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# Ultrametric Smale's $\alpha$ -theory

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## Abstract

We present a version of Smale's  $\alpha$ -theory for ultrametric fields, such as the  $p$ -adics and their extensions, which gives us a multivariate version of Hensel's lemma.

Hensel's lemma [4, §3.4] gives us sufficient condition for lifting roots mod  $p^k$  to roots in  $\mathbb{Z}_p$ . Alternatevely, Hensel's lemma gives us sufficient conditions for Newton's method convergence towards an approximate root. Unfortunately, in the multivariate setting, versions of Hensel's lemma are scarce [2]. However, in the real/complex world, Smale's  $\alpha$ -theory [3] gives us a clean sufficient criterion for deciding if Newton's method will converge quadratically. In the  $p$ -adic setting, Breiding [1] proved a version of the  $\gamma$ -theorem, but he didn't provide a full  $\alpha$ -theory. In this short communication, we provide an ultrametric version of Smale's  $\alpha$ -theory for square systems—initially presented as an appendix in [5]—, together with an easy proof.

In what follows, and for simplicity<sup>1</sup>,  $\mathbb{F}$  is a non-archimedean complete field of characteristic zero with (ultrametric) absolute value  $|\cdot|$  and  $\mathcal{P}_{n,d}[n]$  the set of polynomial maps  $f : \mathbb{F}^n \rightarrow \mathbb{F}^n$  where  $f_i$  is of degree  $d_i$ . In this setting, we will consider on  $\mathbb{F}^n$  the ultranorm given by  $\|x\| := \max\{|x_1|, \dots, |x_n|\}$ , its associated distance  $\text{dist}(x, y) := \|x - y\|$ , and on  $k$ -multilinear maps  $A : (\mathbb{F}^n)^k \rightarrow \mathbb{F}^q$  the induced ultranorm, which is given by

$$\|A\| := \sup_{v_1, \dots, v_k \neq 0} \frac{\|A(v_1, \dots, v_k)\|}{\|v_1\| \cdots \|v_k\|}. \quad (1)$$

In this context, we can define Smale's parameters as follows. Below  $D_x f$  denotes the differential map of  $f$  at  $x$  and  $D_x^k f$  the  $k$ -linear map induced by the  $k$ th order partial derivatives of  $f$  at  $x$ .

**Definition 1** (Smale's parameters). Let  $f \in \mathcal{P}_{n,d}[n]$  and  $x \in \mathbb{F}^n$ . We define the following:

- (a) *Smale's  $\alpha$* :  $\alpha(f, x) := \beta(f, x)\gamma(f, x)$ , if  $D_x f$  is non-singular, and  $\alpha(f, x) := \infty$ , otherwise.
- (b) *Smale's  $\beta$* :  $\beta(f, x) := \|D_x f^{-1} f(x)\|$ , if  $D_x f$  is non-singular, and  $\alpha(f, x) := \infty$ , otherwise.
- (c) *Smale's  $\gamma$* :  $\gamma(f, x) := \sup_{k \geq 2} \left\| D_x f^{-1} \frac{D_x^k f}{k!} \right\|^{\frac{1}{k-1}}$ , if  $D_x f$  is invertible, and  $\gamma(f, x) := \infty$ , otherwise.

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<sup>1</sup>We focus on characteristic zero and polynomials to avoid technical details related to Taylor series.

If  $D_x f$  is invertible, then the *Newton operator*,

$$N_f : x \mapsto x - D_x f^{-1} f(x),$$

is well-defined at  $x$ . For a point  $x$ , the *Newton sequence* is the sequence  $\{N_f^k(x)\}$ . Note that this sequence is well-defined (i.e.,  $N_f^k(x)$  makes sense for all  $k$ ) if and only if  $D_{N_x^k} f$  is invertible at every  $k$ . Also note that

$$\beta(f, x) = \|x - N_f(x)\|,$$

so  $\beta$  measures the length of a Newton step.

**Theorem 1** (Ultrametric  $\alpha/\gamma$ -theorem). *Let  $f \in \mathcal{P}_{n,d}[n]$  and  $x \in \mathbb{F}^n$ . Then the following are equivalent:*

$$(\alpha) \alpha(f, x) < 1 \quad \text{and} \quad (\gamma) \text{dist}(x, f^{-1}(0)) < 1/\gamma(f, x)$$

Moreover, if any of the above equivalent conditions holds, then the Newton sequence,  $\{N_f^k(x)\}$ , is well-defined and it converges quadratically to a non-singular zero  $\zeta$  of  $f$ . More specifically, for all  $k$ , the following holds:

$$(a) \alpha(f, N_f^k(x)) \leq \alpha(f, x)^{2^k}. \quad (b) \beta(f, N_f^k(x)) \leq \beta(f, x) \alpha(f, x)^{2^k - 1}. \quad (c) \gamma(f, N_f^k(x)) \leq \gamma(f, x). \\ (Q) \|N_f^k(x) - \zeta\| = \beta(f, N_f^k(x)) \leq \alpha(f, x)^{2^k - 1} \beta(f, x) < \alpha(f, x)^{2^k} / \gamma(f, x).$$

In the univariate  $p$ -adic setting, we have that for  $f \in \mathbb{Z}_p[X]$  and  $x \in \mathbb{Z}_p$ ,

$$\gamma(f, x) \leq 1/|f'(x)|,$$

since  $|1/k! f^{(k)}(x)| \leq 1$  and  $|f'(x)| \leq 1$ . Therefore we can see that the condition  $|f(x)| < |f'(x)|^2$  of Hensel's lemma implies  $\alpha(f, x) < 1$  for a  $p$ -adic integer polynomial. In this way, we can see that Theorem 1 generalizes Hensel's lemma to the multivariate case.

Moreover, in the univariate setting, we can show the following proposition which gives a precise characterization of Smale's  $\gamma$  in the ultrametric setting as the separation between 'complex' roots—not only a bound as it happens in the complex/real setting.

**Proposition 2** (Ultrametric separation theorem for  $\gamma$ ). [5, Theorem 3.15] *Fix an algebraic closure  $\overline{\mathbb{F}}$  of  $\mathbb{F}$  with the corresponding extension of the ultranorm. Let  $f \in \mathbb{F}[X]$  and  $\zeta \in \overline{\mathbb{F}}$  a simple root, then*

$$\frac{1}{\gamma(f, \zeta)} = \text{dist}(\zeta, f^{-1}(0) \setminus \{\zeta\}). \quad \square$$

## Proof of Theorem 1

The proof of the theorem relies in the following three lemmas, stated for  $f \in \mathcal{P}_{n,d}[n]$  and  $x, y \in \mathbb{F}^n$ .

**Lemma 3.** *If  $\gamma(f, x)\|x - y\| < 1$ , then  $D_y f$  is invertible and  $\|D_y f^{-1} D_x f\| = 1$ .*

**Lemma 4** (Variations of Smale's parameters). *If  $\rho := \gamma(f, x)\|x - y\| < 1$ , then:*

$$(a) \alpha(f, y) \leq \max\{\alpha(f, x), \rho\}. \quad (b) \beta(f, y) \leq \max\{\beta(f, x), \|y - x\|\}. \quad (c) \gamma(f, y) = \gamma(f, x).$$

Moreover, if  $\|y - x\| < \beta(f, x)$ , all are equalities.

**Lemma 5** (Variations along Newton step). *If  $\alpha(f, x) < 1$ , then:*

$$(a) \alpha(f, N_f(x)) \leq \alpha(f, x)^2. \quad (b) \beta(f, N_f(x)) \leq \alpha(f, x)\beta(f, x). \quad (c) \gamma(f, N_f(x)) = \gamma(f, x).$$

*In particular,  $N_f(N_f(x))$  is well-defined.*

*Proof of Theorem 1.* If  $\alpha(f, x) < 1$ , then, using induction and Lemma 5, we obtain that (a), (b) and (c) hold. But then the sequence  $\{N_f^k(x)\}$  converges since  $\lim_{k \rightarrow \infty} \|N_f^{k+1}(x) - N_f^k(x)\| = 0$  and so it is a Cauchy sequence. Finally, (Q) follows from noting that for  $l \geq k$

$$\|N_f^l(x) - N_f^k(x)\| \leq \alpha(f, x)^{2^{l-k}} \beta(f, N_f^k(x))$$

and taking infinite sum together with the equality case of the ultrametric inequality. In particular, we have  $\text{dist}(x, f^{-1}(0)) = \|x - \zeta\| = \beta(f, x) < 1/\gamma(f, x)$ . This shows that  $(\alpha)$  implies  $(\gamma)$ .

For the other direction, assume that  $\text{dist}(x, f^{-1}(0)) < 1/\gamma(f, x)$ . Then  $\gamma(f, x)$  is finite, since otherwise  $\text{dist}(x, f^{-1}(0)) < 0$ , which is impossible. Let  $\zeta \in \mathbb{F}^n$  be a zero of  $f$  such that  $\text{dist}(x, \zeta) < 1/\gamma(f, x)$ . Then  $0 = f(\zeta) = f(x) + D_x f(\zeta - x) + \sum_{k=2}^{\infty} \frac{D_x^k f}{k!}(\zeta - x, \dots, \zeta - x)$ , and so

$$-D_x f^{-1} f(x) = \zeta - x + \sum_{k=2}^{\infty} D_x f^{-1} \frac{D_x^k f}{k!}(\zeta - x, \dots, \zeta - x).$$

Now, the higher order terms satisfy that  $\left\| D_x f^{-1} \frac{D_x^k f}{k!}(\zeta - x, \dots, \zeta - x) \right\| \leq (\gamma(f, x) \|\zeta - x\|)^{k-1} \|\zeta - x\| < \|\zeta - z\|$  and so, by the equality case of the ultrametric inequality,  $\beta(f, x) = \|\zeta - x\| < 1/\gamma(f, x)$ , as desired.  $\square$

Now, we prove the auxiliary lemmas 3, 4 and 5

*Proof of Lemma 3.* We have that  $D_x f^{-1} D_y f = \mathbb{I} + \sum_{k=1}^{\infty} D_x f^{-1} \frac{D_x^{k+1} f(y-x, \dots, y-x)}{k!}$ . Now, under the given assumption,  $\left\| D_x f^{-1} \frac{D_x^{k+1} f(y-x, \dots, y-x)}{k!} \right\| \leq (\gamma(f, x) \|y - x\|)^{k-1} < 1$  for  $k \geq 2$ , and so, by the ultrametric inequality,  $\|D_x f^{-1} D_y f - \mathbb{I}\| < 1$ . Therefore  $\sum_{k=0}^{\infty} (\mathbb{I} - D_x f^{-1} D_y f)^k$  converges, and it does so to the inverse of  $D_x f^{-1} D_y f$ . Since, by assumption  $D_x f$  is invertible, so it is  $D_y f$ .

Finally, by the invertibility of  $D_y f$ , we have that  $D_y f^{-1} D_x f = \sum_{k=0}^{\infty} (\mathbb{I} - D_x f^{-1} D_y f)^k$ , and so, by the equality case of the ultrametric inequality,  $\|D_y f^{-1} D_x f\| = 1$ , as desired.  $\square$

*Proof of Lemma 4.* We first prove (c) and then (b). (a) follows from (b) and (c) immediately.

(c) We note that under the given assumption, for  $k \geq 2$ ,

$$\left\| D_x f^{-1} \frac{D_y^k f}{k!} \right\| \leq \gamma(f, x)^{k-1}. \quad (2)$$

For this, we expand the Taylor series of  $\frac{D_y^k f}{k!}$  (with respect  $y$ ) and note that its  $l$ th term is dominated by

$$\gamma(f, x)^{k+l-1} \|y - x\|^l,$$

which, by the ultrametric inequality, gives the above inequality. In this way, for  $k \geq 2$ ,

$$\left\| D_y f^{-1} \frac{D_y^k f}{k!} \right\| \leq \|D_y f^{-1} D_x f\| \left\| D_x f^{-1} \frac{D_y^k f}{k!} \right\| \leq \gamma(f, x)^{k-1}$$

by Lemma 3 and (2). Thus  $\gamma(f, y) \leq \gamma(f, x)$ . Now, due to this, the hypothesis  $\gamma(f, y)\|x - y\| < 1$  holds, and so, by the same argument,  $\gamma(f, x) \leq \gamma(f, y)$ , which is the desired equality.

(b) Arguing as in (c), we can show that

$$\|D_x f^{-1} f(y)\| \leq \max\{\|D_x f^{-1} f(x) + y - x\|, \gamma(f, x)\|y - x\|^2\} \quad (3)$$

by noting that the general term (of the Taylor series of  $D_x f^{-1} f(y)$  with respect  $y$ ) is dominated by  $\gamma(f, x)^{k-1}\|y - x\|^k < \gamma(f, x)\|y - x\|^2$ . Now,  $\beta(f, y) \leq \|D_y f^{-1} D_x f\| \|D_x f^{-1} f(y)\|$ , and so, by Lemma 3 and (3),

$$\beta(f, y) \leq \max\{\|D_x f^{-1} f(x) + y - x\|, \gamma(f, x)\|y - x\|^2\} \leq \max\{\beta(f, x), \|y - x\|\}.$$

For the equality case, note that, by the same argument, we have  $\beta(f, x) \leq \max\{\beta(f, y), \|y - x\|\} = \beta(f, y)$  where the equality on the right-hand side follows from  $\beta(f, x) > \|y - x\|$ .  $\square$

*Proof of Lemma 5.* (a) follows from combining (b) and (c), and (c) from Lemma 4 (c). We only need to show (b). We use equation (3) in the proof of Lemma 4 with  $y = N_f(x)$ . By (3) and Lemma 3,

$$\beta(f, N_f(x)) \leq \max\{\|D_x f^{-1} f(x) + N_f(x) - x\|, \gamma(f, x)\|N_f(x) - x\|^2\}.$$

Now,  $N_f(x) - x = -D_x f^{-1} f(x)$ , so the above becomes  $\beta(f, N_f(x)) \leq \max\{0, \gamma(f, x)\beta(f, x)^2\}$ , which gives the desired claim.  $\square$

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## References

- [1] P. Breiding. On a  $p$ -adic Newton Method. Master’s thesis, Universität Göttingen, 2013.
- [2] Keith Conrad. A multivariate Hensel’s lemma. Manuscript at <https://kconrad.math.uconn.edu/blurbs/gradnumthy/multivarhensel.pdf>.
- [3] J.-P. Dedieu. *Points fixes, zéros et la méthode de Newton*, volume 54 of *Mathématiques & Applications*. Springer, 2006.
- [4] F. Q. Gouvêa.  *$p$ -adic numbers. An Introduction*. Universitext. Springer, 2nd edition, 1997.
- [5] J. Tonelli-Cueto. A  $p$ -adic Descartes solver: the Strassman solver, 3 2022. arXiv:2203.07016.