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# A Result on Controllability with Constrained Controls

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## Motivation

Control system with state  $x$  in smooth  $\mathcal{M}$  of dimension  $n$ :

$$(\Sigma^U) : \dot{x} = X^0(x) + \sum_{k=1}^m u_k X^k(x), \quad u = (u_1 \dots u_m) \in U \subset \mathbb{R}^m,$$

**Theorem (Bonnard, 1981, see also Crouch, Brockett...)**

System  $(\Sigma^U)$  is controllable if

- (i) the vector field  $X^0$  is Poisson stable,
- (ii) the convex hull of the set  $U$  is a neighbourhood of 0 in  $\mathbb{R}^m$ ,
- (iii) the family  $\{X^0, X^1, \dots, X^m\}$  is bracket generating.

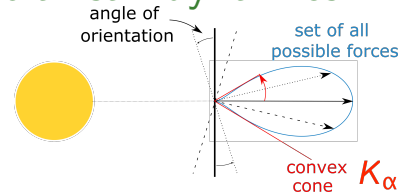
What if  $0 \in \partial(\text{conv } U)$  instead of (ii) ?

Keeping (i), or even: all solutions of  $\dot{x} = X^0(x)$  are periodic, assuming  $U$  convex for simplicity.

What instead of (iii) ?

## Solar sail dynamics

$$\ddot{q} = \underbrace{-\mu \frac{r}{\|r\|^3}}_{\text{Newtonian}} + \underbrace{\varepsilon \vec{\Gamma}(\vec{n})}_{\text{Solar sail}}$$



$\vec{n}$ : orientation of the sail (control). Approximation:

$$\ddot{q} = -\mu \frac{r}{\|r\|^3} + \varepsilon u, \quad u \in K_\alpha \quad \text{cone, } \alpha < \frac{\pi}{2}$$

Gauss equations

( $I$ : orbital elements (1st integrals),  $\varphi \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ ):

$$\dot{I} = \varepsilon \sum_{k=1}^m u_k F^k(I, \varphi), \quad u \in U. \quad \text{Controllable?}$$

$$\dot{\varphi} = \omega(I) + \varepsilon \sum_{k=1}^m u_k f^k(I, \varphi),$$

$U$  convex, bounded.  $0 \in U$ ,  $0 \notin \text{interior of } U$ .

Obviously,

$$\begin{aligned} \dot{I} &= \varepsilon \sum_{k=1}^m u_k F^k(I, \varphi), & u \in U \\ \dot{\varphi} &= \omega(I) + \varepsilon \sum_{k=1}^m u_k f^k(I, \varphi), \end{aligned}$$

is a special case of

$$(\Sigma^U): \quad \dot{x} = X^0(x) + \sum_{k=1}^m u_k X^k(x), \quad u = (u_1 \dots u_m) \in U \subset \mathbb{R}^m,$$

with  $X^0 = \omega \frac{\partial}{\partial \varphi}$ :

for any  $x = (I, \varphi)$ ,

$t \mapsto \exp(tX^0)(x)$  is periodic with period  $T(x) = 2\pi/\omega(x)$ .

## Sufficient condition for controllability

Velocity set:

$$E_0^U(x) = \left\{ \sum_{k=1}^m u_k X^k(x), u \in U \right\} \subset T_x \mathcal{M}.$$

Pullback along  $t \mapsto \exp(tX^0)(x)$ :

$$E_\tau^U(x) = \left\{ \sum_{k=1}^m u_k \left( \exp(-\tau X^0)_* X^k \right)(x), u \in U \right\} \subset T_x \mathcal{M}$$

### Theorem

$(\Sigma^U)$  is controllable if

(i)  $t \mapsto \exp(tX^0)(x)$  periodic with period  $T(x)$  for all  $x$ ,

(ii)  $0 \in U$ ,

(iii)  $\text{conv} \left( \bigcup_{t \in [0, T(x)]} E_t^U(x) \right)$  is a neighbd of 0 in  $T_x \mathcal{M}$  for all  $x$ .

## Controllability in prescribed time

$$(\Sigma^U) : \dot{x} = X^0(x) + \sum_{k=1}^m u_k X^k(x), \quad u \in U \subset \mathbb{R}^m,$$

Accessible sets:

$$\mathcal{A}_T^U(x_0) = \{x(T), \text{ with } t \mapsto (x(t), u(t)) \text{ solution of } (\Sigma^U), x(0) = x_0\}.$$

Let  $t \mapsto \bar{x}(t)$ ,  $[0, t_f] \rightarrow \mathcal{M}$  be a solution of  $\dot{x} = X^0(x)$ .

### Theorem

If  $U$  bounded convex, and  $0 \in U$ , and

$$\text{conv} \left( \bigcup_{t \in [0, t_f]} E_t^U(\bar{x}(0)) \right) \text{ is a neighborhood of } 0 \text{ in } T_{\bar{x}(0)} \mathcal{M} \quad (C)$$

then the set  $\mathcal{A}_{t_f}^{\varepsilon U}(\bar{x}(0))$  is a neighborhood of  $\bar{x}(t_f)$  for any  $\varepsilon > 0$ .

## This is a 1st order condition

If  $U$  is a neighbd of the origin, then (C) is equivalent controllability of the linear approximation, i.e. to

$$\text{Rank}\{\text{ad}_{X_0}^j X^k, k \in \{1, \dots, m\}, j \in \mathbb{N}\} = n$$

(at least in the analytic case).

However non classical when  $U$  is not a neighbd of 0.

## Linear time-varying controllability with constraint

$$\dot{z} = A(t)z + B(t)v, \quad v \in V, \quad (*)$$

$V$  compact,  $0 \in V$ .

$(t_1, t_2) \mapsto \phi(t_2, t_1) \in \mathbb{R}^{d \times d}$  transition matrix.

### Theorem (Schmittendorf Barmish 1980)

$A_{t_0, t_f}^V(0)$  is a neighbourhood of the origin if one of the two following equivalent properties hold.

- For  $p \in (\mathbb{R}^d)^*$  (line vector),

$$\left( p \phi(t_0, s) B(s)v \geq 0 \text{ for all } v \in V, s \in [t_0, t_f] \right) \Rightarrow p = 0. \quad (*)$$

- the convex hull of  $\{\phi(t_0, s) B(s)v, s \in [t_0, t_f], v \in V\}$  is a neighbourhood of the origin in  $\mathbb{R}^d$ .

## Idea of the proof

Linearize along  $t \mapsto \bar{x}(t)$ :

$$\dot{\delta x} = \frac{\partial X^0}{\partial x}(\bar{x}(t)) \delta x + \sum_1^m \delta u_k X^k(\bar{x}(t)) \quad (*)$$

or

$$\begin{aligned} \dot{\xi} &= \sum_1^m \delta u_k \exp(-tX^0)_* X^k(\bar{x}(0)), \\ \delta x(t) &= \exp(tX^0)'(\bar{x}(0)) \xi(t). \end{aligned} \quad (*)$$

with initial condition  $\xi(0) = 0$ .

**Condition (C) is exactly condition for previous linear theorem, hence implies controllability of (\*).**

## Idea of the proof cont'd

$\bar{x}(\cdot)$  corresponds to control zero, on the border of  $U$ , hence it is difficult to use linear approximation.

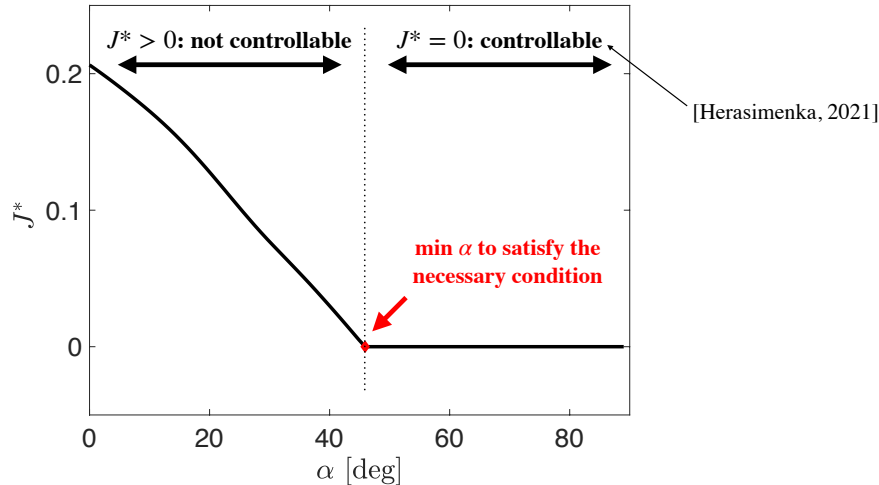
However, one may build a family of  $t \mapsto x^\varepsilon(t)$  and corresponding controls  $t \mapsto u^\varepsilon(t)$  that remain in the interior of  $U$  and satisfy  $x^\varepsilon(0) = \bar{x}(0)$ ,  $x^\varepsilon(t_f) = \bar{x}(t_f)$ .

for the linear approximation *and* for original system.

Then end-point mapping is a submersion around these controls (linear controllability is preserved).

## Application to solar sails

In that case, it is also a necessary condition.  
 Checking condition (C) is more difficult than checking the rank of a family of vector fields: but we manage to set it as convex optimisation problem, computable in that special case.



See e.g. *European Control Conf. 2022*.

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## Extensions?

If other brackets that  $\text{ad}_{X^0}^j X^k$ ,  $j \in \mathbb{N}$  (1st order approximation) are needed for  $\{X^0, \dots, X^m\}$  to be bracket generating, we see no natural extension in terms of a condition of the same type as (C) to replace rank of Lie brackets when  $U$  is not a neighborhood of  $U$ .

In that case,

$$\text{conv} \left( \bigcup_{t \in [0, t_f]} E_t^U(\bar{x}(0)) \right) \subset \text{some nontrivial vector subspace of } T_{\bar{x}(0)}$$

whatever  $U$  is.