Ultrametric Smale’s $\alpha$-theory
Jazz G. Suchen, Josué Tonelli-Cueto

To cite this version:
Jazz G. Suchen, Josué Tonelli-Cueto. Ultrametric Smale’s $\alpha$-theory. International Symposium on Symbolic and Algebraic Computation, Jul 2022, Lille, France. ACM Communications of Computer Algebra, 56 (2), pp.56-59, 2022, 10.1145/3572867.3572875. hal-03738191

HAL Id: hal-03738191
https://inria.hal.science/hal-03738191
Submitted on 25 Jul 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Ultrametric Smale’s $\alpha$-theory

Jazz G. Suchen
Institute for Interspecies Studies
Berlin, Germany
jazz.g.suchen@outlook.com

Josué Tonelli-Cueto
Inria Paris & IMJ-PRG
Paris, France
josue.tonelli.cueto@bizkaia.eu

Abstract

We present a version of Smale’s $\alpha$-theory for ultrametric fields, such as the $p$-adics and their extensions, which gives us a multivariate version of Hensel’s lemma.

Hensel’s lemma [4, §3.4] gives us sufficient condition for lifting roots mod $p^k$ to roots in $\mathbb{Z}_p$. Alternatively, Hensel’s lemma gives us sufficient conditions for Newton’s method convergence towards an approximate root. Unfortunately, in the multivariate setting, versions of Hensel’s lemma are scarce [2]. However, in the real/complex world, Smale’s $\alpha$-theory [3] gives us a clean sufficient criterion for deciding if Newton’s method will converge quadratically. In the $p$-adic setting, Breiding [1] proved a version of the $\gamma$-theorem, but he didn’t provide a full $\alpha$-theory. In this short communication, we provide an ultrametric version of Smale’s $\alpha$-theory for square systems—initially presented as an appendix in [5]—, together with an easy proof.

In what follows, and for simplicity\footnote{We focus on characteristic zero and polynomials to avoid technical details related to Taylor series.}, $\mathbb{F}$ is a non-archimedian complete field of characteristic zero with (ultrametric) absolute value $| |$ and $\mathcal{P}_{n,d}[n]$ the set of polynomial maps $f: \mathbb{F}^n \to \mathbb{F}^n$ where $f_i$ is of degree $d_i$. In this setting, we will consider on $\mathbb{F}^n$ the ultranorm given by $\|x\| := \max\{x_1, \ldots, x_n\}$, its associated distance $\text{dist}(x, y) := \|x - y\|$, and on $k$-multilinear maps $A: (\mathbb{F}^n)^k \to \mathbb{F}^q$ the induced ultranorm, which is given by

$$\|A\| := \sup_{v_1, \ldots, v_k \neq 0} \frac{\|A(v_1, \ldots, v_k)\|}{\|v_1\| \cdots \|v_k\|}. \quad (1)$$

In this context, we can define Smale’s parameters as follows. Below $D_x f$ denotes the differential map of $f$ at $x$ and $D_x^k f$ the $k$-linear map induced by the $k$th order partial derivatives of $f$ at $x$.

**Definition 1** (Smale’s parameters). Let $f \in \mathcal{P}_{n,d}[n]$ and $x \in \mathbb{F}^n$. We define the following:

(a) *Smale’s $\alpha$:* $\alpha(f, x) := \beta(f, x) \gamma(f, x)$, if $D_x f$ is non-singular, and $\alpha(f, x) := \infty$, otherwise.

(b) *Smale’s $\beta$:* $\beta(f, x) := \|D_x f^{-1} f(x)\|$, if $D_x f$ is non-singular, and $\alpha(f, x) := \infty$, otherwise.

(c) *Smale’s $\gamma$:* $\gamma(f, x) := \sup_{k \geq 2} \|D_x f^{-1} D_x^k f\|^\frac{1}{k-1}$, if $D_x f$ is invertible, and $\gamma(f, x) := \infty$, otherwise.
If $D_x f$ is invertible, then the Newton operator,

$$N_f : x \mapsto x - D_x f^{-1} f(x),$$

is well-defined at $x$. For a point $x$, the Newton sequence is the sequence $\{N^k_f(x)\}$. Note that this sequence is well-defined (i.e., $N^k_f(x)$ makes sense for all $k$) if and only if $D_{N^k_f(x)} f$ is invertible at every $k$. Also note that

$$\beta(f, x) = \|x - N_f(x)\|,$$

so $\beta$ measures the length of a Newton step.

**Theorem 1** (Ultrametric $\alpha/\gamma$-theorem). Let $f \in \mathcal{P}_{n,d}[n]$ and $x \in \mathbb{F}^n$. Then the following are equivalent:

(a) $\alpha(f, x) < 1$ and (b) $\gamma(f, f^{-1}(0)) < 1/\gamma(f, x)$

Moreover, if any of the above equivalent conditions holds, then the Newton sequence, $\{N^k_f(x)\}$, is well-defined and it converges quadratically to a non-singular zero $\zeta$ of $f$. More specifically, for all $k$, the following holds:

(a) $\alpha(f, N^k_f(x)) \leq \alpha(f, x)2^k$. 
(b) $\beta(f, N^k_f(x)) \leq \beta(f, x)\alpha(f, x)2^{k-1}$. 
(c) $\gamma(f, N^k_f(x)) \leq \gamma(f, x)$.

(Q) $\|N^k_f(x) - \zeta\| = \beta(f, N^k_f(x)) \leq \alpha(f, x)2^{k-1}\beta(f, x) < \alpha(f, x)2^k/\gamma(f, x)$.

In the univariate $p$-adic setting, we have that for $f \in \mathbb{Z}_p[X]$ and $x \in \mathbb{Z}_p$,

$$\gamma(f, x) \leq 1/|f'(x)|,$$

since $|1/k! f^{(k)}(x)| \leq 1$ and $|f'(x)| \leq 1$. Therefore we can see that the condition $|f(x)| < |f'(x)|^2$ of Hensel's lemma implies $\alpha(f, x) < 1$ for a $p$-adic integer polynomial. In this way, we can see that Theorem 1 generalizes Hensel's lemma to the multivariate case.

Moreover, in the univariate setting, we can show the following proposition which gives a precise characterization of Smale’s $\gamma$ in the ultrametric setting as the separation between 'complex' roots—not only a bound as it happens in the complex/real setting.

**Proposition 2** (Ultrametric separation theorem for $\gamma$). [5, Theorem 3.15] Fix an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$ with the corresponding extension of the ultranorm. Let $f \in \mathbb{F}[X]$ and $\zeta \in \overline{\mathbb{F}}$ a simple root, then

$$\frac{1}{\gamma(f, \zeta)} = \text{dist}(\zeta, f^{-1}(0) \setminus \{\zeta\}).$$

\[\square\]

**Proof of Theorem 1**

The proof of the theorem relies in the following three lemmas, stated for $f \in \mathcal{P}_{n,d}[n]$ and $x, y \in \mathbb{F}^n$.

**Lemma 3.** If $\gamma(f, x)\|x - y\| < 1$, then $D_y f$ is invertible and $\|D_y f^{-1} D_x f\| = 1$.

**Lemma 4** (Variations of Smale's parameters). If $\rho := \gamma(f, x)\|x - y\| < 1$, then:

(a) $\alpha(f, y) \leq \max\{\alpha(f, x), \rho\}$. (b) $\beta(f, y) \leq \max\{\beta(f, x), \|y - x\|\}$. (c) $\gamma(f, y) = \gamma(f, x)$.

Moreover, if $\|y - x\| < \beta(f, x)$, all are equalities.
Lemma 5 (Variations along Newton step). If $\alpha(f, x) < 1$, then:

(a) $\alpha(f, N_f(x)) \leq \alpha(f, x)^2$.

(b) $\beta(f, N_f(x)) \leq \alpha(f, x)\beta(f, x)$.

(c) $\gamma(f, N_f(x)) = \gamma(f, x)$.

In particular, $N_f(N_f(x))$ is well-defined.

Proof of Lemma 4. We have that $D_x f$ is invertible, so it is $D_x f^{-1}$.

Proof of Theorem 1. If $\alpha(f, x) < 1$, then, using induction and Lemma 5, we obtain that (a), (b) and (c) hold. But then the sequence $\{N_f(x)\}$ converges since $\lim_{k \to \infty} \|N_{f+1}^k(x) - N_f^k(x)\| = 0$ and so it is a Cauchy sequence. Finally, (Q) follows from noting that for $l \geq k$

$$\|N_f^l(x) - N_f^k(x)\| \leq \alpha(f, x)^{2l-k} \beta(f, N_f^k(x))$$

and taking infinite sum together with the equality case of the ultrametric inequality. In particular, we have $\operatorname{dist}(x, f^{-1}(0)) = \|x - \zeta\| = \beta(f, x) < 1/\gamma(f, x)$. This shows that $(a)$ implies $(\gamma)$.

For the other direction, assume that $\operatorname{dist}(x, f^{-1}(0)) < 1/\gamma(f, x)$. Then $(f, x)$ is finite, since otherwise $\operatorname{dist}(x, f^{-1}(0)) < 0$, which is impossible. Let $\zeta \in \mathbb{F}^n$ be a zero of $f$ such that $\operatorname{dist}(x, \zeta) < 1/\gamma(f, x)$. Then $0 = f(\zeta) = f(x) + D_x f(\zeta - x) + \sum_{k=2}^{\infty} \frac{D^k f}{k!} (\zeta - x, \ldots, \zeta - x)$, and so

$$-D_x f^{-1} f(x) = \zeta - x + \sum_{k=2}^{\infty} D_x f^{-1} \frac{D^k f}{k!} (\zeta - x, \ldots, \zeta - x).$$

Now, the higher order terms satisfy that $\|D_x f^{-1} \frac{D^k f}{k!} (\zeta - x, \ldots, \zeta - x)\| \leq (\gamma(f, x)\|\zeta - x\|)^{k-1} \|\zeta - x\| < \|\zeta - z\|$ and so, by the equality case of the ultrametric inequality, $\beta(f, x) = \|\zeta - x\| < 1/\gamma(f, x)$, as desired.

Now, we prove the auxiliary lemmas 3, 4 and 5.

Proof of Lemma 3. We have that $D_x f^{-1} D_y f = I + \sum_{k=1}^{\infty} D_x f^{-1} \frac{D^{k+1} f(y, y, \ldots, y, x)}{k!}$. Now, under the given assumption, $\left\|D_x f^{-1} \frac{D^{k+1} f(y, y, \ldots, y, x)}{k!}\right\| \leq (\gamma(f, x)\|y - x\|)^{k-1} < 1$ for $k \geq 2$, and so, by the ultrametric inequality, $\|D_x f^{-1} D_y f - I\| < 1$. Therefore $\sum_{k=0}^{\infty} (I - D_x f^{-1} D_y f)^k$ converges, and it does so to the inverse of $D_x f^{-1} D_y f$. Since, by assumption $D_x f$ is invertible, so it is $D_y f$.

Finally, by the invertibility of $D_y f$, we have that $D_y f^{-1} D_x f = \sum_{k=0}^{\infty} (I - D_x f^{-1} D_y f)^k$, and so, by the equality case of the ultrametric inequality, $\|D_y f^{-1} D_x f\| = 1$, as desired.

Proof of Lemma 4. We first prove (c) and then (b). (a) follows from (b) and (c) immediately.

(c) We note that under the given assumption, for $k \geq 2$,

$$\left\|D_x f^{-1} \frac{D^k f}{k!}\right\| \leq \gamma(f, x)^{k-1}. \quad (2)$$

For this, we expand the Taylor series of $\frac{D^k f}{k!}$ (with respect $y$) and note that its $l$th term is dominated by

$$\gamma(f, x)^{k+l-1} \|y - x\|^l,$$

which, by the ultrametric inequality, gives the above inequality. In this way, for $k \geq 2$,

$$\left\|D_y f^{-1} \frac{D^k f}{k!}\right\| \leq \|D_y f^{-1} D_x f\| \left\|D_x f^{-1} \frac{D^k f}{k!}\right\| \leq \gamma(f, x)^{k-1}$$
by Lemma 3 and (2). Thus $\gamma(f, y) \leq \gamma(f, x)$. Now, due to this, the hypothesis $\gamma(f, y)\|x - y\| < 1$ holds, and so, by the same argument, $\gamma(f, x) \leq \gamma(f, y)$, which is the desired equality.

(b) Arguing as in (c), we can show that

$$\beta(f, y) \leq \max\{\|D_x f^{-1} f(x) + y - x\|, \gamma(f, x)\|y - x\|^2\}$$

by noting that the general term (of the Taylor series of $D_x f^{-1} f(y)$ with respect $y$) is dominated by $\gamma(f, x)^{k-1}\|y - x\|^k < \gamma(f, x)\|y - x\|^2$. Now, $\beta(f, y) \leq \|D_y f^{-1} D_x f||D_x f^{-1} f(y)||$, and so, by Lemma 3 and (3),

$$\beta(f, y) \leq \max\{\|D_x f^{-1} f(x) + y - x\|, \gamma(f, x)\|y - x\|^2\} \leq \max\{\beta(f, x), \|y - x\|\}.$$

For the equality case, note that, by the same argument, we have $\beta(f, x) \leq \max\{\beta(f, y), \|y - x\]\} = \beta(f, y)$ where the equality on the right-hand side follows from $\beta(f, x) > \|y - x\|$. \hfill \qed

**Proof of Lemma 5.** (a) follows from combining (b) and (c), and (c) from Lemma 4 (c). We only need to show (b). We use equation (3) in the proof of Lemma 4 with $y = N_f(x)$. By (3) and Lemma 3,

$$\beta(f, N_f(x)) \leq \max\{\|D_x f^{-1} f(x) + N_f(x) - x\|, \gamma(f, x)\|N_f(x) - x\|^2\}.$$

Now, $N_f(x) - x = -D_x f^{-1} f(x)$, so the above becomes $\beta(f, N_f(x)) \leq \max\{0, \gamma(f, x)\beta(f, x)^2\}$, which gives the desired claim. \hfill \qed

**Acknowledgements**

J.G.S. is supported by a Companion Species Research Fellowship funded by the Durlacher Foundation. J.T.-C. is supported by a postdoctoral fellowship of the 2020 “Interaction” program of the Fondation Sciences Mathématiques de Paris, and partially supported by ANR JCJC GALOP (ANR-17-CE40-0009).

J.G.S. and J.T.-C. are thankful to Elias Tsigaridas for useful suggestions, and to Evgenia Lagoda for moral support.

**References**


