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Homogeneous Nonovershooting Stabilizers and Safety Filters Rejecting Matched Disturbances

Andrey Polyakov and Miroslav Krstic

***Abstract*— Non-overshooting stabilization is a form of safe control where the setpoint chosen by the user is at the boundary of the safe set. Exponential non-overshooting stabilization, including suitable extensions to systems with deterministic and stochastic disturbances, has been solved by the second author and his coauthors. In this paper we develop homogeneous feedback laws for fixed-time nonovershooting stabilization for nonlinear systems that are input-output linearizable with a full relative degree, i.e., for systems that are diffeomorphically equivalent to the chain of integrators. These homogeneous feedback laws can also assume the secondary role of 'fixed-time safety filters' (FxTSf filters) which keep the system within the closed safe set for all time but, in the case where the user's nominal control commands approach to the unsafe set, allow the system to reach the boundary of the safe set no later than a desired time that is independent of nominal control and independent of the value of the state at the time the nominal control begins to be overridden.**

I. Introduction

Nonovershooting control is aimed at solving a tracking problem under certain state/output constrains (see, e.g. [36], [17]). Such a control can be employed in a "safety filter" framework to override a potentially unsafe nominal controller [1]. Nonovershooting and safety controllers have been designed for both linear [24], [21], [8] and nonlinear plants [17], [20], [10]. Inspired by robust non-overshooting control under deterministic disturbances in [17], i.e., by stabilization to an equilibrium at the barrier along with input-to-state safety (ISSf), mean-square stabilization of stochastic nonlinear systems to an equilibrium at the barrier, along with a guarantee of non-violation of the barrier in the mean sense, is solved in [19].

In many cases, the corresponding synthesis was based on the so-called control barrier functions [3], [14], which have been extensively used in various control applications such as automotive systems [2], [31] and multi-agent robotics [34], [32]. The exponential control barrier functions were reported in [23] and allowed the use of simple linear tools for a control design. The nonovershooting prescribed-time control has been developed recently [1] for the integrator chain based on a novel time-varying backstepping procedure [33], [16]. Such a control allows safety requirements to be fulfilled for a prescribed time interval $[0, T]$ with $T > 0$. Prescribed-time controllers [33], which have been developed even for PDEs [9], form a special subset of both finite-time [6] and fixed-time [28] control algorithms. The concepts of finite-time and fixed-time barrier functions were introduced in [18] and [10], respectively.

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A. Homogeneous nonovershooting stabilization

Inspired by the mentioned results, this paper deals with generalized homogeneous nonovershooting controller design. The goal is to synthesize feedback laws that ensure finite-time stabilization (FnTS) and fixed-time stabilization (FxTS) without the overshoot of the system's output.

Homogeneity is a dilation symmetry studied in many branches of pure and applied mathematics. For instance, homogeneous differential equations and homogeneous control systems are well studied in the literature [38], [12], [15], [7], [11], [13]. Being similar (in some sense) to linear control systems, they may demonstrate faster convergence [5], better robustness [4] and smaller "peaking effect" [25, Chapter 1]. To the best of the authors' knowledge, a homogeneous nonovershooting control has never been designed even for the integrator chain.

Finite-time stabilization (FnTS) without an overshoot is formulated as the problem of constructing a feedback law (for example, of a homogeneous type) such that, given an initial state $x_0 \in \mathbb{R}^n$, the feedback guarantees that the trajectory of the closed-loop system initiated in x_0 at the time instant $t = 0$ reaches the origin no later than a time instant $T(x_0)$ without overshoot in the first coordinate. If, given arbitrary $T > 0$, the origin is reached by the time T for any x_0, such a non-overshooting feedback is referred to as fixed-time stabilizing (FxTS).

B. Homogeneous safety filters

To prevent overshoots that may result from the user's nominal control, we also design what are commonly referred to as 'safety filters' and have a particularly simple structure for integrator chains [1].

a) Finite- and fixed-time safety: Since the notion of fixed-time barrier functions has already appeared in the literature [10], let us explain what *fixed-time safety* (FxTSf) means to us. While the precise meaning of this property and the weaker *finite-time safety* (FnTSf) property will become clear in the statements of Theorems 3 and 2, respectively, here we only give a descriptive, intuitive definition. We consider control systems subject to a nominal control $u_{\rm nom}$ and equipped with a "safety filter" feedback law $u = \mathcal{F}(u_{\rm nom}, x)$. The safety filter is said to ensure FxTSf with fixed time T if it ensures safety *and* the following holds for all initial conditions inside the safe set: whenever the applied control $\mathcal{F}(u_{\rm nom}(t), x(t))$ is different from the nominal control $u_{\rm nom}(t)$ over an entire time interval of duration of

T, the solution reaches the boundary of the safe set no later than at the end of that time interval.

Clearly, FnTSf and FxTSf impart *less safety* on the system than the conventional exponential and asymptotic safety properties, under which the trajectories remain away from the boundary for all finite times. This is precisely the point as FnTSf/FxTSf filters are less conservative. They allow the nominal performance to be achieved sooner, with less distortion. In the language of the "safety vs. agility" tradeoff [3], the FxTSf filters are the most agile filters available.

A comparison with the *prescribed-time safety* (PTSf) filters in [1] is in order. Both the PTSf and FxTSf filters ensure safety while allowing convergence to the safety boundary in an amount of time that is independent of the initial condition. The difference is that FxTSf are applicable in infinite-time safety operation, whereas PTSf are ideal when a safety prohibition is only of finite duration.

b) Release time and restraint time: The time T figures in both the PTSf and FnTSf/FxTSf approaches. It is important to understand the different roles that T plays in the two approaches. In PTSf, T is the time after which the prohibition of entering the unsafe set is lifted. Hence, T should be understood in PTSf as *release time* (from the safety constraint). In FnTSf/FxTSf, the safety constraint applies in perpetuity—the unsafe set is prohibited for all time. So, in FnTSf/FxTSf, T is the length of a period that can commence at any moment and, if $u_{\rm nom}$ causes itself to be overridden by the safety filter $\mathcal{F}$ over the entire interval, the trajectory will necessarily be permitted to reach the boundary of the unsafe set, but not enter it. Hence in FnTSf/FxTSf, T should be understood as *restraint time* since the safety filter restrains the trajectory away from the boundary for no longer than T. The release time is a property bestowed upon the system by a PTSf safety filter. Likewise, the restraint time is a property bestowed upon the system by the FnTSf/FxTSf safety filter $\mathcal{F}$. In FnTSf the restraint time T may depend on the value of x at the moment the safety override kicks in and on the nominal control $u_{\rm nom}$, whereas in FxTSf the restraint time can be arbitrarily assigned with the filter and is independent of x and $u_{\rm nom}$.

C. Contributions and organization

To summarize, this paper's contributions are the design and the associated stability and safety analyses for

1) homogeneous finite/fixed-time nonovershooting stabilizing controllers;
2) homogeneous finite/fixed-time safety filters;

and for systems with bounded matched disturbances.

The paper is organized as follows. Section II presents the problem statement. Short preliminaries about the homogeneity are given in Section III. A homogeneous nonovershooting controller design and issues of its tuning are discussed in Section IV. Section V is devoted to homogeneous safety filters. As an example, linear, homogeneous and prescribed-time safety filters for the double integrator are compared in Section VI. Section VII presents concluding remarks.

D. Notation

- $\mathbb{R}$ is the field of reals and $\mathbb{R}_+ = \{\alpha \in \mathbb{R} : \alpha \geq 0\}$.
- $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^{\rm T} \in \mathbb{R}^n$ is the i-th element of the canonical Euclidean basis in $\mathbb{R}^n$.
- $\Sigma_- = \{x \in \mathbb{R}^n : e_1^{\rm T} x \leq 0\}$ is the half-space in $\mathbb{R}^n$ that specifies the safety zone of the system.
- ${\rm int} S$ denotes an interior of the set $S \subset \mathbb{R}^n$.
- $P \succ 0$ (resp. $P \prec 0$) means that the symmetric matrix $P \in \mathbb{R}^{n\times n}$ is positive (resp. negative) definite.
- The matrix $P^{\frac{1}{2}} = M$ is such that $M^2 = P$. By definition, $P^{-\frac{1}{2}} = \left(P^{\frac{1}{2}}\right)^{-1}$. Notice, for any matrix $P \succ 0$ there always exists a unique matrix $P^{\frac{1}{2}} \succ 0$.
- $\lambda_{\max}(P)$ (resp. $\lambda_{\min}(P)$) denotes a maximal (resp. minimal) eigenvalue of the symmetric matrix P.
- ${\rm diag}\{p_1, \ldots, p_n\} \in \mathbb{R}^{n\times n}$ denotes a diagonal matrix with the elements p_1, p_2,...,p_n on the main diagonal.

II. Problem statement

Consider the system

$$\dot{x} = Ax + B(u + q), \quad t > 0, \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $A = (0, e_1, e_2, ..., e_{n-1})$, $B = e_n$ and a continuous function q models matched disturbances:

$$-q_1 \leq q \leq q_2, \quad q_1, q_2 \in \mathbb{R}_+. \tag{2}$$

Given $x_0 \in {\rm int}\Sigma_-$ and $T > 0$, we need to design a state feedback control $u = u_{\rm h}(x)$ such that

1) $u_{\rm h} \in C(\mathbb{R}^n \backslash \{\mathbf{0}\}, \mathbb{R})$ is a uniformly bounded function of the state;
2) the closed-loop system is globally uniformly finite-time stable[1] with
$$x(t, x_0) = \mathbf{0}, \quad \forall t \geq T; \tag{3}$$
3) $x(t, x_0) \in \Sigma_-$ for all $t \geq 0$.

Such a controller (if it exists) steers the state vector at zero without overshoot in the first component, i.e., $e_1^{\rm T} x(t, x_0) \leq 0, \forall t \geq 0$. In this paper we design a *homogeneous nonovershooting finite-time controller* $u_{\rm h}$ allowing also a stabilization in a fixed-time $T > 0$ independently of x_0.

III. Preliminaries: Homogeneity

A. Dilation symmetry of functions and vector fields

Homogeneity is a symmetry of a set or a mapping with respect to a group of transformations called dilation (see, e.g. [38], [12], [15], [5], [4], [25]). In this paper we deal only with the following weighted dilations:

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}}, \quad G_{\mathbf{d}} = {\rm diag}\{n - i + 1\}_{i=1}^n, \tag{4}$$

In a more general case, a linear dilation [25] in $\mathbb{R}^n$ is defined by an arbitrary anti-Hurwitz matrix $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$. Recall [15]

[1] A system $\dot{x} = f(x), t > 0, x(0) = x_0$ is said to be globally uniformly finite-time stable if it is Lyapunov stable and there exists a locally bounded function $\tilde{T} : \mathbb{R}^n \to [0, +\infty)$ such that $x(t, x_0) = \mathbf{0}, \forall t \geq \tilde{T}(x_0)$ for any solution $x(t, x_0)$ of the system.

that *a vector field* $f:\mathbb{R}^n\to\mathbb{R}^n$ *(a function* $h:\mathbb{R}^n\to\mathbb{R}$*) is said to be* $\mathbf{d}$*-homogeneous of degree* $\mu\in\mathbb{R}^n$ *if*

$$f(\mathbf{d}(s)x)=e^{\mu s}\mathbf{d}(s)f(x),\quad \forall x\in\mathbb{R}^n,\quad \forall s\in\mathbb{R} \tag{5}$$

$$\textit{(resp. } h(\mathbf{d}(s)x)=e^{\mu s}h(x),\quad \forall x\in\mathbb{R}^n,\quad \forall s\in\mathbb{R}\textit{)}. \tag{6}$$

Simple computations show that the linear vector field $x\mapsto Ax$ is $\mathbf{d}$-homogeneous of degree -1:

$$A\mathbf{d}(s)=e^{-s}\mathbf{d}(s)A,\quad \forall s\in\mathbb{R} \tag{7}$$

where the dilation $\mathbf{d}$ is given by (4) and A is defined above.

Homogeneous control system are similar (in some sense) to linear control systems, but they may have some additional useful properties such as better robustness, faster convergence and smaller overshoot [25, Chapter 1]. *Any asymptotically stable homogeneous system*

$$\dot{x}=f(x), t>0,\qquad x(0)=x_0, \tag{8}$$

of negative (positive) degree is globally uniformly finite-time (nearly fixed-time[2]*) stable* [37], [22], [5], where $f\in C(\mathbb{R}^n\backslash\{\mathbf{0}\})$. To guarantee a finite-time stabilization of the system (1), it is sufficient to design an asymptotically stabilizing homogeneous control $u:\mathbb{R}^n\to\mathbb{R}$ such that $u(\mathbf{d}(s)x)=u(x),\forall x\in\mathbb{R}^n,\forall s\in\mathbb{R}$. Several schemes of homogeneous and locally homogeneous control design for the chain of integrators are presented in the literature [7], [5], [4]. In this paper we utilize the technique proposed in [28], since it allows simple rules for control parameters tuning and the settling time adjustment to be applied.

B. Canonical Homogeneous Norm

Inspired by [26] let us define the so-called canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}:\mathbb{R}^n\to\mathbb{R}_+$ as follows $\|\mathbf{0}\|_{\mathbf{d}}=0$ and

$$\|x\|_{\mathbf{d}}=e^{s_x}\quad:\quad \|\mathbf{d}(-s_x)x\|=1,\quad x\neq\mathbf{0}, \tag{9}$$

where $\|x\|=\sqrt{x^{\mathrm{T}}Px}$ is the weighted Euclidean norm,

$$P=P^{\mathrm{T}}\in\mathbb{R}^{n\times n}\quad:\quad PG_{\mathbf{d}}+G_{\mathbf{d}}P\succ 0,\ \ P\succ 0. \tag{10}$$

The latter matrix inequalities guarantee that the linear dilation $\mathbf{d}$ is monotone, i.e., the function $s\mapsto\|\mathbf{d}(s)x\|$ is monotone for any $x\in\mathbb{R}^n$. One can be shown [26] that $\|\cdot\|_{\mathbf{d}}\in C(\mathbb{R}^n)\cap C^1(\mathbb{R}^n\backslash\{\mathbf{0}\})$ and

- $\|\mathbf{d}(s)x\|_{\mathbf{d}}=e^s\|x\|_{\mathbf{d}}$ for all $x\in\mathbb{R}^n$ and all $s\in\mathbb{R}$.
- $\|x\|=1\Leftrightarrow\|x\|_{\mathbf{d}}=1$ and $\|x\|\leq 1\Leftrightarrow\|x\|_{\mathbf{d}}\leq 1$.

Moreover, for $x\neq\mathbf{0}$ we have

$$\frac{\partial\|x\|_{\mathbf{d}}}{\partial x}=\|x\|_{\mathbf{d}}\frac{x^{\mathrm{T}}\mathbf{d}^{\mathrm{T}}(-\ln\|x\|_{\mathbf{d}})P\mathbf{d}(-\ln\|x\|_{\mathbf{d}})}{x^{\mathrm{T}}\mathbf{d}^{\mathrm{T}}(-\ln\|x\|_{\mathbf{d}})PG_{\mathbf{d}}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}. \tag{11}$$

Below we utilize the homogeneous norm $\|x\|_{\mathbf{d}}$ for both control design and stability analysis (as a Lyapunov function). Since the homogeneous norm is defined implicitly, then, to compute $\|x\|_{\mathbf{d}}$ for a given $x\neq\mathbf{0}$, the equation $\|\mathbf{d}(-s_x)x\|=1$ needs to be solved with respect to $s_x\in\mathbb{R}$.

[2]The system is globally uniformly nearly fixed-time stable if it is globally uniformly asymptotically stable and for any $r>0$ there exists $T_r>0$ such that $\|x(t,x_0)\|\leq r,\forall t\geq T,\forall x_0\in\mathbb{R}^n$.

Example 1. For $G_{\mathbf{d}}=\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)$, by definition, $\|x\|_{\mathbf{d}}=V$ is a positive definite solution

$$\left(\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right)^{\mathrm{T}}\begin{pmatrix}V^{-2}&0\\0&V^{-1}\end{pmatrix}P\begin{pmatrix}V^{-2}&0\\0&V^{-1}\end{pmatrix}\left(\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right)=1, \tag{12}$$

where $P=\left(\begin{smallmatrix}p_{11}&p_{12}\\p_{12}&p_{22}\end{smallmatrix}\right)\succ 0$. For $x\neq 0$ the solution always exists and unique if

$$P\succ 0,\quad PG_{\mathbf{d}}+G_{\mathbf{d}}P\succ 0. \tag{13}$$

Hence, the homogeneous norm $\|x\|_{\mathbf{d}}$ is a unique positive root of the quartic equation $V^4-p_{22}x_2^2V^2-2p_{12}x_1x_2V-x_1^2p_{11}=0$, which can be found using the Ferrari formulas (see [29]).

Therefore, for $n=2$ the canonical homogeneous norm $\|x\|_{\mathbf{d}}$ can be calculated explicitly. In other cases, a special numerical algorithm may be utilized in order to compute $\|x\|_{\mathbf{d}}$ in practice, (see, e.g. [28], [27], [35] for more details).

IV. Homogeneous Stabilization without Overshoot

This section presents a two step procedure for a homogeneous nonovershooting control design. First, inspired by [17] we design a linear controller which stabilizes the system exponentially at zero without overshoot. Next, inspired by [25, Chapter 9] we transform ("upgrade") a linear controller to a homogeneous one such that an upper estimate of the settling time for the system with the given initial state $x_0\in\mathrm{int}\Sigma_-$ can be assigned a priori.

A. Linear nonovershoting control design for $q=0$

For $\lambda>0, i=1,\ldots,n-1$, let us introduce the vectors

$$h_1=-e_1^{\mathrm{T}},\quad h_i=-e_1^{\mathrm{T}}(A+\lambda I)^{i-1},\quad i=2,\ldots,n, \tag{14}$$

and consider the linear cone

$$\Omega=\{x\in\mathbb{R}^n: h_ix\geq 0,\ i=1,\ldots,n\}\subset\Sigma_-. \tag{15}$$

The following lemma is a corollary of the results from [17].

Lemma 1. *[29] Let* $q=0$*. The linear feedback*

$$u_{\mathrm{lin}}=Kx,\quad K=h_n(A+\lambda I)=-e_1^{\mathrm{T}}(A+\lambda I)^n, \tag{16}$$

stabilizes exponentially the system (1) *at* $x=0$ *and renders the set* Ω *strictly positively invariant for the closed-loop system* (1)*,* (16)*. Moreover, given initial value* $x_0\in\mathrm{int}\Sigma_-$*, choosing* $\lambda>0$ *such that*

$$\sum_{j=0}^{i-1}C_j^{i-1}e_{i-j}^{\mathrm{T}}x_0\lambda^j\leq 0, C_j^{i-1}=\frac{(i-1)!}{(i-j-1)!j!}, i=2,\ldots,n, \tag{17}$$

guarantees that $x_0\in\Omega$*.*

The proof is inspired by [17]. Considering the following barrier functions $\varphi_i:\mathbb{R}^n\to\mathbb{R}$ defined as

$$\varphi_1=-x_1,\quad \varphi_i=\dot{\varphi}_{i-1}+\lambda\varphi_{i-1},\quad i=2,\ldots,n, \tag{18}$$

we derive

$$\dot{\Phi}=(-\lambda I+A)\Phi, \tag{19}$$

where $\Phi=(\varphi_1,\ldots,\varphi_n)^{\mathrm{T}}$. Since the matrix $-\lambda I+A$ is Metzler, the Φ-system is positive and the set $\{\Phi\in\mathbb{R}^n: \varphi_i\geq 0,\ i=1,\ldots,n\}=\Omega$ is strictly positively invariant.

B. Upgrading linear controller to a homogeneous one

In this subsection we design a finite-time nonovershooting control by means of an "upgrade" (a transformation) of a linear "nonovershooting" feedback to a homogeneous one.

Theorem 1. *Let $K \in \mathbb{R}^{1\times n}$ be the gain of the linear controller* (16) *defined by the formula* (16) *and the distrubance function q satisfy the estimate* (2) *for some $q_1 \geq q_2 \geq 0$. Then*

- *the system of linear matrix inequalities (LMIs)*

$$\begin{cases} P(A+BK)+(A+BK)^{\mathrm{T}}P \prec 0, \\ PG_{\mathbf{d}}+G_{\mathbf{d}}P \succ 0, \quad P \succ 0, \end{cases} \tag{20}$$

  has a solution $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$;
- *for any given $T > 0$ the homogeneous controller*

$$u_{\mathrm{h}}(x) = \tilde{K}\mathbf{d}\left(-\ln\|x/r\|_{\mathbf{d}}\right)x, \quad \tilde{K} = K\mathbf{d}(\tilde{s}), \tag{21}$$

$$\tilde{s} \geq \ln\max\left\{\frac{1}{\left(\rho-\frac{2q_1\sqrt{B^{\mathrm{T}}PB}}{r\varkappa}\right)T}, 1\right\}, r > \frac{2q_1\sqrt{B^{\mathrm{T}}PB}}{\rho\varkappa} \tag{22}$$

  stabilizes the system (1) *at $x = \mathbf{0}$ in a finite time and*

$$\|x_0\| \leq r \quad \Rightarrow \quad x(t,x_0) = \mathbf{0}, \quad \forall t \geq T, \tag{23}$$

  where $\mathbf{d}(s)$ is given by (4)*,*

$$\rho = -\lambda_{\max}\left(Q^{\frac{1}{2}}ZQ^{-\frac{1}{2}}\right) > 0, \tag{24}$$

$$Z = P(A+BK)+(A+BK)^{\mathrm{T}}P \prec 0, \tag{25}$$

$$Q = PG_{\mathbf{d}}+G_{\mathbf{d}}P \succ 0, \tag{26}$$

$$\varkappa = \lambda_{\min}(P^{1/2}G_{\mathbf{d}}P^{-1/2}+P^{-1/2}G_{\mathbf{d}}P^{1/2}), \tag{27}$$

  the homogeneous norm $\|x\|_{\mathbf{d}}$ is induced by the weighted Euclidean norm

$$\|x\| = \sqrt{x^{\mathrm{T}}\mathbf{d}(\tilde{s})P\mathbf{d}(\tilde{s})x}, \quad x \in \mathbb{R}^n; \tag{28}$$

- *if $q_2 = 0$ or $n = 2, r \geq \frac{(1-\xi_1\xi_2)q_2}{\rho\xi_2}$,*

$$\xi_1 = \inf_{y\in\mathbb{R}^2_+ : y^{\mathrm{T}}\tilde{P}y=1} \frac{B^{\mathrm{T}}\tilde{P}y}{y^{\mathrm{T}}\tilde{P}HG_{\mathbf{d}}H^{-1}y} \geq 0, \tag{29}$$

$$\xi_2 = \inf_{y\in\mathbb{R}^2_+ : y^{\mathrm{T}}\tilde{P}y=1} \begin{bmatrix}\lambda & 1\end{bmatrix} y > 0, \quad \tilde{P} = \begin{bmatrix}h_1\\h_2\end{bmatrix}^{-\mathrm{T}} P \begin{bmatrix}h_1\\h_2\end{bmatrix}^{-1}, \tag{30}$$

  then the homogeneous cone

$$\Omega_r = \left\{x \in \mathbb{R}^n : \tilde{h}_i\mathbf{d}(-\ln\|x/r\|_{\mathbf{d}})x \geq 0, i = 1,...,n\right\} \tag{31}$$

  is a positively invariant set for the closed-loop system (1)*,* (21) *and*

$$x_0 \in \Omega \quad \Rightarrow \quad x_0 \in \Omega_r \subset \Sigma_- \tag{32}$$

  provided that $\|x_0/r\|_{\mathbf{d}} \leq e^{\tilde{s}}$, where $\Omega \subset \Sigma_-$ is given by (15)*, $\tilde{h}_i = h_i\mathbf{d}(\tilde{s})$ and h_i are defined by* (14)*.*

Proof: 1) The feasibility of LMI (20) is proven in [29].

2) Let us show that the homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is a Lyapunov function of the closed-loop system (1), (21). Indeed, using the formula (11) and the identities $\mathbf{d}(s)A = e^s\mathbf{d}(s)A, \mathbf{d}(s)B = e^sB, \forall s \in \mathbb{R}$ we derive

$$\begin{aligned} &\frac{d\|x/r\|_{\mathbf{d}}}{dt} \\ &= \frac{\|x/r\|_{\mathbf{d}}(x/r)^{\mathrm{T}}\mathbf{d}(-\ln\|x/r\|_{\mathbf{d}})\mathbf{d}(\tilde{s})P\mathbf{d}(\tilde{s})\mathbf{d}(-\ln\|x/r\|_{\mathbf{d}})(\dot{x}/r)}{(x/r)^{\mathrm{T}}\mathbf{d}(-\ln\|x/r\|_{\mathbf{d}})\mathbf{d}(\tilde{s})P\mathbf{d}(\tilde{s})G_{\mathbf{d}}\mathbf{d}(-\ln\|x/r\|_{\mathbf{d}})(x/r)} \\ &= e^{\tilde{s}}\frac{v^{\mathrm{T}}\mathbf{d}(\tilde{s})P(A+BK)\mathbf{d}(\tilde{s})v+v^{\mathrm{T}}\mathbf{d}(\tilde{s})PBq/r}{v^{\mathrm{T}}\mathbf{d}(\tilde{s})PG_{\mathbf{d}}\mathbf{d}(\tilde{s})v} \\ &\leq -e^{\tilde{s}}\left(\rho-\frac{v^{\mathrm{T}}\mathbf{d}(\tilde{s})PBq/r}{v^{\mathrm{T}}\mathbf{d}(\tilde{s})PG_{\mathbf{d}}\mathbf{d}(\tilde{s})v}\right) \end{aligned} \tag{33}$$

where $v = \mathbf{d}(-\ln\|x/r\|_{\mathbf{d}})x/r$ and the linear matrix inequalities (20) as well as the identity (24) are utilized on the last step.

On the one hand, by definition of the canonical homogeneous norm, we have $v^{\mathrm{T}}Pv = 1$, so using the second matrix inequality from (20) we derive $v^{\mathrm{T}}\mathbf{d}(s)P\mathbf{d}(s)v \geq 1, \forall s \geq 0$. One the other hand, since $v^{\mathrm{T}}\mathbf{d}(\tilde{s})PG_{\mathbf{d}}\mathbf{d}(\tilde{s})v \geq 0.5\varkappa v^{\mathrm{T}}\mathbf{d}(\tilde{s})P\mathbf{d}(\tilde{s})v$ then

$$\frac{v^{\mathrm{T}}\mathbf{d}(\tilde{s})PBq/r}{v^{\mathrm{T}}\mathbf{d}(\tilde{s})PG_{\mathbf{d}}\mathbf{d}(\tilde{s})v} \leq \frac{2q_{\min}}{r\varkappa}\frac{|y^{\mathrm{T}}P^{1/2}B|}{y^{\mathrm{T}}y}$$

where $y = P^{1/2}\mathbf{d}(\tilde{s})v$ and the inequality $y^{\mathrm{T}}y \geq 1$ is utilized on the last step. Therefore, using the Cauchy-Schwartz inequality and the inequality (22) we derive

$$\frac{d\|x/r\|_{\mathbf{d}}}{dt} \leq -e^{\tilde{s}}\left(\rho-\frac{2q_{min}\sqrt{B^{\mathrm{T}}PB}}{r\varkappa}\right) \leq -\frac{1}{T}.$$

The latter means that the closed-loop system (1), (21) is globally uniformly finite-time stable and

$$\|x(t,x_0)/r\|_{\mathbf{d}} \leq \|x_0/r\|_{\mathbf{d}} - \frac{t}{T} \tag{34}$$

as long as $x(t,x_0) \neq \mathbf{0}$. If $\|x\| \leq r$ then $\|x/r\|_{\mathbf{d}} \leq 1$ and $x(t,x_0) = \mathbf{0}$ for all $t \geq T$.

3) Let us consider the vector-valued homogeneous barrier functions $\phi : \mathbb{R}^n \to \mathbb{R}^n$ defined as follows $\phi(x) = H\mathbf{d}(\tilde{s})\mathbf{d}(-\ln\|x/r\|_{\mathbf{d}})x$. We have

$$\begin{aligned} \dot{\phi} = &-H\mathbf{d}(\tilde{s})\Big(\tfrac{d\|x/r\|_{\mathbf{d}}}{dt}\|x/r\|_{\mathbf{d}}^{-1}G_{\mathbf{d}}\mathbf{d}(-\ln\|x/r\|_{\mathbf{d}})x \\ &+\mathbf{d}\big(-\ln\|\tfrac{x}{r}\|_{\mathbf{d}}\big)\Big(Ax+B\tilde{K}\mathbf{d}\big(-\ln\|\tfrac{x}{r}\|_{\mathbf{d}}\big)x+Bq\Big)\Big). \end{aligned} \tag{35}$$

Denoting $\gamma = -e^{-\tilde{s}}\frac{d\|x/r\|_{\mathbf{d}}}{dt}$ and using the identities $\mathbf{d}(s)A = e^s\mathbf{d}(s)A, \mathbf{d}(s)B = e^sB, \forall s \in \mathbb{R}$ we derive

$$\dot{\phi} = \frac{e^{\tilde{s}}}{\|x/r\|_{\mathbf{d}}}H\left((A+BK+\gamma G_{\mathbf{d}})\mathbf{d}(\tilde{s})\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x+Bq\right). \tag{36}$$

Using [29, Corollary 2] and the identity $HB = -B$ we obtain

$$\dot{\phi} = \frac{e^{\tilde{s}}}{\|x/r\|_{\mathbf{d}}}\left(\Pi\phi - Bq\right), \tag{37}$$

where $\Pi = -\lambda I_n + A + \gamma G_{\mathbf{d}} + \gamma\lambda(nI_n - G_{\mathbf{d}})A^{\mathrm{T}}$. Since $\gamma > 0$ and the diagonal matrix $nI_n - G_{\mathbf{d}}$ is non-negative then the matrix Π is a three diagonal Metzler matrix. The system (37) is positive for any $q \leq 0$. For the case $n = 2$ we have

$$\begin{cases} \dot{\phi}_1 = \frac{e^{\tilde{s}}}{\|x/r\|_{\mathbf{d}}}((-\lambda+2\gamma)\phi_1+\phi_2), \\ \dot{\phi}_2 = \frac{e^{\tilde{s}}}{\|x/r\|_{\mathbf{d}}}(\gamma\lambda\phi_1+(-\lambda+\gamma)\phi_2-q). \end{cases} \tag{38}$$

Since $P = \left(\begin{smallmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{smallmatrix}\right)$ satisfies (20) then $p_{12} < 0$, all elements of the matrix $H^{-\mathrm{T}}PH^{-1}$ are positive and $-H^{-T}PB = -H^{-T}PH^{-1}HB = H^{-T}PH^{-1}B \in \mathbb{R}^2_+$. Using the representations (33) and the identity $\phi^{\mathrm{T}}\tilde{P}\phi = r^2$ we derive

$$\gamma \geq \rho - q\frac{\phi^{\mathrm{T}}H^{-\mathrm{T}}PB}{\phi^{\mathrm{T}}H^{-\mathrm{T}}PG_{\mathbf{d}}H^{-1}\phi} \geq \rho + \frac{q\xi_1}{r} \tag{39}$$

provided that ϕ is nonnegative and $q \geq 0$. Taking into account $\phi^{\mathrm{T}}\tilde{P}\phi = r^2$ we conclude

$$\gamma(\lambda\phi_1 + \phi_2) \geq \gamma r\xi_2 \geq \rho r\xi_2 + q\xi_1\xi_2 \quad \forall q : 0 \leq q \leq q_2. \tag{40}$$

Therefore, for $n = 2$ and $r \geq \frac{(1-\xi_1\xi_2)q_2}{\rho\xi_2}$ the system (37) is positive even if $q_2 > 0$. Therefore, the homogeneous cone Ω_r is a strictly positively invariant set of the the closed-loop system (1), (21) and, by construction, $\Omega_r \subset \Sigma_-$.

Finally, since $\tilde{s} \geq 0$ then $\|\mathbf{d}(-\tilde{s})x_0/r\|_{\mathbf{d}} \leq e^{-\tilde{s}}\|x_0/r\|_{\mathbf{d}} \leq 1$ provided that $\|x_0/r\|_{\mathbf{d}} \leq e^{\tilde{s}}$. If $x_0 \in \Omega$ then $h_i x_0 \geq 0, i = 1, \ldots, n$. Taking into account the representation $\phi_i(x_0) = h_i e^{-D_i \ln \|\mathbf{d}(-\tilde{s})x_0\|_{\mathbf{d}}} x_0$ we derive $\phi_i(x_0) \geq 0$ due to [29, Corollary 2], i.e., $x_0 \in \Omega_r$.

Theorem 1 presents a scheme for a transformation ("upgrade") of a linear nonovershooting control a homogeneous one. The homogeneous controller guarantees a finite-time stabilization of the system state at zero without overshoot in the first component. The main advantage of the finite-time controller (21) is the simplicity of the parameters tuning. The parameter r in (21) defines the maximum norm of the initial state x_0 for which the settling time of the system is bounded by the arbitrary fixed number $T > 0$.

Remark 1. The controller (21) is uniformly bounded

$$|u_{\mathrm{h}}(x)|^2 \leq r^2\lambda_{\max}\left(P^{-\frac{1}{2}}K^{\mathrm{T}}KP^{-\frac{1}{2}}\right), \forall x \in \mathbb{R}^n. \tag{41}$$

The positive cone[3] $\Omega \subset \Sigma_-$ defines a set of initial states of the system (1) for which the linear control (16) stabilizes the system without overshoot in the first coordinate. In the case of the homogeneous nonovershooting control (21) such a positively invariant set is the $\mathbf{d}$-homogeneous cone[4] $\Omega_r \subset \Sigma_-$. Since $|u_{\mathrm{lin}}(x)| = |u_{\mathrm{h}}(x)|$ for $\|x/r\|_{\mathbf{d}} = e^{\tilde{s}}$ then linear and homogeneous controllers have the same maximum magnitude on the homogeneous ball

$$B_r = \{x \in \mathbb{R}^n : \|x/r\|_{\mathbf{d}} \leq e^{\tilde{s}}\}. \tag{42}$$

Theorem 1 implies that the positively invariant compact set $\Omega_r \cap B_r$ of the homogeneous control system is larger than the positively invariant set $\Omega \cap B_r$ of the linear control system despite that both controllers have the same magnitude on B_r.

The differential equations (19) and (37) for linear and homogeneous barrier functions are slightly different. In the linear case, the equation has the constant Metzler matrix $-\lambda I_n + A$, while the tridiagonal Metzler matrix Π in the homogeneous case depends on the time derivative of the Lyapunov function $\|x/r\|_{\mathbf{d}}$. This dependence complicates the analysis and design of the homogeneous safety filters.

[3]A set $\mathcal{D} \subset \mathbb{R}^n$ is said to be a positive cone if $x \in \mathcal{D} \Rightarrow e^s x \in \mathcal{D}, \forall s \in \mathbb{R}$.

[4]A set $\mathcal{D} \subset \mathbb{R}^n$ is said to be a $\mathbf{d}$-homogeneous cone if $x \in \mathcal{D} \Rightarrow \mathbf{d}(s)x \in \mathcal{D}, \forall s \in \mathbb{R}$.

Despite the obvious difference of positively invariant sets of linear and homogeneous controllers, their mathematical descriptions in polar homogeneous coordinates are identical provided that we do not care about tuning of the settling time of the homogeneous system, namely, if $\tilde{s} = 0$ and $r = 1$. Indeed, the polar homogeneous coordinates [30] in $\mathbb{R}^n$ are given by the polar radius $\rho = \|x\|_{\mathbf{d}}$ and the unit vector $y = \mathbf{d}(-\ln\|x\|_{\mathbf{d}})x$, which defines the so-called homogeneous projection on the unit sphere. In the linear case we deal with the standard dilation $\tilde{\mathbf{d}}(s) := e^{I_n s}$, so the polar radius is just the Euclidean norm $\tilde{\rho} := \|x\|_{\tilde{\mathbf{d}}} = \|x\|$ and the homogeneous projection is $\tilde{y} := \tilde{\mathbf{d}}(-\ln\|x\|_{\tilde{\mathbf{d}}})x = x/\|x\|$. Hence, in the polar coordinates we have $\Omega = \{(\tilde{\rho}, \tilde{y}) : h_i\tilde{y} \geq 0, \forall i\}$ and $\Omega_r = \{(\rho, y) : h_i y \geq 0, \forall i\}$ provided that $r = 1$ and $\tilde{s} = 0$.

The barrier functions as well as the structure of the system (37) depend on the particular control law. Some interesting homogeneous control algorithms for the integrator chain are introduced in [7], [5], [4]. However, a design of barrier functions and a selection of control parameters allowing a nonovershooting finite-time/fixed-time stabilization based on these algorithms are unclear yet.

V. Homogeneous Safety Filters

A nonovershooting controller can be implemented in practice as the so-called 'safety filter' [1]:

$$u = \min(u_{\mathrm{nom}}, u_{\mathrm{lin}}) \tag{43}$$

where u_{nom} is a nominal controller which may have overshoots and u_{lin} is a non-overshooting linear controller whose gain can be time-invariant as in Lemma 1 or time-varying as in [1]. Such an implementation allows overshoots of the nominal controller u_{nom} to be eliminated. For $n \leq 2$ the same safety filter is applicable in the case of the homogeneous nonovershooting controller (21). For $n \geq 3$ we show that the overshoots of the nominal controller can be eliminated by the homogeneous controller using a modified filter.

Let us define $\Delta_r = +\infty$ for $n \leq 2$ and

$$\Delta_r(x) = \frac{\gamma_r}{\gamma_u} + \min_{i=2,\ldots,n-1} \frac{c_i\phi_i + \phi_{i+1}}{\lambda(i-1)\gamma_u\phi_{i-1}}, \tag{44}$$

$$\gamma_r = \frac{\phi^{\mathrm{T}}\tilde{P}(\lambda I_n - A)\phi}{\phi^{\mathrm{T}}\tilde{P}HG_{\mathbf{d}}H^{-1}\phi} > 0, \quad \forall x \in \mathbb{R}^n \tag{45}$$

$$\gamma_u = \frac{\phi^{\mathrm{T}}\tilde{P}B}{\phi^{\mathrm{T}}\tilde{P}HG_{\mathbf{d}}H^{-1}\phi} \geq 0, \quad \forall x \in \Omega_r \tag{46}$$

where $n \geq 3$, $x \in \operatorname{int}\Omega_r$, $c_i > 0$ are arbitrary constants, $\phi_i = \tilde{h}_i\mathbf{d}(-\ln\|x/r\|_{\mathbf{d}})x$, $i = 1, ..., n$ are the homogeneous barrier functions and $\tilde{P} \succ 0$ is a diagonal matrix defined in the proof of Theorem 1. The positive function Δ_r may have infinite values on the boundary of the set Ω_r:

$$\Delta_r(x) := \liminf_{z \to x} \Delta_r(z) \in (0, +\infty], \; z \in \Omega_r, x \in \partial\Omega_r. \tag{47}$$

Finally, let us define $\Delta_r(x) = +\infty$ for $x \in \mathbb{R}^n \backslash \Omega_r$.

Theorem 2 (Finite-time Safety Filter)**.** *Let* $u_{\mathrm{nom}} : [0, +\infty) \mapsto \mathbb{R}$ *be a continuous nominal control signal and* $u_{\mathrm{h}} : \mathbb{R}^n \mapsto \mathbb{R}$ *be a nonovershooting homogeneous*

feedback defined by Theorem 1 with $P = H^{\mathrm{T}}\tilde{P}H$, $\tilde{P} = \mathrm{diag}\{p_1, ..., p_n\}, p_n = 1$.

Let the safety filter be defined on $\mathbb{R}^n \backslash \{\mathbf{0}\}$ *by the formula*

$$u = \max\{u_{\mathrm{h}} - \Delta_r,\ \min\{u_{\mathrm{nom}}, u_{\mathrm{h}}\}\}. \tag{48}$$

- *The* $\mathbf{d}$*-homogeneous cone* Ω_r *is a strictly positively invariant set of the closed-loop system* (1), (48).
- *If* $x(\tau, x_0) = \mathbf{0}$ *then*

$$u_{\mathrm{nom}}(t) < 0, \forall t \in (\tau, \tau+\varepsilon] \ \Rightarrow \ -x(t, x_0) \in \mathbb{R}^n_+, \tag{49}$$

$$u_{\mathrm{nom}}(t) \geq 0, \forall t \in [\tau, \tau+\varepsilon] \ \Rightarrow \ x(t, x_0) = \mathbf{0}, \tag{50}$$

  where $\varepsilon > 0$ *is a positive number.*
- *For any values of time* $\tau > 0$ *for which*

$$u_{\mathrm{nom}}(t) \geq u_{\mathrm{h}}(x(t, x_0)), \ \ \forall t \in [\tau, \tau+T_0] : x(t, x_0) \neq \mathbf{0}, \tag{51}$$

  it holds that

$$x(\tau + T_0, x_0) = \mathbf{0}, \tag{52}$$

  where $T_0 = \|x(\tau, x_0)/r\|_{\mathbf{d}} T$.

The proof repeats the proof of [29, Theorem 2].

The key result of this theorem is (51), (52), which guarantees that the safety boundary is reached by the "release time" T_0. Note that τ is not necessarily a single time but it may represent multiple times. Likewise, $\tau + T_0$ may represent multiple times. Depending on the nominal control, the safety filter may kick in and out intermittently and, if the safety override persists, the safety boundary will always be reached in some finite time T_0, which is not the same at each of the recurrent occasions of the boundary being reached.

Theorem 2 proves that the safety-filter (43) keeps the system within the set $\Omega_r \subset \Sigma_-$ for all time but, in the case where the user's nominal control u_{nom} commands operation in the unsafe set, allows the system to reach the boundary of the safe set Σ_- no later than a finite-time $T_0 > 0$. The time $T_0 = \|x(\tau, x_0)/r\|_{\mathbf{d}} T$ is unknown a-priori. It depends implicitly on the user's nominal control u_{nom} and the initial state $x_0 \in \Omega_r$. To specify the settling time independently of u_{nom} and x_0, we slightly modify the homogeneous controller (21) making the parameter r dependent of the system trajectory.

Theorem 3 (Fixed-time Safety Filter)**.** *If*

$$r = r(t) := \max\left\{r_{\min}, \max_{\tau \in [0,t]} \|\mathbf{d}(-\tilde{s})x(\tau)\|\right\}, \quad r_{\min} > 0 \tag{53}$$

then under conditions of Theorem 2 the following holds.

- *If* $x_0 \in \Omega$ *then* $x(t, x_0) \subset \Theta_{r(t)} \subset \Sigma_-$, $\forall t \geq 0$, *where*

$$\Theta_r = \Omega_r \cap B_r \tag{54}$$

  and Ω_r, B_r *are given by* (31), (42), *respectively* .
- *If* $x(\tau, x_0) = \mathbf{0}$ *then the implications* (49), (50) *hold.*
- *For any values of time* $\tau > 0$ *for which* $x(\tau, x_0) \in \Theta_{r(\tau)}$ *and*

$$u_{\mathrm{nom}}(t) \geq u_{\mathrm{h}}(x(t, x_0)), \ \ \forall t \in [\tau, \tau+T] : x(t, x_0) \neq \mathbf{0}, \tag{55}$$

  it holds that $x(\tau + T, x_0) = \mathbf{0}$, *where* $T > 0$ *is a parameter of the homogeneous control* (21).

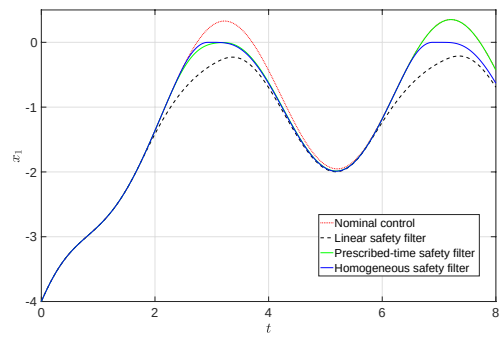

Fig. 1. Trajectories in (t, x_1)-plane without disturbances.

The settling time estimate $T > 0$ of the safety filter (48), (21), (53) is independent of the nominal control u_{nom} and the initial state $x_0 \in \mathbb{R}^n$. Such a 'fixed-time safety' is guaranteed by means of an adaptation of the parameter r of the homogeneous controller (21) (see, the formula (53)). The parameter r depends implicitly on the user's nominal controller u_{nom} and on the initial state $x_0 \in \Omega$ of the system. However, this parameter also specifies the maximum magnitude of the homogeneous controller (see, the formula (41)) as well as a positively invariant set of the closed-loop system (see, the formula (31)). In practice, the maximum value of r should be bounded by some $r_{\max} > 0$ due to physical restrictions to admissible control signals. In this case, the safety filtering with the fixed settling time estimate $T > 0$ may be ensured only in the zone $\Theta_{r_{\max}} \subset \Sigma_-$.

VI. Example: Double Integrator

A. Disturbance-free control

Let $n = 2$. For $x_0 = (-4, 2)$ we select $\lambda = 2$ and $K = (-4 \ \ -4)$. Then $h_1 = (-1 \ \ 0), h_2 = (-2 \ \ -1)$ the matrix

$$P = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix},$$

satisfies the system of LMI (20) for any $\alpha > \frac{1}{2}$. Selecting $\alpha = 0.50125$ we derive $\rho \approx 0.7495$, so for $\bar{q} = \mathbf{0}$ and any $r > 0$ the closed-loop system (1) with the homogeneous nonovershooting controller (21) has the following settling time estimate: $T = \frac{1}{\rho} \approx 1.3342$: $\|x_0\| \leq r \Rightarrow x(t, x_0) = \mathbf{0}, \forall t \geq T$.

A nonovershooting controller for the chain of integrators can assume the role of a "safety filter" of a nominal controller u_{nom} by being implemented as [1]

$$u = \min(u_{\mathrm{nom}}, u_{\mathrm{s}}) \tag{56}$$

where the nominal control u_{nom} may demand overshoots, while the override of such an "unsafe" nominal control is performed using a "safe" control u_{s}, which may be either of the linear kind, denoted as $u_{\mathrm{s}} = u_{\mathrm{lin}}$, or of the homogeneous kind, denoted as $u_{\mathrm{s}} = u_{\mathrm{h}}$).

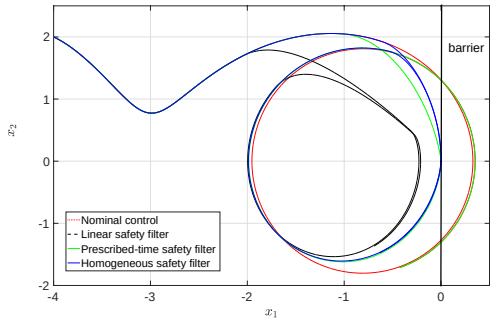

Fig. 2. Trajectories in (x_1, x_2)-plane without disturbances.

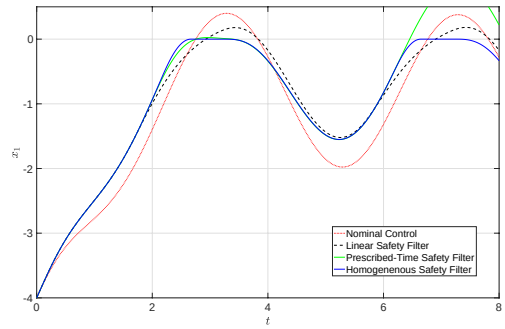

Fig. 3. Trajectories in (t, x_1)-plane with disturbance.

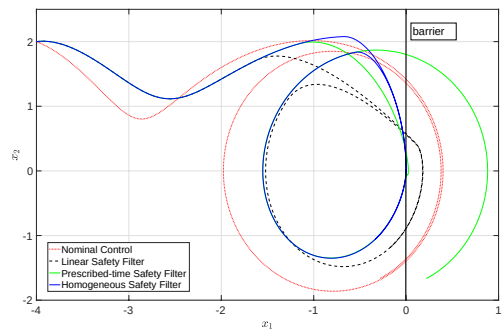

Fig. 4. Trajectories in (x_1, x_2)-plane with disturbance.

To illustrate such "safety filtering" through a simulation, we take a nominal control given by

$$u_{\text{nom}} = -4\left(x_1 + \sin\left(\tfrac{\pi t}{2}\right) + 0.8\right) - 4\left(x_2 + \tfrac{\pi}{2}\cos\left(\tfrac{\pi t}{2}\right)\right), \tag{57}$$

which periodically and persistently attempts to violate the safety condition $x_1 \leq 0$, while also periodically retreating from such an attempt. In this simulation we compare three safety filters: a linear one based on using [17] with (56), a homogeneous one based on the results of this paper, and a prescribed-time (PT) safety filter introduced in [1]. The simulation results are shown in Figures 1 and 2.

But let us first examine Figure 5, which presents the positively invariant sets Ω and Ω_r for, respectively, the linear nonovershooting controller and homogeneous nonovershooting controller with $r = \|x_0\| = \sqrt{x_0^{\mathrm{T}} P x_0} \approx 6.6348$. The sets define the "safety zones" outside of which the corresponding controllers cannot guarantee the absence of overshoots. In the case of the fixed-time safety filter this zone is adaptive and depends of $\max_{\tau\in[0,t]} \|r(\tau)\|$. The larger a zone, the less conservative the override of the nominal controller. Obviously, the homogeneous nonovershooting controller has a larger positively invariant set than the linear controller (at least close to the origin). The PT controller is a time-varying linear feedback [1], so its positively invariant set is not defined.

The numerical simulations in Figures 1 and 2 show that, in comparison with the linear safety filter, both the homogeneous and the PT safety filters perform highly unconservative overrides of the nominal controller. The main difference between the PT and homogeneous safety filters is that the operation time of the PT safety filter terminates after the prohibition on x_1 being positive ends, which is at $T = 4$ seconds in our case, whereas the homogeneous safety filter continues to restrict the operation to $x_1 \leq 0$ for all time.

However, it is remarkable, and evident from Figure 1, that the homogeneous safety filter keeps $x_1(t)$ strictly negative for no longer than the fixed time $T = 4$ sec (and, in fact, for little more than 1 sec), in response to the attempts of the nominal control to make x_1 positive, whereas the more conservative linear safety filter keeps $x_1(t)$ negative all the time, greatly distorting the the resulting (dashed) solution $x_1(t)$ compared to the nominal solution (red).

It is evident from Figure 1 that the safety override with the homogeneous safety filter recurs (twice during the time shown; note the two flat tops of the blue curve). This is the result of the fact that the periodic nominal control keeps attempting to violate the safety boundary and keeps retreating.

B. Control of disturbed system

It is easy to check that all considered filters guarantee safety in the case of non-positive disturbance $q \leq 0$. In the planar case, the homogeneous safety filter is proven to be robust with respect to positive disturbances as well. The numerical simulations illustrate this on Figures 3, 4, where the evolution of the system in the case of the constant disturbance $q = 1.5$ is presented. The nominal controller was simulated for the nominal system (without perturbations). The linear and prescribed-time filters cannot guarantee safety in the perturbed case, while the motion of the system with the homogeneous filter remains safe.

VII. Conclusions

The paper proposes a two-step procedure for a nonovershooting homogeneous control design for the integrator chain. First, we construct a linear nonovershooting controller using a backstepping procedure [17]. Next, we transform ("upgrade") the linear feedback law to a generalized homogeneous one. The obtained homogeneous controller is globally uniformly bounded by a number dependent on control parameters T and r, which defines a convergence time of the system initiated in the ball of radius r. The main advantage of the proposed scheme is the simplicity of the control design and parameter tuning. Moreover, the

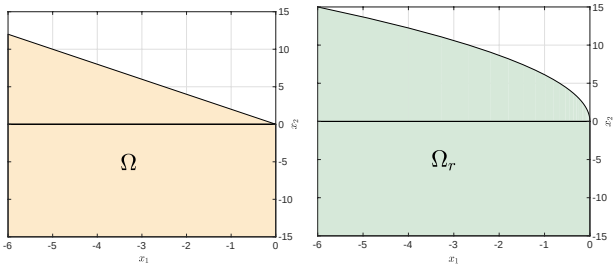

Fig. 5. Positively invariant sets Ω and Ω_r for linear and homogeneous nonovershooting controllers, respectively (without disturbance)

proposed procedure allows a simple upgrade of an existing linear nonovershooting controller to a homogeneous ones. The main disadvantage is the necessity to use a special computational procedure (see, e.g. [28], [35]) to implement the homogeneous controller in practice for high order systems ($n \geq 3$). The positively invariant set of the homogeneous control system is larger than the positively invariant set of the linear control system utilized for the "upgrade". This allows a better (less conservative) overriding of a potentially unsafe nominal controller for the safety filter design. For the planar case ($n = 2$) the homogeneous nonovershooting controllers/safety filters are shown to be robust with respect to matched bound sign-indefinite perturbations. For $n \geq 3$ the robustness is proven only for the sign-definite perturbation ($q \leq 0$).

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