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# Logical treatment of incomplete/uncertain information relying on different systems of rough sets

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**Abstract.** The different systems of rough set theory treat uncertainty in special ways. There is a very important common property: all systems rely on given background knowledge and we cannot say more about an arbitrary set or about its members than its lower and upper approximations make possible. New membership relations have to be introduced. The semantics of classical logic is based on classical set theory. An important question: Is it possible to build logical systems with semantics relying on different versions of the theory of rough sets by using new membership relations? The paper gives an overview of first-order logical systems with partial semantics in order to show the influence of incomplete/uncertain information in logically valid inferences.

**Keywords:** Rough set theory · Partial first-order logic · Multivalued logic

## 1 Introduction

Pawlak's original theory of rough sets (see in e.g. [8,9,11]), covering systems relying on tolerance relations [12], general covering systems [15,10], decision-theoretic rough set theory [14], general partial approximation spaces [1], similarity based approximation spaces [7] are different systems of rough set theory. There is a very important common property: all systems rely on given background knowledge and one cannot say more about an arbitrary set (representing a 'new' property) or about its members than the lower and upper approximations of the set make possible. The system of base sets represents background knowledge.

The different versions of rough set theory have been focusing on objects: What can one say (what can one know) — relying on the knowledge embedded in an information system — about an object: whether it belongs to an arbitrary given set? But background knowledge embedded in an information system contains much more information: base sets represent properties, and therefore they circumscribe a conceptual structure. An arbitrary set of objects may be considered as the representation of a property. A new important question appears: What can one say — relying on the knowledge embedded in an information system — about the property represented by a set of objects? What is more: What can one say about a relation between objects on the basis of background knowledge?

In the semantics of classical first-order logic classical set theory plays a crucial role. Classical first-order logic is not a logic of set theory, it is a logic *on* sets, i.e. a logic with semantics relying *on* sets. Classical first-order logic reveals the logical laws of general (not singular) propositions, therefore it gives possibilities to get general knowledge (it shows how to achieve new general knowledge) by using quantified propositions. The most important tool in the semantics of classical first-order logic is the membership relation of classical set theory. The truth values of atomic formulae containing predicate parameter(s) can be defined with the help of the membership relation of classical set theory. In various versions of rough set theory different characteristic functions, and so different rough membership relations can be introduced. These functions rely on background knowledge represented by the systems of base sets, and if they are used in the semantics of first-order logic, then the logical system is able to reveal the logical laws of general propositions relying on embedded background knowledge, the nature and the consequences of conceptual structure (conceptual knowledge) embedded in the system of base sets. The main goal of the paper is to give a general and flexible first-order logical system based on generalized characteristic functions of theories of rough sets.

After giving a general picture of approximation spaces the role of base sets is surveyed. The influences of embedded knowledge on membership relations result in generalized characteristic functions. Finally, a partial first-order logical system relying on different generalized characteristic functions is presented precisely.<sup>1</sup>

## 2 Background

### 2.1 General Approximation Space

**Definition 1.** *The ordered 5-tuple  $\langle U, \mathfrak{B}, \mathcal{D}_{\mathfrak{B}}, l, u \rangle$  is a general approximation space if*

1.  $U$  is a nonempty set;
2.  $\mathfrak{B} \subseteq 2^U \setminus \emptyset$ ,  $\mathfrak{B} \neq \emptyset$  ( $\mathfrak{B}$  is the set of base sets);
3.  $\mathcal{D}_{\mathfrak{B}}$  is the set of definable sets and it is given by the following inductive definition:
  - (a)  $\emptyset \in \mathcal{D}_{\mathfrak{B}}$ ;
  - (b)  $\mathfrak{B} \subseteq \mathcal{D}_{\mathfrak{B}}$ ;
  - (c) if  $D_1, D_2 \in \mathcal{D}_{\mathfrak{B}}$ , then  $D_1 \cup D_2 \in \mathcal{D}_{\mathfrak{B}}$
4.  $\langle l, u \rangle$  is a Pawlakian approximation pair i.e.
  - (a)  $l(S) = \cup C^l(S)$ , where  $C^l(S) = \{B \mid B \in \mathfrak{B} \text{ and } B \subseteq S\}$ ;
  - (b)  $u(S) = \cup C^u(S)$  where  $C^u(S) = \{B \mid B \in \mathfrak{B} \text{ and } B \cap S \neq \emptyset\}$ .

**Definition 2.**

*Different types of general approximation space  $\langle U, \mathfrak{B}, \mathcal{D}_{\mathfrak{B}}, l, u \rangle$  are the followings:*

1. *A general approximation space is Pawlakian if  $\mathfrak{B}$  is a partition of  $U$ .*

<sup>1</sup> The logical properties of different systems of rough sets are investigated extensively (e.g. [2] and [13]). Typically the investigations are on propositional level and focus on the general properties of rough sets. Arguments, inferences, consequences of embedded knowledge are not in the center of considerations.

2. A general approximation space is a covering approximation space generated by a tolerance relation  $\mathcal{R}$  if  $\mathfrak{B} = \{[u]_{\mathcal{R}} \mid u \in U\}$ , where  $[u]_{\mathcal{R}} = \{u' \mid u\mathcal{R}u'\}$ .
3. A general approximation space is a covering approximation space if  $\bigcup \mathfrak{B} = U$ .
4. A general approximation space is a partial approximation space if  $\bigcup \mathfrak{B} \neq U$ .
5. A general approximation space is a similarity based (partial) approximation space generated by a tolerance relation  $\mathcal{R}$  if
 
$$\mathfrak{B} = \{B \mid B \in \mathfrak{C}l(\mathcal{R}) \text{ and } B \text{ is not a singleton}\},$$
 where  $\mathfrak{C}l(\mathcal{R})$  is the set of clusters determined by a correlation clustering process based on the tolerance relation  $\mathcal{R}$ .

## 2.2 Role of Base Sets

From the philosophical point of view the most important question of the different versions of rough set theory is the following: What can one say (what can one know) — relying on the knowledge embedded in an information system — about an object: whether it belongs to an arbitrary given set? Knowledge embedded in an information system is the background knowledge behind all propositions, which can be made by using the information system. Therefore background knowledge plays a crucial role in rough set theory: background knowledge specifies ‘the world of certainty’. Background knowledge is represented by a system of base sets. It means that the membership relation of classical set theory can only be used with base sets, and so background knowledge restricts possibilities: the members of a base set have to be treated in the same way at least in a special sense.)

The next step is to make clear the ‘nature’, the usage, and the influences of background (and embedded) knowledge. The ‘meanings’ (which determine the usage) of base sets are the followings:

1. In Pawlak’s original system: if an object belongs to a given base set, then it is indiscernible from the members of the base set. In this case, the limit of represented knowledge appears explicitly: base sets consist of indiscernible objects, there is no way to distinguish them from each other.
2. In a covering approximation space generated by a tolerance relation: if an object belongs to a given base set, then it is in the tolerance relation with the generator member of the base set (similar to the generator member of the base set), therefore it may (or has to) be treated in the same way as the other objects which are similar to the generator member of the base set.
3. In a covering approximation space: if an object belongs to a given base set, then it has a common property represented by the base set with the members of the base set. Objects with the same property (members of a base set) may (or have to) be handled in the same way.<sup>2</sup>
4. General partial spaces are similar to general covering ones, but it is not supposed that all objects have at least one property represented by a base set. In practical

<sup>2</sup> In point 2 and 3 ‘may or have to’ are used: if an object is a member of more than one base sets, then a base set has to be chosen whose members are handled in the same way. It depends on the given situation.

cases information systems are not total, there is no relevant information about an object: it may be in our database but some information is missing, and so it does not have any property represented by a base set.

5. In a similarity based (partial) approximation space: if an object belongs to a given base set, then it is similar (at least according to a correlation clustering process) to the members of the base set. In general case similarity based approximation spaces are partial and the base sets are pairwise disjoint ones. Therefore there is no way to distinguish the members of a base set from each other.

The definition of definable sets expands ‘the word of certainty’ by showing the possible usage of base sets and so the possible usage of background knowledge. An arbitrary set of objects belonging to an information system can be approximated by definable sets. One of the most important points of rough set theory is its approximative aspect.

### 3 Influences of Embedded Knowledge on Membership Relations

From the theoretical point of view the system of base sets represents a sort of limit of our knowledge embedded in an information system. In some situation, it makes our judgment of the membership relation uncertain — making a set vague — because a decision about a given object affects the decision about all other objects which are in the same base set containing the given object.

The main source of uncertainty is in our background knowledge, but according to represented background knowledge there are some special sets: uncertainty does not appear in their cases:

#### Definition 3.

Let  $\langle U, \mathfrak{B}, \mathcal{D}_{\mathfrak{B}}, l, u \rangle$  be a general approximation space, and  $S \subseteq U$ ,  $\bar{S} = U \setminus S$ . Then

1. the set  $S$  is lower script if  $S = l(S)$ , and upper script if  $S = u(S)$ ;
2. the set  $S$  is strictly lower/upper script if both  $S$  and  $\bar{S}$  (i.e. the complement of set  $S$ ) are lower/upper script respectively;
3. the set  $S$  is partially lower script if  $S \cap \cup \mathfrak{B} = l(S)$  and partially upper script if  $S \cap \cup \mathfrak{B} = u(S)$ ;
4. the set  $S$  is strictly partially lower/upper script if both  $S$  and  $\bar{S}$  are partially lower/upper script respectively.

In general approximation spaces all definable sets (and therefore  $l(S)$  and  $u(S)$  for all  $S \subseteq U$ ) are lower script (partially lower script in partial spaces), but they are not strictly lower script (strictly partially lower script in partial spaces) necessarily, and if a set is (strictly) upper script, then it is (strictly) lower script. If the approximation space is Pawlakian, then if a set  $S$  is lower script, then it is strictly lower script.

**Theorem 1.** A general approximation space is Pawlakian if and only if all lower script sets are strictly lower script.

A general approximation space is one-layered (i.e. its base sets are pairwise disjoint ones) if and only if all partially lower script sets are strictly partially lower script.

Let  $S$  be a subset of  $U$ , and  $x, y \in U$ . What is the consequence of embedded and limited background knowledge? What can be said about  $y$  with respect to  $x$ ?

1. In an original Pawlakian space relying on an equivalence relation  $\mathcal{R}$ :
  - (a) If  $x \in l(S)$ , then  $y \in S$  for all  $y, x\mathcal{R}y$  (i.e.  $x$  and  $y$  belong to the same base set).
  - (b) If  $x \in l(\bar{S})$ , where  $\bar{S} = U \setminus S$ , then  $y \notin S$  for all  $y, x\mathcal{R}y$ .
  - (c) If  $x \notin l(S)$  and  $x \notin l(\bar{S})$ , then  $x \in u(S) \setminus l(S)$ , and there are  $y_1, y_2$  such that  $x\mathcal{R}y_1, y_1 \in S$ , and  $x\mathcal{R}y_2, y_2 \notin S$ .
  - (d)  $x \in u(S) \setminus l(S)$ , if and only if  $x \notin l(S)$  and  $x \notin l(\bar{S})$ , therefore the point 1. (c) and this case are the same.
2. In a covering space generated by a tolerance relation  $\mathcal{R}$ :
  - (a) If  $x \in l(S)$ , then there is an  $x' \in U$  such that  $x \in [x']_{\mathcal{R}}$  and  $[x'] \in \mathcal{C}^l(S)$ . Therefore  $y \in S$  for all  $y$  if  $y \in \cup\{[x']_{\mathcal{R}} \mid x \in [x']_{\mathcal{R}} \text{ and } [x'] \in \mathcal{C}^l(S)\}$ .
  - (b) If  $x \in l(\bar{S})$ , then there is an  $x' \in U$  such that  $x \in [x']_{\mathcal{R}}$  and  $[x'] \in \mathcal{C}^l(\bar{S})$ . Therefore  $y \notin S$  for all  $y$ , if  $y \in \cup\{[x']_{\mathcal{R}} \mid x \in [x']_{\mathcal{R}} \text{ and } [x'] \in \mathcal{C}^l(\bar{S})\}$ .
  - (c) If  $x \notin l(S) \cup l(\bar{S})$ , then
    - i. there is no base set  $[x']_{\mathcal{R}}$  such that  $x \in [x']_{\mathcal{R}}$ , and  $[x']_{\mathcal{R}} \subseteq S$ ,
    - ii. there is no base set  $[x'']_{\mathcal{R}}$  such that  $x \in [x'']_{\mathcal{R}}$ , and  $[x'']_{\mathcal{R}} \subseteq \bar{S}$ ,
    - iii. there is a base set  $[x''']_{\mathcal{R}}$  such that  $x \in [x''']_{\mathcal{R}}$ ,  $[x''']_{\mathcal{R}} \not\subseteq S, [x''']_{\mathcal{R}} \not\subseteq \bar{S}$  but in this case one cannot say anything about any other object with respect to  $x$  connected with  $S$  or  $\bar{S}$ .
  - (d) If  $x \in u(S) \setminus l(S)$ , then there is a base set  $[x']_{\mathcal{R}}$  such that  $x \in [x']_{\mathcal{R}}, [x']_{\mathcal{R}} \cap S \neq \emptyset$  and  $[x']_{\mathcal{R}} \not\subseteq S$ , but in this case one cannot say anything about any other object with respect to  $x$  connected with  $S$  or  $\bar{S}$ .
3. In a general covering space:
  - (a) If  $x \in l(S)$ , then there is a base set  $B$ , such that  $x \in B$  and  $B \in \mathcal{C}^l(S)$ . Therefore  $y \in S$  for all  $y \in \cup\{B \mid x \in B \text{ and } B \in \mathcal{C}^l(S)\}$ .
  - (b) If  $x \in l(\bar{S})$ , then there is a base set  $B$  such that  $x \in B$  and  $B \in \mathcal{C}^l(\bar{S})$ . Therefore  $y \notin S$  for all  $y \in \cup\{B \mid x \in B \text{ and } B \in \mathcal{C}^l(\bar{S})\}$ .
  - (c) If  $U \setminus (l(S) \cup l(\bar{S})) \neq \emptyset$  (i.e.  $S$  is not strictly script) and  $x \notin l(S) \cup l(\bar{S})$ , then there is a base set  $B$  such that  $x \in B$  and  $B \cap S \neq \emptyset, B \not\subseteq S, B \cap \bar{S} \neq \emptyset, B \not\subseteq \bar{S}$ . In this case one cannot say anything about any other object belonging to  $B$  with respect to  $x$  connected with  $S$  or  $\bar{S}$ .
4. In a general partial space:
  - (a) If  $x \in l(S)$ , then there is a base set  $B$ , such that  $x \in B$  and  $B \in \mathcal{C}^l(S)$ . Therefore  $y \in S$  for all  $y \in \cup\{B \mid x \in B \text{ and } B \in \mathcal{C}^l(S)\}$ .
  - (b) If  $x \in l(\bar{S})$ , then there is a base set  $B$  such that  $x \in B$  and  $B \in \mathcal{C}^l(\bar{S})$ . Therefore  $y \notin S$  for all  $y \in \cup\{B \mid x \in B \text{ and } B \in \mathcal{C}^l(\bar{S})\}$ .
  - (c) If  $x \in \cup\mathfrak{B}, x \notin l(S)$  and  $x \notin l(\bar{S})$ , then there is a base set  $B$  such that  $x \in B, B \cap S \neq \emptyset$  and  $B \not\subseteq S$ , but in this case one cannot say anything about any other object with respect to  $x$  connected with  $S$  or  $\bar{S}$ .
  - (d) Otherwise one does not know anything about  $x$  (i.e. there is no base set  $B$  such that  $x \in B$ ), therefore one cannot say anything about any other object with respect to  $x$ .

### 3.1 Rough Membership Relations

Classical membership relations (with the answer yes/no) can only be used without any restriction in the case of definable sets (more precisely in the case of base sets, but definable sets are introduced relying on base sets), therefore a new membership relation, the rough membership relation ( $\in^{rough}$ ) has to be introduced. The rough membership relation is not a yes/no relation, in the general case it is a relation with four possible answers:

Let  $\langle U, \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, l, u \rangle$  be a general approximation space,  $S$  be an arbitrary subset of the universe  $U$ . The answers for the question whether  $x \in^{rough} S$  are the followings:

- ‘yes’ if  $x \in l(S)$  (i.e.  $x$  belongs to  $S$  necessarily);
- ‘no’ if  $x \in l(\overline{S})$  (i.e.  $x$  does not belong to  $S$  necessarily);
- ‘maybe or maybe not’ if  $x \in \cup \mathfrak{B}$ ,  $x \notin l(S) \cup l(\overline{S})$  (i.e. the embedded background knowledge says something about the object  $x$ , but it is not enough to decide whether  $x$  belongs to or does not belong to  $S$  necessarily);
- ‘there is no information (embedded in base sets)’ if  $x \notin \cup \mathfrak{B}$  (it is possible only in partial cases).

Note that the set  $\{x \mid x \in^{rough} S \text{ is ‘maybe or maybe not’}\}$  is not a definable set necessarily, but the answer ‘maybe or maybe not’ can be determined by using only base sets (and the corresponding classical yes/no membership relation.) If 1, 0, 1/2, 2 are used to represent ‘yes’, ‘no’, ‘maybe or maybe not’ and ‘there is no information (embedded in base sets)’ respectively, then a generalized characteristic function of set  $S$  (called rough characteristic function) can be introduced:

$$\mu^r(S, x) = \begin{cases} 1 & \text{if } x \in l(S) \\ 0 & \text{if } x \in l(\overline{S}) \\ 1/2 & \text{if } x \in \cup \mathfrak{B}, x \notin l(S) \cup l(\overline{S}) \\ 2 & \text{otherwise (i.e. } x \notin \cup \mathfrak{B}) \end{cases}$$

Connection between the rough membership relation  $x \in^{rough} S$  and the rough characteristic function  $\mu^r(S, x)$  is the following:

$$x \in^{rough} S : \begin{cases} \text{yes} & \text{if } \mu^r(S, x) = 1 \\ \text{no} & \text{if } \mu^r(S, x) = 0 \\ \text{maybe or maybe not} & \text{if } \mu^r(S, x) = 1/2 \\ \text{there is no information} & \text{if } \mu^r(S, x) = 2 \end{cases}$$

### 3.2 Rough Membership Measurement on Finite Universe

In practical applications, universes are finite non-empty sets. Therefore, as in DTRS, some special functions can be introduced. These functions show the measurements of ‘belonging to a set’, and say something about the level of ‘maybe or maybe not’ from the side of ‘maybe’.

Relying on a given general approximation space  $\langle U, \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, l, u \rangle$  three different partial characteristic functions ( $\mu^o$  for optimistic,  $\mu^a$  for average and  $\mu^p$  for pessimistic) can be introduced. For the sake of simplicity we use a null entity (the number 2) to show

that a function is undefined for an object  $x$ , i.e. to represent partiality of characteristic functions.<sup>3</sup>

Let  $\langle U, \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, l, u \rangle$  be a general approximation space,  $U$  be a finite nonempty set,  $S \subseteq U$ ,  $x \in U$  and  $\mathcal{C}(x) = \{B \mid B \in \mathfrak{B} \text{ and } x \in B\}$ . Then

1. If  $x \in \cup \mathfrak{B}$ , then  $\mathcal{V}(x, S) = \left\{ \frac{|B \cap S|}{|B|} \mid B \in \mathcal{C}(x) \right\}$ .

2. Optimistic rough membership measurement:

$$\mu^o(S, x) = \begin{cases} 1 & \text{if } x \in l(S) \\ 0 & \text{if } x \in l(\bar{S}) \\ \max(\mathcal{V}(x, S)) & \text{if } x \in \cup \mathfrak{B}, x \notin l(S) \cup l(\bar{S}) \\ 2 & \text{otherwise (i.e. } x \notin \cup \mathfrak{B}) \end{cases}$$

3. Average rough membership measurement:

$$\mu^a(S, x) = \begin{cases} 1 & \text{if } x \in l(S) \\ 0 & \text{if } x \in l(\bar{S}) \\ \text{avg}(\mathcal{V}(x, S)) & \text{if } x \in \cup \mathfrak{B}, x \notin l(S) \cup l(\bar{S}) \\ 2 & \text{otherwise (i.e. } x \notin \cup \mathfrak{B}) \end{cases}$$

4. Pessimistic rough membership measurement:

$$\mu^p(S, x) = \begin{cases} 1 & \text{if } x \in l(S) \\ 0 & \text{if } x \in l(\bar{S}) \\ \min(\mathcal{V}(x, S)) & \text{if } x \in \cup \mathfrak{B}, x \notin l(S) \cup l(\bar{S}) \\ 2 & \text{otherwise (i.e. } x \notin \cup \mathfrak{B}) \end{cases}$$

5. For the sake of simplicity it is useful to introduce the partial characteristic function:

$$\mu^c(S, x) = \begin{cases} 1 & \text{if } x \in S \cap \cup \mathfrak{B} \\ 0 & \text{if } x \in \cup \mathfrak{B} \setminus S \\ 2 & \text{otherwise} \end{cases}$$

Optimistic, average, and pessimistic rough membership measurements are partial functions in the general case, and these are total ones when the approximation space is total. These functions are generalized characteristic functions of a set  $S$  relying on a given general approximation space.

Rough membership relation and rough membership measurements (as generalized characteristic functions) can be considered as generalizations of the membership relation of classical set theory.

## 4 Partial First-order Logic (PFoL) Relying on Different Generalized Characteristic Functions

In the previous subsections, different characteristic functions, and so different rough membership relations have been introduced. These functions rely on background knowledge represented by the systems of base sets. If new characteristic functions are used in

<sup>3</sup> In the definition three different functions (on finite sets of numbers) are used: function min for the minimum, function max for the maximum and function avg for the average value of a finite set of numbers.



the semantics of first-order logic, then the logical system can reveal the logical laws of general propositions relying on embedded background knowledge, the nature and the consequences of conceptual structure (conceptual knowledge) embedded in the system of base sets.

#### 4.1 Language of PFoL

At first, the language of PFoL has to be defined. The language of classical first-order logic must be modified to treat background knowledge represented by base sets. A finite non-empty set of one-argument predicates stands for the system of base sets. The members of this finite non-empty set ( $\mathcal{T}$ ) are called tools. Two new logical constants are introduced:  $\uparrow, \downarrow$ . Their intended meanings are to determine whether upper or lower approximations of a predicate is used.

**Definition 4.** *L is a language of PFoL with the set  $\mathcal{T}$  of tools, if*

1.  $L = \langle LC \cup \{\uparrow, \downarrow\}, Var, Con, Term, \mathcal{T}, Form \rangle$
2.  $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$  is a language of classical first-order logic with the following extension of the set *Form*
  - If  $P$  is an  $n$ -argument ( $n \geq 1$ ) predicate parameter and  $t_1, \dots, t_n \in Term$ , then  $P^\uparrow(t_1, \dots, t_n), P^\downarrow(t_1, \dots, t_n) \in Form$
3.  $\mathcal{T}$  is a finite non-empty set of one-argument predicate parameters.

#### 4.2 Semantics of PFoL

The semantic values of tools (i.e. the members of set  $\mathcal{T}$ ) play a crucial role in giving different types of generalized characteristic functions because their semantic values (as sets) generate the base sets of a logically relevant general partial approximation space.

**Definition 5.** *Let L be a language of PFoL with the set  $\mathcal{T}$  of tools. The ordered pair  $\langle U, \varrho \rangle$  is a tool-based interpretation of L, if*

1.  $U$  is a finite nonempty set;
2.  $\varrho$  is a function such that  $Dom(\varrho) = Con$  and
  - (a) if  $a \in \mathcal{N}$  ( $\mathcal{N}$  is the set of name parameters), then  $\varrho(a) \in U$ ;
  - (b) if  $p \in \mathcal{P}(0)$  ( $\mathcal{P}(0)$  is the set of proposition parameters), then  $\varrho(p) \in \{0, 1\}$ ;
  - (c) if  $P \in \mathcal{P}(n)$  ( $n = 1, 2, \dots$ ) ( $\mathcal{P}(n)$  is the set of  $n$ -argument predicate parameters), then  $\varrho(P) \subseteq U^{(n)}$ , where  $U^{(1)} = U$ ,  $U^{(n+1)} = U^{(n)} \times U$ ;
  - (d) if  $T \in \mathcal{T}$ , then  $\varrho(T) \neq \emptyset$ .

In order to give semantic rules we only need the notions of assignment and modified assignment:

**Definition 6.**

1. Function  $v$  is an assignment relying on the interpretation  $\langle U, \varrho \rangle$  if  $v : Var \rightarrow U$ .
2. Let  $v$  be an assignment relying on the interpretation  $\langle U, \varrho \rangle$ ,  $x \in Var$  and  $u \in U$ . Then  $v[x : u]$  is a modified assignment of  $v$ , if  $v[x : u]$  is an assignment,  $v[x : u](y) = v(y)$  if  $x \neq y$ , and  $v[x : u](x) = u$ .

The semantic values of tools (the members of set  $\mathcal{T}$ ) determine a general (maybe partial) approximation space for the given interpretation. The generated approximation space is logically relevant in the sense, that it gives the lower and upper approximations (what is more, the different generalized characteristic functions) of any predicate  $P$  to be taken into consideration in the definition of semantic rules.

**Definition 7.** Let  $L$  be a language of PFoL with the set  $\mathcal{T}$  of tools and  $\langle U, \varrho \rangle$  be a tool-based interpretation of  $L$ .

The ordered 5-tuple

$$\text{GAS}(\mathcal{T}) = \langle \mathcal{PR}(U), \mathfrak{B}(\mathcal{T}), \mathfrak{D}_{\mathfrak{B}(\mathcal{T})}, l, u \rangle$$

is a logically relevant general partial approximation space generated by set  $\mathcal{T}$  of tools with respect to the interpretation  $\langle U, \varrho \rangle$  if

1.  $\mathcal{PR}(U) = \bigcup_{n=1}^{\infty} U^{(n)}$ ;
2.  $\mathfrak{B}(\mathcal{T}) = \bigcup_{n=1}^{\infty} \mathfrak{B}_n(\mathcal{T})$  where  $\mathfrak{B}_n(\mathcal{T}) = \{\varrho(T_1) \times \dots \times \varrho(T_n) \mid T_i \in \mathcal{T}\}$ ;

The semantic values of tools (given by the interpretation) generate the set  $\mathfrak{B}(\mathcal{T})$ . It contains those sets by which the semantic value of any predicate parameter is approximated.

In the semantics of PFoL, the semantic value of an expression depends on a given interpretation, and a given logically relevant general partial approximation space generated by set of tools with respect to the interpretation. For the sake of simplicity, we use a null entity to represent the partiality of semantic rules. We use number 0 for falsity, number 1 for truth, numbers greater than 0 and less than 1 for true degree and number 2 for the null entity. In many cases, five possibly different semantic values can be given: rough, optimistic, average, pessimistic, and crisp ones. The forms of semantic rules are similar in different cases and so the superscript  $*$  can be used to denote one of them ( $*$   $\in \{r, o, a, p, c\}$ ). The semantic value of an expression  $A$  is denoted by  $\llbracket A \rrbracket_v^*$ .

The most important semantic rules are the following:

1. If  $P \in P(n)$  ( $n \neq 0$ ), i.e.  $P$  is an  $n$ -argument predicate parameter and  $t_1, t_2, \dots, t_n \in Term$ , then
 
$$\begin{aligned} \llbracket P(t_1, \dots, t_n) \rrbracket_v^* &= \mu^*(\varrho(P), \langle \llbracket t_1 \rrbracket_v^*, \dots, \llbracket t_n \rrbracket_v^* \rangle); \\ \llbracket P^\uparrow(t_1, \dots, t_n) \rrbracket_v^* &= \mu^*(u(\varrho(P)), \langle \llbracket t_1 \rrbracket_v^*, \dots, \llbracket t_n \rrbracket_v^* \rangle); \\ \llbracket P^\downarrow(t_1, \dots, t_n) \rrbracket_v^* &= \mu^*(l(\varrho(P)), \langle \llbracket t_1 \rrbracket_v^*, \dots, \llbracket t_n \rrbracket_v^* \rangle). \end{aligned}$$
2. If  $A \in Form$ , then
 
$$\llbracket \neg A \rrbracket_v^* = \begin{cases} 2 & \text{if } \llbracket A \rrbracket_v^* = 2 \\ 1 - \llbracket A \rrbracket_v^* & \text{otherwise} \end{cases}$$
3. If  $A, B \in Form$ , then
 
$$\begin{aligned} \llbracket (A \wedge B) \rrbracket_v^* &= \begin{cases} 2 & \text{if } \llbracket A \rrbracket_v^* = 2, \text{ or } \llbracket B \rrbracket_v^* = 2; \\ \min\{\llbracket A \rrbracket_v^*, \llbracket B \rrbracket_v^*\} & \text{otherwise} \end{cases} \\ \llbracket (A \vee B) \rrbracket_v^* &= \begin{cases} 2 & \text{if } \llbracket A \rrbracket_v^* = 2, \text{ or } \llbracket B \rrbracket_v^* = 2; \\ \max\{\llbracket A \rrbracket_v^*, \llbracket B \rrbracket_v^*\} & \text{otherwise} \end{cases} \\ \llbracket (A \supset B) \rrbracket_v^* &= \begin{cases} 2 & \text{if } \llbracket A \rrbracket_v^* = 2, \text{ or } \llbracket B \rrbracket_v^* = 2; \\ \max\{\llbracket \neg A \rrbracket_v^*, \llbracket B \rrbracket_v^*\} & \text{otherwise} \end{cases} \end{aligned}$$

4. If  $A \in Form, x \in Var$  and  $\mathcal{V}(A) = \left\{ \llbracket A \rrbracket_{v[x:u]}^* \mid u \in U, \llbracket A \rrbracket_{v[x:u]}^* \neq 2 \right\}$ , then
- $$\llbracket \forall x A \rrbracket_v^* = \begin{cases} 2 & \text{if } \mathcal{V}(A) = \emptyset, \\ \min\{\mathcal{V}(A)\} & \text{otherwise} \end{cases}$$
- $$\llbracket \exists x A \rrbracket_v^* = \begin{cases} 2 & \text{if } \mathcal{V}(A) = \emptyset, \\ \max\{\mathcal{V}(A)\} & \text{otherwise} \end{cases}$$

From the logical point of view, flexibility is the main advantage of defined general logical framework: according to the first semantic rule different characteristic functions can be taken into consideration to determine the semantic value of an atomic formula. One of its consequences is that the semantic value of any formula can be determined by the help of different general characteristic functions and so (as it can be seen in the definition of models and consequence relations) the user can decide which general characteristic function is applied for a formula.

### 4.3 Central Semantic Notions

The notion of models plays a fundamental role in the semantic definition of the consequence relation. The standard notion of models has to be generalized:

1. In order to take into consideration different general characteristic functions for formulae, models are defined for ordered  $n$ -tuples of formulae (not for the set of formulae). Decision types determine which general characteristic function is used for a formula.
2. Apart from the null entity the values of general characteristic functions are between 0 and 1. Therefore only a parametrized notion of models is possible.

**Definition 8.** Let  $L$  be a language of PFoL with the set  $\mathcal{T}$  of tools, and  $\Gamma = \langle A_1, A_2, \dots, A_n \rangle$  be an ordered  $n$ -tuple of formulae ( $A_1, A_2, \dots, A_n \in Form$ ).

1. The ordered  $n$ -tuple  $\Delta = \langle \delta_1, \dots, \delta_n \rangle$  is a decision type of  $\Gamma$  if  $\delta_1, \dots, \delta_n \in \{r, o, a, p, c\}$ .
2. Let  $\Delta = \langle \delta_1, \dots, \delta_n \rangle$  be a decision type  $\Gamma$ . Then  $\langle U, \varrho, v \rangle$  is a  $\Delta$ -type model of  $\Gamma$  with parameter  $\alpha$  ( $0 < \alpha \leq 1$ ), if
  - (a)  $\langle U, \varrho \rangle$  is an interpretation of  $L$ ;  $v$  is an assignment relying on  $\langle U, \varrho \rangle$ ;
  - (b)  $\llbracket A_i \rrbracket_v^{\delta_i} \neq 2$  for all  $i$  ( $i = 1, 2, \dots, n$ )
  - (c)  $\llbracket A_i \rrbracket_v^{\delta_i} \geq \alpha$  for all  $i$  ( $i = 1, 2, \dots, n$ ).

**Definition 9.** Let  $L$  be a language of PFoL with the set  $\mathcal{T}$  of tools,  $\Gamma = \langle A_1, \dots, A_n \rangle$  be an ordered  $n$ -tuple of formulae ( $A_1, \dots, A_n \in Form$ ) and  $B \in Form$  be a formula.

1.  $\Delta \rightarrow \delta$  is a decision driven consequence type from  $\Gamma$  to  $B$  if  $\Delta$  is a decision type of  $\Gamma$  and  $\delta$  is a decision type of  $\{B\}$ .
2. Let  $\Delta \rightarrow \delta$  is a decision driven consequence type from  $\Gamma$  to  $B$ .  $B$  is a parametrized consequence of  $\Gamma$  driven by  $\Delta \rightarrow \delta$  with the parameter pair  $\langle \alpha, \beta \rangle$  if all  $\Delta$ -type models of  $\Gamma$  with the parameter  $\alpha$  are  $\delta$ -type models of  $B$  with the parameter  $\beta$  ( $\Gamma \models_{\Delta \rightarrow \delta}^{(\alpha, \beta)} B$ )

The introduced notion of decision-driven consequence relations may be useful in many practical cases. It can be determined

1. how premisses and conclusions are evaluated: all generalized characteristic functions can be used in the evaluation process;
2. the expected level of truth of premisses and conclusions can be determined.

In this paper there is not enough space to give all proved theorems. In [6,3,4,5] the most important properties are investigated.

For example in [4], Aristotle's syllogisms of the first figure are investigated. These consequence relations represent the most typical usages of quantified propositions containing one-argument predicates. A general observation: in any decision driven consequence of Aristotle's syllogisms of the first figure the parameters which give the level of truth of premisses have to be greater than  $1/2$ , therefore there is no valid consequence relation with parameters less than or equal to  $1/2$ . It means that in order to say something about the conclusion the premisses have to be closer to truth than to falsity. Proved theorems show that the validity of the consequence relation (in the case of optimistic, average and pessimistic rough membership measurements) requires that:

- Barbara and Celarent syllogisms:
  1. evaluate premisses and conclusion by using the same (optimistic, average or pessimistic) membership function and so there is no real freedom for making different decisions concerning two premisses and the conclusion;
  2. suppose that the level of truth of premisses is greater than  $1/2$ , i.e. the premisses have to be closer to truth than falsity.
- Darii syllogism:
  1. the decision for the first premiss can not be stronger than the decision for the second one;
  2. the decisions for the first premiss and the conclusion have to be the same;
  3. the first premiss plays a more important role than the second one.
- Ferio syllogism:
  1. the decision for the first premiss can not be stronger than the decision for the second one;
  2. the decisions for the second premiss and the conclusion have to be the same;
  3. the second premiss plays a more important role than the first one.

## 5 Conclusion and Future Work

The main result of the paper is to give a (very general) partial first-order logic using generalized characteristic functions received from different systems of rough sets. An important advantage of the logical system is that in one integrated system any user can determine

- the evaluation processes of premisses and conclusions;
- the expected level of the truth of premisses and conclusions.

The next step is to use the introduced logical system in practice to solve some problems in data mining connected with incomplete/uncertain information.

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