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Ultrametric Smale’s α-theory

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Abstract

We present a version of Smale’s α-theory for ultrametric fields, such as the $p$-adics and their extensions, which gives us a multivariate version of Hensel’s lemma.

Hensel’s lemma [4, §3.4] gives us sufficient condition for lifting roots mod $p^k$ to roots in $\mathbb{Z}_p$. Alternatively, Hensel’s lemma gives us sufficient conditions for Newton’s method convergence towards an approximate root. Unfortunately, in the multivariate setting, versions of Hensel’s lemma are scarce [2]. However, in the real/complex world, Smale’s α-theory [3] gives us a clean sufficient criterion for deciding if Newton’s method will converge quadratically. In the $p$-adic setting, Breiding [1] proved a version of the γ-theorem, but he didn’t provide a full α-theory. In this short communication, we provide an ultrametric version of Smale’s α-theory for square systems—initially presented as an appendix in [5]—, together with an easy proof.

In what follows, and for simplicity

$$\|f\| := \max\{x_1, \ldots, x_n\},$$

its associated distance $\text{dist}(x,y) := \|x - y\|$, and on $k$-multilinear maps $A : (\mathbb{F}^n)^k \to \mathbb{F}^q$ the induced ultranorm, which is given by

$$\|A\| := \sup_{v_1,\ldots,v_k \neq 0} \frac{\|A(v_1, \ldots, v_k)\|}{\|v_1\| \cdots \|v_k\|}.$$  \hspace{1cm} (1)

In this context, we can define Smale’s parameters as follows. Below $D_x f$ denotes the differential map of $f$ at $x$ and $D_x^k f$ the $k$-linear map induced by the $k$th order partial derivatives of $f$ at $x$.

**Definition 1** (Smale’s parameters). Let $f \in \mathcal{P}_{n,d}[n]$ and $x \in \mathbb{F}^n$. We define the following:

(a) Smale’s $\alpha$: $\alpha(f, x) := \beta(f, x) \gamma(f, x)$, if $D_x f$ is non-singular, and $\alpha(f, x) := \infty$, otherwise.

(b) Smale’s $\beta$: $\beta(f, x) := \|D_x f^{-1} f(x)\|$, if $D_x f$ is non-singular, and $\alpha(f, x) := \infty$, otherwise.

(c) Smale’s $\gamma$: $\gamma(f, x) := \sup_{k \geq 2} \left\| D_x f^{-1} \frac{D_x f}{x^k} \right\|^{\frac{1}{k-1}}$, if $D_x f$ is invertible, and $\gamma(f, x) := \infty$, otherwise.

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$^1$We focus on characteristic zero and polynomials to avoid technical details related to Taylor series.
If $D_x f$ is invertible, then the Newton operator,
\[ N_f : x \mapsto x - D_x f^{-1} f(x), \]
is well-defined at $x$. For a point $x$, the Newton sequence is the sequence \( \{ N_f^k(x) \} \). Note that this sequence is well-defined (i.e., $N_f^k(x)$ makes sense for all $k$) if and only if $D_{N_f^k(x)} f$ is invertible at every $k$. Also note that
\[ \beta(f, x) = \| x - N_f(x) \|, \]
so $\beta$ measures the length of a Newton step.

**Theorem 1 (Ultrametric $\alpha/\gamma$-theorem).** Let $f \in \mathcal{P}_{n,d}[n]$ and $x \in \mathbb{F}^n$. Then the following are equivalent:

\[ \begin{align*}
(\alpha) & \quad \alpha(f, x) < 1 \quad \text{and} \quad (\gamma) \quad \text{dist}(x, f^{-1}(0)) < 1/\gamma(f, x) \\
\end{align*} \]

Moreover, if any of the above equivalent conditions holds, then the Newton sequence, \( \{ N_f^k(x) \} \), is well-defined and it converges quadratically to a non-singular zero $\zeta$ of $f$. More specifically, for all $k$, the following holds:

\[ \begin{align*}
(a) & \quad \alpha(f, N_f^k(x)) \leq \alpha(f, x)^{2^k}. \\
(b) & \quad \beta(f, N_f^k(x)) \leq \beta(f, x)\alpha(f, x)^{2^k - 1}. \\
(c) & \quad \gamma(f, N_f^k(x)) \leq \gamma(f, x). \\
(Q) & \quad \| N_f^k(x) - \zeta \| = \beta(f, N_f^k(x)) \leq \alpha(f, x)^{2^k - 1}\beta(f, x) < \alpha(f, x)^{2^k}/\gamma(f, x). \\
\end{align*} \]

In the univariate $p$-adic setting, we have that for $f \in \mathbb{Z}_p[X]$ and $x \in \mathbb{Z}_p$,
\[ \gamma(f, x) \leq 1/|f'(x)|, \]
since $|1/k!f^{(k)}(x)| \leq 1$ and $|f'(x)| \leq 1$. Therefore we can see that the condition $|f(x)| < |f'(x)|^2$ of Hensel’s lemma implies $\alpha(f, x) < 1$ for a $p$-adic integer polynomial. In this way, we can see that Theorem 1 generalizes Hensel’s lemma to the multivariate case.

Moreover, in the univariate setting, we can show the following proposition which gives a precise characterization of Smale’s $\gamma$ in the ultrametric setting as the separation between ‘complex’ roots—not only a bound as it happens in the complex/real setting.

**Proposition 2 (Ultrametric separation theorem for $\gamma$).** [5, Theorem 3.15] Fix an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$ with the corresponding extension of the ultranorm. Let $f \in \mathbb{F}[X]$ and $\zeta \in \overline{\mathbb{F}}$ a simple root, then
\[ \frac{1}{\gamma(f, \zeta)} = \text{dist}(\zeta, f^{-1}(0) \setminus \{ \zeta \}). \]

**Proof of Theorem 1**

The proof of the theorem relies in the following three lemmas, stated for $f \in \mathcal{P}_{n,d}[n]$ and $x, y \in \mathbb{F}^n$.

**Lemma 3.** If $\gamma(f, x)\| x - y \| < 1$, then $D_y f$ is invertible and $\| D_y f^{-1} D_x f \| = 1$.

**Lemma 4 (Variations of Smale’s parameters).** If $\rho := \gamma(f, x)\| x - y \| < 1$, then:

\[ \begin{align*}
(a) & \quad \alpha(f, y) \leq \max\{ \alpha(f, x), \rho \}. \\
(b) & \quad \beta(f, y) \leq \max\{ \beta(f, x), \| y - x \| \}. \\
(c) & \quad \gamma(f, y) = \gamma(f, x). \\
\end{align*} \]

Moreover, if $\| y - x \| < \beta(f, x)$, all are equalities.
Lemma 5 (Variations along Newton step). If $\alpha(f, x) < 1$, then:

(a) $\alpha(f, N_f(x)) \leq \alpha(f, x)^2$.
(b) $\beta(f, N_f(x)) \leq \alpha(f, x)\beta(f, x)$.
(c) $\gamma(f, N_f(x)) = \gamma(f, x)$.

In particular, $N_f(N_f(x))$ is well-defined.

Proof of Theorem 1. If $\alpha(f, x) < 1$, then, using induction and Lemma 5, we obtain that (a), (b) and (c) hold. But then the sequence $\{N_f^k(x)\}$ converges since $\lim_{k \to \infty} \|N_f^{k+1}(x) - N_f^k(x)\| = 0$ and so it is a Cauchy sequence. Finally, (Q) follows from noting that for $l \geq k$

$$
\|N_f^k(x) - N_f^k(x)\| \leq \alpha(f, x)^{2l-k} \beta(f, N_f^k(x))
$$

and taking infinite sum together with the equality case of the ultrametric inequality. In particular, we have $\text{dist}(x, f^{-1}(0)) = \|x - \zeta\| = \beta(f, x) < 1/\gamma(f, x)$. This shows that (α) implies (γ).

For the other direction, assume that $\text{dist}(x, f^{-1}(0)) < 1/\gamma(f, x)$. Then $\gamma(f, x)$ is finite, since otherwise $\text{dist}(x, f^{-1}(0)) < 0$, which is impossible. Let $\zeta \in \mathbb{F}_q$ be a zero of $f$ such that $\text{dist}(x, \zeta) < 1/\gamma(f, x)$. Then $0 = f(\zeta) = f(x) + D_x f(\zeta - x) + \sum_{k=2}^{\infty} \frac{D^k f}{k!}(\zeta - x, \ldots, \zeta - x)$, and so

$$
-D_x f^{-1}f(x) = \zeta - x + \sum_{k=2}^{\infty} D_x f^{-1} \frac{D^k f}{k!}(\zeta - x, \ldots, \zeta - x).
$$

Now, the higher order terms satisfy that $\|D_x f^{-1} \frac{D^k f}{k!}(\zeta - x, \ldots, \zeta - x)\| \leq (\gamma(f, x)\|\zeta - x\|)^{k-1} \|\zeta - x\| < \|\zeta - x\|$ and so, by the equality case of the ultrametric inequality, $\beta(f, x) = \|\zeta - x\| < 1/\gamma(f, x)$, as desired. \qed

Now, we prove the auxiliary lemmas 3, 4 and 5.

Proof of Lemma 3. We have that $D_x f^{-1} D_y f = \mathbb{I} + \sum_{k=1}^{\infty} D_x f^{-1} \frac{D^k f}{k!}(y-x, \ldots, y-x)$. Now, under the given assumption, $\|D_x f^{-1} \frac{D^k f}{k!}(y-x, \ldots, y-x)\| \leq (\gamma(f, x)\|y - x\|)^{k-1} < 1$ for $k \geq 2$, and so, by the ultrametric inequality, $\|D_x f^{-1} D_y f - \mathbb{I}\| < 1$. Therefore $\sum_{k=0}^{\infty} (\mathbb{I} - D_x f^{-1} D_y f)^k$ converges, and it does so to the inverse of $D_x f^{-1} D_y f$. Since, by assumption $D_x f$ is invertible, so it is $D_y f$.

Finally, by the invertibility of $D_y f$, we have that $D_y f^{-1} D_x f = \sum_{k=0}^{\infty} (\mathbb{I} - D_x f^{-1} D_y f)^k$, and so, by the equality case of the ultrametric inequality, $\|D_y f^{-1} D_x f\| = 1$, as desired. \qed

Proof of Lemma 4. We first prove (c) and then (b). (a) follows from (b) and (c) immediately. (c) We note that under the given assumption, for $k \geq 2$,

$$
\left\|D_x f^{-1} \frac{D^k f}{k!}\right\| \leq \gamma(f, x)^{k-1}. \quad (2)
$$

For this, we expand the Taylor series of $\frac{D^k f}{k!}$ (with respect $y$) and note that its $l$th term is dominated by

$$
\gamma(f, x)^{k+l-1}\|y - x\|^l,
$$

which, by the ultrametric inequality, gives the above inequality. In this way, for $k \geq 2$,

$$
\left\|D_y f^{-1} \frac{D^k f}{k!}\right\| \leq \|D_y f^{-1} D_x f\| \left\|D_x f^{-1} \frac{D^k f}{k!}\right\| \leq \gamma(f, x)^{k-1}
$$

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by Lemma 3 and (2). Thus \( \gamma(f, y) \leq \gamma(f, x) \). Now, due to this, the hypothesis \( \gamma(f, y)\|x - y\| < 1 \) holds, and so, by the same argument, \( \gamma(f, x) \leq \gamma(f, y) \), which is the desired equality.

(b) Arguing as in (c), we can show that
\[
\|D_x f^{-1} f(y)\| \leq \max\{\|D_x f^{-1} f(x) + y - x\|, \gamma(f, x)\|y - x\|^2\}
\]
by noting that the general term (of the Taylor series of \( D_x f^{-1} f(y) \) with respect \( y \)) is dominated
by \( \gamma(f, x)^{k-1}\|y - x\|^k < \gamma(f, x)\|y - x\|^2 \). Now, \( \beta(f, y) \leq \|D_y f^{-1} D_x f\|\|D_x f^{-1} f(y)\| \), and so, by Lemma 3 and (3),
\[
\beta(f, y) \leq \max\{\|D_x f^{-1} f(x) + y - x\|, \gamma(f, x)\|y - x\|^2\} \leq \max\{\beta(f, x), \|y - x\|^2\}.
\]
For the equality case, note that, by the same argument, we have \( \beta(f, x) \leq \max\{\beta(f, y), \|y - x\|\} = \beta(f, y) \) where the equality on the right-hand side follows from \( \beta(f, x) > \|y - x\| \).

Proof of Lemma 5. (a) follows from combining (b) and (c), and (c) from Lemma 4 (c). We only need to show (b). We use equation (3) in the proof of Lemma 4 with \( y = N_f(x) \). By (3) and Lemma 3,
\[
\beta(f, N_f(x)) \leq \max\{\|D_x f^{-1} f(x) + N_f(x) - x\|, \gamma(f, x)\|N_f(x) - x\|^2\}.
\]
Now, \( N_f(x) - x = -D_x f^{-1} f(x) \), so the above becomes \( \beta(f, N_f(x)) \leq \max\{0, \gamma(f, x)\beta(f, x)^2\} \), which gives the desired claim.

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