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DECAY OF COEFFICIENTS AND APPROXIMATION RATES IN GABOR GAUSSIAN FRAMES

THÉOPHILE CHAUMONT-FRELET^{*,†} AND MAXIME INGREMEAU[†]

ABSTRACT. Gabor frames are a standard tool to decompose functions into a discrete sum of “coherent states”, which are localised both in position and Fourier spaces. Such expansions are somehow similar to Fourier expansions, but are more subtle, as Gabor frames do not form orthonormal bases. In this work, we analyze decay properties of the coefficients of functions in these frames in terms of the regularity of the functions and their decay at infinity. These results are analogous to the standard decay properties of Fourier coefficients, and permit to show that a finite number of coherent states provide a good approximation to any smooth rapidly decaying function. Specifically, we provide explicit convergence rates in Sobolev norms, as the number of selected coherent states increases. Our results are especially useful in numerical analysis, when Gabor wavelets are employed to discretize PDE problems.

KEYWORDS: Approximation theory, Gabor frames, Phase-space analysis

1. INTRODUCTION

A central tool in analysis is Fourier analysis, which allows to decompose any function as a sum of plane waves, i.e., of functions $e^{ix \cdot \xi_0}$ for some parameter ξ_0 . Crucially, the regularity of the function dictates the decay rate of its Fourier transform (or coefficients): the more regular the function is, the faster its Fourier transform/coefficients decay.

Considering for instance the segment $(-\pi, \pi)$, the family $(e^{inx})_{n \in \mathbb{Z}}$ is an orthonormal Hilbert basis of $L^2(-\pi, \pi)$. As a result, the expansion formula

$$(1.1) \quad u = \sum_{n \in \mathbb{Z}} (u, e^{inx}) e^{inx}$$

holds true for all $u \in L^2(-\pi, \pi)$. It is also well-known that if u is regular, say in the Sobolev space $H^q(-\pi, \pi)$, then its Fourier coefficients decay in the sense that

$$(1.2) \quad \sum_{n \in \mathbb{Z}} |n|^{2q} |(u, e^{inx})|^2 \leq C_q \|u\|_{H^q(-\pi, \pi)}^2,$$

where the constant C_q only depends on q . A direct consequence is that if we form the partial sum

$$(1.3) \quad u_N := \sum_{|n| \leq N} (u, e^{inx}) e^{inx},$$

then

$$(1.4) \quad \|u - u_N\|_{H^p(-\pi, \pi)} \leq C_{p,r} N^{-r} \|u\|_{H^{p+r}(-\pi, \pi)},$$

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whenever $p + r \leq q$ with a constant $C_{p,r}$ only depending on p and r . Among other things, (1.4) is especially interesting in signal theory and numerical analysis, since it means that a smooth function can be well-approximated by a finite number of degrees of freedom.

In many situations, it is more relevant to use other families of functions as a “basis” instead of plane waves. In this work, we are especially interested in decompositions into “coherent states” of the form

$$(1.5) \quad \psi_{\mathbf{x}_0, \boldsymbol{\xi}_0}(\mathbf{x}) := \pi^{-d/4} e^{i(\mathbf{x} - \mathbf{x}_0) \cdot \boldsymbol{\xi}_0} e^{-\frac{|\mathbf{x} - \mathbf{x}_0|^2}{2}},$$

for some parameters $\mathbf{x}_0, \boldsymbol{\xi}_0 \in \mathbb{R}^d$. The appeal of such functions is that they are *localised in phase space*, in the sense that $|\psi_{\mathbf{x}_0, \boldsymbol{\xi}_0}(\mathbf{x})|$ becomes very small when $|\mathbf{x} - \mathbf{x}_0|$ is large, while the Fourier transform $|\mathcal{F}\psi_{\mathbf{x}_0, \boldsymbol{\xi}_0}(\boldsymbol{\xi})|$ becomes very small when $|\boldsymbol{\xi} - \boldsymbol{\xi}_0|$ is large. Actually, Gaussian states are the functions that are the best localised in phase space, in the sense that they saturate the Heisenberg uncertainty principle.

Continuous decompositions of functions as Gaussian states are a common tool in PDEs, and in particular in semiclassical analysis. Namely, for any function $u \in L^2(\mathbb{R}^d)$, we have

$$u = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u, \psi_{\mathbf{x}_0, \boldsymbol{\xi}_0}) \psi_{\mathbf{x}_0, \boldsymbol{\xi}_0} d\mathbf{x}_0 d\boldsymbol{\xi}_0.$$

We refer, for instance, to [9, Chapter 1] for a proof of this formula. However, in many frameworks such as numerical analysis, it is more natural to decompose functions as *discrete* sums of coherent states. To this end, we set, for any $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$,

$$(1.6) \quad \Psi_{\mathbf{m}, \mathbf{n}} := \psi_{\mathbf{m}\sqrt{\pi}, \mathbf{n}\sqrt{\pi}}.$$

The idea of using discrete sums of coherent states first appeared in the seminal work of Gabor [8]. It was then showed that any function can be decomposed using the functions $\Psi_{\mathbf{m}, \mathbf{n}}$ [6]. Specifically, there exists a set of functions $(\Psi_{\mathbf{m}, \mathbf{n}}^*)_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \subset L^2(\mathbb{R}^d)$ such that for any $u \in L^2(\mathbb{R}^d)$

$$(1.7) \quad u = \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} (u, \Psi_{\mathbf{m}, \mathbf{n}}^*) \Psi_{\mathbf{m}, \mathbf{n}},$$

where the series converges in the L^2 sense. The decomposition in (1.7) is to be compared with (1.1), with the major difference that the expansion in (1.7) is by no means the unique expansion of u as a linear combination of $\Psi_{\mathbf{m}, \mathbf{n}}$.

To the best of our knowledge, the equivalent of (1.2) and (1.4) for Gabor frames is not available in the literature, and it is the purpose of this work to rigorously establish it. Unlike for Fourier coefficients, these decay properties will not only depend on the regularity of u , but also on its decay at infinity. We actually show that

$$(1.8) \quad \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} |[\mathbf{m}, \mathbf{n}]|^{2q} |(u, \Psi_{\mathbf{m}, \mathbf{n}}^*)|^2 \leq C_q \|u\|_{\widehat{H}^q(\mathbb{R}^d)},$$

where $\|\cdot\|_{\widehat{H}^q(\mathbb{R}^d)}$ is a weighted Sobolev norm introduced in Section 2. We also prove approximation results with explicit convergence rates, namely

$$(1.9) \quad \|u - \Pi_D u\|_{\widehat{H}^p(\mathbb{R}^d)} \leq C_{p,r} D^{-r} \|u\|_{\widehat{H}^{p+r}(\mathbb{R}^d)}.$$

for all $p, r \in \mathbb{N}$ whenever $u \in \widehat{H}^{p+r}(\mathbb{R}^d)$, where

$$\Pi_D u := \sum_{\substack{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d} \\ |[\mathbf{m}, \mathbf{n}]| \leq D}} (u, \Psi_{\mathbf{m}, \mathbf{n}}^*) \Psi_{\mathbf{m}, \mathbf{n}}.$$

Estimate (1.9) is sharp, meaning that for each p, r and for all D “large enough”, we may find a function u that saturates the inequality. We also stress that not only does $\Pi_D u$ converges to u , but also that its coefficient in the frame are bounded, which turns out to be crucial, for instance for stable numerical applications in floating point arithmetic [2].

The idea of decomposing a function into parts that are localised in phase space (that is, both in position and Fourier space) has been often used in signal theory, as well as in theoretical aspects of PDEs, such as semiclassical analysis [11]. Our motivation comes from [4], where we use Gabor frames to numerically solve high-frequency Helmholtz problems, taking into account the a priori knowledge we have of the microlocal properties (i.e., phase-space localisation) of the solution. The results proven in the present paper play a key role in the proofs of the convergence of the algorithms presented in [4]. Besides, we believe the present results will be useful in a wider range of problems in numerical analysis, including adaptive wavelet approximations [10] and the approximation of Schrödinger equations by Gaussian beams [3].

The remainder of the document is organized as follows. In Section 2, we introduce the main notations we will use in this work. In Section 3, we recall standard properties of Gabor frames and state our main result. In Section 4, we prove properties of decay of the coefficients, which will be essential in the proof of our main result, given in Section 5. Finally, Appendices A, B and C collect necessary technical results.

2. SETTINGS

2.1. Scaling parameter. In the sequel, all our quantities will depend on a parameter $k \geq 1$, which is an arbitrary but fixed real number. This is more convenient for applications to PDEs, where it is useful to choose k related to the wavelength of the solution.

Although we may not indicate it explicitly, all our results hold true for any value of k , with the estimates holding uniformly. Besides, $d \in \mathbb{N}^*$ denotes the number of space dimensions.

2.2. Multi-indices. For a multi-index $\mathbf{a} \in \mathbb{N}^d$, $[\mathbf{a}] := a_1 + \dots + a_d$ denotes its usual ℓ_1 norm. If $v : \mathbb{R}^d \rightarrow \mathbb{C}$, the notation

$$\partial^{\mathbf{a}} v := \frac{\partial^{a_1}}{\partial x_1} \dots \frac{\partial^{a_d}}{\partial x_d} v$$

is employed for the partial derivatives in the sense of distributions, whereas $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} \dots x_d^{a_d}$.

2.3. Functional spaces. $L^2(\mathbb{R}^d)$ is the usual space of complex-valued square integrable functions over \mathbb{R}^d . We respectively denote by $\|\cdot\|_{L^2(\mathbb{R}^d)}$ and (\cdot, \cdot) its norm and inner-product. If $\omega \subset \mathbb{R}^d$ is a measurable subset and $v \in L^2(\mathbb{R}^d)$, we set $\|v\|_{L^2(\omega)} = \|\chi_\omega v\|_{L^2(\mathbb{R}^d)}$, where χ_ω is the set function of ω . We also denote by $H^p(\mathbb{R}^d)$ the usual Sobolev space of order $p \in \mathbb{N}$, that we equip with the norm

$$\|v\|_{H_k^p(\mathbb{R}^d)}^2 := \sum_{[\mathbf{a}] \leq p} k^{-2[\mathbf{a}]} \|\partial^{\mathbf{a}} v\|_{L^2(\mathbb{R}^d)}^2 \quad \forall v \in H^p(\mathbb{R}^d).$$

Notice that the above norm is equivalent to the usual Sobolev norm, but with equivalence constant depending on k . We refer the reader to, e.g., [1] for an in-depth presentation of the above spaces.

We will also need a family of weighted Sobolev spaces. We first introduce the norms

$$\|v\|_{\widehat{H}_k^p(\mathbb{R}^d)}^2 := \sum_{|\mathbf{a}|\leq p} \sum_{q=0}^{p-|\mathbf{a}|} k^{-2|\mathbf{a}|} \|\mathbf{x}^q \partial^{\mathbf{a}} v\|_{L^2(\mathbb{R}^d)}^2 \quad \forall v \in \mathcal{S}(\mathbb{R}^d)$$

for all $p \in \mathbb{N}$, and we define $\widehat{H}^p(\mathbb{R}^d)$ as the closure of $\mathcal{S}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$ with respect to $\|\cdot\|_{\widehat{H}_k^p(\mathbb{R}^d)}$.

For future references, we note that if $u \in \widehat{H}^{p+1}(\mathbb{R}^d)$, we have $x_j u \in \widehat{H}^p(\mathbb{R}^d)$ with

$$(2.1) \quad \|x_j u\|_{\widehat{H}_k^p(\mathbb{R}^d)} \leq C_p \|u\|_{\widehat{H}_k^{p+1}(\mathbb{R}^d)}$$

for $1 \leq j \leq d$.

2.4. Fourier transform. For $v \in L^2(\mathbb{R}^d)$, we define a weighted Fourier transform by

$$\mathcal{F}_k(v)(\boldsymbol{\xi}) := \left(\frac{k}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} v(x) e^{-ik\mathbf{x}\cdot\boldsymbol{\xi}} d\mathbf{x}$$

for a.e. $\boldsymbol{\xi} \in \mathbb{R}^d$. For all $v \in L^2(\mathbb{R}^d)$, we then have

$$\|\mathcal{F}_k(v)\|_{L^2(\mathbb{R}^d)} = \|v\|_{L^2(\mathbb{R}^d)}.$$

In addition, if $v \in H^1(\mathbb{R}^d)$ or respectively if $|\mathbf{x}|v \in L^2(\mathbb{R}^d)$, then

$$\mathcal{F}_k(\partial_j v) = ik\xi_j \mathcal{F}_k(v), \quad ik\mathcal{F}_k(x_j v) = \partial_j \mathcal{F}_k(v)$$

for $j \in \{1, \dots, d\}$. In particular, for any $p \in \mathbb{N}$, we have for every $v \in H^p(\mathbb{R}^d)$

$$(2.2) \quad \|v\|_{H_k^p(\mathbb{R}^d)}^2 = \sum_{|\mathbf{a}|\leq p} \|\boldsymbol{\xi}^{\mathbf{a}} \mathcal{F}_k v\|_{L^2(\mathbb{R}^d)}^2.$$

2.5. Generic constants. Throughout the manuscript, C denotes a constant that may vary from one occurrence to the other and only depends on the space dimension d . If p, q, \dots are previously introduced symbols, then the notation $C_{p,q,\dots}$ is similarly used to indicate a constant that solely depends on d and p, q, \dots .

3. THE FRAME OF GAUSSIAN STATES

3.1. The Gabor frame. For $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$, we set

$$(3.1) \quad \mathbf{x}^{k,\mathbf{m}} := \sqrt{\frac{\pi}{k}} \mathbf{m}, \quad \boldsymbol{\xi}^{k,\mathbf{n}} := \sqrt{\frac{\pi}{k}} \mathbf{n},$$

so that, the couples $[\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}}]$ form a lattice of the phase space \mathbb{Z}^{2d} . We associate with each point $[\mathbf{m}, \mathbf{n}]$ in the lattice the Gaussian state

$$\Psi_{k,\mathbf{m},\mathbf{n}}(\mathbf{x}) := \left(\frac{k}{\pi}\right)^{d/4} e^{-\frac{k}{2}|\mathbf{x}-\mathbf{x}^{k,\mathbf{m}}|^2} e^{ik(\mathbf{x}-\mathbf{x}^{k,\mathbf{m}})\cdot\boldsymbol{\xi}^{k,\mathbf{n}}}.$$

The set $(\Psi_{k,\mathbf{m},\mathbf{n}})_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}}$ forms a *frame*, i.e., there exist two constants $\alpha, \beta > 0$ that only depends on d , such that

$$(3.2) \quad \alpha^2 \|u\|_{L^2(\mathbb{R}^d)}^2 \leq \sum_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}} |(u, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 \leq \beta^2 \|u\|_{L^2(\mathbb{R}^d)}^2 \quad \forall u \in L^2(\mathbb{R}^d).$$

This fact was first shown in [6] when $d = 1$ and $k = 1$ (from which the case $d > 1$ and $k \neq 1$ follow easily). We refer the reader to [5] for generalisations of this result.

3.2. The dual frame. Given $u \in L^2(\mathbb{R}^d)$, we call the sequence $((u, \Psi_{k,m,n}))_{[m,n] \in \mathbb{Z}^{2d}}$ the coefficients of u in the frame, and we introduce the operator $T_k : L^2(\mathbb{R}^d) \rightarrow \ell^2(\mathbb{Z}^{2d})$ mapping a function u to its coefficients:

$$(T_k u)_{\mathbf{m}, \mathbf{n}} := (u, \Psi_{k, \mathbf{m}, \mathbf{n}}) \quad [\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}.$$

The fact that T_k maps into $\ell^2(\mathbb{Z}^{2d})$ is a direct consequence of (3.2). Straightforward computations show that the adjoint $T_k^* : \ell^2(\mathbb{Z}^{2d}) \rightarrow L^2(\mathbb{R}^d)$ of T_k is given by

$$T_k^* U = \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} U_{\mathbf{m}, \mathbf{n}} \Psi_{k, \mathbf{m}, \mathbf{n}},$$

for all $U \in \ell^2(\mathbb{Z}^{2d})$ and, in particular, we have

$$(3.3) \quad (T_k^* T_k)u = \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} (u, \Psi_{k, \mathbf{m}, \mathbf{n}}) \Psi_{k, \mathbf{m}, \mathbf{n}} \quad \forall u \in L^2(\mathbb{R}^d).$$

The operator $(T_k^* T_k)$ is self-adjoint, and the estimates in (3.2) imply that it is bounded from above and from below. As a result, $(T_k^* T_k)$ is invertible, and we will denote by S_k its inverse.

We call set of functions $(\Psi_{k, \mathbf{m}, \mathbf{n}}^*)_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d}$ defined by

$$\Psi_{k, \mathbf{m}, \mathbf{n}}^* := S_k \Psi_{k, \mathbf{m}, \mathbf{n}} \quad \forall [\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d},$$

the ‘‘dual frame’’ of $(\Psi_{k, \mathbf{m}, \mathbf{n}})_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}}$. It is indeed a frame, since, for any $u \in L^2(\mathbb{R}^d)$, we have

$$\sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} |(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)|^2 = \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} |(u, S_k \Psi_{k, \mathbf{m}, \mathbf{n}})|^2 = \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} |(S_k u, \Psi_{k, \mathbf{m}, \mathbf{n}})|^2 = \|T_k S_k u\|^2,$$

and it suffices to use (3.2) and to note that S_k is continuous with a continuous inverse.

Importantly, recalling (3.3), we see that for all $u \in L^2(\mathbb{R}^d)$, we have

$$(3.4) \quad u = \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} (u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*) \Psi_{k, \mathbf{m}, \mathbf{n}}$$

and

$$(3.5) \quad u = \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} (u, \Psi_{k, \mathbf{m}, \mathbf{n}}) \Psi_{k, \mathbf{m}, \mathbf{n}}^*.$$

3.3. Main result. Our key result is a generalization of (3.4) when the series is truncated to a finite number of terms. Specifically, considering $D > 0$ such that $k^{1/2} D > \sqrt{\pi}$, we define an ‘‘interpolation operator’’ $\Pi_D : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ by

$$(3.6) \quad \Pi_D u := \sum_{|(\mathbf{x}^{k, \mathbf{m}}, \boldsymbol{\xi}^{k, \mathbf{n}})| \leq D} (u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*) \Psi_{k, \mathbf{m}, \mathbf{n}}.$$

Notice that, recalling the definitions of $\mathbf{x}^{k, \mathbf{m}}$ and $\boldsymbol{\xi}^{k, \mathbf{n}}$ in (3.1), the requirement that $k^{1/2} D > \sqrt{\pi}$ simply means that there is at least one point in the summation set, so that it is not a restrictive assumption.

Theorem 3.1 (Approximation in the Gabor frame). *Consider two non-negative integers p and r . For all $u \in \widehat{H}^{p+r}(\mathbb{R}^d)$, we have $\Pi_D u \in \widehat{H}^p(\mathbb{R}^d)$. In addition, the following estimate holds true:*

$$(3.7) \quad \|u - \Pi_D u\|_{\widehat{H}_k^p(\mathbb{R}^d)} \leq C_{p,r} D^{-r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}.$$

The estimate in Theorem 3.1 is sharp. Specifically, we have the following result.

Proposition 3.2 (Sharp estimate). *For all $p, r \in \mathbb{N}$, there exists a constant $D_\star > 0$ depending on p, r and d , such that for all $D \geq D_\star$, there exists $u \in \widehat{H}^{p+r}(\mathbb{R}^d)$ with*

$$(3.8) \quad \|u - \Pi_D u\|_{\widehat{H}_k^p(\mathbb{R}^d)} \geq C_{p,r} D^{-r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}.$$

4. DECAY OF COEFFICIENTS IN THE FRAMES

4.1. Decay of coefficients in the dual frame. Recall that the estimates in (3.2) imply that whenever $u \in L^2(\mathbb{R}^d)$, its coefficients belongs to ℓ^2 , with an ℓ^2 norm comparable to $\|u\|_{L^2(\mathbb{R}^d)}$. In Proposition 4.1 we generalize this idea by showing that if u exhibits more regularity, stronger decay properties hold true for its coefficients sequence.

Proposition 4.1 (Coefficients' decay). *For all $p \in \mathbb{N}$ and $u \in \widehat{H}^p(\mathbb{R}^d)$, we have*

$$(4.1) \quad \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \left(|\mathbf{x}^{k, \mathbf{m}}|^2 + |\boldsymbol{\xi}^{k, \mathbf{n}}|^2 \right)^p |(u, \Psi_{k, \mathbf{m}, \mathbf{n}})|^2 \leq C_p \|u\|_{\widehat{H}_k^p(\mathbb{R}^d)}^2.$$

Proof. We will prove (4.1) by induction. Note that, for $p = 0$, it is an immediate consequence of (3.2). Then, suppose that there exists $p \in \mathbb{N}$ such that (4.1) holds for all $q \leq p - 1$, and let us show that it also holds for $q = p$.

Let us fix $j \in \{1, \dots, d\}$. The polynomials $(Q_\ell)_{\ell=0}^p$ from Lemma A.2 all have different degrees, so that they form a basis of $\mathbb{R}_p[X]$. In particular, we may find real numbers $(a_\ell)_{\ell=0}^p$ solely depending on p such that

$$z^p = \sum_{\ell=0}^p a_\ell Q_\ell(z),$$

and in particular, introducing for the sake of shortness the function $y : \mathbf{x} \rightarrow (\mathbf{x}^{k, \mathbf{m}} - \mathbf{x} + j \boldsymbol{\xi}^{k, \mathbf{n}})_j$, we have

$$(4.2) \quad k^{p/2} y^p = \sum_{\ell=0}^p a_\ell Q_\ell(k^{1/2} y).$$

Then, using (4.2), and recalling (A.2), we obtain

$$k^{p/2} (\bar{y}^p u, \Psi_{k, \mathbf{m}, \mathbf{n}}) = \sum_{\ell=0}^p a_\ell (u, Q_\ell(k^{1/2} y) \Psi_{k, \mathbf{m}, \mathbf{n}}) = \sum_{\ell=0}^p a_\ell k^{-\ell/2} (u, \partial_j^\ell \Psi_{k, \mathbf{m}, \mathbf{n}}).$$

and integration by parts shows that

$$(\bar{y}^p u, \Psi_{k, \mathbf{m}, \mathbf{n}}) = \sum_{\ell=0}^p (-1)^\ell a_\ell k^{-(p+\ell)/2} (\partial_j^\ell u, \Psi_{k, \mathbf{m}, \mathbf{n}}),$$

where we also multiplied both sides by $k^{-p/2}$. Recalling the definition of y and using the upper-bound in (3.2), after summing over $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$, we see that

$$(4.3) \quad \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} |((x_j^{k, \mathbf{m}} - i\xi_j^{k, \mathbf{n}} + x_j)^p u, \Psi_{k, \mathbf{m}, \mathbf{n}})|^2 \\ \leq C_p \sum_{\ell=0}^p a_\ell^2 k^{-(\ell+p)} \|\partial_j^\ell u\|_{L^2(\mathbb{R}^d)}^2 \leq C_p \sum_{\ell=0}^p a_\ell^2 k^{-2\ell} \|\partial_j^\ell u\|_{L^2(\mathbb{R}^d)}^2 \leq C_p \|u\|_{\dot{H}_k^p(\mathbb{R}^d)}^2,$$

where we have used the facts that the numbers a_ℓ only depend on p and that $k \geq 1$. Since we have

$$\left(x_j^{k, \mathbf{m}} - i\xi_j^{k, \mathbf{n}}\right)^p = \left(x_j^{k, \mathbf{m}} - x_j - i\xi_j^{k, \mathbf{n}}\right)^p - \sum_{q=0}^{p-1} \binom{p}{q} (-x_j)^{p-q} \left(x_j^{k, \mathbf{m}} - i\xi_j^{k, \mathbf{n}}\right)^q,$$

and

$$\left((x_j^{k, \mathbf{m}})^2 + (\xi_j^{k, \mathbf{n}})^2\right)^p |(u, \Psi_{k, \mathbf{m}, \mathbf{n}})|^2 = \left| \left((x_j^{k, \mathbf{m}} - i\xi_j^{k, \mathbf{n}})^p u, \Psi_{k, \mathbf{m}, \mathbf{n}} \right) \right|^2,$$

we may write that

$$(4.4) \quad \left((x_j^{k, \mathbf{m}})^2 + (\xi_j^{k, \mathbf{n}})^2\right)^p |(u, \Psi_{k, \mathbf{m}, \mathbf{n}})|^2 \leq \\ C_p \left(\left| \left((x_j^{k, \mathbf{m}} - x_j - i\xi_j^{k, \mathbf{n}})^p u, \Psi_{k, \mathbf{m}, \mathbf{n}} \right) \right|^2 + \sum_{q=0}^{p-1} \left((x_j^{k, \mathbf{m}})^2 + (\xi_j^{k, \mathbf{n}})^2\right)^q |(x_j^{p-q} u, \Psi_{k, \mathbf{m}, \mathbf{n}})|^2 \right).$$

Thanks to the induction hypothesis, we have

$$\sum_{q=0}^{p-1} \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \left((x_j^{k, \mathbf{m}})^2 + (\xi_j^{k, \mathbf{n}})^2\right)^q |(x_j^{p-q} u, \Psi_{k, \mathbf{m}, \mathbf{n}})|^2 \leq C_p \sum_{q=0}^{p-1} \|x^{p-q} u\|_{\dot{H}_k^q(\mathbb{R}^d)}^2,$$

and (2.1) then ensures that

$$(4.5) \quad \sum_{q=0}^{p-1} \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \left((x_j^{k, \mathbf{m}})^2 + (\xi_j^{k, \mathbf{n}})^2\right)^q |(x_j^{p-q} u, \Psi_{k, \mathbf{m}, \mathbf{n}})|^2 \leq C_p \|u\|_{\dot{H}_k^p(\mathbb{R}^d)}^2.$$

We then sum (4.4) over $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$. Using (4.3) and (4.5) to respectively bound the first and second term in the right-hand side, we obtain

$$\sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \left((x_j^{k, \mathbf{m}})^2 + (\xi_j^{k, \mathbf{n}})^2\right)^p |(u, \Psi_{k, \mathbf{m}, \mathbf{n}})|^2 \leq C_p \|u\|_{\dot{H}_k^p(\mathbb{R}^d)}^2,$$

and (4.1) follows by summing over $j \in \{1, \dots, d\}$, which concludes the proof by induction. \square

4.2. Decay of coefficients in the initial frame. We now prove an analogue of Proposition 4.1 for the coefficients of the dual coefficients $(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)$, which are used for the expansion in the initial frame $(\Psi_{k, \mathbf{m}, \mathbf{n}})_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}}$.

We start with a preliminary result concerning the inner product of elements of the dual frame. Identity (4.6) follows directly from (3.4), while (4.7) can be deduced from [7, Corollary 3.7]. For the reader's convenience, we give a self-contained elementary proof of (4.7) in Appendix B.

Proposition 4.2 (Expansion of the dual frame). *For all $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$, we have*

$$(4.6) \quad \Psi_{k, \mathbf{m}, \mathbf{n}}^* = \sum_{[\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}} (\Psi_{k, \mathbf{m}, \mathbf{n}}^*, \Psi_{k, \mathbf{m}', \mathbf{n}'}^*) \Psi_{k, \mathbf{m}', \mathbf{n}'}$$

In addition, for all $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$(4.7) \quad |(\Psi_{k, \mathbf{m}, \mathbf{n}}^*, \Psi_{k, \mathbf{m}', \mathbf{n}'}^*)| \leq C_\varepsilon e^{-\|\mathbf{m}, \mathbf{n}\| - \|\mathbf{m}', \mathbf{n}'\|^{1-\varepsilon}}.$$

We are now ready to establish our result concerning the coefficients' decay.

Proposition 4.3 (Dual coefficients' decay). *For all $p \in \mathbb{N}$ and $u \in \widehat{H}^p(\mathbb{R}^d)$, we have*

$$(4.8) \quad \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \left(|\mathbf{x}^{k, \mathbf{m}}|^2 + |\boldsymbol{\xi}^{k, \mathbf{n}}|^2 \right)^p |(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)|^2 \leq C_p \|u\|_{\widehat{H}_k^p(\mathbb{R}^d)}^2.$$

Proof. To lighten notations, let us write $\mathbf{z}^{k, \mathbf{m}, \mathbf{n}} := \mathbf{x}^{k, \mathbf{m}} + i\boldsymbol{\xi}^{k, \mathbf{n}}$. We have

$$\begin{aligned} & \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \left(|\mathbf{x}^{k, \mathbf{m}}|^2 + |\boldsymbol{\xi}^{k, \mathbf{n}}|^2 \right)^p |(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)|^2 \\ &= \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} |\mathbf{z}^{k, \mathbf{m}, \mathbf{n}}|^{2p} \left| \sum_{[\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}} (u, \Psi_{k, \mathbf{m}', \mathbf{n}'}) (\Psi_{k, \mathbf{m}, \mathbf{n}}^*, \Psi_{k, \mathbf{m}', \mathbf{n}'}^*) \right|^2 \quad \text{thanks to (4.6)} \\ &= \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \left| \sum_{[\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}} \left(1 + |\mathbf{z}^{k, \mathbf{m}', \mathbf{n}'}|^p \right) \frac{|\mathbf{z}^{k, \mathbf{m}, \mathbf{n}}|^p}{1 + |\mathbf{z}^{k, \mathbf{m}', \mathbf{n}'}|^p} (u, \Psi_{k, \mathbf{m}', \mathbf{n}'}) (\Psi_{k, \mathbf{m}, \mathbf{n}}^*, \Psi_{k, \mathbf{m}', \mathbf{n}'}^*) \right|^2 \\ &\leq \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \left(\sum_{[\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}} \left| \left(1 + |\mathbf{z}^{k, \mathbf{m}', \mathbf{n}'}|^p \right) (u, \Psi_{k, \mathbf{m}', \mathbf{n}'}) \right|^2 |(\Psi_{k, \mathbf{m}, \mathbf{n}}^*, \Psi_{k, \mathbf{m}', \mathbf{n}'}^*)|^2 \right) \\ &\quad \times \left(\sum_{[\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}} \left| \frac{|\mathbf{z}^{k, \mathbf{m}, \mathbf{n}}|^p}{1 + |\mathbf{z}^{k, \mathbf{m}', \mathbf{n}'}|^p} \right|^2 |(\Psi_{k, \mathbf{m}, \mathbf{n}}^*, \Psi_{k, \mathbf{m}', \mathbf{n}'}^*)|^2 \right), \end{aligned}$$

thanks to Cauchy-Schwarz inequality. Let us bound the second factor using (4.7). For every $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$, we have

$$\begin{aligned} & \sum_{[\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}} \left| \frac{|\mathbf{z}^{k, \mathbf{m}, \mathbf{n}}|^p}{1 + |\mathbf{z}^{k, \mathbf{m}', \mathbf{n}'}|^p} \right|^2 |(\Psi_{k, \mathbf{m}, \mathbf{n}}^*, \Psi_{k, \mathbf{m}', \mathbf{n}'}^*)|^2 \\ &\leq C_p \sum_{[\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}} \left| \frac{|\mathbf{z}^{k, \mathbf{m}', \mathbf{n}'}|^p + |\mathbf{z}^{k, \mathbf{m}, \mathbf{n}} - \mathbf{z}^{k, \mathbf{m}', \mathbf{n}'}|^p}{1 + |\mathbf{z}^{k, \mathbf{m}', \mathbf{n}'}|^p} \right|^2 e^{-\|\mathbf{m}, \mathbf{n}\| - \|\mathbf{m}', \mathbf{n}'\|^{1/2}} \\ &\leq C_p \sum_{[\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}} \left| 1 + |\mathbf{z}^{k, \mathbf{m}, \mathbf{n}} - \mathbf{z}^{k, \mathbf{m}', \mathbf{n}'}|^p \right|^2 e^{-\|\mathbf{m}, \mathbf{n}\| - \|\mathbf{m}', \mathbf{n}'\|^{1/2}} \\ &\leq C_p. \end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \left(|\mathbf{x}^{k, \mathbf{m}}|^2 + |\boldsymbol{\xi}^{k, \mathbf{n}}|^2 \right)^p |(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)|^2 \\
& \leq C_p \sum_{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}} \sum_{[\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}} \left| \left(1 + |\mathbf{z}^{k, \mathbf{m}', \mathbf{n}'}|^p \right) (u, \Psi_{k, \mathbf{m}', \mathbf{n}'}) \right|^2 e^{-[|\mathbf{m}, \mathbf{n}] - [\mathbf{m}', \mathbf{n}']|^{1/2}} \\
& \leq C_p \sum_{[\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}} \left| \left(1 + |\mathbf{z}^{k, \mathbf{m}', \mathbf{n}'}|^p \right) (u, \Psi_{k, \mathbf{m}', \mathbf{n}'}) \right|^2,
\end{aligned}$$

and (4.8) follows from (4.1). \square

5. PROOF OF THEOREM 3.1

5.1. Approximation in $L^2(\mathbb{R}^d)$. We start by proving Theorem 3.1 in the simplest case, when $p = 0$.

Lemma 5.1 ($L^2(\mathbb{R}^d)$ approximation). *For all $u \in \widehat{H}^r(\mathbb{R}^d)$, we have*

$$(5.1) \quad \|u - \Pi_D u\|_{L^2(\mathbb{R}^d)} \leq C_r D^{-r} \|u\|_{\widehat{H}_k^r(\mathbb{R}^d)}.$$

Proof. By definition of Π_D , we have

$$\mathcal{E} := u - \Pi_D u = \sum_{\substack{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d} \\ |(\mathbf{x}^{k, \mathbf{m}}, \boldsymbol{\xi}^{k, \mathbf{n}})| > D}} (u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*) \Psi_{k, \mathbf{m}, \mathbf{n}},$$

and the frame property ensures that

$$\begin{aligned}
\|\mathcal{E}\|_{L^2(\mathbb{R}^d)}^2 & \leq \sum_{\substack{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d} \\ |(\mathbf{x}^{k, \mathbf{m}}, \boldsymbol{\xi}^{k, \mathbf{n}})| > D}} |(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)|^2 \\
& \leq D^{-2r} \sum_{\substack{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d} \\ |(\mathbf{x}^{k, \mathbf{m}}, \boldsymbol{\xi}^{k, \mathbf{n}})| > D}} |(\mathbf{x}^{k, \mathbf{m}}, \boldsymbol{\xi}^{k, \mathbf{n}})|^{2r} |(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)|^2,
\end{aligned}$$

and (5.1) follows from Proposition 4.3. \square

5.2. Technical preliminary results. To prove Theorem 3.1, that is, to generalize (5.1) to the case where $p > 0$, we must first show several preliminary technical results.

Lemma 5.2 (Smallness far from the origin). *For all $\mathbf{a} \in \mathbb{N}^d$, we have*

$$(5.2) \quad |\partial^{\mathbf{a}}(\Pi_D u)(\mathbf{x})| \leq C_{\mathbf{a}} k^{d/4} \left(1 + k^{1/2} D \right)^{d/2} (k^{1/2} |\mathbf{x}|)^{[\mathbf{a}]} e^{-\frac{k}{8} |\mathbf{x}|^2} \|u\|_{L^2(\mathbb{R}^d)}$$

for all $\mathbf{x} \in \mathbb{R}^d$ with $|\mathbf{x}| \geq 2D$. In addition, for all $q \geq 0$, we have

$$(5.3) \quad k^{q/2} \| |\mathbf{x}|^q \partial^{\mathbf{a}}(\Pi_D u) \|_{L^2(\{|\mathbf{x}| > 2D\})} \leq C_{p, q} k^{[\mathbf{a}]/2} e^{-\frac{kD^2}{16}} \|u\|_{L^2(\mathbb{R}^d)}.$$

Proof. If $|\mathbf{x}| \geq 2D$ and $|\mathbf{x}^{k, \mathbf{m}}| \leq D$, we have $|\mathbf{x} - \mathbf{x}^{k, \mathbf{m}}|^2 \geq \frac{|\mathbf{x}|^2}{4}$, so that $e^{-\frac{k}{2} |\mathbf{x} - \mathbf{x}^{k, \mathbf{m}}|^2} \leq e^{-\frac{k}{8} |\mathbf{x}|^2}$. It follows from (A.2) that there exists a polynomial $Q_{\mathbf{a}}$ of total degree $[\mathbf{a}]$ such that

$$(\partial^{\mathbf{a}} \Psi_{k, \mathbf{m}, \mathbf{n}})(\mathbf{x}) = k^{[\mathbf{a}]/2} Q_{\mathbf{a}}(k^{1/2} (\mathbf{x} - \mathbf{x}^{k, \mathbf{m}} + i \boldsymbol{\xi}^{k, \mathbf{n}})) \Psi_{k, \mathbf{m}, \mathbf{n}}(\mathbf{x})$$

so that

$$|\partial^\alpha(\Pi_D u)(\mathbf{x})| \leq C k^{d/4} \sum_{\substack{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d} \\ |(\mathbf{x}^{k, \mathbf{m}}, \boldsymbol{\xi}^{k, \mathbf{n}})| \leq D}} |(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)| |Q_\alpha(k^{1/2}(\mathbf{x} - \mathbf{x}^{k, \mathbf{m}} + i\boldsymbol{\xi}^{k, \mathbf{n}}))| e^{-\frac{k}{8}|\mathbf{x}|^2}.$$

We then observe that since $|\mathbf{x}| \geq 2D$ and $|x_j^{k, \mathbf{m}}| + |\xi_j^{k, \mathbf{n}}| \leq CD$ for all terms in the sum, we have

$$|\mathbf{x} - \mathbf{x}^{k, \mathbf{m}} + i\boldsymbol{\xi}^{k, \mathbf{n}}| \leq C|\mathbf{x}|.$$

As a result, we have

$$|\partial^\alpha(\Pi_D u)(\mathbf{x})| \leq C_\alpha k^{d/4} \left(\sum_{\substack{[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d} \\ |(\mathbf{x}^{k, \mathbf{m}}, \boldsymbol{\xi}^{k, \mathbf{n}})| \leq D}} |(u, \Psi_{k, \mathbf{m}, \mathbf{n}}^*)| \right) (k^{1/2}|\mathbf{x}|)^{|\alpha|} e^{-\frac{k}{8}|\mathbf{x}|^2}$$

and (5.2) follows from (3.2) and from the Cauchy-Schwarz inequality, as the sum is over a set of cardinal smaller than $C(1 + k^{1/2}D)^d$.

Next, we set $p = [\alpha]$, and we consider

$$I_q := \int_{|\mathbf{x}| \geq 2D} (|\mathbf{x}|^{p+q} e^{-\frac{k}{8}|\mathbf{x}|^2})^2 d\mathbf{x} = C \int_{2D}^{+\infty} r^{d-1+2(p+q)} e^{-\frac{k}{4}r^2} dr,$$

using radial coordinates. Introducing $s = \sqrt{kr}$, we get

$$\begin{aligned} I_q &= \int_{2k^{1/2}D}^{+\infty} (k^{-1/2}s)^{d-1+2(p+q)} e^{-\frac{s^2}{4}} k^{-1/2} ds \\ &= k^{-(d+2(p+q))/2} \int_{2k^{1/2}D}^{+\infty} s^{d-1+2(p+q)} e^{-\frac{s^2}{4}} \sqrt{s} ds \\ &\leq k^{-(d+2(p+q))/2} e^{-\frac{kD^2}{2}} \int_0^{+\infty} s^{d-1+2(p+q)} e^{-\frac{s^2}{8}} ds \\ &= C_p k^{-(d+2(p+q))/2} e^{-\frac{kD^2}{2}}. \end{aligned}$$

It then follows from (5.2) that

$$\begin{aligned} \|\|\mathbf{x}\|^q \partial^\alpha(\Pi_D u)\|_{L^2(\{|\mathbf{x}| > 2D\})} &\leq C k^{d/4} (1 + k^{1/2}D)^{d/2} k^p \sqrt{I_q} \|u\|_{L^2(\mathbb{R}^d)} \\ &\leq C_q k^{d/4} (1 + k^{1/2}D)^{d/2} k^p k^{-(d+2(p+q))/4} e^{-\frac{kD^2}{8}} \|u\|_{L^2(\mathbb{R}^d)} \\ &= C_q k^{-q/2} k^{p/2} (1 + k^{1/2}D)^{d/2} e^{-\frac{kD^2}{8}} \|u\|_{L^2(\mathbb{R}^d)} \\ &\leq C_q k^{-q/2} k^{p/2} e^{-\frac{kD^2}{16}} \|u\|_{L^2(\mathbb{R}^d)}, \end{aligned}$$

and (5.3) follows. □

We then prove a preliminary result about the decay properties of $u - \Pi_D u$.

Corollary 5.3. *For any $p, q, r \in \mathbb{N}$, any $\mathbf{a} \in \mathbb{N}^d$ with $q + [\mathbf{a}] \leq p$ and for any $u \in \widehat{H}^{p+r}(\mathbb{R}^d)$, we have*

$$(5.4) \quad k^{-[\mathbf{a}]} \|\mathbf{x}^q \partial^{\mathbf{a}} (u - \Pi_D u)\|_{L^2(\mathbb{R}^d)} \leq C_{\mathbf{a}, q, r} D^q \left(k^{-[\mathbf{a}]} \|\partial^{\mathbf{a}} (u - \Pi_D u)\|_{L^2(\mathbb{R}^d)} + D^{-(p+r)} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)} \right).$$

Proof. We first write that

$$\begin{aligned} & \|\mathbf{x}^q \partial^{\mathbf{a}} (u - \Pi_D u)\|_{L^2(\mathbb{R}^d)} \\ &= \|\mathbf{x}^q \partial^{\mathbf{a}} (u - \Pi_D u)\|_{L^2(\{|\mathbf{x}| < 2D\})} + \|\mathbf{x}^q \partial^{\mathbf{a}} (u - \Pi_D u)\|_{L^2(\{|\mathbf{x}| > 2D\})} \\ &\leq (2D)^q \|\partial^{\mathbf{a}} (u - \Pi_D u)\|_{L^2(\mathbb{R}^d)} + \|\mathbf{x}^q \partial^{\mathbf{a}} u\|_{L^2(\{|\mathbf{x}| > 2D\})} + \|\mathbf{x}^q \partial^{\mathbf{a}} (\Pi_D u)\|_{L^2(\{|\mathbf{x}| > 2D\})}. \end{aligned}$$

To deal with the second term, we note that

$$\|\mathbf{x}^q \partial^{\mathbf{a}} u\|_{L^2(\{|\mathbf{x}| > 2D\})} \leq (2D)^{q-p-r} \|\mathbf{x}^{q-p+r} \partial^{\mathbf{a}} u\|_{L^2(\mathbb{R}^d)} \leq (2D)^{q-p-r} k^{[\mathbf{a}]} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}.$$

On the other hand, for the last term, we use (5.3), to obtain

$$\begin{aligned} k^{-[\mathbf{a}]} \|\mathbf{x}^q \partial^{\mathbf{a}} (\Pi_D u)\|_{L^2(\{|\mathbf{x}| > 2D\})} &\leq k^{(-[\mathbf{a}]-q)/2} e^{-\frac{kD^2}{16}} \|u\|_{L^2(\mathbb{R}^d)} \\ &\leq C_r k^{(-[\mathbf{a}]-q)/2} k^{2(q-p-r)} D^{q-p-r} \|u\|_{L^2} \\ &\leq C_r D^{(q-p)-r} \|u\|_{L^2}. \end{aligned}$$

Here, we used the fact that $e^{-\frac{x^2}{16}} \leq C_r x^{-r}$ for all $x \geq 1$, and the fact that $k \geq 1$. The result follows. \square

5.3. Approximation in $H^p(\mathbb{R}^d)$. The next key result needed to establish Theorem 3.1 concerns approximation in the space $H^p(\mathbb{R}^d)$.

Lemma 5.4 ($H^p(\mathbb{R}^d)$ approximation). *For all $p, r \geq 0$, if $u \in \widehat{H}^{p+r}(\mathbb{R}^d)$, we have*

$$(5.5) \quad k^{-p} \|\partial_j^p (u - \Pi_D u)\|_{L^2(\mathbb{R}^d)} \leq C_{p,r} D^{-r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}.$$

Proof. Consider $u \in \widehat{H}^{p+r}(\mathbb{R}^d)$, fix $D > 0$ and let $\mathcal{E} := u - \Pi_D u$, so that

$$\mathcal{E} = \sum_{|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| > D} (u, \Psi_{k,\mathbf{m},\mathbf{n}}^*) \Psi_{k,\mathbf{m},\mathbf{n}}.$$

Recall that the family $(\Psi_{k,\mathbf{m},\mathbf{n}})_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}}$ is a frame, so that

$$k^{-2p} \|\partial_j^p \mathcal{E}\|_{L^2(\mathbb{R}^d)}^2 \leq C k^{-2p} \sum_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}} |(\partial_j^p \mathcal{E}, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 = C(\sigma_{\text{small}} + \sigma_{\text{large}})$$

with

$$\sigma_{\text{small}} := k^{-2p} \sum_{|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| \leq 2D} |(\partial_j^p \mathcal{E}, \Psi_{k,\mathbf{m},\mathbf{n}})|^2, \quad \sigma_{\text{large}} := k^{-2p} \sum_{|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| > 2D} |(\partial_j^p \mathcal{E}, \Psi_{k,\mathbf{m},\mathbf{n}})|^2.$$

We start by estimating the first term. Recalling (A.2) from Lemma A.2, we have

$$|(\partial_j^p \mathcal{E}, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 = |(\mathcal{E}, \partial_j^p \Psi_{k,\mathbf{m},\mathbf{n}})|^2 = k^p \left| \left(\mathcal{E}, Q_p(\sqrt{k}(x_j^{k,\mathbf{m}} - x_j + i\xi_j^{k,\mathbf{n}})) \Psi_{k,\mathbf{m},\mathbf{n}} \right) \right|^2.$$

The binomial theorem yields that

$$\begin{aligned} |Q_p(k^{1/2}(x_j^{k,\mathbf{m}} - x_j + i\xi_j^{k,\mathbf{n}}))|^2 &\leq C_p \sum_{q=0}^p (k^{1/2}|x_j^{k,\mathbf{m}} + i\xi_j^{k,\mathbf{n}}|)^{2(p-q)} (k^{1/2}x_j)^{2q} \\ &\leq C_p \sum_{q=0}^p (k^{1/2}D)^{2(p-q)} k^q |\mathbf{x}|^{2q}, \end{aligned}$$

so that

$$|(\partial_j^p \mathcal{E}, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 \leq C_p k^p \sum_{q=0}^p (k^{1/2}D)^{2(p-q)} k^q (|\mathbf{x}|^q \mathcal{E}, \Psi_{k,\mathbf{m},\mathbf{n}})^2$$

for all $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$ with $|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| \leq 2D$. Since the coherent states form a frame, after summation over $[\mathbf{m}, \mathbf{n}]$, we may write

$$\begin{aligned} \sigma_{\text{small}} &\leq C_p k^{-p} \sum_{q=0}^p (k^{1/2}D)^{2(p-q)} k^q \|\mathbf{x}|^q \mathcal{E}\|_{L^2(\mathbb{R}^d)}^2 \\ &= C_p D^{2p} \sum_{q=0}^p D^{-2q} \|\mathbf{x}|^q \mathcal{E}\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

Since $q \leq p$, we may employ Corollary 5.3 with $\mathbf{a} = 0$, leading to

$$\begin{aligned} \sigma_{\text{small}} &\leq C_{p,r} D^{2p} \sum_{q=0}^p \left(\|\mathcal{E}\|_{L^2(\mathbb{R}^d)}^2 + D^{-2(p+r)} \|u\|_{\widehat{H}_k^{q+r}(\mathbb{R}^d)}^2 \right) \\ &\leq C_{p,r} \left(D^{2p} \|\mathcal{E}\|_{L^2(\mathbb{R}^d)}^2 + D^{-2r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}^2 \right), \end{aligned}$$

and it follows from (5.1) that

$$\sigma_{\text{small}} \leq C_{p,r} D^{-2r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}^2,$$

which is the desired result.

We now turn σ_{large} , that we further split as

$$\sigma_{\text{large}} \leq 2(\sigma'_{\text{large}} + \sigma''_{\text{large}}),$$

where

$$\begin{aligned} \sigma'_{\text{large}} &:= k^{-2p} \sum_{|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| > 2D} |(\partial_j^p u, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 \\ \sigma''_{\text{large}} &:= k^{-2p} \sum_{|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| > 2D} |(\partial_j^p \Pi_D u, \Psi_{k,\mathbf{m},\mathbf{n}})|^2. \end{aligned}$$

We handle the first component with Proposition 4.1. Indeed, it follows from (4.1) that

$$\begin{aligned} \sigma'_{\text{large}} &\leq k^{-2p} \sum_{|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| > 2D} |(\partial_j^p u, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 \\ &\leq C_r k^{-2p} D^{-2r} \sum_{|(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})| > 2D} |(\mathbf{x}^{k,\mathbf{m}}, \boldsymbol{\xi}^{k,\mathbf{n}})|^{2r} |(\partial_j^p u, \Psi_{k,\mathbf{m},\mathbf{n}})|^2 \\ &\leq C_r k^{-2p} D^{-2r} \|\partial_j^p u\|_{\widehat{H}_k^r(\mathbb{R}^d)}^2 \leq C_r D^{-2r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}^2. \end{aligned}$$

For the second component, we invoke Lemma A.3. On the one hand, recalling the definition of $\Pi_D u$ in (3.6), we have

$$\begin{aligned} |(\partial_j^p(\Pi_D u), \Psi_{k,m,n})|^2 &= \left| \sum_{|(\mathbf{x}^{k,m'}, \boldsymbol{\xi}^{k,n'})| \leq D} (u, \Psi_{k,m',n'}^*) (\partial_j^p \Psi_{k,m',n'}, \Psi_{k,m,n}) \right|^2 \\ &\leq C(k^{1/2}D)^{2d} \sum_{|(\mathbf{x}^{k,m'}, \boldsymbol{\xi}^{k,n'})| \leq D} |(u, \Psi_{k,m',n'}^*)|^2 |(\partial_j^p \Psi_{k,m',n'}, \Psi_{k,m,n})|^2, \end{aligned}$$

since the summation happens on a set of cardinal less than $C(k^{1/2}D)^d$. On the other hand, by Lemma A.3,

$$\begin{aligned} |(\partial_j^p \Psi_{k,m,n}, \Psi_{k,m',n'})| &\leq C_p k^{p/2} (1 + |\mathbf{n}|^p) e^{-\frac{\pi}{8}kD^2} \\ &= C_p k^{p/2} \left(1 + (k^{1/2}D)^p\right) e^{-\frac{\pi}{8}kD^2} \leq C_{p,r} k^{p/2} (k^{1/2}D)^{-r-d} \leq C_{p,r} k^{p/2} (k^{1/2}D)^{-d} D^{-r} \end{aligned}$$

since $k \geq 1$.

Using the frame property, we obtain

$$\begin{aligned} \sigma''_{\text{large}} &\leq k^{-2p} \left(C_{p,r} k^p D^{-2r} \sum_{|(\mathbf{x}^{k,m'}, \boldsymbol{\xi}^{k,n'})| \leq D} |(u, \Psi_{k,m',n'}^*)|^2 \right) \\ &\leq C_{p,r} D^{-2r} \|u\|_{L^2(\mathbb{R}^d)}^2 \\ &\leq C_{p,r} D^{-2r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}^2. \end{aligned}$$

□

We are now ready to conclude the proof of our main result.

Proof of Theorem 3.1. As previously, we set $\mathcal{E} := u - \Pi_D u$. Using Fourier transform, we start by observing that

$$\|\partial^\alpha \mathcal{E}\|_{L^2(\mathbb{R}^d)}^2 = \|(i\xi)^\alpha \mathcal{F}\mathcal{E}\|_{L^2(\mathbb{R}^d)}^2 \leq \|\xi\|^{[\alpha]} \|\mathcal{F}\mathcal{E}\|_{L^2(\mathbb{R}^d)}^2 \leq C \sum_{j=1}^d \|\partial_j^{[\alpha]} \mathcal{E}\|_{L^2(\mathbb{R}^d)}^2.$$

As a result, using (5.5) (with $p = [\mathbf{a}]$ and $r = (p+r) - [\mathbf{a}]$), we have

$$k^{-[\mathbf{a}]} \|\partial^\alpha \mathcal{E}\|_{L^2(\mathbb{R}^d)} \leq C_{p,r} D^{[\mathbf{a}] - (p+r)} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)}.$$

for all $\mathbf{a} \in \mathbb{N}^d$ with $[\mathbf{a}] \leq p$.

Let $q \leq p$, and let $\mathbf{a} \in \mathbb{N}^d$ with $q + |\mathbf{a}| \leq p$. Using (5.4), we have

$$\begin{aligned} &k^{-[\mathbf{a}]} \| |\mathbf{x}|^q \partial^\alpha \mathcal{E} \|_{L^2(\mathbb{R}^d)} \\ &\leq C_{p,r} D^q \left(k^{-[\mathbf{a}]} \|\partial^\alpha \mathcal{E}\|_{L^2(\mathbb{R}^d)} + D^{-(p+r)} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)} \right) \\ &\leq C_{p,r} \left(D^{[\mathbf{a}] + q - p - r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)} + D^{(q-p) - r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)} \right) \\ &\leq C_{p,r} D^{-r} \|u\|_{\widehat{H}_k^{p+r}(\mathbb{R}^d)} \end{aligned}$$

since $q + [\mathbf{a}] \leq p$ and $D \geq 1$, which concludes the proof. □

APPENDIX A. ELEMENTARY FACTS CONCERNING GAUSSIAN STATES

A.1. Fourier transform and derivatives. We start by recalling the expression of the Fourier transform of Gaussian states.

Lemma A.1 (Fourier transform). *For all $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$, we have*

$$(A.1) \quad \mathcal{F}_k(\Psi_{k,\mathbf{m},\mathbf{n}}) = e^{-ik\mathbf{x}^{k,\mathbf{m}} \cdot \boldsymbol{\xi}^{k,\mathbf{n}}} \Psi_{k,\mathbf{n},-\mathbf{m}}.$$

Proof. Let $\mathbf{y} = k^{1/2}(\mathbf{x} - \mathbf{x}^{k,\mathbf{m}})$, we have

$$\begin{aligned} \mathcal{F}_k(\Psi_{k,\mathbf{m},\mathbf{n}}) &= 2^{d/2} \left(\frac{k}{2\pi}\right)^{3d/4} \int_{\mathbb{R}^d} e^{-\frac{k}{2}|\mathbf{x}-\mathbf{x}^{k,\mathbf{m}}|^2} e^{ik(\mathbf{x}-\mathbf{x}^{k,\mathbf{m}}) \cdot \boldsymbol{\xi}^{k,\mathbf{n}}} e^{-ik\mathbf{x} \cdot \boldsymbol{\xi}} d\mathbf{x} \\ &= 2^{d/2} \left(\frac{k}{2\pi}\right)^{3d/4} e^{-i\mathbf{x}^{k,\mathbf{m}} \cdot \boldsymbol{\xi}} \int_{\mathbb{R}^d} e^{-\frac{|\mathbf{y}|^2}{2}} e^{ik^{1/2}\mathbf{y} \cdot \boldsymbol{\xi}^{k,\mathbf{n}}} e^{-ik^{1/2}\mathbf{y} \cdot \boldsymbol{\xi}} d\mathbf{y} \\ &= 2^{d/2} \left(\frac{k}{2\pi}\right)^{3d/4} e^{-i\mathbf{x}^{k,\mathbf{m}} \cdot \boldsymbol{\xi}} \int_{\mathbb{R}^d} e^{-\frac{|\mathbf{y}|^2}{2}} e^{ik^{1/2}\mathbf{y} \cdot (\boldsymbol{\xi}^{k,\mathbf{n}} - \boldsymbol{\xi})} d\mathbf{y} \\ &= \left(\frac{k}{\pi}\right)^{d/4} e^{-\frac{k}{2}|\boldsymbol{\xi} - \boldsymbol{\xi}^{k,\mathbf{n}}|^2} e^{-i\mathbf{x}^{k,\mathbf{m}} \cdot \boldsymbol{\xi}}. \end{aligned}$$

□

The following lemma gives an elementary expression for the derivatives of $\Psi_{k,\mathbf{m},\mathbf{n}}$.

Lemma A.2 (Differentiation). *For all $p \in \mathbb{N}$, there exists a polynomial Q_p of degree p , with real coefficients, and leading coefficient z^p , such that for all $j \in \{1, \dots, d\}$ and all $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$, we have*

$$(A.2) \quad (\partial_j^p \Psi_{k,\mathbf{m},\mathbf{n}})(\mathbf{x}) = k^{p/2} Q_p(k^{1/2}(x_j^{k,\mathbf{m}} - x_j + i\xi_j^{k,\mathbf{n}})) \Psi_{k,\mathbf{m},\mathbf{n}}(\mathbf{x}).$$

Proof. We define the polynomials by induction, setting $Q_0 = 1$, and

$$Q_{p+1}(z) = Q'_p(z) + zQ_p(z),$$

for $p \geq 0$. An elementary induction shows that Q_p is a polynomial of degree p whose term of degree p is z^p .

Our aim is now to show inductively that (A.2) holds. This is trivially the case for $p = 0$ and for $p = 1$, noting that

$$\partial_j \Psi_{k,\mathbf{m},\mathbf{n}} = k \left(x_j^{k,\mathbf{m}} - x_j + i\xi_j^{k,\mathbf{n}} \right) \Psi_{k,\mathbf{m},\mathbf{n}}.$$

Now, suppose the result holds for some $p \in \mathbb{N}$. We then have

$$\begin{aligned} \partial_j^{p+1} \Psi_{k,\mathbf{m},\mathbf{n}} &= k^{p/2} \left[k^{1/2} \partial_j Q_p(\sqrt{k}(x_j^{k,\mathbf{m}} - x_j + i\xi_j^{k,\mathbf{n}})) \Psi_{k,\mathbf{m},\mathbf{n}} \right. \\ &\quad \left. + Q_p(k^{1/2}(x_j^{k,\mathbf{m}} - x_j + i\xi_j^{k,\mathbf{n}})) \times k(x_j^{k,\mathbf{m}} - x_j + i\xi_j^{k,\mathbf{n}}) \Psi_{k,\mathbf{m},\mathbf{n}} \right] \\ &= k^{(p+1)/2} Q_{p+1}(k^{1/2}(x_j^{k,\mathbf{m}} - x_j + i\xi_j^{k,\mathbf{n}})) \Psi_{k,\mathbf{m},\mathbf{n}}, \end{aligned}$$

which proves the result. □

A.2. Scalar products between Gaussian states. We then provide expression and upper-bounds of inner-products involving Gaussian states and their derivatives.

Proposition A.3 (L^2 products). *For all $[\mathbf{m}, \mathbf{n}], [\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}$, we have*

$$(A.3) \quad |(\Psi_{k,\mathbf{m},\mathbf{n}}, \Psi_{k,\mathbf{m}',\mathbf{n}'})| = e^{-\frac{\pi}{4} \|[\mathbf{m},\mathbf{n}] - [\mathbf{m}',\mathbf{n}']\|^2}.$$

In addition, for all $p \in \mathbb{N}$, there exists a constant C_p such that

$$(A.4) \quad |(\partial_j^p \Psi_{k,\mathbf{m},\mathbf{n}}, \Psi_{k,\mathbf{m}',\mathbf{n}'})| \leq C_p k^{p/2} (1 + |\mathbf{n}|^p) e^{-\frac{\pi}{8} \|[\mathbf{m},\mathbf{n}] - [\mathbf{m}',\mathbf{n}']\|^2}$$

for all $[\mathbf{m}, \mathbf{n}], [\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}$.

To prove this proposition, we will need the following elementary result.

Lemma A.4 (A useful identity). *Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^d$. We have*

$$(A.5) \quad \frac{1}{2} (|\mathbf{x} - \mathbf{y}|^2 + |\mathbf{x} - \mathbf{z}|^2) = \left| \mathbf{x} - \frac{1}{2}(\mathbf{y} + \mathbf{z}) \right|^2 + \left| \frac{1}{2}(\mathbf{y} - \mathbf{z}) \right|^2.$$

Proof. On the one hand, we have

$$\frac{1}{2} (|\mathbf{x} - \mathbf{y}|^2 + |\mathbf{x} - \mathbf{z}|^2) = |\mathbf{x}|^2 - \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) + \frac{1}{2} |\mathbf{y}|^2 + \frac{1}{2} |\mathbf{z}|^2.$$

On the other hand, we have

$$\left| \mathbf{x} - \frac{1}{2}(\mathbf{y} + \mathbf{z}) \right|^2 + \left| \frac{1}{2}(\mathbf{y} - \mathbf{z}) \right|^2 = |\mathbf{x}|^2 - \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) + \frac{1}{4} (|\mathbf{y} + \mathbf{z}|^2 + |\mathbf{y} - \mathbf{z}|^2)$$

and the proof follows since

$$|\mathbf{y} + \mathbf{z}|^2 + |\mathbf{y} - \mathbf{z}|^2 = 2|\mathbf{y}|^2 + 2|\mathbf{z}|^2.$$

□

Recall that, if we write, for $k > 0$ and $\mathbf{x}^* \in \mathbb{R}^d$

$$g_{k,\mathbf{x}^*}(\mathbf{x}) := e^{-k|\mathbf{x} - \frac{1}{2}\mathbf{x}^*|^2},$$

then we have for all $\boldsymbol{\xi}^* \in \mathbb{R}^d$

$$(A.6) \quad \mathcal{F}_k(g_{k,\mathbf{x}^*})(\boldsymbol{\xi}^*) = 2^{-d/2} e^{-\frac{k}{4}|\boldsymbol{\xi}^*|^2} e^{-\frac{ik}{2}\mathbf{x}^* \cdot \boldsymbol{\xi}^*}.$$

We will use the following notations

$$\begin{aligned} \mathbf{x}^{k,m,m'} &:= \frac{\mathbf{x}^{k,m} + \mathbf{x}^{k,m'}}{2} \\ \widehat{\mathbf{x}}^{k,m,m'} &:= \mathbf{x}^{k,m} - \mathbf{x}^{k,m'} \\ \widehat{\boldsymbol{\xi}}^{k,n,n'} &:= \boldsymbol{\xi}^{k,n} - \boldsymbol{\xi}^{k,n'}. \end{aligned}$$

Proof of Proposition A.3. We have

$$\begin{aligned} \Psi_{k,\mathbf{m},\mathbf{n}}(\mathbf{x}) \overline{\Psi_{k,\mathbf{m}',\mathbf{n}'}(\mathbf{x})} &= \left(\frac{k}{\pi}\right)^{d/2} e^{-\frac{k}{2}|\mathbf{x} - \mathbf{x}^{k,m}|^2} e^{ik(\mathbf{x} - \mathbf{x}^{k,m}) \cdot \boldsymbol{\xi}^{k,n}} e^{-\frac{k}{2}|\mathbf{x} - \mathbf{x}^{k,m'}|^2} e^{-ik(\mathbf{x} - \mathbf{x}^{k,m'}) \cdot \boldsymbol{\xi}^{k,n'}} \\ &= \left(\frac{k}{\pi}\right)^{d/2} e^{-\frac{k}{2}(|\mathbf{x} - \mathbf{x}^{k,m}|^2 + |\mathbf{x} - \mathbf{x}^{k,m'}|^2)} e^{ik\mathbf{x} \cdot \widehat{\boldsymbol{\xi}}^{k,n,n'}} e^{ik\mathbf{x}^{k,m'} \cdot \widehat{\boldsymbol{\xi}}^{k,n,n'}}. \end{aligned}$$

Using (A.5), we have

$$-\frac{k}{2} \left(|\mathbf{x} - \mathbf{x}^{k,m}|^2 + |\mathbf{x} - \mathbf{x}^{k,m'}|^2 \right) = -k \left| \mathbf{x} - \mathbf{x}^{k,m,m'} \right|^2 - \frac{k}{4} |\widehat{\mathbf{x}}^{k,m,m'}|^2,$$

leading to

(A.7)

$$\Psi_{k,m,n}(\mathbf{x}) \overline{\Psi_{k,m',n'}(\mathbf{x})} = \left(\frac{k}{\pi} \right)^{d/2} e^{ik(\mathbf{x}^{k,m'} \cdot \boldsymbol{\xi}^{k,n'} - \mathbf{x}^{k,m} \cdot \boldsymbol{\xi}^{k,n})} e^{-\frac{k}{4} |\widehat{\mathbf{x}}^{k,m,m'}|^2} e^{-k|\mathbf{x} - \mathbf{x}^{k,m,m'}|^2} e^{ik\mathbf{x} \cdot \widehat{\boldsymbol{\xi}}^{k,n,n'}}.$$

Recalling (A.6), we have

$$\begin{aligned} & (\Psi_{k,m,n}, \Psi_{k,m',n'}) \\ (A.8) \quad &= 2^{d/2} e^{ik(\mathbf{x}^{k,m'} \cdot \boldsymbol{\xi}^{k,n'} - \mathbf{x}^{k,m} \cdot \boldsymbol{\xi}^{k,n})} e^{-\frac{k^2}{4} |\widehat{\mathbf{x}}^{k,m,m'}|^2} \mathcal{F}_k(g_{k,2\mathbf{x}^{k,m,m'}})(-\widehat{\boldsymbol{\xi}}^{k,n,n'}) \\ &= e^{-i\mathbf{x}^{k,m,m'} \cdot \widehat{\boldsymbol{\xi}}^{k,n,n'}} e^{ik(\mathbf{x}^{k,m'} \cdot \boldsymbol{\xi}^{k,n'} - \mathbf{x}^{k,m} \cdot \boldsymbol{\xi}^{k,n})} e^{-\frac{k^2}{4} |\widehat{\mathbf{x}}^{k,m,m'}|^2} e^{-\frac{k^2}{4} |\widehat{\boldsymbol{\xi}}^{k,n,n'}|^2}, \end{aligned}$$

which gives (A.3).

Noting that $\mathbf{x}^{k,m} = \mathbf{x}^{k,m,m'} + \frac{\widehat{\mathbf{x}}^{k,m,m'}}{2}$ and using Lemma A.2, we get

$$\begin{aligned} \partial_j^p \Psi_{k,m,n}(\mathbf{x}) &= k^{p/2} Q_p \left(\sqrt{k}(x_j^{k,m,m'} - x_j + \frac{\widehat{x}_j^{k,m,m'}}{2} + i\xi_j^{k,n}) \right) \Psi_{k,m,n}(\mathbf{x}) \\ &= \sum_{\ell=0}^p k^{\frac{p+\ell}{2}} \sum_{\ell'=0}^{\ell} c_{\ell,\ell'} \left(\frac{\widehat{x}_j^{k,m,m'}}{2} + i\xi_j^{k,n} \right)^{\ell-\ell'} (x_j^{k,m,m'} - x_j)^{\ell'} \Psi_{k,m,n}(\mathbf{x}), \end{aligned}$$

with all the coefficients $c_{\ell,\ell'}$ independent of k . Therefore, we have

$$(A.9) \quad \left| (\partial_j^p \Psi_{k,m,n}, \Psi_{k,m',n'}) \right| \leq C_p \sum_{\ell=0}^p k^{\frac{p+\ell}{2}} \sum_{\ell'=0}^{\ell} \left(|\widehat{x}_j^{k,m,m'}| + |\xi_j^{k,n}| \right)^{\ell-\ell'} \left| \left((x_j^{k,m,m'} - x_j)^{\ell'} \Psi_{k,m,n}, \Psi_{k,m',n'} \right) \right|.$$

Recalling (A.7), we have

$$(A.10) \quad \left(x_j^{k,m,m'} - x_j \right)^{\ell'} \Psi_{k,m,n}(\mathbf{x}) \overline{\Psi_{k,m',n'}(\mathbf{x})} = \left(\frac{k}{\pi} \right)^{d/2} e^{ik(\mathbf{x}^{k,m'} \cdot \boldsymbol{\xi}^{k,n'} - \mathbf{x}^{k,m} \cdot \boldsymbol{\xi}^{k,n})} e^{-\frac{k}{4} |\widehat{\mathbf{x}}^{k,m,m'}|^2} \left(x_j^{k,m,m'} - x_j \right)^{\ell'} e^{-k|\mathbf{x} - \mathbf{x}^{k,m,m'}|^2} e^{ik\mathbf{x} \cdot \widehat{\boldsymbol{\xi}}^{k,n,n'}}.$$

We then compute

$$\begin{aligned} & \int_{\mathbb{R}^d} (\mathbf{x} - \mathbf{x}^{k,m,m'})_j^{\ell'} e^{-k|\mathbf{x} - \mathbf{x}^{k,m,m'}|^2} e^{ik\mathbf{x} \cdot \widehat{\boldsymbol{\xi}}^{k,n,n'}} d\mathbf{x} \\ &= e^{ik\mathbf{x}^{k,m,m'} \cdot \widehat{\boldsymbol{\xi}}^{k,n,n'}} \int_{\mathbb{R}^d} k^{\ell'/2} \mathbf{y}_j^{\ell'} e^{-|\mathbf{y}|^2} e^{i\sqrt{k}\mathbf{y} \cdot \widehat{\boldsymbol{\xi}}^{k,n,n'}} d\mathbf{y} \quad \text{setting } \mathbf{y} = k^{-1/2}(\mathbf{x} - \mathbf{x}^{k,m,m'}) \\ &= e^{ik\mathbf{x}^{k,m,m'} \cdot \widehat{\boldsymbol{\xi}}^{k,n,n'}} (i\sqrt{k}\partial_j)^{\ell'} (e^{-\frac{1}{4}|\boldsymbol{\xi}|^2}) \left(\sqrt{k}\widehat{\boldsymbol{\xi}}^{k,n,n'} \right). \end{aligned}$$

Now, $\partial_j^{\ell'}(e^{-\frac{1}{4}|\xi|^2})$ is a polynomial of degree ℓ' times $e^{-\frac{1}{4}|\xi|^2}$, so that its modulus is bounded by $C_{\ell'}(1 + |\xi|^{\ell'})e^{-\frac{1}{4}|\xi|^2}$. Therefore, we have

$$\left| \left(\left(x_j^{k,m,m'} - x_j \right)^{\ell'} \Psi_{k,m,n}, \Psi_{k,m',n'} \right) \right| \leq C k^{\frac{\ell'}{2}} |\widehat{\xi}^{k,n,n'}|^{\ell'} e^{-\frac{k}{4}|\widehat{x}^{k,m,m'}|^2} e^{-\frac{k}{4}|\widehat{\xi}^{k,n,n'}|^2}.$$

Combining this with (A.9), we get

$$\begin{aligned} \left| (\partial_j^p \Psi_{k,m,n}, \Psi_{k,m',n'}) \right| &\leq C_p k^{\frac{p}{2}} e^{-\frac{k}{4}|\widehat{x}^{k,m,m'}|^2} e^{-\frac{k}{4}|\widehat{\xi}^{k,n,n'}|^2} \sum_{\ell=0}^p \sum_{\ell'=0}^{\ell} k^{\frac{\ell+\ell'}{2}} \left(|\widehat{x}^{k,m,m'}| + |\xi^{k,n}| \right)^{\ell-\ell'} |\widehat{\xi}^{k,n,n'}|^{\ell'} \\ &\leq C_p k^{\frac{p}{2}} e^{-\frac{k}{4}|\widehat{x}^{k,m,m'}|^2} e^{-\frac{k}{4}|\widehat{\xi}^{k,n,n'}|^2} \sum_{\ell=0}^p \left(|\sqrt{k}\widehat{x}^{k,m,m'}| + |\sqrt{k}\xi^{k,n}| + |\sqrt{k}\widehat{\xi}^{k,n,n'}| \right)^{\ell} \\ &\leq C_p k^{\frac{p}{2}} e^{-\frac{k}{4}|\widehat{x}^{k,m,m'}|^2} e^{-\frac{k}{4}|\widehat{\xi}^{k,n,n'}|^2} \left[1 + \left(|\sqrt{k}\widehat{x}^{k,m,m'}| + |\sqrt{k}\xi^{k,n}| + |\sqrt{k}\widehat{\xi}^{k,n,n'}| \right)^p \right], \end{aligned}$$

as announced. \square

APPENDIX B. LOCALISATION PROPERTIES OF THE DUAL FRAME

The aim of this appendix is to give an elementary proof of (4.7). Recalling the constants α and β from the frame inequalities in (3.2), we set

$$(B.1) \quad Y := I - \frac{2}{\alpha + \beta} T_k^* T_k,$$

so that Y is self-adjoint with $\|Y\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq \gamma < 1$. In particular,

$$S_k = (T_k^* T_k)^{-1} = \frac{2}{\alpha + \beta} (I - Y)^{-1} = \frac{2}{\alpha + \beta} \sum_{\ell=0}^{\infty} Y^{\ell},$$

where the sum converges for the norm of bounded linear operators acting on $L^2(\mathbb{R}^d)$. Therefore, we have

$$\begin{aligned} (\Psi_{k,m,n}^*, \Psi_{k,m',n'}^*) &= \left(\frac{2}{\alpha + \beta} \right)^2 \left(\sum_{\ell=0}^{+\infty} Y^{\ell} \Psi_{k,m,n}, \sum_{\ell'=0}^{+\infty} Y^{\ell'} \Psi_{k,m',n'} \right) \\ &= \left(\frac{2}{\alpha + \beta} \right)^2 \sum_{\ell,\ell'=0}^{+\infty} \left(\Psi_{k,m,n}, Y^{\ell+\ell'} \Psi_{k,m',n'} \right) \\ &= \left(\frac{2}{\alpha + \beta} \right)^2 \sum_{p=0}^{+\infty} (p+1) \left(\Psi_{k,m,n}, Y^p \Psi_{k,m',n'} \right). \end{aligned}$$

We claim that there exists $A > 1$ such that, for all $p \in \mathbb{N}$ and all $[m, n], [m', n'] \in \mathbb{Z}^{2d}$, we have

$$(B.2) \quad \left| \left(\Psi_{k,m,n}, Y^p \Psi_{k,m',n'} \right) \right| \leq A^p e^{-\frac{\pi}{4} |[m,n] - [m',n']|}.$$

Let us prove this result by induction. It trivially holds for $p = 0$, due to (A.3). Assume now that (B.2) holds for $p - 1$. Recalling (B.1), we have

$$Y^p = Y^{p-1} - \frac{2}{\alpha + \beta} Y^{p-1} (T_k^* T_k)$$

and it follows from (3.3) that

$$\begin{aligned} (\Psi_{k,\mathbf{m},\mathbf{n}}, Y^p \Psi_{k,\mathbf{m}',\mathbf{n}'}) &= (\Psi_{k,\mathbf{m},\mathbf{n}}, Y^{p-1} \Psi_{k,\mathbf{m}',\mathbf{n}'}) \\ &\quad - \frac{2}{\alpha + \beta} \sum_{[\mathbf{m}'',\mathbf{n}''] \in \mathbb{Z}^{2d}} (\Psi_{k,\mathbf{m},\mathbf{n}}, \Psi_{k,\mathbf{m}'',\mathbf{n}''}) (\Psi_{k,\mathbf{m}'',\mathbf{n}''}, Y^{p-1} \Psi_{k,\mathbf{m}',\mathbf{n}'}), \end{aligned}$$

so that, by our induction hypothesis and (A.3), we have

$$\begin{aligned} |(\Psi_{k,\mathbf{m},\mathbf{n}}, Y^p \Psi_{k,\mathbf{m}',\mathbf{n}'})| &\leq A^{p-1} e^{-\frac{\pi}{4} |[\mathbf{m},\mathbf{n}] - [\mathbf{m}',\mathbf{n}']|^2} \\ &\quad + \frac{2A^{p-1}}{\alpha + \beta} \sum_{[\mathbf{m}'',\mathbf{n}''] \in \mathbb{Z}^{2d}} e^{-\frac{\pi}{4} |[\mathbf{m},\mathbf{n}] - [\mathbf{m}'',\mathbf{n}'']|^2} e^{-\frac{\pi}{4} |[\mathbf{m}'',\mathbf{n}'''] - [\mathbf{m}',\mathbf{n}']|^2}. \end{aligned}$$

We now simplify the sum. For the sake of simplicity, we introduce $\mathbf{q} = [\mathbf{m}, \mathbf{n}]$, $\mathbf{q}' = [\mathbf{m}', \mathbf{n}']$ and $\mathbf{q}'' = [\mathbf{m}'', \mathbf{n}'']$. The triangle inequality reveals that

$$\begin{aligned} |\mathbf{q} - \mathbf{q}''|^2 + |\mathbf{q}' - \mathbf{q}''| &= |\mathbf{q} - \mathbf{q}''|^2 - |\mathbf{q} - \mathbf{q}''| + |\mathbf{q} - \mathbf{q}''| + |\mathbf{q}' - \mathbf{q}''| \\ &\geq |\mathbf{q} - \mathbf{q}''|^2 - |\mathbf{q} - \mathbf{q}''| + |\mathbf{q} - \mathbf{q}'|, \end{aligned}$$

and as a result

$$\sum_{\mathbf{q}'' \in \mathbb{Z}^{2d}} e^{-\frac{\pi}{4} |\mathbf{q} - \mathbf{q}''|^2} e^{-\frac{\pi}{4} |\mathbf{q}' - \mathbf{q}''|} \leq e^{-\frac{\pi}{4} |\mathbf{q} - \mathbf{q}'|} \sum_{\mathbf{q}'' \in \mathbb{Z}^{2d}} e^{\frac{\pi}{4} (|\mathbf{q} - \mathbf{q}''| - |\mathbf{q} - \mathbf{q}''|^2)} \leq \sigma e^{-\frac{\pi}{4} |\mathbf{q} - \mathbf{q}'|},$$

where we easily see that

$$\sigma := \sum_{\mathbf{q}'' \in \mathbb{Z}^{2d}} e^{\frac{\pi}{4} (|\mathbf{q} - \mathbf{q}''| - |\mathbf{q} - \mathbf{q}''|^2)}$$

is finite and does not depend on \mathbf{q} .

Therefore, we have

$$|(\Psi_{k,\mathbf{m},\mathbf{n}}, Y^p \Psi_{k,\mathbf{m}',\mathbf{n}'})| \leq A^{p-1} \left(1 + \frac{2\sigma}{\alpha + \beta} \right) e^{-\frac{\pi}{4} |[\mathbf{m},\mathbf{n}] - [\mathbf{m}',\mathbf{n}']|^2}$$

and (B.2) holds, provided we take

$$A \geq 1 + \frac{2\sigma}{\alpha + \beta}.$$

To conclude, we write $\delta := \|[\mathbf{m}, \mathbf{n}] - [\mathbf{m}', \mathbf{n}']\|$, and we let P denote the largest integer such that $P \leq \delta^{1-\varepsilon}$. We have, for any $\gamma' \in (\gamma, 1)$

$$\begin{aligned} |(\Psi_{k,\mathbf{m},\mathbf{n}}, Y^p \Psi_{k,\mathbf{m}',\mathbf{n}'})| &\leq \sum_{p=0}^P (p+1) (\Psi_{k,\mathbf{m},\mathbf{n}}, Y^p \Psi_{k,\mathbf{m}',\mathbf{n}'}) + \sum_{p=P+1}^{+\infty} (p+1) \gamma^p \\ &\leq e^{-\frac{\pi}{4}\delta} \sum_{p=0}^P (p+1) A^p + C \sum_{p=P+1}^{+\infty} (\gamma')^p \\ &\leq P(P+1) A^P e^{-\frac{\pi}{4}\delta} + C(\gamma')^{P+1} \\ &\leq 2\delta^2 A^{\delta^{1-\varepsilon}} e^{-\frac{\pi}{4}\varepsilon} + C(\gamma')^{\delta^{1-\varepsilon}}. \end{aligned}$$

One readily checks that both terms are bounded by $Ce^{-\delta^{1-\varepsilon}}$, which concludes the proof.

APPENDIX C. SHARPNESS OF THEOREM 3.1

The aim of this appendix is to show the sharpness of Theorem 3.1, as asserted in Proposition 3.2. We start with a preliminary result concerning the norm of a coherent Gaussian state.

Lemma C.1 ($\widehat{H}_k^p(\mathbb{R}^d)$ norm). *Let $p \geq 1$. For all $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$, we have*

$$(C.1) \quad \mu_p \left(1 + |\mathbf{x}^{k,\mathbf{m}}|^2 + |\boldsymbol{\xi}^{k,\mathbf{n}}|^2\right)^p \leq \|\Psi_{k,\mathbf{m},\mathbf{n}}\|_{\widehat{H}_k^p(\mathbb{R}^d)}^2 \leq \lambda_p \left(1 + |\mathbf{x}^{k,\mathbf{m}}|^2 + |\boldsymbol{\xi}^{k,\mathbf{n}}|^2\right)^p,$$

where the constants $\mu_p, \lambda_p > 0$ only depend on p and d .

Proof. Fix $[\mathbf{m}, \mathbf{n}] \in \mathbb{Z}^{2d}$ and let $j \in \{1, \dots, d\}$ be such that $n_j \geq \lfloor \mathbf{n} \rfloor / d$. We have

$$(C.2) \quad \|\Psi_{k,\mathbf{m},\mathbf{n}}\|_{\widehat{H}_k^p(\mathbb{R}^d)}^2 \geq \|\Psi_{k,\mathbf{m},\mathbf{n}}\|_{L^2(\mathbb{R}^d)}^2 + \|\mathbf{x}^p \Psi_{k,\mathbf{m},\mathbf{n}}\|_{L^2(\mathbb{R}^d)}^2 + k^{-2p} \|\partial_{x_j}^p \Psi_{k,\mathbf{m},\mathbf{n}}\|_{L^2(\mathbb{R}^d)}^2.$$

The first term is equal to 1. The second is

$$\begin{aligned} \|\mathbf{x}^p \Psi_{k,\mathbf{m},\mathbf{n}}\|_{L^2(\mathbb{R}^d)}^2 &= \left(\frac{k}{\pi}\right)^{d/2} \int_{\mathbb{R}^d} |\mathbf{x}|^{2p} e^{-k|\mathbf{x} - \mathbf{x}^{k,\mathbf{m}}|^2} d\mathbf{x} \\ &= \pi^{-d/2} \int_{\mathbb{R}^d} \left|\mathbf{x}^{k,\mathbf{m}} + k^{-1/2}\mathbf{y}\right|^{2p} e^{-|\mathbf{y}|^2} d\mathbf{y} \geq ck^p |\mathbf{x}^{k,\mathbf{m}}|^{2p}. \end{aligned}$$

The last term in (C.2) is equal to

$$\begin{aligned} k^{-2p} \|\partial_{x_j}^p \Psi_{k,\mathbf{m},\mathbf{n}}\|_{L^2(\mathbb{R}^d)}^2 &= k^{-p} \left\| Q_p \left(k^{1/2}(x_j^{k,\mathbf{m}} - x_j + i\xi_j^{k,\mathbf{n}}) \right) \Psi_{k,\mathbf{m},\mathbf{n}} \right\|_{L^2(\mathbb{R}^d)}^2 \\ &= k^{-p} \pi^{-d/2} \int_{\mathbb{R}^d} \left| Q_p(y_j + ik^{1/2}\xi_j^{k,\mathbf{n}}) \right|^2 e^{-|\mathbf{y}|^2} d\mathbf{y} \\ &\geq (k^{-p} + |\boldsymbol{\xi}^{k,\mathbf{n}}|^{2p}). \end{aligned}$$

This gives us the lower bound in (C.1).

On the other hand, it holds that

$$\begin{aligned} \|\Psi_{k,\mathbf{m},\mathbf{n}}\|_{\widehat{H}_k^p(\mathbb{R}^d)}^2 &= \sum_{[\mathbf{a}] \leq p} \sum_{q=0}^p k^{-[\mathbf{a}]} \left\| |\mathbf{x}|^q \prod_{j=1}^d Q_{a_j} \left(k^{1/2}(x_j^{k,\mathbf{m}} - x_j + i\xi_j^{k,\mathbf{n}}) \right) \Psi_{k,\mathbf{m},\mathbf{n}} \right\|_{L^2(\mathbb{R}^d)}^2 \\ &= \sum_{[\mathbf{a}] \leq p} \sum_{q=0}^p k^{-[\mathbf{a}]} \pi^{-d/2} \int_{\mathbb{R}^d} |\mathbf{x}^{k,\mathbf{m}} + k^{-1/2}\mathbf{y}|^{2q} \left| \prod_{j=1}^d Q_{a_j}(y_j + ik^{1/2}\xi_j^{k,\mathbf{n}}) \right|^2 e^{-|\mathbf{y}|^2} d\mathbf{y}. \end{aligned}$$

We then observe that

$$\begin{aligned} k^{-[\mathbf{a}]} \int_{\mathbb{R}^d} |\mathbf{x}^{k,\mathbf{m}} + k^{-1/2}\mathbf{y}|^{2q} \left| \prod_{j=1}^d Q_{a_j}(y_j + ik^{1/2}\xi_j^{k,\mathbf{n}}) \right|^2 e^{-|\mathbf{y}|^2} d\mathbf{y} \\ \leq C_{d,p} k^{-[\mathbf{a}]} |\mathbf{x}^{k,\mathbf{m}}|^{2q} \prod_{j=1}^d \left(1 + |k^{1/2}\xi_j^{k,\mathbf{n}}|^{a_j} \right)^2 \leq C_{d,p} \left(1 + |\mathbf{x}^{k,\mathbf{m}}|^2 + |\boldsymbol{\xi}^{k,\mathbf{n}}|^2 \right)^p, \end{aligned}$$

which lead to the upper bound in (C.1) after summation. \square

We then need another intermediate result concerning the inner product of elements of the Gabor and dual frames.

Lemma C.2 (L^2 product of the dual frame). *For all $[\mathbf{m}, \mathbf{n}], [\mathbf{m}', \mathbf{n}'] \in \mathbb{Z}^{2d}$, we have*

$$(C.3) \quad |(\Psi_{k,\mathbf{m},\mathbf{n}}^*, \Psi_{k,\mathbf{m}',\mathbf{n}'})| \leq C e^{-\frac{\pi}{8}([\mathbf{m},\mathbf{n}] - [\mathbf{m}',\mathbf{n}'])^{1/2}}.$$

Proof. Using (A.4) and (4.7), we get

$$\begin{aligned} |(\Psi_{k,\mathbf{m},\mathbf{n}}^*, \Psi_{k,\mathbf{m}',\mathbf{n}'})| &\leq \sum_{[\mathbf{m}'', \mathbf{n}''] \in \mathbb{Z}^{2d}} |(\Psi_{k,\mathbf{m},\mathbf{n}}^*, \Psi_{k,\mathbf{m}'',\mathbf{n}''}^*)| |(\Psi_{k,\mathbf{m}'',\mathbf{n}''}, \Psi_{k,\mathbf{m}',\mathbf{n}'})| \\ &\leq C \sum_{[\mathbf{m}'', \mathbf{n}''] \in \mathbb{Z}^{2d}} e^{-|[\mathbf{m}'', \mathbf{n}''] - [\mathbf{m}, \mathbf{n}]|^{1/2}} e^{-\frac{\pi}{8}([\mathbf{m}'', \mathbf{n}'''] - [\mathbf{m}', \mathbf{n}'])^2} \\ &= C \sum_{[\mathbf{m}'', \mathbf{n}''] \in \mathbb{Z}^{2d}} e^{-|\mathbf{q}'' - \mathbf{q}|^{1/2} - \frac{\pi}{8}|\mathbf{q}'' - \mathbf{q}'|^2}, \end{aligned}$$

where we introduced $\mathbf{q} := [\mathbf{m}, \mathbf{n}]$, $\mathbf{q}' := [\mathbf{m}', \mathbf{n}']$ and $\mathbf{q}'' := [\mathbf{m}'', \mathbf{n}'']$ for the sake of shortness. Since $\mathbf{q}', \mathbf{q}'' \in \mathbb{Z}^{2d}$, we have $|\mathbf{q}'' - \mathbf{q}'|^2 \geq |\mathbf{q}'' - \mathbf{q}'|^{1/2}$, and hence

$$e^{-|\mathbf{q}'' - \mathbf{q}|^{1/2} - \frac{\pi}{8}|\mathbf{q}'' - \mathbf{q}'|^2} \leq e^{-|\mathbf{q}'' - \mathbf{q}|^{1/2} - \frac{\pi}{8}|\mathbf{q}'' - \mathbf{q}'|^{1/2}} = e^{-\frac{\pi}{8}(|\mathbf{q}'' - \mathbf{q}|^{1/2} + |\mathbf{q}'' - \mathbf{q}'|^{1/2})} e^{-(1 - \frac{\pi}{8})|\mathbf{q}'' - \mathbf{q}|^{1/2}},$$

where we note that $1 - \pi/8 > 0$. Then, since

$$\sqrt{|\mathbf{x} - \mathbf{y}|} \leq \sqrt{|\mathbf{x}|} + \sqrt{|\mathbf{y}|} \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^{2d},$$

we have

$$e^{-|\mathbf{q}'' - \mathbf{q}|^{1/2} - \frac{\pi}{8}|\mathbf{q}'' - \mathbf{q}'|^2} \leq e^{-\frac{\pi}{8}|\mathbf{q}' - \mathbf{q}|^{1/2}} e^{-(1 - \frac{\pi}{8})|\mathbf{q}'' - \mathbf{q}|^{1/2}}.$$

Therefore, we get

$$\begin{aligned} |(\Psi_{k,\mathbf{m},\mathbf{n}}^*, \Psi_{k,\mathbf{m}',\mathbf{n}'})| &\leq C e^{-\frac{\pi}{8}|\mathbf{q} - \mathbf{q}'|^{1/2}} \sum_{[\mathbf{m}'', \mathbf{n}''] \in \mathbb{Z}^{2d}} e^{-(1 - \frac{\pi}{8})|\mathbf{q}'' - \mathbf{q}|^{1/2}} \\ &\leq C e^{-\frac{\pi}{8}|\mathbf{q} - \mathbf{q}'|^{1/2}} \end{aligned}$$

as announced. \square

Proof of Proposition 3.2. Let $D \geq k^{-1/2}$. Pick $\mathbf{m} = (2\lceil k^{1/2}D \rceil, 0, 0)$ and let $u = \Psi_{k,\mathbf{m},\mathbf{0}}$. For any $p \geq 0$, from Lemma C.1, we have

$$\|u\|_{\widehat{H}_k^p(\mathbb{R}^d)} \geq \mu_p(2D)^{-p}.$$

On the other hand since

$$\Pi_D \Psi_{k,\mathbf{m},\mathbf{0}} = \sum_{\substack{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d} \\ \|[\mathbf{m}',\mathbf{n}']\| \leq k^{1/2}D}} (\Psi_{k,\mathbf{m},\mathbf{0}}, \Psi_{k,\mathbf{m}',\mathbf{n}'}^*) \Psi_{k,\mathbf{m}',\mathbf{n}'}$$

we have

$$\begin{aligned} \|\Pi_D \Psi_{k,\mathbf{m},\mathbf{0}}\|_{\widehat{H}_k^p(\mathbb{R}^d)} &\leq \sum_{\substack{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d} \\ \|[\mathbf{m}',\mathbf{n}']\| \leq k^{1/2}D}} |(\Psi_{k,\mathbf{m},\mathbf{0}}, \Psi_{k,\mathbf{m}',\mathbf{n}'}^*)| \|\Psi_{k,\mathbf{m}',\mathbf{n}'}\|_{\widehat{H}_k^p(\mathbb{R}^d)} \\ &\leq C \sum_{\substack{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d} \\ \|[\mathbf{m}',\mathbf{n}']\| \leq k^{1/2}D}} e^{-\frac{\pi}{8}|\mathbf{m}-\mathbf{m}'|^{1/2}} \|\Psi_{k,\mathbf{m}',\mathbf{n}'}\|_{\widehat{H}_k^p(\mathbb{R}^d)} \\ &\leq C \sum_{\substack{[\mathbf{m}',\mathbf{n}'] \in \mathbb{Z}^{2d} \\ \|[\mathbf{m}',\mathbf{n}']\| \leq k^{1/2}D}} e^{-\frac{\pi}{8}|\mathbf{m}-\mathbf{m}'|^{1/2}} D^p \\ &\leq CD^p M(k^{1/2}D)^{2d+1} e^{-a\sqrt{k^{1/2}D}} \\ &\leq CD^p e^{-a'\sqrt{k^{1/2}D}} \\ &\leq CD^p e^{-a'\sqrt{D}}, \end{aligned}$$

since $k \geq 1$.

In particular, there exists D_* such that for all $D \geq D_*$, this quantity is smaller than $\mu_p D^{-p}$. We thus have, for all $D \geq D_*$

$$\|u - \Pi_D u\|_{\widehat{H}_k^p(\mathbb{R}^d)} \geq \|u\|_{\widehat{H}_k^p(\mathbb{R}^d)} - \|\Pi_D u\|_{\widehat{H}_k^p(\mathbb{R}^d)} \geq \mu_p D^{-p},$$

and thanks to Lemma C.1, this quantity is larger than $\frac{\mu_{p+r}}{\lambda_p} D^{-r} \|u\|_{\widehat{H}^{p+r}(\mathbb{R}^d)}$. The result follows. \square

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