lower bound in (3.2).

5. DECAY OF COEFFICIENTS IN THE FRAMES

5.1. Decay of coefficients in the dual frame. Recall that the estimates in (3.2) imply that whenever $u \in L^2(\mathbb{R}^d)$, its coefficients belongs to ℓ^2 , with an ℓ^2 norm comparable to $\|u\|_{L^2(\mathbb{R}^d)}$. In Proposition 5.1 we generalize this idea by showing that if u exhibits more regularity, stronger decay properties hold true for its coefficients sequence.

Proposition 5.1 (Coefficients' decay). *For all $p \in \mathbb{N}$ and $u \in \widehat{H}^p(\mathbb{R}^d)$, we have*

$$(5.1) \quad \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \left(|\mathbf{x}^{k, \mathbf{m}}|^2 + |\boldsymbol{\xi}^{k, \mathbf{n}}|^2 \right)^p |(u, \Psi_{k, \mathbf{m}, \mathbf{n}})|^2 \leq C_p \|u\|_{\widehat{H}_k^p(\mathbb{R}^d)}^2.$$

Proof. We will prove (5.1) by induction. Note that, for $p = 0$, it is an immediate consequence of (3.2). Then, suppose that there exists $p \in \mathbb{N}$ such that (5.1) holds for all $q \leq p - 1$, and let us show that it also holds for $q = p$.

Let us fix $j \in \{1, \dots, d\}$. The polynomials $(Q_\ell)_{\ell=0}^p$ from Lemma A.2 all have different degrees, so that they form a basis of $\mathbb{R}_p[X]$. In particular, we may find real numbers $(a_\ell)_{\ell=0}^p$

solely depending on p such that

$$z^p = \sum_{\ell=0}^p a_\ell Q_\ell(z),$$

and in particular, introducing for the sake of shortness the function $y : \mathbf{x} \rightarrow (\mathbf{x}^{k,\mathbf{m}} - \mathbf{x} + i\xi^{k,\mathbf{n}})_j$, we have

$$(5.2) \quad k^{p/2} y^p = \sum_{\ell=0}^p a_\ell Q_\ell(k^{1/2} y).$$

Then, using (5.2), and recalling (A.2), we obtain

$$k^{p/2} (\bar{y}^p u, \Psi_{k,\mathbf{m},\mathbf{n}}) = \sum_{\ell=0}^p a_\ell (u, Q_\ell(k^{1/2} y) \Psi_{k,\mathbf{m},\mathbf{n}}) = \sum_{\ell=0}^p a_\ell k^{-\ell/2} (u, \partial_j^\ell \Psi_{k,\mathbf{m},\mathbf{n}}).$$

and integration by parts shows that

$$(\bar{y}^p u, \Psi_{k,\mathbf{m},\mathbf{n}}) = \sum_{\ell=0}^p (-1)^\ell a_\ell k^{-(p+\ell)/2} (\partial_j^\ell u, \Psi_{k,\mathbf{m},\mathbf{n}}),$$

where we also multiplied both sides by $k^{-p/2}$. Recalling the definition of y and using the upper-bound in (3.2), after summing over $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$, we see that

$$(5.3) \quad \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} |((x_j^{k,\mathbf{m}} - i\xi_j^{k,\mathbf{n}} + x_j)^p u, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 \\ \leq C_p \sum_{\ell=0}^p a_\ell^2 k^{-(\ell+p)} \|\partial_j^\ell u\|_{L^2(\mathbb{R}^d)}^2 \leq C_p \sum_{\ell=0}^p a_\ell^2 k^{-2\ell} \|\partial_j^\ell u\|_{L^2(\mathbb{R}^d)}^2 \leq C_p \|u\|_{H_k^p(\mathbb{R}^d)}^2,$$

where we have used the facts that the numbers a_ℓ only depend on p and that $k \geq 1$. Since we have

$$(x_j^{k,\mathbf{m}} - i\xi_j^{k,\mathbf{n}})^p = (x_j^{k,\mathbf{m}} - x_j - i\xi_j^{k,\mathbf{n}})^p - \sum_{q=0}^{p-1} \binom{p}{q} (-x_j)^{p-q} (x_j^{k,\mathbf{m}} - i\xi_j^{k,\mathbf{n}})^q,$$

and

$$\left((x_j^{k,\mathbf{m}})^2 + (\xi_j^{k,\mathbf{n}})^2 \right)^p |(u, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 = \left| \left((x_j^{k,\mathbf{m}} - i\xi_j^{k,\mathbf{n}})^p u, \Psi_{k,\mathbf{m},\mathbf{n}} \right) \right|^2,$$

we may write that

$$(5.4) \quad \left((x_j^{k,\mathbf{m}})^2 + (\xi_j^{k,\mathbf{n}})^2 \right)^p |(u, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 \leq \\ C_p \left(\left| \left((x_j^{k,\mathbf{m}} - x_j - i\xi_j^{k,\mathbf{n}})^p u, \Psi_{k,\mathbf{m},\mathbf{n}} \right) \right|^2 + \sum_{q=0}^{p-1} \left((x_j^{k,\mathbf{m}})^2 + (\xi_j^{k,\mathbf{n}})^2 \right)^q |(x_j^{p-q} u, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 \right).$$

Thanks to the induction hypothesis, we have

$$\sum_{q=0}^{p-1} \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \left((x_j^{k,\mathbf{m}})^2 + (\xi_j^{k,\mathbf{n}})^2 \right)^q |(x_j^{p-q} u, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 \leq C_p \sum_{q=0}^{p-1} \|x^{p-q} u\|_{\hat{H}_k^q(\mathbb{R}^d)}^2,$$

and (2.1) then ensures that

$$(5.5) \quad \sum_{q=0}^{p-1} \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \left((x_j^{k, \mathbf{m}})^2 + (\xi_j^{k, \mathbf{n}})^2 \right)^q |(x_j^{p-q} u, \Psi_{k, \mathbf{m}, \mathbf{n}})|^2 \leq C_p \|u\|_{\widehat{H}_k^p(\mathbb{R}^d)}^2.$$

We then sum (5.4) over $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$. Using (5.3) and (5.5) to respectively bound the first and second term in the right-hand side, we obtain

$$\sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \left((x_j^{k, \mathbf{m}})^2 + (\xi_j^{k, \mathbf{n}})^2 \right)^p |(u, \Psi_{k, \mathbf{m}, \mathbf{n}})|^2 \leq C_p \|u\|_{\widehat{H}_k^p(\mathbb{R}^d)}^2,$$

and (5.1) follows by summing over $j \in \{1, \dots, d\}$, which concludes the proof by induction. \square

5.2. Decay of coefficients in the initial frame. We now prove an analogue of Proposition 5.1 for the coefficients of the dual coefficients $(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)$, which are used for the expansion in the initial frame $(\Psi_{k, \mathbf{m}, \mathbf{n}})_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}}$.

We start with a preliminary result concerning the inner product of elements of the dual frame. Identity (5.6) follows directly from (3.5), while (5.7) can be deduced from [6, Corollary 3.7]. For the reader's convenience, we give a self-contained elementary proof of (5.7) in Appendix B.

Proposition 5.2 (Expansion of the dual frame). *For all $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$, we have*

$$(5.6) \quad \Psi_{k, \mathbf{m}, \mathbf{n}}^* = \sum_{[\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}} (\Psi_{k, \mathbf{m}, \mathbf{n}}^*, \Psi_{k, \mathbf{m}', \mathbf{n}'}^*) \Psi_{k, \mathbf{m}', \mathbf{n}'}.$$

In addition, for all $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$(5.7) \quad |(\Psi_{k, \mathbf{m}, \mathbf{n}}^*, \Psi_{k, \mathbf{m}', \mathbf{n}'}^*)| \leq C_\varepsilon e^{-\|\mathbf{m}, \mathbf{n}\| - \|\mathbf{m}', \mathbf{n}'\|^{1-\varepsilon}}.$$

We are now ready to establish our result concerning the coefficients' decay.

Proposition 5.3 (Dual coefficients' decay). *For all $p \in \mathbb{N}$ and $u \in \widehat{H}^p(\mathbb{R}^d)$, we have*

$$(5.8) \quad \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \left(|\mathbf{x}^{k, \mathbf{m}}|^2 + |\boldsymbol{\xi}^{k, \mathbf{n}}|^2 \right)^p |(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)|^2 \leq C_p \|u\|_{\widehat{H}_k^p(\mathbb{R}^d)}^2.$$

Proof. To lighten notations, let us write $\mathbf{z}^{k,\mathbf{m},\mathbf{n}} := \mathbf{x}^{k,\mathbf{m}} + i\xi^{k,\mathbf{n}}$. We have

$$\begin{aligned}
& \sum_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}} \left(|\mathbf{x}^{k,\mathbf{m}}|^2 + |\xi^{k,\mathbf{n}}|^2 \right)^p |(u, \Psi_{k,\mathbf{m},\mathbf{n}}^*)|^2 \\
&= \sum_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}} |\mathbf{z}^{k,\mathbf{m},\mathbf{n}}|^{2p} \left| \sum_{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d}} (u, \Psi_{k,\mathbf{m}',\mathbf{n}'})(\Psi_{k,\mathbf{m},\mathbf{n}}^*, \Psi_{k,\mathbf{m}',\mathbf{n}'}^*) \right|^2 \quad \text{thanks to (5.6)} \\
&= \sum_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}} \left| \sum_{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d}} \left(1 + |\mathbf{z}^{k,\mathbf{m}',\mathbf{n}'}|^p \right) \frac{|\mathbf{z}^{k,\mathbf{m},\mathbf{n}}|^p}{1 + |\mathbf{z}^{k,\mathbf{m}',\mathbf{n}'}|^p} (u, \Psi_{k,\mathbf{m}',\mathbf{n}'})(\Psi_{k,\mathbf{m},\mathbf{n}}^*, \Psi_{k,\mathbf{m}',\mathbf{n}'}^*) \right|^2 \\
&\leq \sum_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}} \left(\sum_{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d}} \left| \left(1 + |\mathbf{z}^{k,\mathbf{m}',\mathbf{n}'}|^p \right) (u, \Psi_{k,\mathbf{m}',\mathbf{n}'})(\Psi_{k,\mathbf{m},\mathbf{n}}^*, \Psi_{k,\mathbf{m}',\mathbf{n}'}^*) \right|^2 \right) \\
&\quad \times \left(\sum_{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d}} \left| \frac{|\mathbf{z}^{k,\mathbf{m},\mathbf{n}}|^p}{1 + |\mathbf{z}^{k,\mathbf{m}',\mathbf{n}'}|^p} \right|^2 |(\Psi_{k,\mathbf{m},\mathbf{n}}^*, \Psi_{k,\mathbf{m}',\mathbf{n}'}^*)| \right),
\end{aligned}$$

thanks to Cauchy-Schwarz inequality. Let us bound the second factor using (5.7). For every $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$, we have

$$\begin{aligned}
& \sum_{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d}} \left| \frac{|\mathbf{z}^{k,\mathbf{m},\mathbf{n}}|^p}{1 + |\mathbf{z}^{k,\mathbf{m}',\mathbf{n}'}|^p} \right|^2 |(\Psi_{k,\mathbf{m},\mathbf{n}}^*, \Psi_{k,\mathbf{m}',\mathbf{n}'}^*)| \\
&\leq C_p \sum_{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d}} \left| \frac{|\mathbf{z}^{k,\mathbf{m}',\mathbf{n}'}|^p + |\mathbf{z}^{k,\mathbf{m},\mathbf{n}} - \mathbf{z}^{k,\mathbf{m}',\mathbf{n}'}|^p}{1 + |\mathbf{z}^{k,\mathbf{m}',\mathbf{n}'}|^p} \right|^2 e^{-|[m,\mathbf{n}]-[\mathbf{m}',\mathbf{n}']|^{1/2}} \\
&\leq C_p \sum_{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d}} \left| 1 + |\mathbf{z}^{k,\mathbf{m},\mathbf{n}} - \mathbf{z}^{k,\mathbf{m}',\mathbf{n}'}|^p \right|^2 e^{-|[m,\mathbf{n}]-[\mathbf{m}',\mathbf{n}']|^{1/2}} \\
&\leq C_p.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \sum_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}} \left(|\mathbf{x}^{k,\mathbf{m}}|^2 + |\xi^{k,\mathbf{n}}|^2 \right)^p |(u, \Psi_{k,\mathbf{m},\mathbf{n}}^*)|^2 \\
&\leq C_p \sum_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}} \sum_{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d}} \left| \left(1 + |\mathbf{z}^{k,\mathbf{m}',\mathbf{n}'}|^p \right) (u, \Psi_{k,\mathbf{m}',\mathbf{n}'})(\Psi_{k,\mathbf{m},\mathbf{n}}^*, \Psi_{k,\mathbf{m}',\mathbf{n}'}^*) \right|^2 e^{-|[m,\mathbf{n}]-[\mathbf{m}',\mathbf{n}']|^{1/2}} \\
&\leq C_p \sum_{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d}} \left| \left(1 + |\mathbf{z}^{k,\mathbf{m}',\mathbf{n}'}|^p \right) (u, \Psi_{k,\mathbf{m}',\mathbf{n}'})(\Psi_{k,\mathbf{m},\mathbf{n}}^*, \Psi_{k,\mathbf{m}',\mathbf{n}'}^*) \right|^2,
\end{aligned}$$

and (5.8) follows from (5.1). \square

6. PROOF OF THEOREM 3.1

6.1. Approximation in $L^2(\mathbb{R}^d)$. We start by proving Theorem 3.1 in the simplest case, when $p = 0$.

Lemma 6.1 ($L^2(\mathbb{R}^d)$ approximation). *For all $u \in \widehat{H}^r(\mathbb{R}^d)$, we have*

$$(6.1) \quad \|u - \Pi_D u\|_{L^2(\mathbb{R}^d)} \leq C_r D^{-r} \|u\|_{\widehat{H}_k^r(\mathbb{R}^d)}.$$

Proof. By definition of Π_D , we have

$$\mathcal{E} := u - \Pi_D u = \sum_{\substack{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d} \\ |(\mathbf{x}^{k, \mathbf{m}}, \boldsymbol{\xi}^{k, \mathbf{n}})| > D}} (u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*) \Psi_{k, \mathbf{m}, \mathbf{n}},$$

and the frame property ensures that

$$\begin{aligned} \|\mathcal{E}\|_{L^2(\mathbb{R}^d)}^2 &\leq \sum_{\substack{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d} \\ |(\mathbf{x}^{k, \mathbf{m}}, \boldsymbol{\xi}^{k, \mathbf{n}})| > D}} |(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)|^2 \\ &\leq D^{-2r} \sum_{\substack{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d} \\ |(\mathbf{x}^{k, \mathbf{m}}, \boldsymbol{\xi}^{k, \mathbf{n}})| > D}} |(\mathbf{x}^{k, \mathbf{m}}, \boldsymbol{\xi}^{k, \mathbf{n}})|^{2r} |(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)|^2, \end{aligned}$$

and (6.1) follows from Proposition 5.3. \square

6.2. Technical preliminary results. To prove Theorem 3.1, that is, to generalize (6.1) to the case where $p > 0$, we must first show several preliminary technical results.

Lemma 6.2 (Smallness far from the origin). *For all $\mathbf{a} \in \mathbb{N}^d$, we have*

$$(6.2) \quad |\partial^{\mathbf{a}}(\Pi_D u)(\mathbf{x})| \leq C_{\mathbf{a}} k^{d/4} \left(1 + k^{1/2} D\right)^{d/2} (k^{1/2} |\mathbf{x}|)^{[\mathbf{a}]} e^{-\frac{k}{8} |\mathbf{x}|^2} \|u\|_{L^2(\mathbb{R}^d)}$$

for all $\mathbf{x} \in \mathbb{R}^d$ with $|\mathbf{x}| \geq 2D$. In addition, for all $q \geq 0$, we have

$$(6.3) \quad k^{q/2} \|\|\mathbf{x}\|^q \partial^{\mathbf{a}}(\Pi_D u)\|_{L^2(\{|\mathbf{x}| > 2D\})} \leq C_{p, q} k^{[\mathbf{a}]/2} e^{-\frac{kD^2}{16}} \|u\|_{L^2(\mathbb{R}^d)}.$$

Proof. If $|\mathbf{x}| \geq 2D$ and $|\mathbf{x}^{k, \mathbf{m}}| \leq D$, we have $|\mathbf{x} - \mathbf{x}^{k, \mathbf{m}}|^2 \geq \frac{|\mathbf{x}|^2}{4}$, so that $e^{-\frac{k}{2} |\mathbf{x} - \mathbf{x}^{k, \mathbf{m}}|^2} \leq e^{-\frac{k}{8} |\mathbf{x}|^2}$. It follows from (A.2) that there exists a polynomial $Q_{\mathbf{a}}$ of total degree $[\mathbf{a}]$ such that

$$(\partial^{\mathbf{a}} \Psi_{k, \mathbf{m}, \mathbf{n}})(\mathbf{x}) = k^{[\mathbf{a}]/2} Q_{\mathbf{a}}(k^{1/2} (\mathbf{x} - \mathbf{x}^{k, \mathbf{m}} + i \boldsymbol{\xi}^{k, \mathbf{n}})) \Psi_{k, \mathbf{m}, \mathbf{n}}(\mathbf{x})$$

so that

$$|\partial^{\mathbf{a}}(\Pi_D u)(\mathbf{x})| \leq C k^{d/4} \sum_{\substack{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d} \\ |(\mathbf{x}^{k, \mathbf{m}}, \boldsymbol{\xi}^{k, \mathbf{n}})| \leq D}} |(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)| |Q_{\mathbf{a}}(k^{1/2} (\mathbf{x} - \mathbf{x}^{k, \mathbf{m}} + i \boldsymbol{\xi}^{k, \mathbf{n}}))| e^{-\frac{k}{8} |\mathbf{x}|^2}.$$

We then observe that since $|\mathbf{x}| \geq 2D$ and $|\mathbf{x}_j^{k, \mathbf{m}}| + |\xi_j^{k, \mathbf{n}}| \leq CD$ for all terms in the sum, we have

$$|\mathbf{x} - \mathbf{x}^{k, \mathbf{m}} + i \boldsymbol{\xi}^{k, \mathbf{n}}| \leq C |\mathbf{x}|.$$

As a result, we have

$$|\partial^{\mathbf{a}}(\Pi_D u)(\mathbf{x})| \leq C_{\mathbf{a}} k^{d/4} \left(\sum_{\substack{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d} \\ |(\mathbf{x}^{k, \mathbf{m}}, \boldsymbol{\xi}^{k, \mathbf{n}})| \leq D}} |(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)| \right) (k^{1/2} |\mathbf{x}|)^{[\mathbf{a}]} e^{-\frac{k}{8} |\mathbf{x}|^2}$$

and (6.2) follows from (3.2) and from the Cauchy-Schwarz inequality, as the sum is over a set of cardinal smaller than $C(1+k^{1/2}D)^d$.

Next, we set $p = [\mathbf{a}]$, and we consider

$$I_q := \int_{|\mathbf{x}| \geq 2D} \left(|\mathbf{x}|^{p+q} e^{-\frac{k}{8}|\mathbf{x}|^2} \right)^2 d\mathbf{x} = C \int_{2D}^{+\infty} r^{d-1+2(p+q)} e^{-\frac{k}{4}r^2} dr,$$

using radial coordinates. Introducing $s = \sqrt{kr}$, we get

$$\begin{aligned} I_q &= \int_{2k^{1/2}D}^{+\infty} (k^{-1/2}s)^{d-1+2(p+q)} e^{-\frac{s^2}{4}} k^{-1/2} ds \\ &= k^{-(d+2(p+q))/2} \int_{2k^{1/2}D}^{+\infty} s^{d-1+2(p+q)} e^{-\frac{s^2}{4}} \sqrt{s} ds \\ &\leq k^{-(d+2(p+q))/2} e^{-\frac{kD^2}{2}} \int_0^{+\infty} s^{d-1+2(p+q)} e^{-\frac{s^2}{8}} ds \\ &= C_p k^{-(d+2(p+q))/2} e^{-\frac{kD^2}{2}}. \end{aligned}$$

It then follows from (6.2) that

$$\begin{aligned} \|\mathbf{x}^q \partial^{\mathbf{a}}(\Pi_D u)\|_{L^2(\{|\mathbf{x}| > 2D\})} &\leq C k^{d/4} (1+k^{1/2}D)^{d/2} k^p \sqrt{I_q} \|u\|_{L^2(\mathbb{R}^d)} \\ &\leq C_q k^{d/4} (1+k^{1/2}D)^{d/2} k^p k^{-(d+2(p+q))/4} e^{-\frac{kD^2}{8}} \|u\|_{L^2(\mathbb{R}^d)} \\ &= C_q k^{-q/2} k^{p/2} (1+k^{1/2}D)^{d/2} e^{-\frac{kD^2}{8}} \|u\|_{L^2(\mathbb{R}^d)} \\ &\leq C_q k^{-q/2} k^{p/2} e^{-\frac{kD^2}{16}} \|u\|_{L^2(\mathbb{R}^d)}, \end{aligned}$$

and (6.3) follows. \square

We then prove a preliminary result about the decay properties of $u - \Pi_D u$.

Corollary 6.3. *For any $p, q, r \in \mathbb{N}$, any $\mathbf{a} \in \mathbb{N}^d$ with $q + [\mathbf{a}] \leq p$ and for any $u \in \widehat{H}^{p+r}(\mathbb{R}^d)$, we have*

$$(6.4) \quad k^{-[\mathbf{a}]} \|\mathbf{x}^q \partial^{\mathbf{a}}(u - \Pi_D u)\|_{L^2(\mathbb{R}^d)} \leq C_{\mathbf{a}, q, r} D^q \left(k^{-[\mathbf{a}]} \|\partial^{\mathbf{a}}(u - \Pi_D u)\|_{L^2(\mathbb{R}^d)} + D^{-(p+r)} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)} \right).$$

Proof. We first write that

$$\begin{aligned} &\|\mathbf{x}^q \partial^{\mathbf{a}}(u - \Pi_D u)\|_{L^2(\mathbb{R}^d)} \\ &= \|\mathbf{x}^q \partial^{\mathbf{a}}(u - \Pi_D u)\|_{L^2(\{|\mathbf{x}| < 2D\})} + \|\mathbf{x}^q \partial^{\mathbf{a}}(u - \Pi_D u)\|_{L^2(\{|\mathbf{x}| > 2D\})} \\ &\leq (2D)^q \|\partial^{\mathbf{a}}(u - \Pi_D u)\|_{L^2(\mathbb{R}^d)} + \|\mathbf{x}^q \partial^{\mathbf{a}} u\|_{L^2(\{|\mathbf{x}| > 2D\})} + \|\mathbf{x}^q \partial^{\mathbf{a}}(\Pi_D u)\|_{L^2(\{|\mathbf{x}| > 2D\})}. \end{aligned}$$

To deal with the second term, we note that

$$\|\mathbf{x}^q \partial^{\mathbf{a}} u\|_{L^2(\{|\mathbf{x}| > 2D\})} \leq (2D)^{q-p-r} \|\mathbf{x}^{q-q+p} + r \partial^{\mathbf{a}} u\|_{L^2(\mathbb{R}^d)} \leq (2D)^{q-p-r} k^{[\mathbf{a}]} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}.$$

On the other hand, for the last term, we use (6.3), to obtain

$$\begin{aligned} k^{-[\mathbf{a}]}\|\|\mathbf{x}|^q\partial^{\mathbf{a}}(\Pi_D u)\|_{L^2(\{|\mathbf{x}|>2D\})} &\leq k^{(-[\mathbf{a}]-q)/2}e^{-\frac{kD^2}{16}}\|u\|_{L^2(\mathbb{R}^d)} \\ &\leq C_r k^{(-[\mathbf{a}]-q)/2}k^{2(q-p-r)}D^{q-p-r}\|u\|_{L^2} \\ &\leq C_r D^{(q-p)-r}\|u\|_{L^2}. \end{aligned}$$

Here, we used the fact that $e^{-\frac{x^2}{16}} \leq C_r x^{-r}$ for all $x \geq 1$, and the fact that $k \geq 1$. The result follows. \square

6.3. Approximation in $H^p(\mathbb{R}^d)$. The next key result needed to establish Theorem 3.1 concerns approximation in the space $H^p(\mathbb{R}^d)$.

Lemma 6.4 ($H^p(\mathbb{R}^d)$ approximation). *For all $p, r \geq 0$, if $u \in \widehat{H}^{p+r}(\mathbb{R}^d)$, we have*

$$(6.5) \quad k^{-p}\|\partial_j^p(u - \Pi_D u)\|_{L^2(\mathbb{R}^d)} \leq C_{p,r}D^{-r}\|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}.$$

Proof. Consider $u \in \widehat{H}^{p+r}(\mathbb{R}^d)$, fix $D > 0$ and let $\mathcal{E} := u - \Pi_D u$, so that

$$\mathcal{E} = \sum_{|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| > D} (u, \Psi_{k,\mathbf{m},\mathbf{n}}^*) \Psi_{k,\mathbf{m},\mathbf{n}}.$$

Recall that the family $(\Psi_{k,\mathbf{m},\mathbf{n}})_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}}$ is a frame, so that

$$k^{-2p}\|\partial_j^p \mathcal{E}\|_{L^2(\mathbb{R}^d)}^2 \leq Ck^{-2p} \sum_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}} |(\partial_j^p \mathcal{E}, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 = C(\sigma_{\text{small}} + \sigma_{\text{large}})$$

with

$$\sigma_{\text{small}} := k^{-2p} \sum_{|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| \leq 2D} |(\partial_j^p \mathcal{E}, \Psi_{k,\mathbf{m},\mathbf{n}})|^2, \quad \sigma_{\text{large}} := k^{-2p} \sum_{|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| > 2D} |(\partial_j^p \mathcal{E}, \Psi_{k,\mathbf{m},\mathbf{n}})|^2.$$

We start by estimating the first term. Recalling (A.2) from Lemma A.2, we have

$$|(\partial_j^p \mathcal{E}, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 = |(\mathcal{E}, \partial_j^p \Psi_{k,\mathbf{m},\mathbf{n}})|^2 = k^p \left| \left(\mathcal{E}, Q_p(\sqrt{k}(x_j^{k,\mathbf{m}} - x_j + i\xi_j^{k,\mathbf{n}})) \Psi_{k,\mathbf{m},\mathbf{n}} \right) \right|^2.$$

The binomial theorem yields that

$$\begin{aligned} |Q_p(k^{1/2}(x_j^{k,\mathbf{m}} - x_j + i\xi_j^{k,\mathbf{n}}))|^2 &\leq C_p \sum_{q=0}^p (k^{1/2}|x_j^{k,\mathbf{m}} + i\xi_j^{k,\mathbf{n}}|)^{2(p-q)} (k^{1/2}x_j)^{2q} \\ &\leq C_p \sum_{q=0}^p (k^{1/2}D)^{2(p-q)} k^q |\mathbf{x}|^{2q}, \end{aligned}$$

so that

$$|(\partial_j^p \mathcal{E}, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 \leq C_p k^p \sum_{q=0}^p (k^{1/2}D)^{2(p-q)} k^q (|\mathbf{x}|^q \mathcal{E}, \Psi_{k,\mathbf{m},\mathbf{n}})^2$$

for all $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$ with $|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| \leq 2D$. Since the coherent states form a frame, after summation over $[\mathbf{m}, \mathbf{n}]$, we may write

$$\begin{aligned} \sigma_{\text{small}} &\leq C_p k^{-p} \sum_{q=0}^p (k^{1/2} D)^{2(p-q)} k^q \|\mathbf{x}^q \mathcal{E}\|_{L^2(\mathbb{R}^d)}^2 \\ &= C_p D^{2p} \sum_{q=0}^p D^{-2q} \|\mathbf{x}^q \mathcal{E}\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

Since $q \leq p$, we may employ Corollary 6.3 with $\mathbf{a} = 0$, leading to

$$\begin{aligned} \sigma_{\text{small}} &\leq C_{p,r} D^{2p} \sum_{q=0}^p \left(\|\mathcal{E}\|_{L^2(\mathbb{R}^d)}^2 + D^{-2(p+r)} \|u\|_{\widehat{H}_k^{q+r}(\mathbb{R}^d)}^2 \right) \\ &\leq C_{p,r} \left(D^{2p} \|\mathcal{E}\|_{L^2(\mathbb{R}^d)}^2 + D^{-2r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}^2 \right), \end{aligned}$$

and it follows from (6.1) that

$$\sigma_{\text{small}} \leq C_{p,r} D^{-2r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}^2,$$

which is the desired result.

We now turn σ_{large} , that we further split as

$$\sigma_{\text{large}} \leq 2(\sigma'_{\text{large}} + \sigma''_{\text{large}}),$$

where

$$\begin{aligned} \sigma'_{\text{large}} &:= k^{-2p} \sum_{|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| > 2D} |(\partial_j^p u, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 \\ \sigma''_{\text{large}} &:= k^{-2p} \sum_{|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| > 2D} |(\partial_j^p \Pi_D u, \Psi_{k,\mathbf{m},\mathbf{n}})|^2. \end{aligned}$$

We handle the first component with Proposition 5.1. Indeed, it follows from (5.1) that

$$\begin{aligned} \sigma'_{\text{large}} &\leq k^{-2p} \sum_{|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| > 2D} |(\partial_j^p u, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 \\ &\leq C_r k^{-2p} D^{-2r} \sum_{|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| > 2D} |(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})|^{2r} |(\partial_j^p u, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 \\ &\leq C_r k^{-2p} D^{-2r} \|\partial_j^p u\|_{\widehat{H}_k^r(\mathbb{R}^d)}^2 \leq C_r D^{-2r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}^2. \end{aligned}$$

For the second component, we invoke Lemma A.3. On the one hand, recalling the definition of $\Pi_D u$ in (3.7), we have

$$\begin{aligned} |(\partial_j^p (\Pi_D u), \Psi_{k,\mathbf{m},\mathbf{n}})|^2 &= \left| \sum_{|(\mathbf{x}^{k,\mathbf{m}'}, \boldsymbol{\xi}^{k,\mathbf{n}'})| \leq D} (u, \Psi_{k,\mathbf{m}',\mathbf{n}'}^*) (\partial_j^p \Psi_{k,\mathbf{m}',\mathbf{n}'}, \Psi_{k,\mathbf{m},\mathbf{n}}) \right|^2 \\ &\leq C(k^{1/2} D)^{2d} \sum_{|(\mathbf{x}^{k,\mathbf{m}'}, \boldsymbol{\xi}^{k,\mathbf{n}'})| \leq D} |(u, \Psi_{k,\mathbf{m}',\mathbf{n}'}^*)|^2 |(\partial_j^p \Psi_{k,\mathbf{m}',\mathbf{n}'}, \Psi_{k,\mathbf{m},\mathbf{n}})|^2, \end{aligned}$$

since the summation happens on a set of cardinal less than $C(k^{1/2}D)^d$. On the other hand, by Lemma A.3,

$$\begin{aligned} |(\partial_j^p \Psi_{k,\mathbf{m},\mathbf{n}}, \Psi_{k,\mathbf{m}',\mathbf{n}'})| &\leq C_p k^{p/2} (1 + |\mathbf{n}|^p) e^{-\frac{\pi}{8}kD^2} \\ &= C_p k^{p/2} \left(1 + (k^{1/2}D)^p\right) e^{-\frac{\pi}{8}kD^2} \leq C_{p,r} k^{p/2} (k^{1/2}D)^{-r-d} \leq C_{p,r} k^{p/2} (k^{1/2}D)^{-d} D^{-r} \end{aligned}$$

since $k \geq 1$.

Using the frame property, we obtain

$$\begin{aligned} \sigma''_{\text{large}} &\leq k^{-2p} \left(C_{p,r} k^p D^{-2r} \sum_{|(\mathbf{x}^k, \mathbf{m}', \boldsymbol{\xi}^k, \mathbf{n}')| \leq D} |(u, \Psi_{k,\mathbf{m}',\mathbf{n}'})|^2 \right) \\ &\leq C_{p,r} D^{-2r} \|u\|_{L^2(\mathbb{R}^d)}^2 \\ &\leq C_{p,r} D^{-2r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}^2. \end{aligned}$$

□

We are now ready to conclude the proof of our main result.

Proof of Theorem 3.1. As previously, we set $\mathcal{E} := u - \Pi_D u$. Using Fourier transform, we start by observing that

$$\|\partial^\alpha \mathcal{E}\|_{L^2(\mathbb{R}^d)}^2 = \|(i\xi)^\alpha \mathcal{F}\mathcal{E}\|_{L^2(\mathbb{R}^d)}^2 \leq \| |\xi|^{[\alpha]} \mathcal{F}\mathcal{E} \|_{L^2(\mathbb{R}^d)}^2 \leq C \sum_{j=1}^d \|\partial_j^{[\alpha]} \mathcal{E}\|_{L^2(\mathbb{R}^d)}^2.$$

As a result, using (6.5) (with $p = [\mathbf{a}]$ and $r = (p+r) - [\mathbf{a}]$), we have

$$k^{-[\mathbf{a}]} \|\partial^\alpha \mathcal{E}\|_{L^2(\mathbb{R}^d)} \leq C_{p,r} D^{[\mathbf{a}] - (p+r)} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}.$$

for all $\mathbf{a} \in \mathbb{N}^d$ with $[\mathbf{a}] \leq p$.

Let $q \leq p$, and let $\mathbf{a} \in \mathbb{N}^d$ with $q + |\mathbf{a}| \leq p$. Using (6.4), we have

$$\begin{aligned} &k^{-[\mathbf{a}]} \| |\mathbf{x}|^q \partial^\alpha \mathcal{E} \|_{L^2(\mathbb{R}^d)} \\ &\leq C_{p,r} D^q \left(k^{-[\mathbf{a}]} \|\partial^\alpha \mathcal{E}\|_{L^2(\mathbb{R}^d)} + D^{-(p+r)} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)} \right) \\ &\leq C_{p,r} \left(D^{[\mathbf{a}] + q - p - r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)} + D^{(q-p) - r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)} \right) \\ &\leq C_{p,r} D^{-r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)} \end{aligned}$$

since $q + [\mathbf{a}] \leq p$ and $D \geq 1$, which concludes the proof. □

APPENDIX A. ELEMENTARY FACTS CONCERNING GAUSSIAN STATES

A.1. Fourier transform and derivatives. We start by recalling the expression of the Fourier transform of Gaussian states.

Lemma A.1 (Fourier transform). *For all $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$, we have*

$$(A.1) \quad \mathcal{F}_k(\Psi_{k,\mathbf{m},\mathbf{n}}) = e^{-ik\mathbf{x}^k, \mathbf{m} \cdot \boldsymbol{\xi}^k, \mathbf{n}} \Psi_{k,\mathbf{n}, -\mathbf{m}}.$$

Proof. Let $\mathbf{y} = k^{1/2}(\mathbf{x} - \mathbf{x}^{k,m})$, we have

$$\begin{aligned} \mathcal{F}_k(\Psi_{k,m,n}) &= 2^{d/2} \left(\frac{k}{2\pi}\right)^{3d/4} \int_{\mathbb{R}^d} e^{-\frac{k}{2}|\mathbf{x}-\mathbf{x}^{k,m}|^2} e^{ik(\mathbf{x}-\mathbf{x}^{k,m})\cdot\xi^{k,n}} e^{-ik\mathbf{x}\cdot\xi} d\mathbf{x} \\ &= 2^{d/2} \left(\frac{k}{2\pi}\right)^{3d/4} e^{-i\mathbf{x}^{k,m}\cdot\xi} \int_{\mathbb{R}^d} e^{-\frac{|\mathbf{y}|^2}{2}} e^{ik^{1/2}\mathbf{y}\cdot\xi^{k,n}} e^{-ik^{1/2}\mathbf{y}\cdot\xi} d\mathbf{y} \\ &= 2^{d/2} \left(\frac{k}{2\pi}\right)^{3d/4} e^{-i\mathbf{x}^{k,m}\cdot\xi} \int_{\mathbb{R}^d} e^{-\frac{|\mathbf{y}|^2}{2}} e^{ik^{1/2}\mathbf{y}\cdot(\xi^{k,n}-\xi)} d\mathbf{y} \\ &= \left(\frac{k}{\pi}\right)^{d/4} e^{-\frac{k}{2}|\xi-\xi^{k,n}|^2} e^{-i\mathbf{x}^{k,m}\cdot\xi}. \end{aligned}$$

□

The following lemma gives an elementary expression for the derivatives of $\Psi_{k,m,n}$.

Lemma A.2 (Differentiation). *For all $p \in \mathbb{N}$, there exists a polynomial Q_p of degree p , with real coefficients, and leading coefficient z^p , such that for all $j \in \{1, \dots, d\}$ and all $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$, we have*

$$(A.2) \quad (\partial_j^p \Psi_{k,m,n})(\mathbf{x}) = k^{p/2} Q_p(k^{1/2}(x_j^{k,m} - x_j + i\xi_j^{k,n})) \Psi_{k,m,n}(\mathbf{x}).$$

Proof. We define the polynomials by induction, setting $Q_0 = 1$, and

$$Q_{p+1}(z) = Q_p'(z) + zQ_p(z),$$

for $p \geq 0$. An elementary induction shows that Q_p is a polynomial of degree p whose term of degree p is z^p .

Our aim is now to show inductively that (A.2) holds. This is trivially the case for $p = 0$ and for $p = 1$, noting that

$$\partial_j \Psi_{k,m,n} = k \left(x_j^{k,m} - x_j + i\xi_j^{k,n} \right) \Psi_{k,m,n}.$$

Now, suppose the result holds for some $p \in \mathbb{N}$. We then have

$$\begin{aligned} \partial_j^{p+1} \Psi_{k,m,n} &= k^{p/2} \left[k^{1/2} \partial_j Q_p(\sqrt{k}(x_j^{k,m} - x_j + i\xi_j^{k,n})) \Psi_{k,m,n} \right. \\ &\quad \left. + Q_p(k^{1/2}(x_j^{k,m} - x_j + i\xi_j^{k,n})) \times k(x_j^{k,m} - x_j + i\xi_j^{k,n}) \Psi_{k,m,n} \right] \\ &= k^{(p+1)/2} Q_{p+1}(k^{1/2}(x_j^{k,m} - x_j + i\xi_j^{k,n})) \Psi_{k,m,n}, \end{aligned}$$

which proves the result. □

A.2. Scalar products between Gaussian states. We then provide expression and upper-bounds of inner-products involving Gaussian states and their derivatives.

Proposition A.3 (L^2 products). *For all $[\mathbf{m}, \mathbf{n}], [\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}$, we have*

$$(A.3) \quad |(\Psi_{k,m,n}, \Psi_{k,m',n'})| = e^{-\frac{\pi}{4}||[\mathbf{m}, \mathbf{n}] - [\mathbf{m}', \mathbf{n}']|^2}.$$

In addition, for all $p \in \mathbb{N}$, there exists a constant C_p such that

$$(A.4) \quad |(\partial_j^p \Psi_{k,m,n}, \Psi_{k,m',n'})| \leq C_p k^{p/2} (1 + |\mathbf{n}|^p) e^{-\frac{\pi}{8}||[\mathbf{m}, \mathbf{n}] - [\mathbf{m}', \mathbf{n}']|^2}$$

for all $[\mathbf{m}, \mathbf{n}], [\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}$.

To prove this proposition, we will need the following elementary result.

Lemma A.4 (A useful identity). *Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^d$. We have*

$$(A.5) \quad \frac{1}{2} (|\mathbf{x} - \mathbf{y}|^2 + |\mathbf{x} - \mathbf{z}|^2) = \left| \mathbf{x} - \frac{1}{2}(\mathbf{y} + \mathbf{z}) \right|^2 + \left| \frac{1}{2}(\mathbf{y} - \mathbf{z}) \right|^2.$$

Proof. On the one hand, we have

$$\frac{1}{2} (|\mathbf{x} - \mathbf{y}|^2 + |\mathbf{x} - \mathbf{z}|^2) = |\mathbf{x}|^2 - \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) + \frac{1}{2}|\mathbf{y}|^2 + \frac{1}{2}|\mathbf{z}|^2.$$

On the other hand, we have

$$\left| \mathbf{x} - \frac{1}{2}(\mathbf{y} + \mathbf{z}) \right|^2 + \left| \frac{1}{2}(\mathbf{y} - \mathbf{z}) \right|^2 = |\mathbf{x}|^2 - \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) + \frac{1}{4} (|\mathbf{y} + \mathbf{z}|^2 + |\mathbf{y} - \mathbf{z}|^2)$$

and the proof follows since

$$|\mathbf{y} + \mathbf{z}|^2 + |\mathbf{y} - \mathbf{z}|^2 = 2|\mathbf{y}|^2 + 2|\mathbf{z}|^2.$$

□

Recall that, if we write, for $k > 0$ and $\mathbf{x}^* \in \mathbb{R}^d$

$$g_{k, \mathbf{x}^*}(\mathbf{x}) := e^{-k|\mathbf{x} - \frac{1}{2}\mathbf{x}^*|^2},$$

then we have for all $\boldsymbol{\xi}^* \in \mathbb{R}^d$

$$(A.6) \quad \mathcal{F}_k(g_{k, \mathbf{x}^*})(\boldsymbol{\xi}^*) = 2^{-d/2} e^{-\frac{k}{4}|\boldsymbol{\xi}^*|^2} e^{-\frac{ik}{2}\mathbf{x}^* \cdot \boldsymbol{\xi}^*}.$$

We will use the following notations

$$\begin{aligned} \mathbf{x}^{k, m, m'} &:= \frac{\mathbf{x}^{k, m} + \mathbf{x}^{k, m'}}{2} \\ \widehat{\mathbf{x}}^{k, m, m'} &:= \mathbf{x}^{k, m} - \mathbf{x}^{k, m'} \\ \widehat{\boldsymbol{\xi}}^{k, n, n'} &:= \boldsymbol{\xi}^{k, n} - \boldsymbol{\xi}^{k, n'}. \end{aligned}$$

Proof of Proposition A.3. We have

$$\begin{aligned} \Psi_{k, m, n}(\mathbf{x}) \overline{\Psi_{k, m', n'}(\mathbf{x})} &= \left(\frac{k}{\pi}\right)^{d/2} e^{-\frac{k}{2}|\mathbf{x} - \mathbf{x}^{k, m}|^2} e^{ik(\mathbf{x} - \mathbf{x}^{k, m}) \cdot \boldsymbol{\xi}^{k, n}} e^{-\frac{k}{2}|\mathbf{x} - \mathbf{x}^{k, m'}|^2} e^{-ik(\mathbf{x} - \mathbf{x}^{k, m'}) \cdot \boldsymbol{\xi}^{k, n'}} \\ &= \left(\frac{k}{\pi}\right)^{d/2} e^{-\frac{k}{2}(|\mathbf{x} - \mathbf{x}^{k, m}|^2 + |\mathbf{x} - \mathbf{x}^{k, m'}|^2)} e^{ik\mathbf{x} \cdot \widehat{\boldsymbol{\xi}}^{k, n, n'}} e^{ik\mathbf{x}^{k, m'} \cdot \widehat{\boldsymbol{\xi}}^{k, n, n'}}. \end{aligned}$$

Using (A.5), we have

$$-\frac{k}{2} (|\mathbf{x} - \mathbf{x}^{k, m}|^2 + |\mathbf{x} - \mathbf{x}^{k, m'}|^2) = -k \left| \mathbf{x} - \mathbf{x}^{k, m, m'} \right|^2 - \frac{k}{4} |\widehat{\mathbf{x}}^{k, m, m'}|^2,$$

leading to

$$(A.7) \quad \Psi_{k, m, n}(\mathbf{x}) \overline{\Psi_{k, m', n'}(\mathbf{x})} = \left(\frac{k}{\pi}\right)^{d/2} e^{ik(\mathbf{x}^{k, m'} \cdot \boldsymbol{\xi}^{k, n'} - \mathbf{x}^{k, m} \cdot \boldsymbol{\xi}^{k, n})} e^{-\frac{k}{4}|\widehat{\mathbf{x}}^{k, m, m'}|^2} e^{-k|\mathbf{x} - \mathbf{x}^{k, m, m'}|^2} e^{ik\mathbf{x} \cdot \widehat{\boldsymbol{\xi}}^{k, n, n'}}.$$

Recalling (A.6), we have

$$\begin{aligned}
 & (\Psi_{k,m,n}, \Psi_{k,m',n'}) \\
 \text{(A.8)} \quad &= 2^{d/2} e^{ik(\mathbf{x}^{k,m'} \cdot \boldsymbol{\xi}^{k,n'} - \mathbf{x}^{k,m} \cdot \boldsymbol{\xi}^{k,n})} e^{-\frac{k^2}{4} |\widehat{\mathbf{x}}^{k,m,m'}|^2} \mathcal{F}_k(g_{k,2\mathbf{x}^{k,m,m'}})(-\widehat{\boldsymbol{\xi}}^{k,n,n'}) \\
 &= e^{-i\mathbf{x}^{k,m,m'} \cdot \widehat{\boldsymbol{\xi}}^{k,n,n'}} e^{ik(\mathbf{x}^{k,m'} \cdot \boldsymbol{\xi}^{k,n'} - \mathbf{x}^{k,m} \cdot \boldsymbol{\xi}^{k,n})} e^{-\frac{k^2}{4} |\widehat{\mathbf{x}}^{k,m,m'}|^2} e^{-\frac{k^2}{4} |\widehat{\boldsymbol{\xi}}^{k,n,n'}|^2},
 \end{aligned}$$

which gives (A.3).

Noting that $\mathbf{x}^{k,m} = \mathbf{x}^{k,m,m'} + \frac{\widehat{\mathbf{x}}^{k,m,m'}}{2}$ and using Lemma A.2, we get

$$\begin{aligned}
 \partial_j^p \Psi_{k,m,n}(\mathbf{x}) &= k^{p/2} Q_p \left(\sqrt{k} (x_j^{k,m,m'} - x_j + \frac{\widehat{x}_j^{k,m,m'}}{2} + i\xi_j^{k,n}) \right) \Psi_{k,m,n}(\mathbf{x}) \\
 &= \sum_{\ell=0}^p k^{\frac{p+\ell}{2}} \sum_{\ell'=0}^{\ell} c_{\ell,\ell'} \left(\frac{\widehat{x}_j^{k,m,m'}}{2} + i\xi_j^{k,n} \right)^{\ell-\ell'} (x_j^{k,m,m'} - x_j)^{\ell'} \Psi_{k,m,n}(\mathbf{x}),
 \end{aligned}$$

with all the coefficients $c_{\ell,\ell'}$ independent of k . Therefore, we have

$$\begin{aligned}
 \text{(A.9)} \quad & \left| (\partial_j^p \Psi_{k,m,n}, \Psi_{k,m',n'}) \right| \leq \\
 & C_p \sum_{\ell=0}^p k^{\frac{p+\ell}{2}} \sum_{\ell'=0}^{\ell} \left(|\widehat{x}_j^{k,m,m'}| + |\xi_j^{k,n}| \right)^{\ell-\ell'} \left| \left((x_j^{k,m,m'} - x_j)^{\ell'} \Psi_{k,m,n}, \Psi_{k,m',n'} \right) \right|.
 \end{aligned}$$

Recalling (A.7), we have

$$\begin{aligned}
 \text{(A.10)} \quad & \left(x_j^{k,m,m'} - x_j \right)^{\ell'} \Psi_{k,m,n}(\mathbf{x}) \overline{\Psi_{k,m',n'}(\mathbf{x})} = \\
 & \left(\frac{k}{\pi} \right)^{d/2} e^{ik(\mathbf{x}^{k,m'} \cdot \boldsymbol{\xi}^{k,n'} - \mathbf{x}^{k,m} \cdot \boldsymbol{\xi}^{k,n})} e^{-\frac{k}{4} |\widehat{\mathbf{x}}^{k,m,m'}|^2} \left(x_j^{k,m,m'} - x_j \right)^{\ell'} e^{-k|\mathbf{x} - \mathbf{x}^{k,m,m'}|^2} e^{ik\mathbf{x} \cdot \widehat{\boldsymbol{\xi}}^{k,n,n'}}.
 \end{aligned}$$

We then compute

$$\begin{aligned}
 & \int_{\mathbb{R}^d} (\mathbf{x} - \mathbf{x}^{k,m,m'})_j^{\ell'} e^{-k|\mathbf{x} - \mathbf{x}^{k,m,m'}|^2} e^{ik\mathbf{x} \cdot \widehat{\boldsymbol{\xi}}^{k,n,n'}} d\mathbf{x} \\
 &= e^{ik\mathbf{x}^{k,m,m'} \cdot \widehat{\boldsymbol{\xi}}^{k,n,n'}} \int_{\mathbb{R}^d} k^{\ell'/2} \mathbf{y}_j^{\ell'} e^{-|\mathbf{y}|^2} e^{i\sqrt{k}\mathbf{y} \cdot \widehat{\boldsymbol{\xi}}^{k,n,n'}} d\mathbf{y} \quad \text{setting } \mathbf{y} = k^{-1/2}(\mathbf{x} - \mathbf{x}^{k,m,m'}) \\
 &= e^{ik\mathbf{x}^{k,m,m'} \cdot \widehat{\boldsymbol{\xi}}^{k,n,n'}} (i\sqrt{k}\partial_j)^{\ell'} (e^{-\frac{1}{4}|\boldsymbol{\xi}|^2}) \left(\sqrt{k}\widehat{\boldsymbol{\xi}}^{k,n,n'} \right).
 \end{aligned}$$

Now, $\partial_j^{\ell'} (e^{-\frac{1}{4}|\boldsymbol{\xi}|^2})$ is a polynomial of degree ℓ' times $e^{-\frac{1}{4}|\boldsymbol{\xi}|^2}$, so that its modulus is bounded by $C_{\ell'}(1 + |\boldsymbol{\xi}|^{\ell'})e^{-\frac{1}{4}|\boldsymbol{\xi}|^2}$. Therefore, we have

$$\left| \left((x_j^{k,m,m'} - x_j)^{\ell'} \Psi_{k,m,n}, \Psi_{k,m',n'} \right) \right| \leq C k^{\frac{\ell'}{2}} |\widehat{\boldsymbol{\xi}}^{k,n,n'}|^{\ell'} e^{-\frac{k}{4} |\widehat{\mathbf{x}}^{k,m,m'}|^2} e^{-\frac{k}{4} |\widehat{\boldsymbol{\xi}}^{k,n,n'}|^2}.$$

Combining this with (A.9), we get

$$\begin{aligned}
\left| (\partial_j^p \Psi_{k,m,n}, \Psi_{k,m',n'}) \right| &\leq C_p k^{\frac{p}{2}} e^{-\frac{k}{4} |\widehat{\mathbf{x}}^{k,m,m'}|^2} e^{-\frac{k}{4} |\widehat{\boldsymbol{\xi}}^{k,n,n'}|^2} \sum_{\ell=0}^p \sum_{\ell'=0}^{\ell} k^{\frac{\ell+\ell'}{2}} \left(|\widehat{\mathbf{x}}^{k,m,m'}| + |\boldsymbol{\xi}^{k,n}| \right)^{\ell-\ell'} |\widehat{\boldsymbol{\xi}}^{k,n,n'}|^{\ell'} \\
&\leq C_p k^{\frac{p}{2}} e^{-\frac{k}{4} |\widehat{\mathbf{x}}^{k,m,m'}|^2} e^{-\frac{k}{4} |\widehat{\boldsymbol{\xi}}^{k,n,n'}|^2} \sum_{\ell=0}^p \left(|\sqrt{k} \widehat{\mathbf{x}}^{k,m,m'}| + |\sqrt{k} \boldsymbol{\xi}^{k,n}| + |\sqrt{k} \widehat{\boldsymbol{\xi}}^{k,n,n'}| \right)^{\ell} \\
&\leq C_p k^{\frac{p}{2}} e^{-\frac{k}{4} |\widehat{\mathbf{x}}^{k,m,m'}|^2} e^{-\frac{k}{4} |\widehat{\boldsymbol{\xi}}^{k,n,n'}|^2} \left[1 + \left(|\sqrt{k} \widehat{\mathbf{x}}^{k,m,m'}| + |\sqrt{k} \boldsymbol{\xi}^{k,n}| + |\sqrt{k} \widehat{\boldsymbol{\xi}}^{k,n,n'}| \right)^p \right],
\end{aligned}$$

as announced. \square

APPENDIX B. LOCALISATION PROPERTIES OF THE DUAL FRAME

The aim of this appendix is to give an elementary proof of (5.7). Recalling the constants α and β from the frame inequalities in (3.2), we set

$$(B.1) \quad Y := I - \frac{2}{\alpha + \beta} T_k^* T_k,$$

so that Y is self-adjoint with $\|Y\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq \gamma < 1$. In particular,

$$S_k = (T_k^* T_k)^{-1} = \frac{2}{\alpha + \beta} (I - Y)^{-1} = \frac{2}{\alpha + \beta} \sum_{\ell=0}^{\infty} Y^{\ell},$$

where the sum converges for the norm of bounded linear operators acting on $L^2(\mathbb{R}^d)$. Therefore, we have

$$\begin{aligned}
(\Psi_{k,m,n}^*, \Psi_{k,m',n'}^*) &= \left(\frac{2}{\alpha + \beta} \right)^2 \left(\sum_{\ell=0}^{+\infty} Y^{\ell} \Psi_{k,m,n}, \sum_{\ell'=0}^{+\infty} Y^{\ell'} \Psi_{k,m',n'} \right) \\
&= \left(\frac{2}{\alpha + \beta} \right)^2 \sum_{\ell,\ell'=0}^{+\infty} \left(\Psi_{k,m,n}, Y^{\ell+\ell'} \Psi_{k,m',n'} \right) \\
&= \left(\frac{2}{\alpha + \beta} \right)^2 \sum_{p=0}^{+\infty} (p+1) (\Psi_{k,m,n}, Y^p \Psi_{k,m',n'}).
\end{aligned}$$

We claim that there exists $A > 1$ such that, for all $p \in \mathbb{N}$ and all $[\mathbf{m}, \mathbf{n}], [\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}$, we have

$$(B.2) \quad |(\Psi_{k,m,n}, Y^p \Psi_{k,m',n'})| \leq A^p e^{-\frac{\pi}{4} |[\mathbf{m}, \mathbf{n}] - [\mathbf{m}', \mathbf{n}']|}.$$

Let us prove this result by induction. It trivially holds for $p = 0$, due to (A.3). Assume now that (B.2) holds for $p - 1$. Recalling (B.1), we have

$$Y^p = Y^{p-1} - \frac{2}{\alpha + \beta} (T_k^* T_k) Y^{p-1}$$

and its follows from (3.4) that

$$\begin{aligned} (\Psi_{k,\mathbf{m},\mathbf{n}}, Y^p \Psi_{k,\mathbf{m}',\mathbf{n}'}) &= (\Psi_{k,\mathbf{m},\mathbf{n}}, Y^{p-1} \Psi_{k,\mathbf{m}',\mathbf{n}'}) \\ &\quad - \frac{2}{\alpha + \beta} \sum_{[\mathbf{m}'', \mathbf{n}''] \in \mathbb{Z}^{2d}} (\Psi_{k,\mathbf{m},\mathbf{n}}, \Psi_{k,\mathbf{m}'',\mathbf{n}''}) (\Psi_{k,\mathbf{m}'',\mathbf{n}'', Y^{p-1} \Psi_{k,\mathbf{m}',\mathbf{n}'}) , \end{aligned}$$

so that, by our induction hypothesis and (A.3), we have

$$\begin{aligned} |(\Psi_{k,\mathbf{m},\mathbf{n}}, Y^p \Psi_{k,\mathbf{m}',\mathbf{n}'})| &\leq A^{p-1} e^{-\frac{\pi}{4} |[\mathbf{m}, \mathbf{n}] - [\mathbf{m}', \mathbf{n}']|} \\ &\quad + \frac{2A^{p-1}}{\alpha + \beta} \sum_{[\mathbf{m}'', \mathbf{n}''] \in \mathbb{Z}^{2d}} e^{-\frac{\pi}{4} |[\mathbf{m}, \mathbf{n}] - [\mathbf{m}'', \mathbf{n}'']|^2} e^{-\frac{\pi}{4} |[\mathbf{m}'', \mathbf{n}''] - [\mathbf{m}', \mathbf{n}']|} . \end{aligned}$$

We now simplify the sum. For the sake of simplicity, we introduce $\mathbf{q} = [\mathbf{m}, \mathbf{n}]$, $\mathbf{q}' = [\mathbf{m}', \mathbf{n}']$ and $\mathbf{q}'' = [\mathbf{m}'', \mathbf{n}'']$. The triangle inequality reveals that

$$\begin{aligned} |\mathbf{q} - \mathbf{q}''|^2 + |\mathbf{q}' - \mathbf{q}''| &= |\mathbf{q} - \mathbf{q}''|^2 - |\mathbf{q} - \mathbf{q}''| + |\mathbf{q} - \mathbf{q}''| + |\mathbf{q}' - \mathbf{q}''| \\ &\geq |\mathbf{q} - \mathbf{q}''|^2 - |\mathbf{q} - \mathbf{q}''| + |\mathbf{q} - \mathbf{q}'| , \end{aligned}$$

and as a result

$$\sum_{\mathbf{q}'' \in \mathbb{Z}^{2d}} e^{-\frac{\pi}{4} |\mathbf{q} - \mathbf{q}''|^2} e^{-\frac{\pi}{4} |\mathbf{q}' - \mathbf{q}''|} \leq e^{-\frac{\pi}{4} |\mathbf{q} - \mathbf{q}'|} \sum_{\mathbf{q}'' \in \mathbb{Z}^{2d}} e^{\frac{\pi}{4} (|\mathbf{q} - \mathbf{q}''| - |\mathbf{q} - \mathbf{q}'|)} \leq \sigma e^{-\frac{\pi}{4} |\mathbf{q} - \mathbf{q}'|} ,$$

where we easily see that

$$\sigma := \sum_{\mathbf{q}'' \in \mathbb{Z}^{2d}} e^{\frac{\pi}{4} (|\mathbf{q} - \mathbf{q}''| - |\mathbf{q} - \mathbf{q}'|)}$$

is finite and does not depend on \mathbf{q} .

Therefore, we have

$$|(\Psi_{k,\mathbf{m},\mathbf{n}}, Y^p \Psi_{k,\mathbf{m}',\mathbf{n}'})| \leq A^{p-1} \left(1 + \frac{2\sigma}{\alpha + \beta} \right) e^{-\frac{\pi}{4} |[\mathbf{m}, \mathbf{n}] - [\mathbf{m}', \mathbf{n}']|}$$

and (B.2) holds, provided we take

$$A \geq 1 + \frac{2\sigma}{\alpha + \beta} .$$

To conclude, we write $\delta := |[\mathbf{m}, \mathbf{n}] - [\mathbf{m}', \mathbf{n}']|$, and we let P denote the largest integer such that $P \leq \delta^{1-\varepsilon}$. We have, for any $\gamma' \in (\gamma, 1)$

$$\begin{aligned} |(\Psi_{k,\mathbf{m},\mathbf{n}}^*, \Psi_{k,\mathbf{m}',\mathbf{n}'}^*)| &\leq C \sum_{p=0}^P (p+1) |(\Psi_{k,\mathbf{m},\mathbf{n}}, Y^p \Psi_{k,\mathbf{m}',\mathbf{n}'})| + C \sum_{p=P+1}^{+\infty} (p+1) \gamma^p \\ &\leq C e^{-\frac{\pi}{4} \delta} \sum_{p=0}^P (p+1) A^p + C \sum_{p=P+1}^{+\infty} (\gamma')^p \\ &\leq C(P+1)^2 A^P e^{-\frac{\pi}{4} \delta} + C(\gamma')^{P+1} \\ &\leq C\delta^2 A^{\delta^{1-\varepsilon}} e^{-\frac{\pi}{4} \delta} + C(\gamma')^{\delta^{1-\varepsilon}} . \end{aligned}$$

One readily checks that both terms are bounded by $C_\varepsilon e^{-\delta^{1-2\varepsilon}}$ for any $\varepsilon > 0$, which concludes the proof.

APPENDIX C. SHARPNESS OF THEOREM 3.1

The aim of this appendix is to show the sharpness of Theorem 3.1, as asserted in Proposition 3.2. We start with a preliminary result concerning the norm of a coherent Gaussian state.

Lemma C.1 ($\widehat{H}_k^p(\mathbb{R}^d)$ norm). *Let $p \geq 1$. For all $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$, we have*

$$(C.1) \quad \mu_p \left(1 + |\mathbf{x}^{k,\mathbf{m}}|^2 + |\boldsymbol{\xi}^{k,\mathbf{n}}|^2\right)^p \leq \|\Psi_{k,\mathbf{m},\mathbf{n}}\|_{\widehat{H}_k^p(\mathbb{R}^d)}^2 \leq \lambda_p \left(1 + |\mathbf{x}^{k,\mathbf{m}}|^2 + |\boldsymbol{\xi}^{k,\mathbf{n}}|^2\right)^p,$$

where the constants $\mu_p, \lambda_p > 0$ only depend on p and d .

Proof. Fix $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$ and let $j \in \{1, \dots, d\}$ be such that $n_j \geq [\mathbf{n}]/d$. We have

$$(C.2) \quad \|\Psi_{k,\mathbf{m},\mathbf{n}}\|_{\widehat{H}_k^p(\mathbb{R}^d)}^2 \geq \|\Psi_{k,\mathbf{m},\mathbf{n}}\|_{L^2(\mathbb{R}^d)}^2 + \|\mathbf{x}^{k,\mathbf{m}}\Psi_{k,\mathbf{m},\mathbf{n}}\|_{L^2(\mathbb{R}^d)}^2 + k^{-2p}\|\partial_{x_j}^p \Psi_{k,\mathbf{m},\mathbf{n}}\|_{L^2(\mathbb{R}^d)}^2.$$

The first term is equal to 1. The second is

$$\begin{aligned} \|\mathbf{x}^{k,\mathbf{m}}\Psi_{k,\mathbf{m},\mathbf{n}}\|_{L^2(\mathbb{R}^d)}^2 &= \left(\frac{k}{\pi}\right)^{d/2} \int_{\mathbb{R}^d} |\mathbf{x}|^{2p} e^{-k|\mathbf{x}-\mathbf{x}^{k,\mathbf{m}}|^2} d\mathbf{x} \\ &= \pi^{-d/2} \int_{\mathbb{R}^d} |\mathbf{x}^{k,\mathbf{m}} + k^{-1/2}\mathbf{y}|^{2p} e^{-|\mathbf{y}|^2} d\mathbf{y} \geq ck^p |\mathbf{x}^{k,\mathbf{m}}|^{2p}. \end{aligned}$$

The last term in (C.2) is equal to

$$\begin{aligned} k^{-2p}\|\partial_{x_j}^p \Psi_{k,\mathbf{m},\mathbf{n}}\|_{L^2(\mathbb{R}^d)}^2 &= k^{-p} \left\| Q_p \left(k^{1/2}(x_j^{k,\mathbf{m}} - x_j + i\xi_j^{k,\mathbf{n}}) \right) \Psi_{k,\mathbf{m},\mathbf{n}} \right\|_{L^2(\mathbb{R}^d)}^2 \\ &= k^{-p} \pi^{-d/2} \int_{\mathbb{R}^d} \left| Q_p(y_j + ik^{1/2}\xi_j^{k,\mathbf{n}}) \right|^2 e^{-|\mathbf{y}|^2} d\mathbf{y} \\ &\geq (k^{-p} + |\boldsymbol{\xi}^{k,\mathbf{n}}|^{2p}). \end{aligned}$$

This gives us the lower bound in (C.1).

On the other hand, it holds that

$$\begin{aligned} \|\Psi_{k,\mathbf{m},\mathbf{n}}\|_{\widehat{H}_k^p(\mathbb{R}^d)}^2 &= \sum_{[\mathbf{a}] \leq p} \sum_{q=0}^p k^{-[\mathbf{a}]} \left\| |\mathbf{x}|^q \prod_{j=1}^d Q_{a_j} \left(k^{1/2}(x_j^{k,\mathbf{m}} - x_j + i\xi_j^{k,\mathbf{n}}) \right) \Psi_{k,\mathbf{m},\mathbf{n}} \right\|_{L^2(\mathbb{R}^d)}^2 \\ &= \sum_{[\mathbf{a}] \leq p} \sum_{q=0}^p k^{-[\mathbf{a}]} \pi^{-d/2} \int_{\mathbb{R}^d} |\mathbf{x}^{k,\mathbf{m}} + k^{-1/2}\mathbf{y}|^{2q} \left| \prod_{j=1}^d Q_{a_j}(y_j + ik^{1/2}\xi_j^{k,\mathbf{n}}) \right|^2 e^{-|\mathbf{y}|^2} d\mathbf{y}. \end{aligned}$$

We then observe that

$$\begin{aligned} k^{-[\mathbf{a}]} \int_{\mathbb{R}^d} |\mathbf{x}^{k,\mathbf{m}} + k^{-1/2}\mathbf{y}|^{2q} \left| \prod_{j=1}^d Q_{a_j}(y_j + ik^{1/2}\xi_j^{k,\mathbf{n}}) \right|^2 e^{-|\mathbf{y}|^2} d\mathbf{y} \\ \leq C_{d,p} k^{-[\mathbf{a}]} |\mathbf{x}^{k,\mathbf{m}}|^{2q} \prod_{j=1}^d \left(1 + |k^{1/2}\xi_j^{k,\mathbf{n}}|^{a_j}\right)^2 \leq C_{d,p} \left(1 + |\mathbf{x}^{k,\mathbf{m}}|^2 + |\boldsymbol{\xi}^{k,\mathbf{n}}|^2\right)^p, \end{aligned}$$

which lead to the upper bound in (C.1) after summation. \square

We then need another intermediate result concerning the inner product of elements of the Gabor and dual frames.

Lemma C.2 (L^2 product of the dual frame). *For all $[\mathbf{m}, \mathbf{n}], [\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}$, we have*

$$(C.3) \quad |(\Psi_{k, \mathbf{m}, \mathbf{n}}^*, \Psi_{k, \mathbf{m}', \mathbf{n}'})| \leq C e^{-\frac{\pi}{8} |[\mathbf{m}, \mathbf{n}] - [\mathbf{m}', \mathbf{n}']|^{1/2}}.$$

Proof. Using (A.4) and (5.7), we get

$$\begin{aligned} |(\Psi_{k, \mathbf{m}, \mathbf{n}}^*, \Psi_{k, \mathbf{m}', \mathbf{n}'})| &\leq \sum_{[\mathbf{m}'', \mathbf{n}''] \in \mathbb{Z}^{2d}} |(\Psi_{k, \mathbf{m}, \mathbf{n}}^*, \Psi_{k, \mathbf{m}'', \mathbf{n}''}^*)| |(\Psi_{k, \mathbf{m}'', \mathbf{n}''}, \Psi_{k, \mathbf{m}', \mathbf{n}'})| \\ &\leq C \sum_{[\mathbf{m}'', \mathbf{n}''] \in \mathbb{Z}^{2d}} e^{-|[\mathbf{m}'', \mathbf{n}''] - [\mathbf{m}, \mathbf{n}]|^{1/2}} e^{-\frac{\pi}{8} |[\mathbf{m}'', \mathbf{n}''] - [\mathbf{m}', \mathbf{n}']|^2} \\ &= C \sum_{[\mathbf{m}'', \mathbf{n}''] \in \mathbb{Z}^{2d}} e^{-|\mathbf{q}'' - \mathbf{q}|^{1/2} - \frac{\pi}{8} |\mathbf{q}'' - \mathbf{q}'|^2}, \end{aligned}$$

where we introduced $\mathbf{q} := [\mathbf{m}, \mathbf{n}]$, $\mathbf{q}' := [\mathbf{m}', \mathbf{n}']$ and $\mathbf{q}'' := [\mathbf{m}'', \mathbf{n}'']$ for the sake of shortness. Since $\mathbf{q}', \mathbf{q}'' \in \mathbb{Z}^{2d}$, we have $|\mathbf{q}'' - \mathbf{q}'|^2 \geq |\mathbf{q}'' - \mathbf{q}'|^{1/2}$, and hence

$$e^{-|\mathbf{q}'' - \mathbf{q}|^{1/2} - \frac{\pi}{8} |\mathbf{q}'' - \mathbf{q}'|^2} \leq e^{-|\mathbf{q}'' - \mathbf{q}|^{1/2} - \frac{\pi}{8} |\mathbf{q}'' - \mathbf{q}'|^{1/2}} = e^{-\frac{\pi}{8} (|\mathbf{q}'' - \mathbf{q}|^{1/2} + |\mathbf{q}'' - \mathbf{q}'|^{1/2})} e^{-(1 - \frac{\pi}{8}) |\mathbf{q}'' - \mathbf{q}|^{1/2}},$$

where we note that $1 - \pi/8 > 0$. Then, since

$$\sqrt{|\mathbf{x} - \mathbf{y}|} \leq \sqrt{|\mathbf{x}|} + \sqrt{|\mathbf{y}|} \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^{2d},$$

we have

$$e^{-|\mathbf{q}'' - \mathbf{q}|^{1/2} - \frac{\pi}{8} |\mathbf{q}'' - \mathbf{q}'|^2} \leq e^{-\frac{\pi}{8} |\mathbf{q}' - \mathbf{q}|^{1/2}} e^{-(1 - \frac{\pi}{8}) |\mathbf{q}'' - \mathbf{q}|^{1/2}}.$$

Therefore, we get

$$\begin{aligned} |(\Psi_{k, \mathbf{m}, \mathbf{n}}^*, \Psi_{k, \mathbf{m}', \mathbf{n}'})| &\leq C e^{-\frac{\pi}{8} |\mathbf{q} - \mathbf{q}'|^{1/2}} \sum_{[\mathbf{m}'', \mathbf{n}''] \in \mathbb{Z}^{2d}} e^{-(1 - \frac{\pi}{8}) |\mathbf{q}'' - \mathbf{q}|^{1/2}} \\ &\leq C e^{-\frac{\pi}{8} |\mathbf{q} - \mathbf{q}'|^{1/2}} \end{aligned}$$

as announced. □

Proof of Proposition 3.2. Let $D \geq k^{-1/2}$. Pick $\mathbf{m} = (2\lceil k^{1/2} D \rceil, 0, 0)$ and let $u = \Psi_{k, \mathbf{m}, \mathbf{0}}$. For any $p \geq 0$, from Lemma C.1, we have

$$\|u\|_{\hat{H}_k^p(\mathbb{R}^d)} \geq \mu_p (2D)^{-p}.$$

On the other hand since

$$\Pi_D \Psi_{k, \mathbf{m}, \mathbf{0}} = \sum_{\substack{[\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d} \\ \|[\mathbf{m}', \mathbf{n}']\| \leq k^{1/2} D}} (\Psi_{k, \mathbf{m}, \mathbf{0}}, \Psi_{k, \mathbf{m}', \mathbf{n}'}^*) \Psi_{k, \mathbf{m}', \mathbf{n}'}$$

we have

$$\begin{aligned}
\|\Pi_D \Psi_{k,m,0}\|_{\widehat{H}_k^p(\mathbb{R}^d)} &\leq \sum_{\substack{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d} \\ \|[\mathbf{m}',\mathbf{n}']\| \leq k^{1/2}D}} |(\Psi_{k,m,0}, \Psi_{k,m',n'}^*)| \|\Psi_{k,m',n'}\|_{\widehat{H}_k^p(\mathbb{R}^d)} \\
&\leq C \sum_{\substack{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d} \\ \|[\mathbf{m}',\mathbf{n}']\| \leq k^{1/2}D}} e^{-\frac{\pi}{8}|\mathbf{m}-\mathbf{m}'|^{1/2}} \|\Psi_{k,m',n'}\|_{\widehat{H}_k^p(\mathbb{R}^d)} \\
&\leq C \sum_{\substack{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d} \\ \|[\mathbf{m}',\mathbf{n}']\| \leq k^{1/2}D}} e^{-\frac{\pi}{8}|\mathbf{m}-\mathbf{m}'|^{1/2}} D^p \\
&\leq CD^p M(k^{1/2}D)^{2d+1} e^{-a\sqrt{k^{1/2}D}} \\
&\leq CD^p e^{-a'\sqrt{k^{1/2}D}} \\
&\leq CD^p e^{-a'\sqrt{D}},
\end{aligned}$$

since $k \geq 1$.

In particular, there exists D_* such that for all $D \geq D^*$, this quantity is smaller than $\mu_p D^{-p}$. We thus have, for all $D \geq D^*$

$$\|u - \Pi_D u\|_{\widehat{H}_k^p(\mathbb{R}^d)} \geq \|u\|_{\widehat{H}_k^p(\mathbb{R}^d)} - \|\Pi_D u\|_{\widehat{H}_k^p(\mathbb{R}^d)} \geq \mu_p D^{-p},$$

and thanks to Lemma C.1, this quantity is larger than $\frac{\mu_{p+r}}{\lambda_p} D^{-r} \|u\|_{\widehat{H}^{p+r}(\mathbb{R}^d)}$. The result follows. \square

APPENDIX D. PROOF OF THEOREM 3.1 USING MODULATION SPACES

D.1. Rescaling. First of all, let us explain how the proof of Theorem 3.1 can be reduced to the case where $k = 1$. Recall that the isometry δ_k was introduced in (4.2). For any $s \in \mathbb{N}$, we write, for all $\mathbf{a} \in \mathbb{Z}^d$ and $q \in \mathbb{N}$ with $[\mathbf{a}] + q = s$

$$\begin{aligned}
\|\mathbf{x}^q \partial^{\mathbf{a}} u\|_{L^2(\mathbb{R}^d)} &= \|\delta_k(|\mathbf{x}|^q \partial^{\mathbf{a}} u)\|_{L^2(\mathbb{R}^d)} = \|k^{-q/2} |\mathbf{x}|^q \delta_k(\partial^{\mathbf{a}} u)\|_{L^2(\mathbb{R}^d)} \\
&= \|k^{(-q+\mathbf{a})/2} |\mathbf{x}|^q \partial^{\mathbf{a}}(\delta_k u)\|_{L^2(\mathbb{R}^d)} = k^{-s/2} k^{[\mathbf{a}]} \|\mathbf{x}^q \partial^{\mathbf{a}} \delta_k u\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Therefore, if we introduce the norm

$$\|u\|_{\widetilde{H}_k^s(\mathbb{R}^d)}^2 := \sum_{\alpha+\beta=s} k^{-2|\beta|} \|x^\alpha \partial^\beta u\|_{L^2(\mathbb{R}^d)}^2,$$

for any parameter $k \geq 1$, we have

$$\|u\|_{\widetilde{H}_k^s(\mathbb{R}^d)} = k^{-s/2} \|\delta_k u\|_{\widetilde{H}_1^s(\mathbb{R}^d)}.$$

We want to study

$$\begin{aligned} \left\| u - \sum_{|(\mathbf{m}, \mathbf{n})| \leq D\sqrt{k/\pi}} (u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*) \Psi_{k, \mathbf{m}, \mathbf{n}} \right\|_{\tilde{H}_k^s(\mathbb{R}^d)} &= k^{-\frac{s}{2}} \left\| \delta_k u - \sum_{|(\mathbf{m}, \mathbf{n})| \leq D\sqrt{k/\pi}} (\delta_k u, \delta_k \Psi_{k, \mathbf{m}, \mathbf{n}}^*) \delta_k \Psi_{k, \mathbf{m}, \mathbf{n}} \right\|_{\tilde{H}_1^s(\mathbb{R}^d)} \\ &= k^{-\frac{s}{2}} \left\| \delta_k u - \sum_{|(\mathbf{m}, \mathbf{n})| \leq D\sqrt{k/\pi}} (\delta_k u, \Psi_{1, \mathbf{m}, \mathbf{n}}^*) \Psi_{1, \mathbf{m}, \mathbf{n}} \right\|_{\tilde{H}_1^s(\mathbb{R}^d)}. \end{aligned}$$

Suppose that we can show that, for any $v \in \widehat{H}^{s+s'}$, we have

$$(D.1) \quad \left\| v - \sum_{|(\mathbf{m}, \mathbf{n})| \leq D} \langle (v, \Psi_{1, \mathbf{m}, \mathbf{n}}^*) \Psi_{1, \mathbf{m}, \mathbf{n}} \right\|_{\tilde{H}_1^s} \leq C(s, s') D^{-s'} \|v\|_{\tilde{H}_1^{s+s'}}.$$

We will then deduce from what precedes that

$$\begin{aligned} \|u - \Pi_D u\|_{\widehat{H}_k^p(\mathbb{R}^d)} &= \sum_{s=0}^p \|u - \Pi_D u\|_{\tilde{H}_k^s(\mathbb{R}^d)} \\ &\leq \sum_{s=0}^p C(s, s') k^{-\frac{s}{2}} D^{-s'} k^{-\frac{s'}{2}} \|\delta_k u\|_{\tilde{H}_1^{s+s'}} \\ &= D^{-s'} \sum_{s=0}^p C(s, s') \|u\|_{\tilde{H}_k^{s+s'}} \\ &\leq CD^{-s'} \|u\|_{H_k^{p+s'}}, \end{aligned}$$

which gives us (3.8). Therefore, Theorem 3.1 will follow if we prove (D.1).

D.2. Reminder on modulation spaces. If $(\mathbf{x}_0, \boldsymbol{\xi}_0) \in \mathbb{R}^{2d}$, we introduce the function on \mathbb{R}^d

$$(D.2) \quad \psi_{\mathbf{x}_0, \boldsymbol{\xi}_0}(\mathbf{x}) := \pi^{-d/4} e^{i(\mathbf{x} - \mathbf{x}_0) \cdot \boldsymbol{\xi}_0} e^{-\frac{|\mathbf{x} - \mathbf{x}_0|^2}{2}} \quad \forall \mathbf{x} \in \mathbb{R}^d.$$

For every $s \in \mathbb{N}$, we introduce the weight v_s on \mathbb{R}^{2d} given by $v_s(\mathbf{x}, \boldsymbol{\xi}) = (1 + |\mathbf{x}|^2 + |\boldsymbol{\xi}|^2)^{s/2}$, and we define the corresponding modulation spaces $M_{v_s}^p$ for all $p \geq 1$ to be the space of functions such that the following norm is finite:

$$\|f\|_{M_{v_s}^p} := \left(\int_{\mathbb{R}^{2d}} |v_s(\mathbf{x}, \boldsymbol{\xi})|^p |(f, \psi_{\mathbf{x}, \boldsymbol{\xi}})|^p d\mathbf{x} d\boldsymbol{\xi} \right)^{1/p}.$$

Let us discuss another interpretation of these spaces when $p = 2$, following [7, Proposition 11.3.1]. Let $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$ with $[\mathbf{a}] + [\mathbf{b}] \leq s$, so that $|\mathbf{x}^{\mathbf{a}} \boldsymbol{\xi}^{\mathbf{b}}| \leq v_s^2(\mathbf{x}, \boldsymbol{\xi})$. Writing $(\tau_{\mathbf{x}} f)(\mathbf{y}) :=$

$f(\mathbf{y} - \mathbf{x})$ and $g(\mathbf{x}) = e^{-|\mathbf{x}|^2/2}$, we then have

$$\begin{aligned}
\|f\|_{M_{v_s}^2}^2 &\geq \int_{\mathbb{R}^d} |\mathbf{x}^a| \left(\int_{\mathbb{R}^d} |\boldsymbol{\xi}^b| \left| \left(e^{i\mathbf{x}\cdot\boldsymbol{\xi}} \tau_{\mathbf{x}} f, g \right) \right|^2 \right) d\mathbf{x} \\
&= \int_{\mathbb{R}^d} |\mathbf{x}^a| \left(\int_{\mathbb{R}^d} |\boldsymbol{\xi}^b| |(\mathcal{F}(\tau_{\mathbf{x}} f, g))(\boldsymbol{\xi})|^2 \right) d\mathbf{x} \\
&= \int_{\mathbb{R}^d} |\mathbf{x}^a| \left\| \partial_{\mathbf{x}}^b (\tau_{\mathbf{x}} f, g) \right\|_{L^2}^2 d\mathbf{x} \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathbf{x}^a| |\partial_{\mathbf{x}}^b f|^2(\mathbf{x} - \mathbf{y}) |g(\mathbf{y})|^2 d\mathbf{y} d\mathbf{x} \\
&= \int_{\mathbb{R}^d} |\partial_{\mathbf{z}}^b f|^2(\mathbf{z}) \left(\int_{\mathbb{R}^d} |\mathbf{y}^a + \mathbf{z}^a| |g(\mathbf{y})|^2 d\mathbf{z} \right) d\mathbf{y} \\
&\geq C \int_{\mathbb{R}^d} |\partial_{\mathbf{z}}^b f|^2(\mathbf{z}) |\mathbf{z}^a| d\mathbf{z}
\end{aligned}$$

Conversely, [7, Proposition 11.3.1] implies that $\|f\|_{M_{v_s}^2} \leq C \|f\|_{\tilde{H}_1^s(\mathbb{R}^d)}$, so that $\|\cdot\|_{M_{v_s}^2}$ and $\|\cdot\|_{\tilde{H}_1^s(\mathbb{R}^d)}$ are equivalent norms on $M_{v_s}^2$.

D.3. Sequence spaces. Let $s \in \mathbb{N}$. If $c = (c_{\mathbf{m}, \mathbf{n}})_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}}$, we introduce the norm

$$\|c\|_{\ell_s^2} := \left(\sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} |c_{\mathbf{m}, \mathbf{n}}|^2 (1 + |\mathbf{m}|^2 + |\mathbf{n}|^2)^s \right)^{1/2},$$

and denote by ℓ_s^2 the set of sequences such that this norm is finite.

If $D \in \mathbb{N}$, write χ_D for the multiplication by the sequence taking value 1 if $|(\mathbf{m}, \mathbf{n})| \geq D$ and 0 otherwise. We then clearly have, for any $s, s' \geq 0$,

$$\|\chi_D\|_{\ell_{s+s'}^2 \rightarrow \ell_s^2} \leq D^{-s'}.$$

D.4. Proof of Theorem 3.1. First of all, we note that the function $g(\mathbf{x}) = e^{-|\mathbf{x}|^2/2}$ belongs to $M_{v_s}^1$ for any s . Therefore, the canonical dual function $\gamma = \Psi_{1,0,0}^*$ does also belong to $M_{v_s}^1$ for all s , thanks to [7, Theorem 13.2.1].

It follows from [7, Theorem 12.2.4] that the restriction to M_s^2 of the operator T_1 introduced in (3.3) is bounded from M_s^2 to ℓ_s^2 . We shall denote this operator by C_γ .

We also introduce the operator $D_g : c \mapsto \sum_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^{2d}} c_{\mathbf{m}, \mathbf{n}} \Psi_{\mathbf{m}, \mathbf{n}}$. Thanks to [7, Theorem 12.2.4], D_g is a bounded operator from ℓ_s^2 to M_s^2 . Furthermore, thanks to [7, Corollary 12.2.6], for any $u \in M_s^2$, we have $u = D_g C_\gamma u$.

We therefore have, for any $u \in M_s^2$,

$$\begin{aligned}
\left\| u - \sum_{|(\mathbf{m}, \mathbf{n})| \leq \frac{D}{\sqrt{\pi}}} \langle (u, \Psi_{1, \mathbf{m}, \mathbf{n}}^*) \Psi_{1, \mathbf{m}, \mathbf{n}} \right\|_{M_s^2} &= \left\| D_g \chi_{\frac{D}{\sqrt{\pi}}} C_\gamma u \right\|_{M_s^2} \\
&\leq C(s, s') \left\| \chi_{\frac{D}{\sqrt{\pi}}} \right\|_{\ell_{s+s'}^2 \rightarrow \ell^2(s)} \|u\|_{M_{s+s'}^2} \\
&\leq C(s, s') D^{-s'} \|u\|_{M_{s+s'}^2},
\end{aligned}$$

so that

$$\left\| u - \sum_{|(\mathbf{m}, \mathbf{n})| \leq \frac{D}{\sqrt{\pi}}} \langle (u, \Psi_{1, \mathbf{m}, \mathbf{n}}^*) \Psi_{1, \mathbf{m}, \mathbf{n}} \right\|_{\tilde{H}_1^s} \leq C(s, s') D^{-s'} \|u\|_{\tilde{H}_1^{s+s'}},$$

so that (D.1) follows.

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