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Jean-Baptiste Pomet, Jean-Baptiste Caillau, Lamberto Dell'Elce, Alesia Herasimenka

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A Result on Controllability with Constrained Controls

Jean-Baptiste Pomet
INRIA Sophia Antipolis, France
“Centre Inria d’Université Côte d’Azur”

Common work with:
Jean-Baptiste Caillau, Lamberto Dell’Elce, Alesia Herasimenka

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Motivation

Control system with state x in smooth \mathcal{M} of dimension n :

$$(\Sigma^U) : \dot{x} = X^0(x) + \sum_{k=1}^m u_k X^k(x), \quad u = (u_1 \dots u_m) \in U \subset \mathbb{R}^m,$$

Theorem (Bonnard, 1981, see also Crouch, Brockett...)

System (Σ^U) is controllable if

- (i) the vector field X^0 is Poisson stable,
- (ii) the convex hull of the set U is a neighbourhood of 0 in \mathbb{R}^m ,
- (iii) the family $\{X^0, X^1, \dots, X^m\}$ is bracket generating.

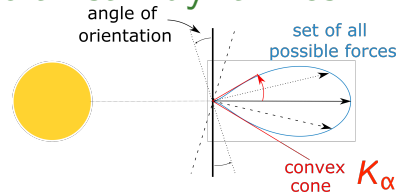
What if $0 \in \partial(\text{conv } U)$ instead of (ii) ?

Keeping (i), or even: all solutions of $\dot{x} = X^0(x)$ are periodic, assuming U convex for simplicity.

What instead of (iii) ?

Solar sail dynamics

$$\ddot{q} = \underbrace{-\mu \frac{r}{\|r\|^3}}_{\text{Newtonian}} + \underbrace{\varepsilon \vec{\Gamma}(\vec{n})}_{\text{Solar sail}}$$



\vec{n} : orientation of the sail (control). Approximation:

$$\ddot{q} = -\mu \frac{r}{\|r\|^3} + \varepsilon u, \quad u \in K_\alpha \quad \text{cone, } \alpha < \frac{\pi}{2}$$

Gauss equations

(I : orbital elements (1st integrals), $\varphi \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$):

$$\dot{I} = \varepsilon \sum_{k=1}^m u_k F^k(I, \varphi), \quad u \in U. \quad \text{Controllable?}$$

$$\dot{\varphi} = \omega(I) + \varepsilon \sum_{k=1}^m u_k f^k(I, \varphi),$$

U convex, bounded. $0 \in U$, $0 \notin \text{interior of } U$.

Obviously,

$$\dot{I} = \varepsilon \sum_{k=1}^m u_k F^k(I, \varphi), \quad u \in U$$

$$\dot{\varphi} = \omega(I) + \varepsilon \sum_{k=1}^m u_k f^k(I, \varphi),$$

is a special case of

$$(\Sigma^U): \quad \dot{x} = X^0(x) + \sum_{k=1}^m u_k X^k(x), \quad u = (u_1 \dots u_m) \in U \subset \mathbb{R}^m,$$

with $X^0 = \omega \frac{\partial}{\partial \varphi}$:

for any $x = (I, \varphi)$,

$t \mapsto \exp(tX^0)(x)$ is periodic with period $T(x) = 2\pi/\omega(x)$.

Sufficient condition for controllability

Velocity set:

$$E_0^U(x) = \left\{ \sum_{k=1}^m u_k X^k(x), u \in U \right\} \subset T_x \mathcal{M}.$$

Pullback along $t \mapsto \exp(tX^0)(x)$:

$$E_\tau^U(x) = \left\{ \sum_{k=1}^m u_k \left(\exp(-\tau X^0)_* X^k \right)(x), u \in U \right\} \subset T_x \mathcal{M}$$

Theorem

(Σ^U) is controllable if

(i) $t \mapsto \exp(tX^0)(x)$ periodic with period $T(x)$ for all x ,

(ii) $0 \in U$,

(iii) $\text{conv} \left(\bigcup_{t \in [0, T(x)]} E_t^U(x) \right)$ is a neighbd of 0 in $T_x \mathcal{M}$ for all x .

Controllability in prescribed time

$$(\Sigma^U) : \dot{x} = X^0(x) + \sum_{k=1}^m u_k X^k(x), \quad u \in U \subset \mathbb{R}^m,$$

Accessible sets:

$$\mathcal{A}_T^U(x_0) = \{x(T), \text{ with } t \mapsto (x(t), u(t)) \text{ solution of } (\Sigma^U), x(0) = x_0\}.$$

Let $t \mapsto \bar{x}(t)$, $[0, t_f] \rightarrow \mathcal{M}$ be a solution of $\dot{x} = X^0(x)$.

Theorem

If U bounded convex, and $0 \in U$, and

$$\text{conv} \left(\bigcup_{t \in [0, t_f]} E_t^U(\bar{x}(0)) \right) \text{ is a neighborhood of } 0 \text{ in } T_{\bar{x}(0)} \mathcal{M} \quad (C)$$

then the set $\mathcal{A}_{t_f}^{\varepsilon U}(\bar{x}(0))$ is a neighborhood of $\bar{x}(t_f)$ for any $\varepsilon > 0$.

This is a 1st order condition

If U is a neighbd of the origin, then (C) is equivalent controllability of the linear approximation, i.e. to

$$\text{Rank}\{\text{ad}_{X_0}^j X^k, k \in \{1, \dots, m\}, j \in \mathbb{N}\} = n$$

(at least in the analytic case).

However non classical when U is not a neighbd of 0.

Linear time-varying controlability with constraint

$$\dot{z} = A(t)z + B(t)v, \quad v \in V, \quad (*)$$

V compact, $0 \in V$.

$(t_1, t_2) \mapsto \phi(t_2, t_1) \in \mathbb{R}^{d \times d}$ transition matrix.

Theorem (Schmittendorf Barmish 1980)

$A_{t_0, t_f}^V(0)$ is a neighbourhood of the origin if one of the two following equivalent properties hold.

- For $p \in (\mathbb{R}^d)^*$ (line vector),

$$\left(p \phi(t_0, s) B(s)v \geq 0 \text{ for all } v \in V, s \in [t_0, t_f] \right) \Rightarrow p = 0. \quad (*)$$

- the convex hull of $\{\phi(t_0, s) B(s)v, s \in [t_0, t_f], v \in V\}$ is a neighbourhood of the origin in \mathbb{R}^d .

Idea of the proof

Linearize along $t \mapsto \bar{x}(t)$:

$$\dot{\delta x} = \frac{\partial X^0}{\partial x}(\bar{x}(t)) \delta x + \sum_1^m \delta u_k X^k(\bar{x}(t)) \quad (*)$$

or

$$\begin{aligned} \dot{\xi} &= \sum_1^m \delta u_k \exp(-tX^0)_* X^k(\bar{x}(0)), \\ \delta x(t) &= \exp(tX^0)'(\bar{x}(0)) \xi(t). \end{aligned} \quad (*)$$

with initial condition $\xi(0) = 0$.

Condition (C) is exactly condition for previous linear theorem, hence implies controllability of (*).

Idea of the proof cont'd

$\bar{x}(\cdot)$ corresponds to control zero, on the border of U , hence it is difficult to use linear approximation.

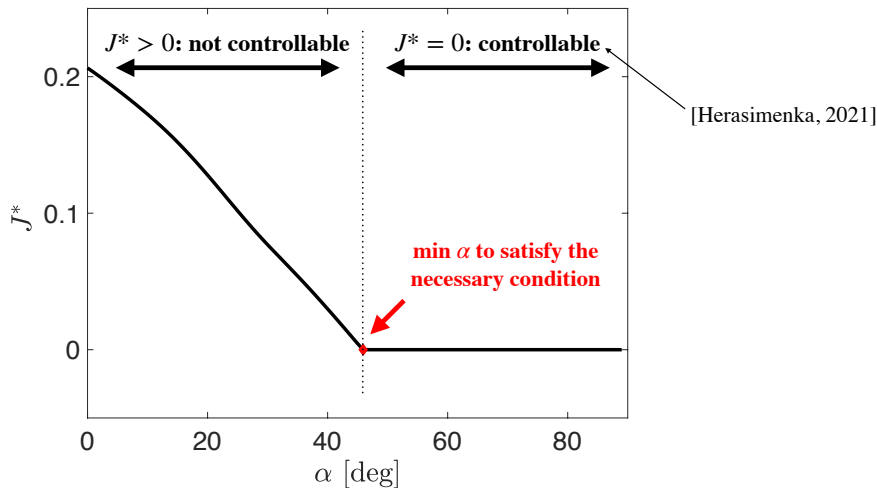
However, one may build a family of $t \mapsto x^\varepsilon(t)$ and corresponding controls $t \mapsto u^\varepsilon(t)$ that remain in the interior of U and satisfy $x^\varepsilon(0) = \bar{x}(0)$, $x^\varepsilon(t_f) = \bar{x}(t_f)$.

for the linear approximation *and* for original system.

Then end-point mapping is a submersion around these controls (linear controllability is preserved).

Application to solar sails

In that case, it is also a necessary condition.
 Checking condition (C) is more difficult than checking the rank of a family of vector fields: but we manage to set it as convex optimisation problem, computable in that special case.



See e.g. *European Control Conf. 2022*.

Workshop on "Optimal Control Theory" – Rouen – June 22 to 24, 2022

J.-B. Pomet

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Extensions?

If other brackets that $\text{ad}_{X^0}^j X^k$, $j \in \mathbb{N}$ (1st order approximation) are needed for $\{X^0, \dots, X^m\}$ to be bracket generating, we see no natural extension in terms of a condition of the same type as (C) to replace rank of Lie brackets when U is not a neighborhood of U .

In that case,

$$\text{conv} \left(\bigcup_{t \in [0, t_f]} E_t^U(\bar{x}(0)) \right) \subset \text{some nontrivial vector subspace of } T_{\bar{x}(0)}$$

whatever U is.