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# A Result on <br> Controllability with Constrained Controls 

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## Motivation

Control system with state $x$ in smooth $\mathcal{M}$ of dimension $n$ :

$$
\left(\Sigma^{U}\right): \quad \dot{x}=X^{0}(x)+\sum_{k=1}^{m} u_{k} X^{k}(x), \quad u=\left(u_{1} \ldots u_{m}\right) \in U \subset \mathbb{R}^{m},
$$

## Theorem (Bonnard, 1981, see also Crouch, Brockett...)

System $\left(\Sigma^{U}\right)$ is controllable if
(i) the vector field $X^{0}$ is Poisson stable,
(ii) the convex hull of the set $U$ is a neighbourhood of 0 in $\mathbb{R}^{m}$,
(iii) the family $\left\{X^{0}, X^{1}, \ldots, X^{m}\right\}$ is bracket generating.

What if $0 \in \partial(\operatorname{conv} U)$ instead of (ii) ?
Keeping (i), or even: all solutions of $\dot{x}=X^{0}(x)$ are periodic,
assuming $U$ convex for simplicity.
What instead of (iii) ?

# Solar sail dynamics 

$$
\ddot{q}=\underbrace{-\mu \frac{r}{\|r\|^{3}}}_{\text {Newtonnian }}+\underbrace{\varepsilon \vec{\Gamma}(\vec{n})}_{\text {Solar sail }}
$$


$\vec{n}$ : orientation of the sail (control). Approximation:

$$
\ddot{q}=-\mu \frac{r}{\|r\|^{3}}+\varepsilon u, \quad u \in K_{\alpha} \quad \text { cone, } \alpha<\frac{\pi}{2}
$$

Gauss equations
(I: orbital elements (1st integrals), $\varphi \in \mathbb{S}^{1}=\mathbb{R} / 2 \pi \mathbb{Z}$ ):

$$
\begin{aligned}
& i=\varepsilon \sum_{k=1}^{m} u_{k} F^{k}(I, \varphi), \quad u \in U . \quad \text { Controllable? } \\
& \dot{\varphi}=\omega(I)+\varepsilon \sum_{k=1}^{m} u_{k} f^{k}(I, \varphi),
\end{aligned}
$$

$U$ convex, bounded. $0 \in U, 0 \notin$ interior of $U$.

Obviously,

$$
\begin{aligned}
i & =\varepsilon \sum_{k=1}^{m} u_{k} F^{k}(I, \varphi), \\
\dot{\varphi} & =\omega(I)+\varepsilon \sum_{k=1}^{m} u_{k} f^{k}(I, \varphi),
\end{aligned} \quad u \in U
$$

is a special case of

$$
\left(\Sigma^{U}\right): \quad \dot{x}=X^{0}(x)+\sum_{k=1}^{m} u_{k} X^{k}(x), \quad u=\left(u_{1} \ldots u_{m}\right) \in U \subset \mathbb{R}^{m}
$$

with $X^{0}=\omega \frac{\partial}{\partial \varphi}$ :
for any $x=(I, \varphi)$,

$$
t \mapsto \exp \left(t X^{0}\right)(x) \text { is periodic with period } T(x)=2 \pi / \omega(x)
$$

## Sufficient condition for controllability

Velocity set:

$$
E_{0}^{U}(x)=\left\{\sum_{k=1}^{m} u_{k} X^{k}(x), u \in U\right\} \subset T_{x} \mathcal{M} .
$$

Pullback along $t \mapsto \exp \left(t X^{0}\right)(x)$ :

$$
E_{\tau}^{U}(x)=\left\{\sum_{k=1}^{m} u_{k}\left(\exp \left(-\tau X^{0}\right)_{\star} X^{k}\right)(x), u \in U\right\} \subset T_{x} \mathcal{M}
$$

## Theorem

$\left(\Sigma^{U}\right)$ is controllable if
(i) $t \mapsto \exp \left(t X^{0}\right)(x)$ periodic with period $T(x)$ for all $x$,
(ii) $0 \in U$,
(iii) $\operatorname{conv}\left(\bigcup_{t \in[0, T(x)]} E_{t}^{U}(x)\right)$ is a neighbd of 0 in $T_{x} \mathcal{M}$ for all $x$.

Controllability in prescribed time

$$
\left(\Sigma^{U}\right): \quad \dot{x}=X^{0}(x)+\sum_{k=1}^{m} u_{k} X^{k}(x), \quad u \in U \subset \mathbb{R}^{m},
$$

Accessible sets:
$\mathcal{A}_{T}^{U}\left(x_{0}\right)=\left\{x(T)\right.$, with $t \mapsto(x(t), u(t))$ solution of $\left.\left(\Sigma^{U}\right), x(0)=x_{0}\right\}$.
Let $t \mapsto \bar{x}(t),\left[0, t_{f}\right] \rightarrow \mathcal{M}$ be a solution of $\dot{x}=X^{0}(x)$.

## Theorem

If $U$ bounded convex, and $0 \in U$, and

$$
\begin{equation*}
\operatorname{conv}\left(\bigcup_{t \in\left[0, t_{t}\right]} E_{t}^{U(\bar{x}(0)))} \text { is a neighborhood of } 0 \text { in } T_{\bar{x}(0)} \mathcal{M}\right. \tag{C}
\end{equation*}
$$

then the set $\mathcal{A}_{t_{f}}^{\varepsilon U}(\bar{x}(0))$ is a neighborhood of $\bar{x}\left(t_{f}\right)$ for any $\varepsilon>0$.

If $U$ is a neighbd of the origin, then $(\mathrm{C})$ is equivalent controllability of the linear approximation, i.e. to

$$
\operatorname{Rank}\left\{\operatorname{ad}_{X^{0}}^{j} X^{k}, k \in\{1, \ldots, m\}, j \in \mathbb{N}\right\}=n
$$

(at least in the analytic case).
However non classical when $U$ is not a neighbd of 0 .

## Linear time-varying controlability with

constraint

$$
\begin{equation*}
\dot{z}=A(t) z+B(t) v, \quad v \in V, \tag{}
\end{equation*}
$$

$V$ compact, $0 \in V$.
$\left(t_{1}, t_{2}\right) \mapsto \phi\left(t_{2}, t_{1}\right) \in \mathbb{R}^{d \times d}$ transition matrix.

## Theorem (Schmittendorf Barmish 1980)

$A_{t_{o}, t_{f}}^{V}(0)$ is a neighbourhood of the origin if one of the two following equivalent properties hold.

- For $p \in\left(\mathbb{R}^{d}\right)^{*}$ (line vector),

$$
\begin{equation*}
\left(p \phi\left(t_{0}, s\right) B(s) v \geq 0 \quad \text { for all } v \in V, s \in\left[t_{0}, t_{f}\right]\right) \Rightarrow p=0 \tag{*}
\end{equation*}
$$

- the convex hull of $\left\{\phi\left(t_{0}, s\right) B(s) v, s \in\left[t_{0}, t_{f}\right], v \in V\right\}$ is a neighbourhood of the origin in $\mathbb{R}^{d}$.

Linearize along $t \mapsto \bar{x}(t)$ :

$$
\begin{equation*}
\dot{\delta x}=\frac{\partial X^{0}}{\partial x}(\bar{x}(t)) \delta x+\sum_{1}^{m} \delta u_{k} X^{k}(\bar{x}(t)) \tag{}
\end{equation*}
$$

or

$$
\begin{align*}
\dot{\xi} & =\sum_{1}^{m} \delta u_{k} \exp \left(-t X^{0}\right)_{\star} X^{k}(\bar{x}(0)),  \tag{*}\\
\delta x(t) & =\exp \left(t X^{0}\right)^{\prime}(\bar{x}(0)) \xi(t) .
\end{align*}
$$

with initial condition $\xi(0)=0$.

## Condition (C) is exactly condition for previous linear theorem, hence implies controlability of $\left(^{*}\right)$.

## Idea of the proof cont'd

$\bar{x}($.$) corresponds to control zero, on the border of U$, hence it is difficult to use linear approximation.

However, one may built a family of $t \mapsto x^{\varepsilon}(t)$ and corresponding controls $t \mapsto u^{\varepsilon}(t)$ that remain in the interior of $U$ and satisfy $x^{\varepsilon}(0)=\bar{x}(0), x^{\varepsilon}\left(t_{f}\right)=\bar{x}\left(t_{f}\right)$.
for the linear approximation and for original system.
Then end-point mapping is a submersion around these controls (linear controllability is preserved.

## Application to solar sails

In that case, it is also a necessary condition. Checking condition ( C ) is more difficult than checking the rank of a family of vector fields: but we manage to set it as convex optimisation problem, computable in that special case.


See e.g. European Control Conf. 2022.

## Extensions?

If other brackets that $\operatorname{ad}_{X^{0}}^{j} X^{k}, j \in \mathbb{N}$ (1st order approximation) are needed for $\left\{X^{0}, \ldots, X^{m}\right\}$ to be bracket generating, we see no natural extension
in terms of a condition of the same type as (C)
to replace rank of Lie brackets when $U$ is not a neighborhood of $U$.
In that case,

$$
\operatorname{conv}\left(\bigcup_{t \in\left[0, t_{f}\right]} E_{t}^{U}(\bar{x}(0))\right) \subset \begin{aligned}
& \text { some nontrivial vector } \\
& \text { subspace of } T_{\bar{x}(0)}
\end{aligned}
$$

whatever $U$ is.

