Boundary stabilization of 1D hyperbolic systems

Amaury Hayat[a]

[a] *CERMICS, Ecole des Ponts ParisTech, 6-8 avenue Blaise Pascal, 77420 Champs-sur-Marne, France*

Abstract

Hyperbolic systems model the phenomena of propagations at finite speeds. They are present in many fields of science and, consequently, in many human applications. For these applications, the question of stability or stabilization of their stationary state is a major issue. In this paper we present state-of-the-art tools to stabilize 1-D nonlinear hyperbolic systems using boundary controls. We review the power and limits of energy-like Lyapunov functions; the particular case of density-velocity systems; a method to stabilize shock steady-states; an extraction method allowing to use the spectral information of the linearized system in order to stabilize the nonlinear system; and some results on proportional-integral boundary control. We also review open questions and perspectives for this field, which is still largely open.

Keywords— Boundary feedback controls; hyperbolic systems; partial differential equations; Lyapunov functions; exponential stability; Input-to-State Stability; shock;

In memory of Federica Bubba

1. Introduction

Hyperbolic systems are involved everywhere in nature and human application. Mathematically, they are known to model the propagation phenomena with finite speed, which explains that they are found in hydrodynamics, engineering, physics, but also biology or economics. To give only a few notorious examples, one can cite Maxwell's equations, Euler isentropic equations, Saint-Venant equations, Klein-Gordon equation, models for the growth of cells [135], the propagation of epidemics [25], or traffic flows [8], but also many human-made systems as the telegrapher's equations, heat exchangers, supply-chains, or the behavior of some production lines [56, 38, 51]. For these examples, a 1-D approximation is relevant in many situations. A generic 1-D hyperbolic system can be written as follows:

$$\partial_t \mathbf{Y} + F(\mathbf{Y})\partial_x \mathbf{Y} + S(\mathbf{Y}, x) = 0. \tag{1.1}$$

where $\mathbf{Y}$ is the state of the system, $F(\mathbf{Y})$ is a diagonalizable matrix with distinct and real eigenvalues – sometimes called the transport term– and $S(\mathbf{Y}, x)$ is the source term. When $S(\mathbf{Y}, x) \equiv 0$, the system is said to be homogeneous. When the system is considered on a bounded domain $[0, L]$, we need to specify in addition some boundary conditions that express the fact that some information enters the system and some information leaves the system (see Section 2). Given their prominence, it is no wonder that hyperbolic systems have a special place in control theory. In particular, the question of their stabilization –essential for practical applications– has received a large interest lately.

For this problem, the typical control goal is to stabilize a target state $\mathbf{Y}^*$ that represents an ideal state or an operating point. For example, in river regulations this represents a state with a desired and fixed water level supposed to ensure a safe navigation and prevent flooding of riverside cities; or a state that has a given velocity profile to ensure a good hydroelectric production or a low velocity gradient in order to

Email address: amaury.hayat@enpc.fr (Amaury Hayat)

respect biodiversity. In traffic flows, this would be, for instance, a uniform flow on a highway, instead of a stop-and-go traffic jam that would appear naturally without control. Under reasonable assumptions on the regularity of F and S and a change of variables (see Section 1), the system (1.1) can be reduced to

$$\partial_t \mathbf{u} + A(\mathbf{u}, x)\partial_x \mathbf{u} + B(\mathbf{u}, x) = 0, \tag{1.2}$$

where $\mathbf{u}$ represents the perturbation between the state $\mathbf{Y}$ and the target state $\mathbf{Y}^*$. This means that the target state to stabilize is now $\mathbf{u}^* \equiv 0$. In other words, the goal –assuming we have a control on the system– is to find a feedback law such that the perturbations $\mathbf{u}$ tend to 0 and, if possible, exponentially quickly. As we will see in Section 2, when the system is nonlinear it is necessary to state in which norm we would like to have this exponential stability, as the stabilities in different norms are not equivalent. In most of this article the control will be supposed to be in the boundary conditions which will have the form

$$\begin{pmatrix} \mathbf{u}_+(t,0) \\ \mathbf{u}_-(t,L) \end{pmatrix} = G \begin{pmatrix} \mathbf{u}_+(t,L) \\ \mathbf{u}_-(t,0) \end{pmatrix}, \tag{1.3}$$

where G is the feedback law. We explain in Section 2 why this condition is quite physical for hyperbolic systems. This is also the most used boundary conditions for this problem [111, 110, 12]. Having the control located in the boundary conditions is often a challenge but, unfortunately, it also represents numerous physical problems where we can only act at the boundaries rather than in the system. For instance, it would be unrealistic to act on the physical dynamics inside a river, but we can consider using the dams at the boundary as means of control.

In some applications the target state might not be steady. For instance, in river regulations if the upstream flow changes during the day, there is no hope to have a steady-state [90]. In such cases, one would like to stabilize is a target state $\mathbf{Y}_1(t,\cdot)$ that is time dependent. To take again the example of river regulations, this could be a state $\mathbf{Y}_1 = (H_1, V_1)$ where the height of the water H_1 remains fixed or within constrained bounds to ensure safety of the navigation, while the velocity V_1 might change a lot to take into account the change of inflow upstream. In this case we can still define the perturbations $\mathbf{u}$ and the system becomes

$$\partial_t \mathbf{u} + A(t, \mathbf{u}, x)\partial_x \mathbf{u} + B(t, \mathbf{u}, x) = 0. \tag{1.4}$$

In this case the system is non-autonomous, which introduces new obstacles in the stabilization (see Section 6 for instance).

In practice the systems could also be subject to unknown disturbances. They could occur for several reasons: external inputs, quantized measurements, but also imprecisions of the model, etc. In these cases, the system (1.1), (1.3) would be replaced by:

$$\partial_t \mathbf{u} + A(\mathbf{u}, x)\partial_x \mathbf{u} + B(\mathbf{u}, x) + \mathbf{d}_1(t,x) = 0, \tag{1.5}$$

$$\begin{pmatrix} \mathbf{u}_+(t,0) \\ \mathbf{u}_-(t,L) \end{pmatrix} = G \begin{pmatrix} \mathbf{u}_+(t,L) \\ \mathbf{u}_-(t,0) \end{pmatrix} + \mathbf{d}_2(t), \tag{1.6}$$

where $\mathbf{d}_1(t,\cdot)$ are the internal disturbances and $\mathbf{d}_2(t)$ are the boundary disturbances. In these cases, there is no steady-state anymore to the system and there is sometimes no hope of stabilizing any target state, given that the disturbances are unknown. Nevertheless, it could be interesting to look at what is the error we make because of the disturbances when trying to stabilize a steady-state. In other words: how robust is the exponential stability with the external disturbances? In such a situation, we would like to show a generic estimate of the form

$$\|\mathbf{u}(t,\cdot)\|_X \leq C_1 e^{-\gamma t}\|\mathbf{u}_0\|_X + C_2\left(\|\mathbf{d}_1\|_{X_t\times X} + \|\mathbf{d}_2\|_{X_t}\right), \tag{1.7}$$

where $\|\cdot\|_X$ is the norm considered over space, $\|\cdot\|_{X_t}$ is the norm considered over time on $[0,t]$ and $\|\cdot\|_{X_t\times X}$ is the norm considered both over time and space. This estimate, as it is, is generic, and we could try to show different estimates in some particular situations. If such an estimate holds, we say that the system is

exponentially Input-to-State Stable (ISS) and C_1 and C_2 are the ISS gains. This is a particular case of a more general notion of ISS

$$\|\mathbf{u}(t,\cdot)\|_X \leq \sigma(\|\mathbf{u}_0\|_X, t) + \|\gamma(\|\mathbf{d}_1(s,\cdot)\|_X + |\mathbf{d}_2(s)|)\|_{X_t}, \tag{1.8}$$

where γ belongs to $\mathcal{K}$, the space of strictly increasing functions $\mathbb{R}_+ \to \mathbb{R}_+$ such that $\gamma(0) = 0$; σ is a function such that for any $t \in \mathbb{R}_+$, $x \to \sigma(x,t)$ belongs to $\mathcal{K}$ and for any $x \in \mathbb{R}_+$, $t \to \sigma(x,t)$ is non-increasing and such that $\lim_{t\to+\infty} \sigma(x,t) = 0$. Some discussion about the different notions of ISS for that can be found in [98, 16] and in Section 3. Of course, if there are no disturbances, i.e. $\mathbf{d}_1 \equiv \mathbf{d}_2 \equiv 0$, the definition of exponential ISS coincides with the definition of exponential stability. Therefore, ISS can be seen as an extension of the notion of exponential stability. It is, in fact, a more relevant notion for practical applications, as there are always uncertainties and disturbances in physical systems. While exponential stability of hyperbolic systems has been studied for decades, fewer results were known concerning ISS of nonlinear hyperbolic systems until the very recent years.

In this article we present cutting-edge tools for stabilizing nonlinear 1D hyperbolic systems with boundary controls and for ensuring their exponential stability, or their ISS in some cases. We will state the results and only give ideas of the proofs, which can be found in detail in [91] and in [94, 89, 88, 18, 17, 47, 90, 95, 92, 16], from which the results presented in this article are taken. While some methods presented have existed for decades, others are only very recent. Some, even, have not been generalized yet and were only applied to practical examples so far. We also list at the end of each section some open questions and interesting perspectives for this field, which is still very open.

In Section 2 we present the rigorous framework and some definitions. In Section 3 we show how to use energy-like Lyapunov functions (also called basic quadratic Lyapunov functions) to design good control feedbacks, and we also investigate the limits of this method. In Section 4 we will look at an important physical example among hyperbolic systems: density-velocity systems. They are the systems that consist in two equations: a mass conservation and a dynamical equation, and are often found in physics. We will see that they have a particular hidden structure that allows to use energy-like Lyapunov functions, whatever the length of the domain. In Section 5 we will see how to stabilize a system with a shock-steady-state. Finally, in Section 6 we will see how to deal with Proportional Integral (PI) controllers for nonlinear hyperbolic systems, a type of controller particularly interesting for its resilience to constant disturbances.

2. Definitions and framework

2.1. Nonlinear hyperbolic systems

Let us consider a system of the form (1.1) and let us derive the system of perturbations (1.2). Consider a steady-state $\mathbf{Y}^*$. As $F(\mathbf{Y}^*)$ is diagonalizable one can define a matrix P such that

$$P(x)F(\mathbf{Y}^*(x))P^{-1}(x) = \Lambda(x), \tag{2.1}$$

where $\Lambda(x)$ is a diagonal matrix with coefficient $(\Lambda_i(x))_{i\in\{1,\dots n\}}$. Note that P and Λ depend on x only if $\mathbf{Y}^*$ does. Setting $\mathbf{u}(t,x) = P(x)(\mathbf{Y}(t,x) - \mathbf{Y}^*(x))$ we obtain,

$$\partial_t \mathbf{u} + A(\mathbf{u},x)\partial_x \mathbf{u} + B(\mathbf{u},x) = 0, \tag{2.2}$$

where

$$\begin{aligned} A(\mathbf{u},x) &= P(x)F(\mathbf{Y})P^{-1}(x) = P(x)F(P^{-1}(x)\mathbf{u} + \mathbf{Y}^*(x))P^{-1}(x), \\ B(\mathbf{u},x) &= P(F(\mathbf{Y})(\mathbf{Y}^*_x + (P^{-1})'\mathbf{u}) + S(\mathbf{Y},x)), \end{aligned} \tag{2.3}$$

and in particular $A(\mathbf{0},x) = \Lambda(x)$, and $B(\mathbf{0},x) = 0$ since $\mathbf{Y}^*$ is a steady-state. Now, the steady-state to stabilize in the new variables is $\mathbf{u}^* = 0$ and, of course, the exponential stability of $\mathbf{Y}^*$ in the original system

is equivalent to the stability of $\mathbf{u}^* = 0$. In the following we will assume that the steady-state $\mathbf{Y}^* \in C^2([0, L])$, except in Section 5 where we will study a shock steady-state. The regularity required on A and B will usually be C^1 or C^2 depending on the cases, as we will see in the next section.

Let us now look at the boundary conditions. It is known that 1-D hyperbolic systems have some quantities that propagate forward and some that propagate backward. By assumption, $F(\mathbf{Y}^*)$ has non-zero and distinct eigenvalues for any x, and therefore so do $\Lambda(x)$. From (2.3), as long as $\mathbf{Y}^*$ and F are continuous, A is continuous with $\mathbf{u}$. As $A(\mathbf{0}, x) = \Lambda(x)$, this means that if $\mathbf{u}$ is small enough in L^∞ norm, $A(\mathbf{u}, x)$ has also distinct and non-zero eigenvalues $\lambda_i(\mathbf{u}, x)$ which have the same sign as $\Lambda_i(x)$. Without loss of generality we can assume that $\Lambda_i > 0$ for $i \in \{1, ..., m\}$ and $\Lambda_i < 0$ for $i \in \{m+1, ..., n\}$, for some $m \in \{1, ..., n\}$. The only boundary conditions we can impose are those corresponding to information entering the system (see [110, 111] for more details). This means that at $x = 0$ we can impose the quantities that have a positive propagation speed, and at $x = L$ the quantities that have a negative propagation speed. This is translated as

$$\begin{pmatrix} u_1(t,0) \\ ... \\ u_m(t,0) \\ u_{m+1}(t,L) \\ ... \\ u_n(t,L) \end{pmatrix} = \mathcal{U}(t), \tag{2.4}$$

where $\mathcal{U}(t)$ is the control we would like to impose. Of course, the u_i are not exactly the quantities propagating with speed $\lambda_i(\mathbf{u}, x)$ given that $A(\mathbf{u}, x)$ is not diagonal. But, assuming that we are close enough to the steady-state $\mathbf{u}^* = 0$, the perturbations are small and one can show that we can still use the boundary condition (2.4) (see [13]). In the following we will denote without loss of generality $\mathbf{u}_+ = (u_1, ..., u_m)^T$ the vector of components associated to positive propagation speeds, and $\mathbf{u}_- = (u_{m+1}, ..., u_n)^T$ the vector of components associated to negative propagation speeds. As a consequence, (2.4) can be written in the compact notation

$$\begin{pmatrix} \mathbf{u}_+(t,0) \\ \mathbf{u}_-(t,L) \end{pmatrix} = \mathcal{U}(t), \tag{2.5}$$

Now, we would like the control $\mathcal{U}(t)$ to be a feedback law. By definition a feedback law is a control that depends on some measurements on the state of the system. This brings the question: what do we allow ourselves to measure? Here we will choose to only measure the information leaving the system at the boundaries. Namely, our control will be

$$\mathcal{U}(t) = G\begin{pmatrix} \mathbf{u}_+(t,L) \\ \mathbf{u}_-(t,0) \end{pmatrix}, \tag{2.6}$$

where G is a feedback law (of class C^1) to be chosen. This gives the boundary conditions

$$\begin{pmatrix} \mathbf{u}_+(0) \\ \mathbf{u}_-(L) \end{pmatrix} = G\begin{pmatrix} \mathbf{u}_+(t,L) \\ \mathbf{u}_-(t,0) \end{pmatrix}. \tag{2.7}$$

This represents that the input information is imposed as a function of the output information. We now introduce the first order compatibility conditions for an initial condition $\mathbf{u}^0 \in C^1([0, L])$. These conditions represent the fact that the initial condition is compatible with the boundary conditions (1.3) and are stated as follows

$$\begin{aligned} &\begin{pmatrix} \mathbf{u}^0_+(0) \\ \mathbf{u}^0_-(L) \end{pmatrix} = G\begin{pmatrix} \mathbf{u}^0_+(L) \\ \mathbf{u}^0_-(t,0) \end{pmatrix}. \\ &\begin{pmatrix} (\mathcal{A}(u^0(0),0)\partial_x\mathbf{u}^0(0) + B(\mathbf{u}^0(0),0))_+ \\ (\mathcal{A}(u^0(L),L)\partial_x\mathbf{u}^0(L) + B(\mathbf{u}^0(L),L))_- \end{pmatrix} \\ &= G'\begin{pmatrix} \mathbf{u}^0_+(L) \\ \mathbf{u}^0_-(0) \end{pmatrix}\begin{pmatrix} (\mathcal{A}(u^0(L),L)\partial_x\mathbf{u}^0(L) + B(\mathbf{u}^0(L),L))_+ \\ (\mathcal{A}(u^0(0),0)\partial_x\mathbf{u}^0(0) + B(\mathbf{u}^0(0),0))_- \end{pmatrix}. \end{aligned} \tag{2.8}$$

One can show that (1.2), (1.3) is well posed in C^1 and H^2 norm for compatible initial condition satisfying (2.8) [111, 132, 12].

Theorem 2.1. *For all $T > 0$ there exist $C(T) > 0$, $\eta(T) > 0$ such that: for every $\mathbf{u_0} \in C^1([0, L]; \mathbb{R}^n)$ (resp. $H^2((0, L); \mathbb{R}^n)$) satisfying the compatibility conditions* (2.8) *and such that*

$$\|\mathbf{u}_0\|_{C^1} \leq \eta(T), \ \textit{(resp.}\|\mathbf{u}_0\|_{H^2} \leq \eta(T)\textit{)}, \tag{2.9}$$

the system (1.2), (1.3)*, with A and B of class C^1 (resp. C^2) and initial condition $\mathbf{u_0}$, has a unique solution $\mathbf{u} \in C^1([0, T] \times [0, L])$ (resp. $C^0([0, T], H^2(0, L))$). Moreover one has*

$$\begin{gathered} \|\mathbf{u}(t, \cdot)\|_{C^1} \leq C(T)\|\mathbf{u}(0, \cdot)\|_{C^1}, \ \forall t \in [0, T] \\ \textit{(resp. } \|\mathbf{u}(t, \cdot)\|_{H^2} \leq C(T)\|\mathbf{u}(0, \cdot)\|_{H^2}, \ \forall t \in [0, T]\textit{).} \end{gathered} \tag{2.10}$$

Note that in (2.6) we could have considered a feedback law with a memory. Namely, a feedback law that also depends on the output information at previous time $s \leq t$. This, in fact, is tacitly exploited when using a proportional-integral (PI) controller (see Section 6).

2.2. Exponential stability and inherent challenges

For a Banach space X with a norm $\|\cdot\|_X$, referred to as X norm in the following, the exponential stability is formally defined as follows.

Definition 2.1. The steady-state $\mathbf{u}^* = 0$ of the system (1.2), (1.3) is exponentially stable for the X norm if there exist $\gamma > 0$, $\eta > 0$, and $C > 0$ such that for every $\mathbf{u}^0 \in X$ satisfying the compatibility conditions[1] and $\|\mathbf{u}^0\|_X \leq \eta$, the Cauchy problem (1.2), (1.3), $(\mathbf{u}(0, x) = \mathbf{u}^0)$ has a unique solution in $C^0([0, +\infty), X)$ and

$$\|\mathbf{u}(t, \cdot)\|_X \leq Ce^{-\gamma t}\|\mathbf{u}^0\|_X, \ \forall t \in [0, +\infty). \tag{2.11}$$

We can remark several features:

- First of all it is a local exponential stability. Indeed, for a nonlinear hyperbolic system even the global well-posedness is usually impossible to guarantee for regular norms such as the C^1 and H^2 that will be studied in the following. However, we will see in Section 3 that under some Lipschitz assumption on the source term is it possible to ensure a global exponential stability for the L^2 norm.
- The norm $\|\cdot\|_X$ is specified. Indeed, for nonlinear systems the stabilities in different norms are not equivalent. In [53] for instance the authors show an example of a simple 2×2 nonlinear system that is exponentially stable for the H^2 norm, but unstable for the C^1 norm.

For linear systems there is an almighty tool to study the exponential stability: the spectral mapping theorems [127, 133, 116]. A version can be stated as[2]

Theorem 2.2 ([116])**.** *Let a system of the form* (1.2), (1.3) *where A does not depend on $\mathbf{u}$, $B = M(x)\mathbf{u}$, and $G(\mathbf{U}) = K\mathbf{U}$, with K a matrix. Let $\mathcal{L} := -A\partial_x\mathbf{u} - M\mathbf{u}$ be defined on the domain $D(\mathcal{L}) := \{\mathbf{u} \in W^{1,2}((0, L), \mathbb{C}^n) | ((\mathbf{u}(0))_+, (\mathbf{u}(L))_-)^T = K((\mathbf{u}(L))_+, (\mathbf{u}(0))_-)^T\}$. Then*

$$\sigma(e^{\mathcal{L}t}) \setminus \{0\} = \overline{e^{\sigma(\mathcal{L})t}} \setminus \{0\}, \ \textit{for } t \geq 0, \tag{2.12}$$

where $\sigma(C)$ refers to the spectrum of the operator C, and $\overline{e^{\sigma(\mathcal{L})t}}$ refers to the closure of $e^{\sigma(\mathcal{L})t}$.

[1] this very formal given that X is not precised. For the H^p or C^p norm this correspond to the p order compatibility condition given in [12, (4.136) (see also (4.137)-(4.142))]

[2] Technically the result stated by [116] only deals with local boundary conditions in the sense where the control in 0 only depends on values measured in 0 and the control in L only depends on values measured in L. Nevertheless, we can always go back to this case using a doubling of variables as shown in [59].

This means that it is sufficient to know the spectrum of the operator $\mathcal{L} = -A\partial_x - BId$ on the domain $D(\mathcal{L})$ to know the spectrum of the semigroup operator $(e^{t\mathcal{L}})_{t>0}$ at each time $t \geq 0$. This allows to conclude on the growth bound of the semigroup operator (see for instance [116]) given by $\omega_0 = \lim_{t\to+\infty} t^{-1}\log(\|e^{\mathcal{L}t}\|_{\mathcal{L}(L^2((0,L);\mathbb{C}^n))})$. Using that there exists $M > 0$ such that, for any $\mathbf{u}_0 \in L^2((0,L);\mathbb{C}^n)$,

$$\|\mathbf{u}(t,\cdot)\|_{L^2} = \|e^{\mathcal{L}t}\mathbf{u}_0\|_{L^2} \leq Me^{\omega_0 t}\|\mathbf{u}_0\|_{L^2}, \tag{2.13}$$

this allows to conclude on the exponential stability of the system.

Unfortunately, when it comes to nonlinear systems, such a theorem is not helpful: it was shown in [53] that the exponential stability of the linearized system around a state does not give any information of the exponential stability of the nonlinear system, even locally. In [53] the authors constructed an example when the linearized system is exponentially stable but the nonlinear system is unstable for the C^1 norm. It is this uselessness of spectral tools for non-linear systems that makes the problem complicated and makes it necessary to use other methods. Other inherent challenges include: the non-uniformity of the steady-states when the system is inhomogeneous, and the possible formation of shocks in some cases. Several methods exist, among which:

- The characteristic method, which leverages the existence of characteristic curves in 1-D hyperbolic systems. This is the most natural and historical method. The exponential stability results from a careful estimation along the characteristics. However, it might become hard to use when the system gets complicated, in particular for coupled inhomogeneous systems.
- The time-delay approach which was used in [53, 37] and allowed to prove the results when just talked about, as well as a stability in the $W^{2,p}$ norm.
- The backstepping approach: this method, very powerful, was introduced in [31, 99, 144] for finite dimensional systems. It was then adapted in [42] and modified in [9, 28, 136, 103] for parabolic partial differential equations and used for linear and then non-linear 1-D hyperbolic systems in [102, 146, 6, 5, 96, 97, 50]. It has also been used for other more complicated operator (e.g. [149]). It consists in using an invertible transform to map the system considered into a system much simpler to study. Then once a feedback is found for this simpler system, it suffices to use the inverse mapping to find a suitable feedback for the original system. Originally the invertible transformation was sought in the form of a Volterra transformation of the second kind. These transforms have a triangular structure recalling the original backstepping for finite dimensional systems, and are directly invertible. This method give rise to full-state feedback control whose

  In the recent years several studies started to look at other more general linear transforms such as Fredholm transforms [52, 49, 46, 154, 155, 48, 75, 74]. These transforms are more general, and therefore potentially more powerful, but they are not always invertible and proving the invertibility of the candidate transform becomes one of the main difficulties. A good overview of the method when using a Volterra transform can be found in [103] or [150], and in [48, Section 2.1] for more general transforms.
- The direct Lyapunov approach, which consists in finding a nonnegative functional decreasing along the trajectories of the system, as we are going to detail it in the following sections.

In this article we focus on results using a direct Lyapunov approach, potentially combined with new methods such as a method for stabilizing a shock (see Section 5) or the extraction methods (see Section 6).

2.3. Input-to-State Stability for the C^q and H^p norms

Finally, we give a definition of exponential Input-to-State Stability (ISS). When there are some internal and/or boundary disturbance, the system (1.2), (1.3) become (1.5), (1.6) and we can define:

Definition 2.2. A system of the form (1.5), (1.6) is exponentially ISS with fading memory for the C^q norm (resp. H^p norm) if there exist positive constants $C_1 > 0$, $C_2 > 0$, $\gamma > 0$, and $\delta > 0$ such that, for any $T > 0$, for any $\mathbf{u_0} \in C^q([0,L];\mathbb{R}^n)$ (resp. $\mathbf{u}_0 \in H^p((0,L);\mathbb{R}^n)$) satisfying the q-th order (resp. $p-1$

order) compatibility conditions (see [12, 4.5.2]), with $\|\mathbf{u}_0\|_{C^q} \leq \delta$ (resp. $\|\mathbf{u}_0\|_{H^p} \leq \delta$) and $\|\mathbf{d}_2\|_{C^q([0,T])} + \|\mathbf{d}_1\|_{C^q([0,T]\times[0,L])} \leq \delta$, (resp. $\|\mathbf{d}_2\|_{H^p} + \|\mathbf{d}_1\|_{H^p([0,T]\times[0,L])} \leq \delta$), there exists a unique solution $\mathbf{u} \in C^q([0,T]\times[0,L])$, (resp. $\mathbf{u} \in C^0([0,T];H^p(0,L))$).

$$\|\mathbf{u}(t,\cdot)\|_{C^q} \leq C_1 e^{-\gamma t}\|\mathbf{u}_0\|_{C^q} + C_2\left(\sum_{k=0}^{q}\sup_{\tau\in[0,t]}\left(e^{-\gamma(t-\tau)}|\mathbf{d}_2^{(k)}(\tau)|\right)\right)$$
$$+ C_3\left(\sup_{(\tau,x)\in[0,t]\times[0,L]}\left(e^{-\gamma(t-\tau)}|\partial_t^q \mathbf{d}_1(\tau,x)|\right) + \sum_{k_1+k_2\leq q-1}\sup_{(\tau,x)\in[0,t]\times[0,L]}\left(e^{-\gamma(t-\tau)}|\partial_t^{k_1}\partial_x^{k_2}\mathbf{d}_1(\tau,x)|\right)\right),$$
$$\Big(\text{resp. } \|\mathbf{u}(t,\cdot)\|_{H^p} \leq C_1 e^{-\gamma t}\|\mathbf{u}_0\|_{H^p} + C_2\left(\sum_{k=0}^{p}\|e^{-\gamma(t-\tau)}\mathbf{d}_2^{(k)}(\tau)\|_{L^2(0,t)}\right)$$
$$+ C_3\left(\|e^{-\gamma(t-\tau)}\partial_t^p \mathbf{d}_1(\tau,x)\|_{L^2((0,t)\times(0,L))} + \sum_{k_1+k_2\leq p-1}\|e^{-\gamma(t-\tau)}\partial_t^{k_1}\partial_x^{k_2}\mathbf{d}_1(\tau,x)\|_{L^2((0,t)\times(0,L))}\right)\Big). \tag{2.14}$$

One can note that this definition is in fact stronger than the exponential ISS estimate (1.7): looking at the last term we impose a fading memory, meaning that the influence of the past disturbance decays exponentially with time. We could have chosen other less restrictive fading factors or weaker notions (see for instance [98, Chapter 7] or [126]).

3. Energy-like Lyapunov functions

Taking physical quantities as Lyapunov functions is a very natural idea. For a pendulum with friction for example the stability can be seen because the mechanical energy is strictly decreasing and tends to 0, and is therefore a perfect candidate to be a Lyapunov function [100, 62]. For other closed systems, the strict increase of the physical entropy (the mathematical entropy is often the opposite) allows one to find a good Lyapunov function [43, 128]. For a hyperbolic system, these physical quantities would have the form

$$V = \int_0^L f(\mathbf{u}(t,x))dx, \tag{3.1}$$

where f is the local physical quantity per unit length. For the kinetic energy, this would be for instance $\rho v^2/2$ where v is the velocity and ρ the density. Looking at the local behavior around a steady-state $\mathbf{u}^*(x)$, these physical Lyapunov function candidates would become

$$V(\mathbf{u}) = V^* + \int_0^L f(\mathbf{u}^*(t,x)) + (\mathbf{u}(t,x)-\mathbf{u}^*(t,x))^T Q(x)(\mathbf{u}(t,x)-\mathbf{u}^*(t,x)) + O(|(\mathbf{u}(t,x)-\mathbf{u}^*(t,x))|^3)dx. \tag{3.2}$$

Note that the first order term of the Taylor expansion has to be 0 otherwise it would jeopardize the local minimum in $\mathbf{u} = \mathbf{u}^*$ of the Lyapunov function candidate. For the same reason, Q should be positive definite (symmetric) to have a proper Lyapunov function. This form (3.2) thus suggests to look locally at Lyapunov functions of the form

$$V(\mathbf{u}) = \int_0^L (\mathbf{u}(t,x) - \mathbf{u}^*(t,x))^T Q(x)(\mathbf{u}(t,x) - \mathbf{u}^*(t,x))dx, \tag{3.3}$$

where Q is a C^1 function from $[0,L]$ to the space of positive definite matrix of $\mathbb{R}^n$. Such an energy-like Lyapunov function was used in many studies (e.g. [43, 66, 11, 73, 148, 92]), and is sometimes called *basic quadratic Lyapunov function* [11, 89]. Of course this Lyapunov function candidate is equivalent to the square

of the L^2 norm and, as it is, it can only be used to show the stability in this norm. Naturally, we would like to extend this to other norms. This definition could be easily extended for the H^p norm for $p \in \mathbb{N}$ by

$$V(\mathbf{u}) = \int_0^L \sum_{k=0}^{p} \partial_x^k (\mathbf{u}(t,x) - \mathbf{u}^*(x))^T Q(x) \partial_x^k (\mathbf{u}(t,x) - \mathbf{u}^*(x)) dx. \tag{3.4}$$

But, in fact, for hyperbolic systems of the from (1.2) it happens to be more convenient to rather extend the definition as follows [12],

$$V(\mathbf{u}) = \int_0^L \sum_{k=0}^{p} (E(\mathbf{u}(t,x),x) D_k \mathbf{u}(t,x))^T Q(x) E(\mathbf{u}(t,x),x) D_k \mathbf{u}(t,x) dx. \tag{3.5}$$

where $E(\mathbf{u}(t,x),x)$ is a matrix diagonalizing $A(\mathbf{u},x)$ and D_k is a differential operator defined iteratively as follows:

$$\begin{aligned} &D_0\mathbf{U} = \mathbf{U}, \quad D_1\mathbf{U} = A(\mathbf{U},x)\partial_x \mathbf{U} + B(\mathbf{U},x), \\ &D_n\mathbf{U} = \partial_{\mathbf{U}}(D_{n-1}\mathbf{U}) D_1 \mathbf{U}, \ \forall\, k \geq 2. \end{aligned} \tag{3.6}$$

This allows the definition of V given in (3.5) to hold for functions $u \in H^k(0,L)$ that only depend on the space variable and guarantees the equivalence with the (square of the) H^p norm when $\mathbf{u}$ is small enough (see [12, Chapter 6]). The interest of such definition is that, for functions $\mathbf{u} \in C^0([0,T], H^k)$ solution to (1.2), the expression of $V(\mathbf{u})$ becomes

$$V(\mathbf{u}) = \int_0^L \sum_{k=0}^{p} (E(\mathbf{u},x)\partial_t^k \mathbf{u}(t,x))^T Q(x) (E(\mathbf{u}(t,x),x)\partial_t^k \mathbf{u}(t,x)) dx. \tag{3.7}$$

In fact, it was shown in [11, 12] that if the system is diagonalized (i.e. $A(\mathbf{0},x)$ is diagonal), then Q has to be a diagonal matrix for V to be a suitable Lyapunov function candidate. Of course this would not hold in the original physical coordinates $\mathbf{Y}$. Similarly, extensions for the C^p norm were given in [40, 89] as:

$$V(\mathbf{u}) = \sum_{k=0}^{p} \sup_{x \in [0,L]} \|Q(\cdot) E(\mathbf{u},\cdot) D_k \mathbf{u}(t,\cdot)\|_\infty^2, \tag{3.8}$$

where $\|\cdot\|_\infty$ refers to the infinite norm in $\mathbb{R}^n$, i.e. $\|v\|_\infty = \max_{i \in \{1,..,n\}}(v_i)$. Overall we can generalize this form and define an energy-like Lyapunov function (or basic quadratic Lyapunov function) for the $W^{p,q}$ norm as

$$V(u) = \sum_{k=0}^{p} \|Q(\cdot) E(\mathbf{u},\cdot) D_k \mathbf{u}(t,\cdot)\|_{L^q}^2. \tag{3.9}$$

In the following we explore what can be done with such Lyapunov functions for the H^p and C^q norms, which were used in many studies (e.g. [44, 64, 32, 141, 71, 147]).

3.1. Exponential stability for the C^q and H^p norms

As energy-like Lyapunov functions are a very natural tool to study physical systems, the first results were obtained in particular cases. For instance, using the mechanical energy or the entropy as Lyapunov functions on systems coming from fluid dynamics [43]. The first general result for hyperbolic systems was given in [40], where the authors used an energy-like Lyapunov function for the C^1 norm to give an alternative proof to a result that had already been shown with a characteristic approach (see [78, 132, 157, 60]).

Theorem 3.1. *If $B(\mathbf{u},x) \equiv 0$ (i.e. the system is homogeneous) then the system* (1.2), (1.3) *is exponentially stable for the C^1 norm if*

$$\rho_\infty(G'(0)) < 1, \tag{3.10}$$

where

$$\rho_\infty(M) = \inf(\|\Delta M \Delta^{-1}\|_\infty, \Delta \in D_n^+), \tag{3.11}$$

with D_n^+ the set of diagonal matrix with positive entries.

This result was shown originally in [78] for a 2×2 system and had been then generalized in [132, 157, 60]. Still using energy-like Lyapunov functions, Coron and Bastin showed later in [41] that the exponential stability is easier to obtain in the more regular (hence more restrictive) H^2 norm. They showed the following

Theorem 3.2. *If $B(\mathbf{u}, x) \equiv 0$ then the system (1.2), (1.3) is exponentially stable for the H^2 norm if*

$$\rho_2(G'(0)) < 1, \tag{3.12}$$

where

$$\rho_2(M) = \inf(\|\Delta M \Delta^{-1}\|_2, \Delta \in D_n^+). \tag{3.13}$$

And it can be shown that $\rho_\infty \leq \rho_2$, hence the condition (3.12) for the stability in H^2 norm is less restrictive than the condition (3.10) for the stability in C^1 norm.

For inhomogeneous systems the first general result was in 2011 in [11] for systems of two equations where the propagation speeds have opposite directions, for instance $\Lambda_1 > 0 > \Lambda_2$. Let us denote

$$M(x) = \partial_{\mathbf{u}} B(\mathbf{0}, x) \tag{3.14}$$

and introduce

$$\begin{aligned} a(x) &= M_{12}(x)\varphi(x) \\ b(x) &= M_{21}(x)\varphi^{-1}(x), \end{aligned} \tag{3.15}$$

where

$$\varphi = \exp\left(\int_0^x \frac{M_{11}(s)}{\Lambda_1(s)} + \frac{M_{22}(s)}{|\Lambda_2(s)|} ds\right). \tag{3.16}$$

The result shown in [11] is the following[3].

Theorem 3.3. *Assume that $n = 2$, A, B and G are of class C^2 and $\Lambda_1 > 0$ and $\Lambda_2 < 0$ on $[0, L]$. There exists a control (1.3) such that there exists an energy-like function for this system if and and only if there exists a solution η on $[0, L]$ to*

$$\eta'(x) = \left|\frac{a(x)}{\Lambda_1} + \frac{b(x)}{|\Lambda_2|}\eta^2\right|, \tag{3.17}$$

where a and b are given by (3.14)–(3.15). Moreover, if this condition is satisfied, then for any $\sigma > 0$ such that there exists a solution η_σ on $[0, L]$ to

$$\begin{aligned} \eta_\sigma' &\geq \left|\frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\eta^2\right|, \\ \eta_\sigma(0) &= \sigma, \end{aligned} \tag{3.18}$$

the system is exponentially stable for the H^2 with the choice of control

$$G'(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix} \text{ with } l_1^2 < \eta_\sigma^2(0) \text{ and } l_2^2 < \frac{\varphi^2(L)}{\eta_\sigma^2(L)}. \tag{3.19}$$

We see here a new phenomenon appearing: not only there is a boundary condition on the control, but there is also an internal condition, independent of the control and therefore intrinsic to the system. Indeed, as the differential equation given by (3.17) is nonlinear, the solution with $\eta(0) = 0$ could cease to exist in finite time if L, the length of the domain, is too large. This would happen, for instance, for $L > \pi/2$ when $M_{11} = M_{22} = 0$, $M_{21} = M_{12} = 1$. Consequently, in such cases, there would not be any control (1.3) such that

[3] In fact the result stated in [11] only look at the linear case but can be extended directly to Theorem 3.3

this result applies. Worse, this result even guarantees that there would not exist any energy-like Lyapunov function, whatever the control (1.3) is, showing the limit of this method. In fact, there are even cases where the control has a form anti-diagonal like in (3.19) with l_1 chosen and the system can be shown to be even unstable whatever the value of l_2 [12, Section 5.6] (see also [80]). The question of whether a system could still be exponentially stable when there does not exist any energy-like Lyapunov function is an open question.

This result was later extended to a generic $n \times n$ hyperbolic system in [12]:

Theorem 3.4. *If A, B and G are of class C^2, the system (1.2), (1.3) is exponentially stable for the H^2 norm if there exists a diagonal matrix Q with positive entries such that,*

- *(internal condition) the matrix*

$$-(Q\Lambda)'(x) + Q(x)M(x) + M(x)^T Q(x)^T \tag{3.20}$$

is positive definite for any $x \in [0, L]$, where M is given by (3.14),

- *(boundary condition) the matrix*

$$\begin{pmatrix} \Lambda_+(L)Q_+(L) & 0 \\ 0 & -\Lambda_-(0)Q_-(0) \end{pmatrix} - G'(0)^T \begin{pmatrix} \Lambda_+(0)Q_+(0) & 0 \\ 0 & -\Lambda_-(L)Q_-(L) \end{pmatrix} G'(0) \tag{3.21}$$

is semi-definite positive.

Here we denote $\Lambda_+ = \mathrm{diag}(\Lambda_1, ..., \Lambda_m)$ which correspond to the positive propagation speeds and $\Lambda_- = \mathrm{diag}(\Lambda_{m+1}, ..., \Lambda_n)$ and similarly for Q. These notations are used to be coherent with the definition of $\mathbf{u}_+$ and $\mathbf{u}_-$ used in Section 2 and given before (2.5). In this result, the same phenomenon as in Theorem 3.3 appears: first there is an internal condition intrinsic to the system, telling whether the stability of this system can be guaranteed with an energy-like Lyapunov function; and second a boundary condition telling which boundary control is suitable to guarantee the exponential stability if the internal condition is satisfied. However, in contrast to the previous 2×2 system and Theorem 3.3, it is not known whether the internal condition is necessary for the existence of an energy-like Lyapunov function for the system.

In the two previous results, we somehow lost the C^1 norm for a more restrictive norm. The reason is that the H^2 norm and the energy-like Lyapunov functions for the H^2 norm are easier to handle. Nevertheless, the C^1 norm is the most natural norm for classical solutions of a hyperbolic system. It was shown in [89] that, in fact, it is possible to have the same type of result for the C^1 norm:

Theorem 3.5. *If A, B and G are of class C^1, the system (1.2), (1.3) is exponentially stable for the C^1 norm if the two following properties are satisfied:*

- *(internal condition) there exists a solution $(f_1, ...f_n) \in C^1([0, L], (0, +\infty))^n$ to the system*

$$\Lambda_i f_i' \leq -2 \left(-M_{ii}(x) f_i + \sum_{k=1, k\neq i}^{n} |M_{ik}(x)| \frac{f_i^{3/2}}{\sqrt{f_k}} \right), \ \forall\, i \in \{1, ..., n\}, \tag{3.22}$$

where M is given by (3.14);

- *(boundary condition) there exists a matrix Δ with positive components such that*

$$\|\Delta G'(0) \Delta^{-1}\|_\infty < \frac{\inf_i \left(\frac{f_i(d_i)}{\Delta_i^2} \right)}{\sup_i \left(\frac{f_i(L-d_i)}{\Delta_i^2} \right)}, \tag{3.23}$$

where $d_i = L$ if $\Lambda_i > 0$ and $d_i = 0$ otherwise.

We observe again the same phenomenon of internal condition, intrinsic to the system. The main problem of the C^1 norm is that it is hard to differentiate, compared to the H^2 norm or other $W^{s,p}$ norm where $(s,p) \in \mathbb{N}$. Therefore, the idea of the proof of [89] is to approximate the C^1 norm by the $W^{1,p}$ norm where p is large enough; to show an exponential stability estimate with the $W^{1,p}$ norm where the constants do not depend on p; and then to let $p \to +\infty$ in a similar way as what was done in [40]. In this case, the condition (3.22) is sharp for the existence of an energy-like Lyapunov function for the C^1 norm.

Theorem 3.6. *Assume that A,B and G are of class C^3. There exists an energy-like Lyapunov function for the C^1 norm to the system* (1.2), (1.3) *if and only if there exists a solution $(f_1, ...f_n) \in C^1([0,L],(0,+\infty))^n$ to the system*

$$\Lambda_i f_i' \leq -2\left(-M_{ii}(x)f_i + \sum_{k=1,k\neq i}^{n} |M_{ik}(x)|\frac{f_i^{3/2}}{\sqrt{f_k}}\right), \quad \forall\, i \in \{1,...,n\}, \tag{3.24}$$

where $M(x) = \partial_{\mathbf{u}} B(\mathbf{0},x)$ and $\Lambda(x) = A(\mathbf{0},x)$.

We can also note that if $B(\mathbf{u},x) \equiv 0$ (i.e. the system is homogeneous) then the two conditions of Theorem 3.5 simplify and we recover exactly the previous Theorem 3.1 shown by Bastin and Coron.

It is easy to see that both the internal and the boundary conditions of Theorem 3.4 and 3.5 are quite different. It would therefore be interesting to compare them. While no comparison result exists yet for $n \times n$ systems, some have been shown for systems of two equations. These systems are interesting not only because they are the simplest systems showing some coupling, but also (and especially) because they correspond to many physical examples: rivers dynamics, gas transport, signal transmission on an electric line, traffic flow, etc. (see Section 4). We denote again $A(\mathbf{0},x) = \mathrm{diag}(\Lambda_1,\Lambda_2)$ and, in the rest of this subsection, we assume that $n = 2$ unless specified otherwise. When Λ_1 and Λ_2 have the same sign, there always exists an energy-like Lyapunov function both for the C^1 norm and the H^2 norm, as shown in [88],

Theorem 3.7. *Assume that A, B and G are of class C^2 and $\Lambda_1\Lambda_2 > 0$ and let $M(x) = \partial_{\mathbf{u}} B(\mathbf{0},x)$. There always exists a control of the form* (1.3) *such that the system* (1.2), (1.3) *has energy-like Lyapunov functions for the C^1 norm and for the H^2 norm and is exponentially stable for the C^1 norm and the H^2 norm. In particular, a suitable choice for the control* (1.3) *is*

$$\begin{aligned} G'(0) &= \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \text{ , where} \\ k_1^2 &< \exp\left(\int_0^L 2\frac{M_{11}(s)}{|\Lambda_1|} - 2\max\left(\left|\frac{a(s)}{\Lambda_1}\right|, \left|\frac{b(s)}{\Lambda_2}\right|\right) ds\right), \\ k_2^2 &< \exp\left(\int_0^L 2\frac{M_{22}(s)}{|\Lambda_2|} - 2\max\left(\left|\frac{a(s)}{\Lambda_1}\right|, \left|\frac{b(s)}{\Lambda_2}\right|\right) ds\right). \end{aligned} \tag{3.25}$$

When Λ_1 and Λ_2 have opposite signs, meaning that the propagation speeds have opposite directions, the situation is more interesting. Without loss of generality we can assume that $\Lambda_1 > 0$ and $\Lambda_2 < 0$. In this case it is possible to show that Theorem 3.5 can take a simpler form and that we have

Theorem 3.8. *Assume that A, B, G are of class C^3 and $\Lambda_1\Lambda_2 < 0$, and let $M(x) = \partial_{\mathbf{u}} B(\mathbf{0},x)$. There exists a control of the form* (1.3) *such that the system* (1.2), (1.3) *has an energy-like Lyapunov function for the C^1 norm if and only if*

$$\begin{aligned} d_1' &= \frac{|a(x)|}{\Lambda_1} d_2, \\ d_2' &= -\frac{|b(x)|}{|\Lambda_2|} d_1, \end{aligned} \tag{3.26}$$

has a solution $(d_1, d_2) \in C^1([0,L],(0,+\infty))^2$*, or equivalently*

$$\begin{aligned} \eta' &= \left|\frac{a}{\Lambda_1}\right| + \left|\frac{b}{\Lambda_2}\right| \eta^2, \\ \eta(0) &= 0, \end{aligned} \tag{3.27}$$

has a solution on $[0,L]$*, where* a *and* b *are given in* (3.15)*.*

In addition, if one of the conditions above is satisfied, then a suitable control is

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ with } k_2^2 < \varphi(L)^2 \left(\frac{d_2(L)}{d_1(L)}\right)^2 \text{ and } k_1^2 < \left(\frac{d_1(0)}{d_2(0)}\right)^2. \tag{3.28}$$

The internal conditions (3.26) and (3.27) are now much simpler than the original condition (3.22): instead of having to look at two nonlinear differential inequalities, one has the choice between looking at a system of two *linear* ODE or a single nonlinear ODE to conclude. Interestingly, this implies that the existence of solutions to the differential inequalities (3.22) is equivalent to the existence of solutions to the associated differential equations for 2×2 systems, while this may not be true for $n \times n$ system. In addition, we can see a strong similarity between the condition (3.27) and the condition (3.17) of Theorem 3.3 dealing with the H^2 norm. This enables the following comparison result [88]

Corollary. *Assume that* A*,* B*,* G *are of class* C^3 *and* $\Lambda_1\Lambda_2 < 0$ *and let* $M = \partial_{\mathbf{u}} B(\mathbf{0},\cdot)$*.*

- *If the system* (1.2)*,* (1.3) *has an energy-like Lyapunov function for the* C^1 *norm, then there exists a control* (1.3) *such that the system* (1.2)*,* (1.3) *has an energy-like Lyapunov function for the* H^2 *norm.*
- *If in addition* $M_{12}(x)M_{21}(x) \geq 0$ *on* $[0,L]$*, then the converse is true.*

In this corollary, the controls (1.3) that are used to obtain an energy-like Lyapunov function (and hence exponential stability) for the C^1 norm and the H^2 norm are not necessarily the same. However, it is possible to use the same control for both as shown in this proposition

Proposition 3.9. *Assume that* A*,* B*,* G *are of class* C^3*,* $\Lambda_1\Lambda_2 < 0$ *and let* $M = \partial_{\mathbf{u}} B(\mathbf{0},\cdot)$*. If there exists an energy-like Lyapunov function for the* C^1 *norm and if* G *has the form*

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ with } k_2^2 < \varphi(L)^2 \left(\frac{d_2(L)}{d_1(L)}\right)^2 \text{ and } k_1^2 < \left(\frac{d_1(0)}{d_2(0)}\right)^2, \tag{3.29}$$

where $(d_1, d_2) \in C^1([0,L],(0,+\infty))^2$ *is a solution of* (3.26)*, then there also exists an energy-like Lyapunov function for the* H^2 *norm. In particular the system is exponentially stable both for the* C^1 *and* H^2 *norm.*

Conversely, if in addition $M_{12}(\cdot)M_{21}(\cdot) \geq 0$ *on* $[0,L]$ *and the system has an energy-like Lyapunov function for the* H^2 *norm with* G *such that*

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ with } k_2^2 < \left(\frac{\varphi(L)}{\eta(L)}\right)^2 \text{ and } k_1^2 < \eta(0)^2, \tag{3.30}$$

where η *is a positive solution to* (3.18) *for some* $\sigma > 0$*, then the system also has an energy-like Lyapunov function for the* C^1 *norm. In particular the system is exponentially stable both for the* C^1 *and* H^2 *norm.*

In practice one might want to construct a Lyapunov function for the H^2 norm from the one for the C^1 norm and vice versa (when possible). To do so we introduce the following notations

$$\varphi_1 = \exp\left(\int_0^x \frac{M_{11}(s)}{\Lambda_1} ds\right),\ \varphi_2 = \exp\left(-\int_0^x \frac{M_{22}(s)}{|\Lambda_2|} ds\right), \tag{3.31}$$

such that φ given by (3.16) becomes $\varphi = \varphi_1/\varphi_2$. The construction is given by the following theorem.

Theorem 3.10. *Assume A, B, G are of class $C^p \cap C^{q+2}$, where $p \geq 2$ and $q \geq 1$ and $\Lambda_1\Lambda_2 < 0$ and let $M = \partial_{\mathbf{u}} B(\mathbf{0}, \cdot)$.. Assume in addition that there exists an energy-like Lyapunov function of the form* (3.8) *for the C^q norm and such that $Q = diag(g_1, g_2)$.*

Then for all $0 < \varepsilon < \min_{[0,L]}(\sqrt{g_1/g_2})/L\varphi$ where φ is given by (3.16)*, there exists a boundary control of the form* (1.3) *such that there exists an energy-like Lyapunov function of the form* (3.5) *for the H^p norm with*

$$Q = \begin{pmatrix} \frac{1}{\Lambda_1}\left(\sqrt{\frac{g_1}{g_2}}\varphi_1\varphi_2 - \varphi_1^2\varepsilon Id\right) & 0 \\ 0 & \frac{1}{|\Lambda_2|}\sqrt{\frac{g_2}{g_1}}\varphi_1\varphi_2 \end{pmatrix}, \tag{3.32}$$

and where φ_1 and φ_2 are given by (3.31) *and Id refers to the identity.*

Conversely if there exists an energy-like Lyapunov function of the form (3.5) *for the H^p norm with $Q = diag(q_1, q_2)$, and if $M_{12}M_{21} \geq 0$, then for any $\varepsilon > 0$ there is a boundary control of the form* (1.3) *such that there exists an energy-like Lyapunov function for the C^q norm of the form* (3.8) *where*

$$Q = \exp\left(2\int_0^x \frac{M_{11}(\cdot)}{\Lambda_1} - \frac{|M_{12}(\cdot)|}{\Lambda_1}\sqrt{\frac{|\Lambda_1|q_1}{|\Lambda_2|q_2}}ds - \varepsilon x\right)\begin{pmatrix} 1 & 0 \\ 0 & \frac{|\Lambda_2|q_2}{\Lambda_1 q_1} \end{pmatrix} \tag{3.33}$$

Corollary 3.1 tells us that the existence of an energy-like Lyapunov function for the H^2 norm implies the existence of an energy-like Lyapunov function for the C^1 norm. It also gives a condition under which the converse is true. Looking at the internal condition (3.27) for the C^1 norm and the internal condition (3.17) for the H^2 norm, we see that it is easy to design a counter-example when this condition $M_{11}M_{22} \geq 0$ is not satisfied: simply take $\Lambda_1 = |\Lambda_2| = 1$, $b = -1$ and $a = 1$, in this case one can find an energy-like Lyapunov function for the H^2 norm for any length L of the domain (with the appropriate control) while no energy-like Lyapunov function for the C^1 norm exists for $L > \pi/2$. One might think that such counterexamples would be purely artificial. Yet, surprisingly, very physical counterexamples do exist. Let us consider the Saint-Venant equations modelling flows under shallow water approximation such as navigable rivers or the atmosphere:

$$\begin{aligned} &\partial_t H + \partial_x(HV) = 0, \\ &\partial_t V + \partial_x\left(\frac{V^2}{2} + gH\right) + \left(\frac{kV^2}{H} - C\right) = 0, \end{aligned} \tag{3.34}$$

These equations will be introduced more properly in Section 4, but (3.34) correspond to a rectangular channel with a constant slope C and a friction coefficient k. We denote by (H^*, V^*) the steady-state we want to stabilize and we introduce the following boundary control

$$\begin{aligned} &(H(t,0) - H^*(0)) = b_1(V(t,0) - V^*(0)), \\ &(H(t,L) - H^*(L)) = b_2(V(t,L) - V^*(L)), \end{aligned} \tag{3.35}$$

where b_1 and b_2 are control parameters that can be chosen, corresponding for instance to hydraulic installations at the boundaries. Even though H and V are not the propagating quantities of this system, one can show that this boundary control is in fact equivalent to a boundary control of the form (1.3) after a change of variable to diagonalize the system [12, 94]. It can also be shown (see Section 4) that, with suitable boundary conditions, this system always have an energy-like Lyapunov function for the H^2 norm, whatever the domain length. On the other hand, when the influence of the slope is higher than the influence of the friction, there can never be a Lyapunov function for the C^1 norm above a certain domain length, as shown in [88].

Theorem 3.11. *Let (H^*, V^*) be a steady-state on $[0, L]$ of* (3.34) *and $kV^{*2}(0)/H^*(0) < C$. Then :*

1. *There exists $L_1 > 0$ such that for any $L < L_1$, there exists a control of the from* (3.35) *such that the system* (3.34), (3.35) *has an energy-like Lyapunov function for the C^1 norm.*

2. *There exists $L_2 > 0$ such that if $L > L_2$, then for any control of the form* (3.35) *the system* (3.34), (3.35) *does not have an energy-like Lyapunov function for the C^1 norm.*

Higher norms. Finally, all the previous results for 2×2 and $n \times n$ systems can be extended at no cost and with the same conditions to the H^p and C^q norm for $p \geq 2$ and $q \geq 1$, provided that A, B and G are regular enough.

Particular case of semilinear systems. Let us consider again a $n \times n$ system of the form (1.2), (1.3) with $n \in \mathbb{N} \setminus \{0\}$. When the system is semilinear, i.e. $A(\mathbf{u}, x) = A(\mathbf{0}, x) = \Lambda(x)$, then the nonlinearity is much less of a problem and the results can be further extended: Theorem 3.3–3.10 even hold true for the H^1 and C^0 norm instead of the H^2 norm and the C^1 norm.

If, in addition, the source term B is Lipschitz in the following sense:

$$\|B(\mathbf{u}, \cdot) - B(\mathbf{v}, \cdot)\|_{L^2} \leq C_B \|\mathbf{u} - \mathbf{v}\|_{L^2}, \ \forall \ (\mathbf{u}, \mathbf{v}) \in L^2(0, L)^2, \tag{3.36}$$

and G is also Lipschitz, i.e. there exists a matrix K such that for any $i \in \{1, ..., n\}$,

$$\left| G_i \begin{pmatrix} \mathbf{u}_+(t, L) \\ \mathbf{u}_-(t, 0) \end{pmatrix} \right| \leq \sum_{j=1}^{m} K_{ij} |u_j(t, L)| + \sum_{j=m+1}^{n} K_{ij} |u_j(t, 0)|, \tag{3.37}$$

then it is possible to show a stronger result: a *global* exponential stabilization for the L^2 norm (which extend to the H^p norm for $p \geq 0$).

Theorem 3.12. *Assume that $\Lambda \in C^1([0, L])$, B and G are Lipschitz with respect to $\mathbf{u}$ in the sense of* (3.36) *and* (3.37)*. If the following conditions are satisfied*

1. *(Interior condition) there exists a diagonal matrix $J \in M_n(\mathbb{R})$ with positive coefficients and a matrix $M \in M_n(\mathbb{R})$ such that one of the two following holds*

$$C_g < \frac{\lambda_m}{\max_{i,x}(J_i^2)}, \quad \text{or} \quad C_g < \mu_m \frac{\max_{i,x}(J_i)}{\inf_{i,x}(J_i)}, \tag{3.38}$$

where C_g is the Lipschitz constant of $B - M$ and λ_m denotes the smallest eigenvalue of

$$-(\Lambda J^2)' + J^2 M + M^T J^2 \tag{3.39}$$

and μ_m denotes the smallest eigenvalue of

$$-J^{-1}(\Lambda J^2)' J^{-1} + JMJ^{-1} + J^{-1} M^T J, \tag{3.40}$$

2. *(Boundary condition) the matrix*

$$\begin{pmatrix} J_+^2(L)\Lambda_+(L) & 0 \\ 0 & J_-^2(0)|\Lambda_-(0)| \end{pmatrix} - K^T \begin{pmatrix} J_+^2(0)\Lambda_+(0) & 0 \\ 0 & J_-^2(L)|\Lambda_-(L)| \end{pmatrix} K \tag{3.41}$$

is positive semidefinite, where K is given by (3.37)*,*

then the system is globally exponentially stable for the L^2 norm. Moreover the gain[4] of the exponential stability estimate is $\|J^{-1}\|_{L^\infty} \|J\|_{L^\infty}$.

[4]The gain of an exponential stability estimate is defined as the constant $C > 0$ such that $\|\mathbf{u}(t, \cdot)\|_X \leq Ce^{-\gamma t} \|\mathbf{u}(\mathbf{0}, \cdot)\|_X$.

Even though the conditions are more complicated to write than the conditions of Theorem 3.4, we see that there are still two conditions: an internal condition intrinsic to the system, and a boundary condition on the control. Of course, it would be possible to derive a much simpler sufficient condition by taking $M = 0$. However, such condition is would be much more conservative. This result, shown in [92], is striking in that it allows to deal with the L^2 norm, and in that it is global. It also has another interesting specificity: it still holds even if the source term B is nonlocal. In this case, the source term B is defined as a continuous function on $L^2(0, L)$ with value in $L^2(0, L)$ such that $B(\mathbf{0}, \cdot) = 0$. This could be useful as nonlocal source terms are found in several phenomena such as material sciences, traffic flow, flocking, population dynamics etc. [24, 139, 20].

3.2. Input-to-State Stability

In recent years, there was an interest to know whether these energy-like Lyapunov functions could also bring ISS estimates and, if so, whether it could improve the ISS estimates that were known so far. The answer turned out to be yes to both.

The notion of ISS was first introduced for finite dimensional systems by Sontag in [138]. It was later extended to time-delay systems, then to PDEs [98, Chapter 1]. Several results were then derived in many different cases [130, 57, 123, 58, 124, 125] (see [125] for a more detailed overview). However, until recently no generic result existed for hyperbolic systems, and the best known result was the shown by Karafyllis and Krstic in [98, Section 9.4] for 2×2 systems using a small-gain analysis (see Theorem 3.16 below).

It turns out that the exponential stability results presented in the previous subsection and obtained with energy-like Lyapunov functions generalize to ISS. In [16] it was shown the following

Theorem 3.13. *Assume that A, B and G are of class C^q with $q \in \mathbb{N} \setminus \{0\}$ and let $M = \partial_{\mathbf{u}} B(\mathbf{0}, \cdot)$. The system (1.5), (1.6) is ISS for the C^q norm if the two following properties are satisfied:*

- *(internal condition) there exists a solution $(f_1, ...f_n) \in C^1([0, L], (0, +\infty))^n$ to the system*

$$\Lambda_i f_i' \leq -2 \left(-M_{ii}(x) f_i + \sum_{k=1, k\neq i}^{n} |M_{ik}(x)| \frac{f_i^{3/2}}{\sqrt{f_k}} \right), \quad \forall\, i \in \{1, ..., n\}, \tag{3.42}$$

- *(boundary condition) there exists a matrix Δ with positive components such that*

$$\|\Delta G'(0) \Delta^{-1}\|_\infty < \frac{\inf_i \left(\frac{f_i(d_i)}{\Delta_i^2} \right)}{\sup_i \left(\frac{f_i(L-d_i)}{\Delta_i^2} \right)}, \tag{3.43}$$

where $d_i = L$ if $\Lambda_i > 0$ and $d_i = 0$ otherwise.

Surprisingly, in this framework one is unable to obtain the usual differential inequality of standard Lyapunov theory, which would be in this case

$$\frac{dV(\mathbf{u}(t, \cdot))}{dt} \leq -CV(\mathbf{u}(t, \cdot)) + \gamma(\sum_{k=0}^{q} \sup_{\tau \in [0,t]} |\mathbf{d}^{(k)}(\tau)|), \tag{3.44}$$

where V denotes the Lyapunov function, C is a positive constant and γ is a class $\mathcal{K}$ function. But, nevertheless, Theorem 3.13 still holds.

Remark 3.1 (Semilinear systems). As previously, this extends under the same condition to the ISS for the C^0 norm when the system is semi-linear, i.e. $A(\mathbf{u}, x) = A(\mathbf{0}, x) = \Lambda(x)$.

The ISS in the H^2 norm using the same basic quadratic Lyapunov function as [12] was studied in [148] where they show the following, as expected[5].

Theorem 3.14. *Assume that A, B and G are of class C^p and let $M = \partial_{\mathbf{u}} B(\mathbf{0}, \cdot)$. The system* (1.5), (1.6) *is ISS for the H^p norm if there exists a diagonal matrix Q with positive entries such that*

- *(internal condition) the matrix*

$$-(Q\Lambda)'(x) + Q(x)M(x) + M(x)^T Q(x)^T \tag{3.45}$$

  is positive definite for any $x \in [0, L]$,

- *(boundary condition) the matrix*

$$\begin{pmatrix} \Lambda_+(L)Q_+(L) & 0 \\ 0 & -\Lambda_-(0)Q_-(0) \end{pmatrix} - G'(0)^T \begin{pmatrix} \Lambda_+(0)Q_+(0) & 0 \\ 0 & -\Lambda_-(L)Q_-(L) \end{pmatrix} G'(0) \tag{3.46}$$

  is semi-definite positive.

This case is simpler than the ISS in C^p norm as, in this case, it is possible to obtain a differential inequality similar to (3.44). Once again, if the system is semilinear the ISS holds in the H^2 norm, and in the L^2 norm if the system is linear. The linear case was treated shown in [73] where they also provide an optimization approach to minimize the ISS gain. Finally, ISS can also be shown in the L^2 norm when the system is semilinear with a (potentially nonlocal) Lipschitz source term and Lipschitz boundary conditions, as shown in [92].

Theorem 3.15. *Let a system of the form* (1.5), (1.6) *where $\Lambda \in C^1([0, L])$, $\mathbf{d}_1 \in L^2((0, T) \times (0, L))$, $\mathbf{d}_2 \in H^1([0, T])$, B and G are Lipschitz with respect to $\mathbf{u}$ in the sense of* (3.36), (3.37). *If the condition* (3.38) *is satisfied and the matrix defined by* (3.41) *is positive definite, then the system is globally ISS for the L^2 norm.*

As expected, in this case the ISS is even global.

Coming back to the ISS in sup norms (i.e. C^q norms), it is possible to compare the conditions of Theorem 3.13 with the previous existing conditions from [98] for linear 2×2 systems with boundary disturbances, which are,

Theorem 3.16 (Karafyllis, Krstic, 2019)**.** *Assume that $n = 2$, $d_1 \equiv 0$, $A(\mathbf{u}, x) = \Lambda$ with $\Lambda_1 > 0 > \Lambda_2$, $B(\mathbf{u}, x) = M(x)\mathbf{u}$, $G(\mathbf{U}) = K\mathbf{U}$ where*

$$M(x) = \begin{pmatrix} 0 & a(x) \\ b(x) & 0 \end{pmatrix}, K = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix}, \tag{3.47}$$

and assume that[6] $L = 1$. The system (1.5), (1.6) *is ISS for the C^0 norm if there exists $K > 0$ such that*

$$\begin{gathered} (|k_1| + |k_2|)\exp(-K) < 1, \\ \left(\sqrt{\frac{\exp(2K) - \exp K}{|\Lambda_2|K} B} + \sqrt{|k_2|}\right)\left(\sqrt{\frac{1 - \exp(-K)}{\Lambda_1 K} A} + \sqrt{|k_1|}\right) < 1, \\ \text{where } A := \max_{0 \le z \le 1} |a(z)\exp(2Kz)| \text{ and } B := \max_{0 \le z \le 1} |b(z)\exp(-2Kz)|. \end{gathered} \tag{3.48}$$

In [16] is shown the following:

[5] In fact they only show it for the linear case and the L^2 norm but this can be directly extended to the nonlinear system and the H^2 as in [41] for instance.

[6] this last assumption can be made without loss of generality by a rescaling.

Proposition 3.17. *Consider the setting of Theorem 3.16 and assume in addition that a and b are constant. Suppose there exists $K > 0$ such that* (3.48) *holds, then the two conditions* (3.42), (3.43) *of Theorem 3.13, are satisfied.*

Interestingly, this is a strict implication and the converse is false in general. However, the comparison of the Theorems 3.16 and 3.13 in general when a and b are not constant remains an open question that it would be interesting to look at, to see how complementary the small-gain analysis and the energy-like Lyapunov functions are.

3.3. Perspectives

In view of the previous results, many open questions remain. Among them, we can list the following

- Is it possible to find a result similar to Theorems 3.4 and 3.5 for other norms such as the $W^{2,p}$ norms for $p \in \mathbb{N}$? On the one hand, the energy-like Lyapunov function would be easier to handle than the energy-like Lyapunov function for the C^1 norm. But, on the other hand, the internal condition would require dealing with a p order polynomial which is different to what was done for the H^2 or C^1 norm in [12] and [89].
- Is it possible to find a condition similar to (3.17), maybe with two equations, for systems of higher degree, namely 3×3 systems?
- Comparing the result obtained using a small gain analysis (Theorem 3.16) and using an energy-like Lyapunov function (Theorem 3.13) in the general case, i.e. when a and b are not constants.
- Is it possible to find a system such that there exists no energy-like Lyapunov function for the H^2 (resp. C^1 norm) but that is still exponentially stable? For the linearized system this is very likely to be true, using spectral tools as shown in [12, Section 5.6] (see also [80, 72])
- What happens for 2D systems? Is it possible to find a result similar to Theorems 3.4 and 3.5, at least when the system in 2×2 and when the two transport terms (in x and y) are co-diagonalizable? If so, what happens if the two transport terms are not co-diagonalizable?
- More generally, are energy-like Lyapunov functions as adapted for 2D systems as they are for 1D systems? And otherwise, what new tools could be developed?. The same question arises for the backstepping method, where the Volterra transform commonly used is fundamentally 1D.
- Is it possible to adapt other promising methods such as the frequency Lyapunov method introduced in [151] to the hyperbolic case?

4. Density-velocity systems

Density-velocity systems are an important example of hyperbolic systems. They consist in two equations: a mass conservation and a momentum equation. They owe their name to the two variables they involve: a density ρ and a velocity V.

$$\begin{aligned} &\partial_t \varrho + \partial_x(\varrho V) = 0, \\ &\partial_t V + V\partial_x V + \partial_x(P(\varrho, x)) + S(\varrho, V, x) = 0, \end{aligned} \tag{4.1}$$

The first equation is the mass conservation. In the second equation $V\partial_x V$ is a kinetic term, $\partial_x(P(A, x))$ is the variation of potential energy and represents the conservative forces, while $S(A, V, x)$ represents the non-conservative forces. When $\partial_\varrho P(\varrho, x)\varrho < V^2$, the system has two propagation speeds with same sign, which means that it can be treated with the results of Section 3. Therefore, in the following we assume that

$$\partial_\varrho P(\varrho, x)\varrho > V^2, \tag{4.2}$$

which means that the system has two propagation speeds with opposite signs. This is usually called subcritical regime. We also assume that S and P are of class C^2, and in the following we will consider steady-state where $\varrho^* \geq 0$ and $V^* > 0$, to be physically acceptable. The case $V^* = 0$ is discussed at the end of this section.

There are many important examples of density-velocity systems in physics and mechanics. For instance the Saint-Venant equations [10]

$$\begin{aligned} &\partial_t A + \partial_x(AV) = 0, \\ &\partial_t(AV) + \partial_x(AV^2) + gA(\partial_x H - S_b(x) + S_f(A, V, x)) = 0, \end{aligned} \tag{4.3}$$

where A is the wet section, V is the (vertically averaged) velocity of the water, g is the gravity acceleration, S_b is the slope and S_f the friction; The isentropic Euler equations [87, 36]

$$\begin{aligned} &\partial_t \varrho + \partial_x(\varrho V) = 0, \\ &\partial_t V + V\partial_x V + \frac{\partial_x(\mathcal{P}(\varrho))}{\varrho} + \frac{1}{2}\theta V|V| + g\sin\alpha(x) = 0, \end{aligned} \tag{4.4}$$

where ϱ is the fluid density, θ is a friction coefficient, $\sin(\alpha)$ is the slope. The water in a rigid pipe [12, Section 1.6]

$$\begin{aligned} \partial_t\left(\exp\left(\frac{g\mathcal{H}}{c^2}\right)\right) + \partial_x\left(V\exp\left(\frac{g\mathcal{H}}{c^2}\right)\right) = 0, \\ \partial_t V + V\partial_x V + \partial_x(g\mathcal{H}) + S_f(V, x) = 0, \end{aligned} \tag{4.5}$$

where $\mathcal{H}$ is called the piezometric head, while c is the sound velocity in water, and S_f is the friction term; Some traffic flow models [112, 95]

$$\begin{aligned} &\partial_t \varrho + \partial_x(\varrho v) = 0, \\ &\partial_t v + v\partial_x v + \varrho(\partial_\varrho \bar{v}(\varrho))^2 \partial_x \varrho = \frac{\bar{v}(\varrho) - v}{\tau}. \end{aligned} \tag{4.6}$$

This is a non exhaustive list, density-velocity systems also represent several other systems such as the Savage-Hutter equations [27, 85], the flow under osmosis [120], etc.

We consider the following boundary conditions

$$\begin{aligned} V(t,0) - V^*(0) = G_1(\varrho(t,0) - \varrho^*(0)), \\ V(t,L) - V^*(L) = G_2(\varrho(t,L) - \varrho^*(L)), \end{aligned} \tag{4.7}$$

where G_1 and G_2 are the boundary controls to be chosen and (ϱ^*, V^*) is the steady-state considered. Our goal is to use an energy-like Lyapunov function for the H^2 norm to stabilize the steady-state (ϱ^*, V^*). This problem was considered in many articles for different particular cases. The first result on the nonlinear homogeneous Saint-Venant equations goes back to [43] in 1999. The approach needs a LaSalle invariance principle to conclude and was later improved in [44] that works for any homogeneous density-velocity system. In [69, 131] the authors looked again at Saint-Venant equations but this time with a source term, although with a bound on this source term in C^1 norm depending on the length L. This bound can also be seen as a bound on the length L instead, and the proof used a careful analysis of the characteristics. Other estimates when the source term is small and the solutions are $W^{1,\infty}$ were given in [85]. The linear inhomogeneous Saint-Venant equations were then studied in [15] in the particular case where the steady-state is constant. In [67], the authors managed to stabilize the isentropic Euler equations for the H^1 norm when the system is semi-linear and adding an isothermal assumption, i.e. assuming that the pressure P is proportional to the density ϱ. These equations with the isothermal assumption were also studied when there are uncertainties on the boundary condition in [86] and in the case of a network in [66, 79, 81, 65]. In [82], the authors dealt

with these equations in the case of a slowly moving gas with a kind of Neumann feedback law. In [84], the authors managed to stabilize the isentropic Euler equations for the H^2 norm but only if the length L of the domain satisfies some bound. In this article they give in addition an estimation on the maximal decay rate. The case of the flow in a rigid pipe was studied in [12]. In [13] the authors managed to stabilize a generic density-velocity systems in the H^2 norm in the case where the source term is purely dissipative and applied it to two examples: the Euler isentropic equations and the Saint-Venant equations without slope. Finally, in [94, 95] were found some results that allowed to deal with density-velocity systems in general, without the previous assumptions or particular cases. This is what we present now.

Our goal is to find an energy-like Lyapunov function for the H^2 norm. From the result of [11] recalled in Section 3 (Theorem 3.3), we know that finding such a Lyapunov function amounts to show that there is a solution to the equation

$$\begin{aligned} \eta'(x) &= \left| \frac{a(x)}{\lambda_1(x)} + \frac{b(x)}{\lambda_2(x)}\eta^2(x) \right|, \\ \eta(0) &> 0, \end{aligned} \tag{4.8}$$

where

$$\lambda_1(x) = \sqrt{\partial_\varrho P(\varrho^*, x)\varrho^*(x)} + V^*,\ \lambda_2 = \sqrt{\partial_\varrho P(\varrho^*, x)\varrho^*(x)} + V^* \tag{4.9}$$

and where a and b are given by (3.15) as in Section 3. Here a and b have the rather complicated expression:

$$\begin{aligned} a(x) &= \varphi(x)\delta_1(x), \\ b(x) &= \varphi(x)^{-1}\gamma_2(x), \end{aligned} \tag{4.10}$$

where

$$\varphi(x) = \exp\left(\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)} + \frac{\delta_2(s)}{\lambda_2(s)} ds \right) \tag{4.11}$$

and

$$\begin{aligned} \gamma_1 =& \frac{1}{4}\left(2S_{V^*} + 2\mathscr{S}_{\varrho^*}\sqrt{\frac{\varrho^*}{\partial_\varrho P(\varrho^*, x)}} + 3\lambda_2\frac{\varrho^*_x}{\varrho^*} - \lambda_1\frac{\partial^2_{\varrho x}P(\varrho^*, x)}{\partial_\varrho P(\varrho^*, x)} + \lambda_2\frac{\partial^2_{\varrho\varrho}P(\varrho^*, x)\varrho^*_x}{\partial_\varrho P(\varrho^*, x)} \right), \\ \gamma_2 =& \frac{1}{4}\left(2S_{V^*} + 2\mathscr{S}_{\varrho^*}\sqrt{\frac{\varrho^*}{\partial_\varrho P(\varrho^*, x)}} - \lambda_1\frac{\varrho^*_x}{\varrho^*} - \lambda_2\frac{\partial^2_{\varrho x}P(\varrho^*, x)}{\partial_\varrho P(\varrho^*, x)} + \lambda_1\frac{\partial^2_{\varrho\varrho}P(\varrho^*, x)\varrho^*_x}{\partial_\varrho P(\varrho^*, x)} \right), \\ \delta_1 =& \frac{1}{4}\left(2S_{V^*} - 2\mathscr{S}_{\varrho^*}\sqrt{\frac{\varrho^*}{\partial_\varrho P(\varrho^*, x)}} + \lambda_2\frac{\varrho^*_x}{\varrho^*} + \lambda_1\frac{\partial^2_{\varrho x}P(\varrho^*, x)}{\partial_\varrho P(\varrho^*, x)} - \lambda_2\frac{\partial^2_{\varrho\varrho}P(\varrho^*, x)\varrho^*_x}{\partial_\varrho P(\varrho^*, x)} \right), \\ \delta_2 =& \frac{1}{4}\left(2S_{V^*} - 2\mathscr{S}_{\varrho^*}\sqrt{\frac{\varrho^*}{\partial_\varrho P(\varrho^*, x)}} - 3\lambda_1\frac{\varrho^*_x}{\varrho^*} + \lambda_2\frac{\partial^2_{\varrho x}P(\varrho^*, x)}{\partial_\varrho P(\varrho^*, x)} - \lambda_1\frac{\partial^2_{\varrho\varrho}P(\varrho^*, x)\varrho^*_x}{\partial_\varrho P(\varrho^*, x)} \right). \end{aligned} \tag{4.12}$$

Looking at (4.8), or recalling Section 3, we see that this equation is quadratic and might *a priori* explode in finite length. Therefore, we understand why most articles on density-velocity systems assume that the source term is not too large or that the length of the system is not too large. We can also understand why, in [13], the authors managed to show that the exponential stability can be guaranteed without bounds on the length of the domain when the source term is purely dissipative: intuitively the source term should only help; moreover the length of the domain cannot be too long anyway since the steady-state cease to exists in finite time (see [12, 94]). The only way to go beyond this would be that the special structure of the density-velocity systems allows to have a solution to (4.8) in any cases. Remarkably, this turns out to be true. And this is what is shown in [94, 95]. The key lemma is the following

Lemma 4.1. *The function $x \to \lambda_2\phi(x)/\lambda_1(x)$ is a solution to* (4.8)*, if*

$$\partial_V S(H^*, V^*, \cdot) - V^* \frac{\partial_H S(H^*, V^*, \cdot)}{\partial_H P(H^*, \cdot)} \geq 0, \quad \forall x \in [0, L]. \tag{4.13}$$

Note that the condition (4.13) is automatically satisfied for a friction term, as friction is non-decreasing with speed. It is also automatically satisfied for a slope or any external force. Surprisingly, for the traffic flow model (4.6) this condition (4.13) corresponds exactly the condition (4.2) of subcritical regime (also known as subsonic flows [81, 83] in fluid mechanics, or congested regime for traffic flow models). This condition (4.13) might have a general physical meaning that we did not see. Overall it is satisfied for all the examples of density-velocity systems mentioned earlier.

With this lemma, it becomes possible to show the following result

Theorem 4.1. *The steady-state (H^*, V^*) of the system* (4.1)*,* (4.7) *is exponentially stable for the H^2 norm if the boundary conditions satisfy:*

$$G_1'(\varrho^*(0)) \in \left[-\frac{\partial_\varrho P(\varrho^*(0), 0)}{V^*(0)}, -\frac{V^*(0)}{\varrho^*(0)} \right], \tag{4.14}$$

$$G_2'(\varrho^*(L)) \in \mathbb{R} \setminus \left[-\frac{\partial_\varrho P(\varrho^*(L), L)}{V^*(L)}, -\frac{V^*(L)}{\varrho^*(L)} \right] \tag{4.15}$$

Note that there is another remarkable thing about this theorem: the conditions (4.14), (4.15) do not depend directly on the source term. This means that the system can be stabilized even with only a partial knowledge of the model itself. This is very interesting as one never really knows what is S in practice. For Saint-Venant equations, for instance, there is often a debate about which model to choose for the friction [35, Section 4.5]. What Theorem 4.1 says is that this does not matter for stabilizing the system.

This result is in some sense optimal when using energy-like Lyapunov function. Indeed, if one wants to preserve the validity for any length of domain and any source term, provided that a steady-state exists, then it is essentially not possible to derive better boundary conditions, as illustrated from the two following theorems [95]. We have in any case

Theorem 4.2. *Let $L > 0$. There exists $G_2'(\varrho^*(L))$ satisfying* (4.15) *such that, if there exists a basic quadratic Lyapunov function for the H^2 norm, then*

$$G_1'(\varrho^*(0)) \in \left[-\frac{\partial_\varrho P(\varrho^*(0), 0)}{V^*(0)}, -\frac{V^*(0)}{\varrho^*(0)} \right]. \tag{4.16}$$

And, when the steady-state (H^*, V^*) exists on $(0, +\infty)$, we have

Theorem 4.3. *For any $\varepsilon > 0$, there exists $L > 0$ such that, if there exists a basic quadratic Lyapunov function for the H^2 norm, then*

$$G_2'(H^*(L)) \in \mathbb{R} \setminus \left[\varepsilon - \frac{\partial_\varrho P(\varrho^*(L), L)}{V^*(L)}, -\varepsilon - \frac{V^*(L)}{\varrho^*(L)} \right]. \tag{4.17}$$

What is more, Lemma 4.1 also allows to stabilize the system with only a single control. For instance, if the boundary condition downstream remains a control but the upstream boundary condition is imposed, we have at the boundaries

$$\begin{aligned} \varrho(t, 0)V(t, 0) &= Q_0, \\ V(t, L) - V^*(L) &= G_2(\varrho(t, L) - \varrho^*(L)), \end{aligned} \tag{4.18}$$

where Q_0 is the constant imposed inflow and G_2 is the single control. In this case, the following theorem holds

Theorem 4.4. *The steady-state (H^*, V^*) is exponentially stable for the H^2 norm if the boundary control satisfies*

$$G_2'(\varrho^*(L)) \in \mathbb{R} \setminus \left[-\frac{\partial_\varrho P(\varrho^*(L), L)}{V^*(L)}, -\frac{V^*(L)}{\varrho^*(L)} \right]. \tag{4.19}$$

In this case the optimality condition is clearer

Theorem 4.5. *If there exists a basic quadratic Lyapunov function for the H^2 norm for system* (4.1), (4.18), *then*

$$\mathcal{B}_2'(\varrho^*(L)) \in \mathbb{R} \setminus \left[-\frac{\partial_\varrho P(\varrho^*(L), L)}{V^*(L)}, -\frac{V^*(L)}{\varrho^*(L)} \right]. \tag{4.20}$$

Note that this theorem holds independently of (4.13).

Particular case $V^* \equiv 0$.. Theorem 4.1–4.5 also hold in the particular case where $V^* \equiv 0$. In such a case, $-\partial_\varrho P(\varrho^*, \cdot)/V^*$ is replaced by $-\infty$ in the conditions (4.14), (4.15), (4.16), (4.17), (4.19), (4.20).

4.1. Perspective

While the Lyapunov function obtained from Lemma 4.1 and used in Theorem 4.1 opens many doors, there are still many open questions

- Is there a physical meaning behind the condition (4.13)? Are there density-velocity systems in mechanics and physics such that the condition (4.13) is not satisfied?
- Is it possible to consider more complicated geometries such as networks? In particular, would it be still possible to control or stabilize a density-velocity system with less control than the number of vertices in the network? This would represent for instance a junction of rivers where we can only control one end of the rivers, or a traffic network where we can only control one red light.
- Would the results be preserved if a small viscosity was added to the density-velocity systems? The nature of the system would change, but this is an interesting question as a small viscosity is likely to occur in many areas.
- What about 2D density-velocity systems, where there are two space variables x and y? Is it still possible in this framework to find a Lyapunov function that allows to stabilize density-velocity systems for any length?
- The basic quadratic Lyapunov function we use has the form (3.5) and could be seen as the linearization of a more complicated function (see Section 3). In the particular case of density-velocity systems, is there such a Lyapunov function that would enable a global stabilization? In other words, is it possible that the Lyapunov function we found is only the local approximation of a more powerful Lyapunov function? The question is worth asking since the system has a global entropy when the system is homogeneous.
- What can be done with these energy-like Lyapunov functions for coupled hyperbolic - parabolic systems, that typically represent interactions between fluid and nonrigid structure ? Several results exist (e.g. [156, 7, 98]), but, to our knowledge, no generic result exist when the system is nonlinear and inhomogeneous, even in 1D.

5. Stabilizing shock steady-states

Nonlinear hyperbolic systems have an important feature: they can make discontinuities appear naturally even when the initial condition is smooth [104, 105, 29]. These discontinuities, called shocks, play an important role in their behaviors and are the source of many difficulties in their analysis [55]. These shocks also correspond to some physically significant phenomena such as the hydraulic jump in rivers, the morning glory cloud in the atmosphere [39], or stop-and-go waves in traffic flows [134].

Shocks can prove complicated to handle to a point where even showing the well-posedness of a hyperbolic system requires involved tools and relatively long proofs [1, 30] (see also [114, 115, 93, 76]). No wonder that stabilizing a system with shocks is a challenge. In the past, several works have considered this problem. In [101, 137], for instance, the authors consider regular solutions and regular steady-states but whose profiles are close to a shock. Other works have addressed systems with potentially discontinuous solutions. In [30], the authors study the controllability of a hyperbolic homogeneous system for BV class solutions (i.e. functions with bounded variations), which is usually the most general class of functions with shocks for hyperbolic systems [77, 34] (see also [45] for a proper definition). In [33], the authors looked at the optimal control problem for the Burgers' equation. The stabilization of a scalar equation is treated in [26, 129]. The stabilization of a homogeneous hyperbolic system is studied in [30, 45]. But, in these articles, the steady-state to stabilize is regular (and even constant). Ideally, one would like to be able to stabilize a steady-state with a shock, such as for example a hydraulic jump. In this section we present a method to stabilize such a steady-state when there is a single shock. Subsection 5.1 illustrate this method on a scalar equation (such as Burgers' equation) and Subsection 5.2 present the results when applied to an example of system, namely Saint-Venant equations.

5.1. Method on a scalar equation

To illustrate this method we first start with Burgers' equation. The system we consider is the following

$$\partial_t y + \partial_x \left(\frac{y^2}{2} \right) = 0 \tag{5.1}$$

with the boundary controls

$$\begin{aligned} y(t, 0^+) &= u_0(t) \\ y(t, L^-) &= u_L(t). \end{aligned} \tag{5.2}$$

As announced, we are interested with states y that have a single shock. If we denote its location by x_s the Rankine-Hugoniot conditions impose the following [105, 55]

$$\dot{x}_s(t) = \frac{y(t, x_s(t)^+) + y(t, x_s(t)^-)}{2}. \tag{5.3}$$

For such a hyperbolic system we usually require an additional entropy condition for the system to be well-posed (see [105, Section 3] for more details). In the simple case (5.1), this condition is given by

$$y(t, x_s(t)^-) \leq y(t, x_s(t)^+). \tag{5.4}$$

Our goal is to stabilize an entropic steady-state with a shock (see Definition 5.1 below). Without loss of generality, we can assume that this steady-state is

$$\begin{aligned} y^*(x) &= \begin{cases} 1, & x \in [0, x_0), \\ -1, & x \in (x_0, L], \end{cases} \\ x_s^* &= x_0. \end{aligned} \tag{5.5}$$

Clearly, if we only want to stabilize the amplitude y before and after the shock, we could choose the constant controls $u_0(t) = -u_L(t) = 1$. The main difficulty is to stabilize completely the steady-state: both the amplitude of y, and the shock location x_s. To do so we choose the feedback

$$\begin{aligned} u_0(t) &= 1 + k_1(y(t, x_s(t)^-) - 1) + b_1(x_0 - x_s(t)), \\ u_L(t) &= -1 + k_2(y(t, x_s(t)^+) + 1) + b_2(x_0 - x_s(t)), \end{aligned} \tag{5.6}$$

where k_1, k_2, b_1, b_2 are design parameters that can be chosen. Note that this is likely to be the simplest possible control for such a task: assume that the system is close to the steady-state, then from (5.1), (5.5) the propagation speed is positive before the shock and negative after the shock, so no information goes through the shock. This means that, to stabilize both sides, we need to measure the state at least at one point on each side, and this is what is used by the first feedback term of (5.6). On the other hand, since we want to stabilize the location of the shock using a feedback, we need at least to know its location[7] x_s. This is what is used by the second feedback term of (5.5).

As shown in [18, Appendix] by a (somehow lengthy) characteristic method, the system is well-posed when starting with a single shock. More importantly, it keeps only a single shock.

Theorem 5.1. *For all $T > 0$, there exists $\delta(T) > 0$ such that, for every initial condition $x_{s,0} \in (0, L)$ and $y_0 \in H^2((0, x_{s0}); \mathbb{R}) \cap H^2((x_{s0}, L); \mathbb{R})$ satisfying the first-order compatibility conditions associated to* (5.1) *with*

$$\begin{aligned} &\|y_0 - 1\|_{H^2((0,x_{s0});\mathbb{R})} + \|y_0 + 1\|_{H^2((x_{s0},L);\mathbb{R})} \leq \delta, \\ &|x_{s0} - x_0| \leq \delta, \end{aligned} \tag{5.7}$$

the system (5.1)–(5.3), (5.6) *has a unique piecewise continuously differentiable entropy solution $y \in C^0([0,T]; H^2((0, x_s(t)); \mathbb{R}) \cap H^2((x_s(t), L); \mathbb{R}))$ satisfying* (5.4) *with $x_s \in C^1([0,T]; \mathbb{R})$ its single shock. Moreover, there exists $C(T)$ such that the following estimate holds for all $t \in [0, T]$*

$$\begin{aligned} &\|y(t,\cdot) - 1\|_{H^2((0,x_s(t));\mathbb{R})} + \|y(t,\cdot) + 1\|_{H^2((x_s(t),L);\mathbb{R})} + |x_s(t) - x_0| \\ &\qquad \leq C(T) \left(\|y_0 - 1\|_{H^2((0,x_{s,0});\mathbb{R})} + \|y_0 + 1\|_{H^2((x_{s,0},L);\mathbb{R})} + |x_{s,0} - x_0| \right). \end{aligned} \tag{5.8}$$

This allows to give the following definition for the exponential stability we would like to achieve:

Definition 5.1. The steady-state $(y^*, x_0) \in (H^2((0, x_0); \mathbb{R}) \cap H^2((x_0, L); \mathbb{R})) \times (0, L)$ is exponentially stable for the H^2 norm with decay rate γ, if there exists $\delta > 0$ and $C > 0$ such that, for any $T > 0$, any initial condition $y_0 \in H^2((0, x_{s0}); \mathbb{R}) \cap H^2((x_{s0}, L); \mathbb{R})$, and any initial shock location $x_{s,0} \in (0, L)$ satisfying

$$\begin{aligned} &\|y_0 - y_1^*(0,\cdot)\|_{H^2((0,x_{s,0});\mathbb{R})} + \|y_0 - y_2^*(0,\cdot)\|_{H^2((x_{s,0},L);\mathbb{R})} \leq \delta, \\ &|x_{s,0} - x_0| \leq \delta, \end{aligned} \tag{5.9}$$

and the first order compatibility conditions associated to (5.1), the system (5.1), (5.3), (5.2) has a unique solution $(y, x_s) \in C^0([0,T]; H^2((0, x_s(t)); \mathbb{R}) \cap H^2((x_s(t), L); \mathbb{R})) \times C^1([0,T]; \mathbb{R})$ and

$$\begin{aligned} &\|y(t,\cdot) - y_1^*(t,\cdot)\|_{H^2((0,x_s(t));\mathbb{R})} + \|y(t,\cdot) - y_2^*(t,\cdot)\|_{H^2((x_s(t),L);\mathbb{R})} + |x_s(t) - x_0| \\ &\quad \leq Ce^{-\gamma t} \left(\|y_0 - y_1^*(0,\cdot)\|_{H^2((0,x_{s,0});\mathbb{R})} + \|y_0 - y_2^*(0,\cdot)\|_{H^2((x_{s,0},L);\mathbb{R})} + |x_{s,0} - x_0| \right), \ \forall t \in [0, T). \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} y_1^*(t,x) &= y^*\left(x \frac{x_0}{x_s(t)} \right), \\ y_2^*(t,x) &= y^*\left(\frac{(x-L)x_0}{x_s(t) - L} \right), \end{aligned} \tag{5.11}$$

[7] In fact, we could also have indirect measurements, in this case the system would also be coupled with another quantity.

This definition might seem a little complicated but it says exactly what we would expect intuitively: the solution has to converge exponentially to the steady-state value before and after the shock and the shock location has to converge exponentially to the shock location of the steady-state. In the following we will still refer to this as "exponential stability for the H^2 norm", given that the solutions are H^2 except at the shock. The main difficulties in showing such a stability are:

- The solutions are not regular.
- We have a direct control on the amplitude of y at each boundary, but we have no direct control on the shock location x_s whose dynamic is imposed by the Rankine-Hugoniot condition (5.3).

We describe now how to tackle these difficulties. Since at any time there is only one shock to deal with, we can divide the problem in two: before and after the shock. We set

$$\xi_1(t,x) = y(t,x) - 1 \ \text{ on } [0, x_s(t)), \quad \xi_2(t,x) = y(t,x) + 1 \ \text{ on } (x_s(t), L]. \tag{5.12}$$

extended with $\xi_1(t, x_s(t)) = \lim_{x_s^-(t)} y(t,\cdot)$ and $\xi_2(t, x_s(t)) = \lim_{x_s^+(t)} y(t,\cdot)$. Now $\xi_1(t,x)$ and $\xi(t,x)$ are regular functions but defined on different moving domains whose dynamics is given by Rankine-Hugoniot conditions (5.3). To remedy this, we rescale the system and define now

$$\mathbf{z}(t,x) = \begin{pmatrix} \xi_1(t, \frac{x_s x}{x_0}) \\ \xi_2(t, L + x\frac{x_s - L}{x_0}) \end{pmatrix} = \begin{pmatrix} y(t, \frac{x_s x}{x_0}) - 1 \\ y(t, L + x\frac{x_s - L}{x_0}) + 1 \end{pmatrix}, \ \text{ for } x \in [0, x_0]. \tag{5.13}$$

Now, z_1 still represent the system before the shock and z_2 the system after the shock, but they are defined on the same fixed domain. Of course, they are still coupled with x_s and (5.3), and the system is now

$$\begin{aligned} &\partial_t z_1 + \left(1 + z_1 - x\frac{\dot{x}_s}{x_0}\right)\partial_x z_1 \frac{x_0}{x_s} = 0, \\ &\partial_t z_2 + \left(1 - z_2 + x\frac{\dot{x}_s}{x_0}\right)\partial_x z_2 \frac{x_0}{L - x_s} = 0, \\ &\dot{x}_s(t) = \frac{z_1(t,x_0) + z_2(t,x_0)}{2}, \end{aligned} \tag{5.14}$$

with the boundary feedback laws

$$\begin{aligned} z_1(t,0) &= k_1 z_1(t,x_0) + b_1(x_0 - x_s(t)), \\ z_2(t,0) &= k_2 z_2(t,x_0) + b_2(x_0 - x_s(t)). \end{aligned} \tag{5.15}$$

This system is now more complicated than the original system, but, at least, it is regular. We still have the problem of having no direct control on x_s. Indeed the feedback laws (5.15) only act on z_1 and z_2 at $x = 0$. Instead, we have a coupling between a system of two PDEs and an ODE. Therefore, we need to use the coupling between the state of the system and the shock location to stabilize the latter, and this should reflect in the Lyapunov function. Such a Lyapunov function is defined in [18] and for a solution of (5.14), (5.15) it reduces to

$$V(\mathbf{z}, x_s) = V_1(\mathbf{z}) + V_2(\mathbf{z}, x_s) + V_3(\mathbf{z}, x_s) + V_4(\mathbf{z}, x_s) + V_5(\mathbf{z}, x_s) + V_6(\mathbf{z}, x_s) \tag{5.16}$$

where the components V_1, V_2, V_3, V_4, V_5, V_6 are defined by

$$\begin{aligned} V_1(\mathbf{z}) &= \int_0^{x_0} p_1 e^{\frac{-\mu x}{\eta_1}} z_1^2 + p_2 e^{\frac{-\mu x}{\eta_2}} z_2^2 dx, \\ V_2(\mathbf{z}, x_s) &= \int_0^{x_0} p_1 e^{\frac{-\mu x}{\eta_1}} (\partial_t z_1)^2 + p_2 e^{\frac{-\mu x}{\eta_2}} (\partial_t z_2)^2 dx, \\ V_3(\mathbf{z}, x_s) &= \int_0^{x_0} p_1 e^{\frac{-\mu x}{\eta_1}} (\partial_{tt}^2 z_1)^2 + p_2 e^{\frac{-\mu x}{\eta_2}} (\partial_{tt}^2 z_2)^2 dx, \end{aligned} \tag{5.17}$$

$$
\begin{aligned}
V_4(\mathbf{z}, x_s) &= \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_1(x_s - x_0)\, dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_2(x_s - x_0)\, dx + \kappa(x_s - x_0)^2, \\
V_5(\mathbf{z}, x_s) &= \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} \partial_t z_1 \dot{x}_s\, dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} \partial_t z_2 \dot{x}_s\, dx + \kappa(\dot{x}_s)^2, \\
V_6(\mathbf{z}, x_s) &= \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} \partial_{tt}^2 z_1 \ddot{x}_s\, dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} \partial_{tt}^2 z_2 \ddot{x}_s\, dx + \kappa(\ddot{x}_s)^2.
\end{aligned} \tag{5.18}
$$

where $\kappa > 1$, p_1, p_2, $\bar{p}_1$, $\bar{p}_2$ and μ are positives constants, and we denoted

$$
\eta_1 = 1, \quad \eta_2 = \frac{x_0}{L - x_0}. \tag{5.19}
$$

Designing carefully κ, p_1, p_2, $\bar{p}_1$, $\bar{p}_2$ and μ leads to the following theorem [18]

Theorem 5.2. *For any $\gamma > 0$, if the following conditions are satisfied :*

$$
b_1 \in \left(\gamma e^{-\gamma x_0}, \frac{\gamma e^{-\gamma x_0}}{1 - e^{-\gamma x_0}} \right), \, b_2 \in \left(\gamma e^{-\gamma (L - x_0)}, \frac{\gamma e^{-\gamma (L - x_0)}}{1 - e^{-\gamma (L - x_0)}} \right), \tag{5.20a}
$$

$$
k_1^2 < e^{-\gamma x_0} \left(1 - \frac{b_1}{\gamma} \left(b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma (L - x_0)}}{\gamma e^{-\gamma (L - x_0)}} \right) \right), \tag{5.20b}
$$

$$
k_2^2 < e^{-\gamma (L - x_0)} \left(1 - \frac{b_2}{\gamma} \left(b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma (L - x_0)}}{\gamma e^{-\gamma (L - x_0)}} \right) \right), \tag{5.20c}
$$

then the steady-state (y^*, x_0) is exponentially stable for the H^2 norm with a decay rate $\gamma/4$.

This is a rapid stabilization result: it holds for any decay rate $\gamma > 0$. The conditions of this theorem might seem complicated, but recall that we are using on purpose the simplest possible feedback laws and we also explicit the dependency in γ. Also, it can be shown that for any $\gamma > 0$ there exist parameters k_1, k_2, b_1, b_2 such that the condition (5.20) hold [18].

We can make an interesting remark on the conditions given by (5.20): it may seem counter intuitive that b_1 and b_2 should tend to 0 when γ goes to $+\infty$. Indeed, if $b_1 = b_2 = 0$, then there is no more feedback on the location of the shock and one cannot stabilize the location of the shock. In other words, for all $\gamma > 0$ the feedback law we define with (5.20) works and the decay rate increase with γ, but the limit feedback law we would get by making γ tend to $+\infty$ does not even ensure the asymptotic stability of the system. In fact, the explanation behind this apparent paradox is that when γ goes to infinity, the Lyapunov function (5.16)-(5.18) is no longer equivalent to the norm of the solution. Therefore, it can no longer guarantee its exponential decay. More details are given in [18] where this remark was originally made.

Finally this result can in fact be extended exactly similarly to any scalar conservation law of the form

$$
\partial_t y + \partial_x (f(y)) = 0, \tag{5.21}
$$

where f is convex of class C^3 and such that $f(1) = f(-1)$ and

$$
\min(f'(1), |f'(-1)|) \geq 1. \tag{5.22}
$$

We see that Burgers equation is in fact a critical case for (5.22). The question of whether the same type of result could be shown with f convex without the requirement (5.22) is an open question.

5.2. Saint-Venant equations and hydraulic jump

Can the previous method be applied to a more complicated system, such as the homogeneous Saint-Venant equations? The answer is essentially yes, even though there are some new difficulties [17].

The system we consider is the following

$$\begin{aligned} &\partial_t H + \partial_x Q = 0, \\ &\partial_t Q + \partial_x \left(\frac{gH^2}{2} + \frac{Q^2}{H} \right) = 0, \end{aligned} \tag{5.23}$$

where $Q = HV$ is the flow. Note that the second equation is a balance of momentum rather than a balance of energy. The two would be equivalent when there is no shock, but they differ when a shock occurs and this is logical: an entropic shock is likely to dissipate energy (see [55] for more details). In this framework a steady-state (H^*, Q^*) with a single entropic shock should satisfy [17]:

1. Q^* is a positive constant and

$$H^* = H_1^* > 0, \quad \text{for } x \in [0, x_s^*), \quad H_2^* > 0, \quad \text{for } x \in (x_s^*, L]. \tag{5.24}$$

2. Supercritical regime before the shock and subcritical after the shock (entropy condition) :

$$\frac{Q^*}{H_1^*} - \sqrt{gH_1^*} > 0, \quad \frac{Q^*}{H_2^*} - \sqrt{gH_2^*} < 0. \tag{5.25}$$

3. Condition at the interface (dynamics of the shock)

$$\frac{H_2^*}{H_1^*} = \frac{-1 + \sqrt{1 + 8\frac{(Q^*)^2}{g(H_1^*)^3}}}{2}. \tag{5.26}$$

The third condition is called Bélanger equation [35] and is a consequence of the Rankine-Hugoniot condition, while the second condition is a consequence of the entropy condition in this case.

We can now understand the new difficulty: there are two positive propagation speeds before the shock and one positive and one negative propagation speed after the shock. This means that there are only three boundary conditions that can be imposed to the system: two in $x = 0$ and one in $x = L$. Therefore, if we divide again the system in two: before and after the shock, we will have a 4×4 system with only 3 controls, whereas with Burgers' equation we had 2 controls for 2 equations. This comes from the fact that only one propagation speed changes sign. As a consequence there are even more quantities which are not directly controlled. This is illustrated in Fig. 1

Nevertheless, the method still works: we select a steady-state (H^*, Q^*, x_s^*) such that (H^*, Q^*) satisfies (5.24)–(5.26). We define

$$\begin{aligned} \lambda_1 = \frac{Q^*}{H_1^*} - \sqrt{gH_1^*}, \quad \lambda_2 = \frac{Q^*}{H_1^*} + \sqrt{gH_1^*} \\ \lambda_3 = -\left(\frac{Q^*}{H_2^*} - \sqrt{gH_2^*} \right), \quad \lambda_4 = \frac{Q^*}{H_2^*} + \sqrt{gH_2^*}. \end{aligned} \tag{5.27}$$

And we set the boundary control as

$$\begin{pmatrix} H(t,0) - H_1^* \\ Q(t,0) - Q^* \\ Q(t,L) - Q^* \end{pmatrix} = G \begin{pmatrix} Q(t,x_s^-) - Q^* \\ Q(t,x_s^+) - Q^* \\ H(t,x_s^-) - H_1^* \\ x_s - x_s^* \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ G_4(H(t,L) - H_2^*) \end{pmatrix}, \tag{5.28}$$

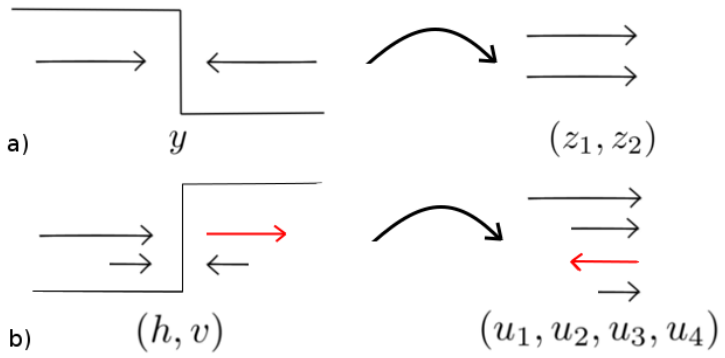


Figure 1: Change of variables to transform the problem into a regular system.**a)** Burgers equation. **b)** Saint-Venant equations. In red, the component corresponding to the propagation speed which does not change sign and on which we have no boundary control.

where $G = (G_1, G_2, G_3)^T : \mathbb{R}^4 \to \mathbb{R}^3$ and $G_4 : \mathbb{R} \to \mathbb{R}$ are of class C^2 and satisfy

$$G(\mathbf{0}) = \mathbf{0}, \quad G_4(0) = 0, \quad G_4'(0) = -\lambda_4. \tag{5.29}$$

Defining

$$\begin{aligned}
D(x,\gamma) &= \operatorname{diag}\left(\frac{s_i(1 - s_i\frac{\lambda_i}{\lambda_4})}{b_i} e^{\frac{\gamma}{x_i\lambda_i}(x_s^* - x)}, i \in \{1,2,3\} \right), \\
\widetilde{D}(\gamma) &= \operatorname{diag}\left(\left(\sum_{j=1}^{3} e^{\frac{\gamma x_s^*}{x_i\lambda_i} - \frac{\gamma x_s^*}{x_j\lambda_j}} \right) \left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)^2, i \in \{1,2,3\} \right), \\
K &= \begin{pmatrix} \frac{\lambda_2\lambda_1}{\lambda_2-\lambda_1} & -\frac{\lambda_1}{\lambda_2-\lambda_1} & 0 \\ \frac{\lambda_2\lambda_1}{\lambda_1-\lambda_2} & -\frac{\lambda_2}{\lambda_1-\lambda_2} & 0 \\ 0 & 0 & \frac{\lambda_3}{\lambda_3+\lambda_4} \end{pmatrix} G'(\mathbf{0}) \begin{pmatrix} 1 & 1 & 0 \\ \frac{\lambda_1}{\lambda_4} & \frac{\lambda_2}{\lambda_4} & 1+\frac{\lambda_3}{\lambda_4} \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} &= \begin{pmatrix} \frac{\lambda_2\lambda_1}{\lambda_2-\lambda_1} & -\frac{\lambda_1}{\lambda_2-\lambda_1} & 0 \\ \frac{\lambda_2\lambda_1}{\lambda_1-\lambda_2} & -\frac{\lambda_2}{\lambda_1-\lambda_2} & 0 \\ 0 & 0 & \frac{\lambda_3}{\lambda_3+\lambda_4} \end{pmatrix} G'(\mathbf{0}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},
\end{aligned} \tag{5.30}$$

where $s_1 = s_2 = 1$, $s_3 = -1$, $x_1 = x_2 = 1$, $x_3 = x_s^*/(L - x_s^*)$, $x_4 = x_s^*/(x_s^* - L)$, the following theorem holds

Theorem 5.3. *For any $\gamma > 0$, if, for $i = 1, 2, 3$,*

$$\begin{aligned}
b_i &\in \left(\frac{-\gamma e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}(H_1^* - H_2^*)}{3s_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)\left(1 - e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}\right)}, \frac{-\gamma e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}(H_1^* - H_2^*)}{3s_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)} \right), \quad \text{if} \quad s_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right) < 0, \\
b_i &\in \left(\frac{-\gamma e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}(H_1^* - H_2^*)}{3s_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)}, \frac{-\gamma x_i e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}(H_1^* - H_2^*)}{3s_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)\left(1 - e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}\right)} \right), \quad \text{if} \quad s_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right) > 0,
\end{aligned} \tag{5.31}$$

and if the matrix

$$D(x_s^*, \gamma) - K^T D(0,\gamma) K - \left(\sum_{k=1}^{3} \frac{2}{\gamma^2(H_1^* - H_2^*)^2} b_k s_k (1 - s_k\frac{\lambda_k}{\lambda_4})(e^{\frac{\gamma x_s^*}{x_k\lambda_k}} - 1) \right) \widetilde{D}(\gamma) \tag{5.32}$$

is positive definite, then the steady-state $((H^*, Q^*), x_s^*)$ *is exponentially stable for the* H^2 *norm with decay rate* $\gamma/4$.

Once again, this is a rapid stabilization result and, for any $\gamma > 0$, there exists G satisfying (5.31)–(5.32) [17].

5.3. Perspective

Stabilizing shock steady-states –or more generally solutions with shocks– is a problem only recently addressed. There are thus many open questions, which are all the more interesting as the solutions with shocks are natural solutions of hyperbolic systems. Without being exhaustive, we can list the following:

- Is it possible to prove a result similar to Theorem 5.20 without the requirement (5.22)?
- Concerning scalar equations, is it possible to obtain a global stabilization ? Some encouraging results about controllability were given in [68].
- Could this method works for generic $n \times n$ system (at least for homogeneous systems)?
- Could this be adapted to BV functions instead of (only) piecewise H^2 functions?
- Could Theorem 5.3 still hold, maybe with different conditions, for inhomogeneous equations, starting with the Saint-Venant equations?
- If we aim at stabilizing a regular steady-state, is it possible to adapt the methods and results of Section 3 to the BV norm ? Some results exist in this direction [45, 70] but, so far, they only deal with the case where the propagation speeds have the same sign and the system is homogeneous.

6. PI-controllers

This section starts by a question: in the results presented in Section 3–4, what would happen if we made a small error on the control? As seen in Section 3, in general when a disturbance occurs we cannot ensure exponential stability anymore, and we have to look at ISS instead. However, when the disturbance is constant, there is a way to recover exponential stability using a tool called Proportional-Integral (or PI) controllers. The general principle is the following: if a unknown constant disturbance δ is added at the boundary control used in Section 4 and defined in (4.7), we obtain

$$V(t, L) - V^*(L) = G_1(\varrho(t, L) - \varrho^*(L)) + \delta. \tag{6.1}$$

Then the exponential stability will not hold anymore, simply because, if it could, the left-hand side would be equal to 0 and, using $G(0) = 0$, we would deduce that $\delta = 0$ which means that there would be in fact no disturbance at all. However, if one manages to show exponential stability with a control of the form

$$\begin{aligned} &V(t, L) - V^*(L) = G_1(\varrho(t, L) - \varrho^*(L)) + k_I I \\ &\dot{I} = G_1(\varrho(t, L) - \varrho^*(L)), \end{aligned} \tag{6.2}$$

where k_I is a design parameter and I is only defined up to a constant, then adding a small constant disturbance δ would simply reduce to redefining $I + \delta$ as I, and the exponential stability would still hold, thanks to this integral term added to the original proportional control. PI control is an old tool, its first apparition seems to date back to the Perier brothers [61, Pages 50-51 and figure 231, Plate 26] (see also [23]). Then Jenkins' regulator working on the same principle was studied by Maxwell in his celebrated article "On governors" [121]. PI controls were then truly mathematized at the beginning of the XXth century by Minorski [122]. They are used today as one of the main controller around the world [63]. To give just one example among many others, the navigable rivers are usually regulated using PI controls [119], [12, Chapitre 8]. Consequently, there exist many results for finite dimensional systems (see for instance [2, 4, 3]). For

linear infinite dimensional systems there exist some results thanks to spectral tools like the spectral mapping theorems. For instance, for Saint-Venant equations –which will be studied in Subsection 6.2– necessary and sufficient conditions for the linearized homogeneous equations were found in [19]. Other results on these linearized equations using a spectral approach can be found in [118, 117, 152, 153]. Other results also exist, for instance, on parabolic equations [109, 106, 107].

For nonlinear infinite dimensional systems, however, very few results exist. This can be explained by the loss of spectral tools together with the difficulty of handling PI controllers. However, being able to deal directly with nonlinear systems has its importance: the stability of the linearized equations does not *a priori* guarantee anything about the "real" non-linear system[8] [53].

Among the nonlinear results one can refer to [142], in the case where the system without PI controller generate an exponentially stable semigroup, [12, 2.2.4.2] where the authors derive sufficient conditions for homogeneous 2×2 systems, [12, 5.4.4,5.5] where the authors find sufficient conditions for the nonlinear Saint-Venant equations in the special case where the steady-state is constant and [14] where the steady-state does not have to be constant, but the source term has to be dissipative. One can also cite [108] which looked at semilinear wave equations. In this section, we first present a new tool called the extraction method, which allows us to give necessary and sufficient conditions for scalar systems. Then we investigate the Saint-Venant equations and give a general result without any restriction on the steady-state, the source term or the length of the domain. We also show that it is possible to ensure the exponential stabilization of non-steady target states using a feedforward approach, and to achieve ISS when there are constant disturbances. This second part follows what was done in Section 4 using energy-like Lyapunov functions.

6.1. The extraction method

In order to illustrate this method, we look at the simplest nonlinear hyperbolic system, namely the nonlinear transport equation. And we use the simplest possible PI control that features this robustness behavior, namely the integral control:

$$\partial_t u + \lambda(u)\partial_x u = 0, \tag{6.3}$$

$$u(0,t) = -k_I I(t), \tag{6.4}$$

$$\dot{I} = u(L,t), \tag{6.5}$$

where λ is a C^1 function such that $\lambda(0) > 0$ and k_I is the control coefficient to be designed. This system, in appearance quite simple, is already very rich. When the system is linear (i.e. $\lambda(u) = \lambda(0) = \lambda_0$) the stability can be found using the spectral mapping theorem. The eigenvalues satisfy

$$k_I + \varrho e^{\frac{\varrho L}{\lambda_0}} = 0. \tag{6.6}$$

and one can show the following [143] (see also [22]):

- Solutions are conjugate: if ρ is a solution to (6.6) then $\bar{\rho}$ is too
- If $k_0 < 0$ there always exists a solution $\rho \in \mathbb{R}_{\geq 0}$
- If $k_0 \in \left(0, \frac{2k\pi\lambda_0}{L} + \frac{\pi\lambda_0}{2L}\right)$ there exist exactly $2k$ solutions $\rho \in \mathbb{C}$ such that $\mathrm{Re}(\rho) \geq 0$.

From this we can deduce the following stability result [143]

Proposition 6.1. *The linear system* (6.3)–(6.5) *with* $\lambda \equiv \lambda_0$ *is exponentially stable if and only if*

$$k_I \in \left(0, \frac{\pi\lambda_0}{2L}\right). \tag{6.7}$$

[8]The fact that the "real" physical system was seldom studied while this control is used in practice can seem surprising. However, this can in part be explained because many applicative studies prefer to approximate the Saint-Venant system by a finite dimensional system, see for instance [145].

While this result is quickly derived, the nonlinear system is harder, precisely because the spectral mapping theorem cannot be used. As surprising as it may seem, the first stability result concerning this simple nonlinear system is very recent [143]:

Theorem 6.2 (Trinh, Andrieu, Xu, 2016)**.** *If $k_I \in (0, \lambda(0)\Pi(2-\sqrt{2})/2L)$, then the nonlinear system* (6.3)–(6.5) *is exponentially stable for the H^2 norm, where*

$$\Pi(x) = \sqrt{x(2-x)}e^{-x/2}. \tag{6.8}$$

Here the exponential stability in H^2 norm is meant in the following sense

Definition 6.1. The system (6.3)–(6.5) is said exponentially stable if there exist $\eta > 0$, $C > 0$ and $\gamma > 0$ such that for any $T > 0$ and initial condition (u^0, I^0) satisfying the first order compatibility condition associated to (6.3) and such that

$$\|u^0\|_{H^2} + |I^0| \leq \eta, \tag{6.9}$$

the system has a unique solution $(u, I) \in C^0([0,T]; H^2(0,L)) \times C^1([0,T])$ and

$$(\|u(t,\cdot)\|_{H^2} + |I(t)|) \leq Ce^{-\gamma t}\left(\|u^0\|_{H^2} + |I^0|\right). \tag{6.10}$$

Theorem 6.2 is a sufficient condition, meaning that it could be too conservative and that the system could be exponentially stable for value of k_I larger than $\Pi(2-\sqrt{2})\lambda(0)/2L$. On the other hand, for $k_I > \pi\lambda(0)/2L$ it is likely that the system is not exponentially stable since even the linearized system is not[9]. Naturally, it is therefore very tempting to investigate what happens between $\Pi(2-\sqrt{2})\lambda(0)/2L \approx 0.34\lambda(0)/L$ and $\pi\lambda(0)/2L$. And, more generally, it is very tempting to see if we can get a necessary and sufficient condition. This is the goal of the extraction method introduced in [47] and that we present now.

Our approach starts by finding a good, but possibly conservative, Lyapunov function. Our Lyapunov function will be slightly different from the one used in [143] but the method may also work with their Lyapunov function if we push the extraction far enough (this will become clearer later). We use

$$V(u,I) = V_0(u(t,\cdot), I(t)) + V_0(\partial_t u(t,\cdot), u(t,L)) + V_0(\partial_t^2 u, \partial_t u(t,L)), \tag{6.11}$$

where V_0 is given by

$$V_0(U,I) := \int_0^L f(x)e^{\frac{-\mu}{\lambda(0)}x}U^2(x)dx + \left(\int_0^L \alpha U(x)dx + \beta I\right)^2, \tag{6.12}$$

where $f(x) = f(0) + (f(L) - f(0))x/L$ and $f(0) > f(L) > 0$ are two positive constants to be chosen. As in the beginning of Section 3, the definition of this Lyapunov function can be extended to $(U,I) \in H^2(0,L)\times\mathbb{R}$ thanks to (6.3). One can show that there exists $C > 0$ such that for any $(u,I) \in H^2(0,L)\times\mathbb{R}$

$$\frac{1}{C}\left(\|u\|_{H^2}^2 + \|I\|^2\right) \leq V(u,I) \leq C\left(\|u\|_{H^2}^2 + \|I\|^2\right). \tag{6.13}$$

With this Lyapunov function one can show the following result [47]

Proposition 6.3. *If $k_I \in \left(0, \frac{\lambda(0)}{L}\right)$, then the system* (6.3)–(6.5) *is exponentially stable for the H^2 norm.*

[9] Of course, this is not obvious as counterexamples exist where the linearized system is not stable but the nonlinear system is [54, 140, 21].

This is closer, but not quite enough to conclude. Thus, we look again at the eigenvalue equation (6.6). Recall that, if $k_I \in (\pi\lambda_0/2L, \frac{2\pi\lambda_0}{L} + \pi\lambda_0/2L)$, there exist exactly two eigenvalues that have a non-negative real part and they are conjugate. Therefore we can denote them by ρ_1 and $\bar{\rho}_1$. This means that, in the linear system, if we could extract from the solution the part that correspond to ρ_1 and $\bar{\rho}_1$ then the solution would be exponentially stable for $k_I \in (0, 2\pi\lambda_0/L + \pi\lambda_0/2L)$. Therefore, even if our Lyapunov function V is conservative, there is a hope it might still allow us to show the exponential stability for $k_I \in (0, \pi\lambda_0/2L)$ because this is a much smaller interval than $(0, 2\pi\lambda_0/L + \pi\lambda_0/2L)$. So, if we could define a projector p on the eigenspace $E_{\rho_1,\bar{\rho}_1}$ associated to ρ_1 and $\bar{\rho}_1$, we could separate any solution (u, I) in two parts:

- The part $p(u, I)$, which belongs to a finite dimensional space. Therefore we might be able to find a good Lyapunov function and to show its exponential stability for $k_I \in (0, \pi\lambda_0/2L)$.
- The part $(u, I) - p(u, I)$, which still belongs to an infinite dimensional space. This part is stable for $k_I \in (0, 2\pi\lambda_0/L + \pi\lambda_0/2L)$ and therefore even our conservative Lyapunov function V might be able to show the exponential stability for $k_I \in (0, \pi\lambda_0/2L)$.

Overall, if our hopes are fulfilled, we are able to show the stability on $k_I \in (0, \pi\lambda_0/2L)$ for the linear system using a basic quadratic Lyapunov function of the form (6.11)–(6.12). And, as this proof relies only on a basic quadratic Lyapunov function, there is a hope that it would also work for the nonlinear system. As shown in [47], it turns out to be true. There are several difficulties to show this:

- Finding the projector p. In this example, finding the form of a potential projector p is not very hard, but showing that it is a projector is more challenging. The expression of the projector is given by:

$$p := \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \tag{6.14}$$

with

$$\begin{aligned} p_1 \begin{pmatrix} \phi \\ I \end{pmatrix} :=& \frac{\varrho_1}{(\varrho_1 L + \lambda_0)} \left(\int_0^L \phi(x) e^{\frac{\varrho_1}{\lambda_0} x} dx + \lambda_0 e^{\frac{\varrho_1}{\lambda_0} L} I \right) e^{-\frac{\varrho_1}{\lambda_0} x} \\ &+ \frac{\bar{\varrho}_1}{(\bar{\varrho}_1 L + \lambda_0)} \left(\int_0^L \phi(x) e^{\frac{\bar{\varrho}_1}{\lambda_0} x} dx + \lambda_0 e^{\frac{\bar{\varrho}_1}{\lambda_0} L} I \right) e^{-\frac{\bar{\varrho}_1}{\lambda_0} x}, \end{aligned} \tag{6.15}$$

$$\begin{aligned} p_2 \begin{pmatrix} \phi \\ I \end{pmatrix} :=& \frac{1}{(\varrho_1 L + \lambda_0)} \left(\int_0^L \phi(x) e^{\frac{\varrho_1}{\lambda_0} x} dx + \lambda_0 e^{\frac{\varrho_1}{\lambda_0} L} I \right) e^{-\frac{\varrho_1}{\lambda_0} L} \\ &+ \frac{1}{(\bar{\varrho}_1 L + \lambda_0)} \left(\int_0^L \phi(x) e^{\frac{\bar{\varrho}_1}{\lambda_0} x} dx + \lambda_0 e^{\frac{\bar{\varrho}_1}{\lambda_0} L} I \right) e^{-\frac{\bar{\varrho}_1}{\lambda_0} L}, \end{aligned} \tag{6.16}$$

where $\bar{z}$ refers for the conjugate of z.

- When using V on $(u, I) - p(u, I)$, we need to find a way to exploit the fact that $(u_2, I_2 := (u, I) - p(u, I)$ is orthogonal to $E_{\rho_1,\bar{\rho}_1}$. To do this, we find a function θ such that $\int_0^L \theta(x) u_2(t, x) dx = 0$ and we can then add $\kappa \int_0^L \theta(x) u_2(t, x) dx$ to the expression of $dV(u(t, \cdot), I(t))/dt$ for a κ that can be chosen and optimized (see [47] for more details).
- When looking at the nonlinear system, $(u_1, I_1) := p(u, I)$ and (u_2, I_2) are coupled and have to be studied together.

Summary. We give here the different steps of the extraction method:

1. Find a good Lyapunov function candidate
2. Isolate the eigenvalues that limit the stability. These eigenvalues do not need to be explicit.

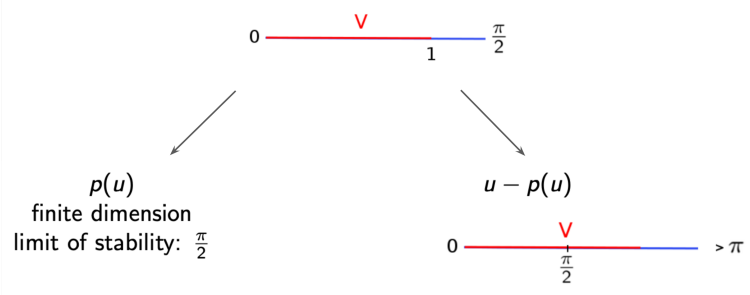


Figure 2: Illustration of the strategy of the extraction method. In red, the stability domain that can be obtained using directly a Lyapunov function. In blue, the total domain of stability for the linear system (obtained with spectral tools). Values are normalized such that $\lambda_0/L = 1$.

3. Find a projector p on the eigenspace generated by these eigenvalues.
4. Find a function θ such that
$$\int_0^L \theta(x)u_2(t,x)dx = 0 \tag{6.17}$$
where $(u_2, I_2) := (u, I) - p(u, I)$.
5. Find an optimal Lyapunov function for the finite dimensional quantity $(u_1, I_1) := p(u, I)$.
6. Use (6.17) in the expression of $dV(u(t,\cdot), I(t))/dt$ to improve the range of stability for the parameters.

If this is not enough, then we can go deeper in the extraction and extract not only the first eigenvalues that limit the stability but also the following ones. The method is illustrated in Figure 2.

Using this method, one can show [47]:

Theorem 6.4. *The system* (6.3)–(6.5) *is exponentially stable for the* H^2 *norm if*
$$k_I \in \left(0, \frac{\pi\lambda(0)}{2L}\right). \tag{6.18}$$

And, as expected, the result is optimal

Proposition 6.5. *There exists* $k_1 > \frac{\pi\lambda(0)}{2L}$ *such that for any* $\frac{\pi\lambda(0)}{2L} < k_I < k_1$, *the system* (6.3)–(6.5) *is instable for the* H^2 *norm.*

We introduced this method on a simple system (6.3). In fact, this example is more general as it may seem: every scalar quasilinear hyperbolic system can be reduced to this case. Indeed, let us consider:
$$\begin{aligned} &\partial_t y + \lambda(y,x)\partial_x y + g(y,x) = 0,\\ &y(t,0) = y^*(0) - k_I I,\\ &\dot{I} = (y(t,L) - y^*(L)), \end{aligned} \tag{6.19}$$
where y^* is the steady-state to be stabilized and $\lambda(y^*(\cdot),\cdot) > 0$. Then we have [91]

Theorem 6.6. *The steady-state* y^* *is exponentially stable for the* H^2 *norm if*
$$k_I \in \left(0, \frac{\pi \exp\left(\int_0^L \frac{\partial_y g(y^*,s) + y_x^* \partial_y \lambda(y^*,s)}{\lambda(y^*,s)} ds\right)}{2l^{-1}(L)}\right), \tag{6.20}$$
where l^{-1} *is the inverse of the strictly increasing function given by*
$$l(x) = \int_0^x \lambda(y^*(s), s)ds. \tag{6.21}$$

and

Proposition 6.7. *There exists* $k_1 > k_c := \pi \exp\left(\int_0^L (\partial_y g(y^*,s) + y_x^* \partial_y \lambda(y^*,s))/\lambda(y^*,s) ds\right)/2l^{-1}(L)$*, such that for any* $k_I \in (k_c, k_1)$ *the steady-state* y^* *of the nonlinear system* (6.19) *is unstable for the* H^2 *norm.*

6.2. Saint-Venant equations and regulation of navigable rivers

In Section 4, we have seen several results about the boundary stabilization of density-systems, which include Saint-Venant equations. However, in real applications the Saint-Venant equations are actually stabilized with PI controllers for robustness purpose. We deal with this system in this subsection. If we consider a rectangular channel with a given friction model, the Saint-Venant system is

$$\begin{aligned} &\partial_t H + \partial_x(HV) = 0, \\ &\partial_t V + V\partial_x V + g\partial_x H + \left(\frac{kV^2}{H} - C(x)\right) = 0. \end{aligned} \tag{6.22}$$

where k is the friction coefficient, C is the influence of the slope. When stabilizing a steady-state (H^*, V^*), the PI control is given by

$$\begin{aligned} H(t,L)V(t,L) - H(L)^*V(L)^* =& k_p(H(t,L) - H^*(L)) - k_I I, \\ \dot{I} =& H(t,L) - H^*(L), \end{aligned} \tag{6.23}$$

while the other boundary condition is given by the upstream flow Q_0 which is usually an unknown constant,

$$H(t,0)V(t,0) = Q_0. \tag{6.24}$$

Until the results of [90], the most advanced results on this nonlinear system was the given in [14]

Theorem 6.8 (Bastin, Coron, 2018)**.** *If* $C = 0$ *(i.e. there is no slope), for any steady-state* $(H^*, V^*) \in H^2([0,L], \mathbb{R}^2)$ *of* (6.22)–(6.24)*, if the following conditions are satisfied*

$$k_p > 1, \quad k_I > 0, \tag{6.25}$$

then the system is exponentially stable for the H^2 *norm.*

Surprisingly, the condition do not depend on the steady-state to stabilize. Looking at (6.23), (6.25), this means that one does not need to know the entire steady-state to be able to stabilize the system, but only its values at $x = L$. The first result of [90] is to show that, in fact, the following holds

Theorem 6.9. *For any steady-state* $(H^*, V^*) \in H^2([0,L], \mathbb{R}^2)$ *of* (6.22)–(6.24)*, if the following conditions are satisfied,*

$$\begin{aligned} &k_p > 0 \;\; k_I > 0, \\ \text{or} \;\; &k_p < -\frac{gH^*(L) - V^{*2}(L)}{V^*(L)}, \;\; k_I < 0, \end{aligned} \tag{6.26}$$

then the system is exponentially stable for the H^2 *norm.*

In addition to improving the conditions given in Theorem 6.8, this theorem also removes the restriction on the slope. Note here that the second set of conditions in (6.26) are quite counter-intuitive. Suppose that $H(t,L)$ is too large compared to the target value $H^*(L)$. The second set of conditions imply that it can be a good strategy to reduce even further the flow leaving the system at $x = L$, although intuitively we would expect this to increase even more $H(t,L)$. Note also that when the system is homogeneous, the conditions (6.26) are optimal [19]. This result is shown using an energy-like Lyapunov function, like in Section 4.

In fact, the results in [90] go further and allow to stabilize also a slowly time-varying target state by adding a feedforward component to the PI controller. This is important in practice as the inflow Q_0 often changes a lot during the day, but slowly. In this case, the goal is to keep a constant height at $x = L$, namely

$$H(t, L) = H_c, \tag{6.27}$$

where H_c is the target height. The first step is to define this slowly time-varying target state that satisfies (6.27). We choose this target state as the solution to

$$\begin{aligned} &\partial_t H_1 + \partial_x(H_1 V_1) = 0, \\ &\partial_t V_1 + V_1 \partial_x V_1 + g\partial_x H_1 + \left(\frac{kV_1^2}{H_1} - C(x)\right) = 0, \\ &H_1(t, 0)V_1(t, 0) = Q_0(t), \\ &H_1(t, L) = H_c, \end{aligned} \tag{6.28}$$

with $H_1(0, \cdot) = H_0^*$ and $V_1(0, \cdot) = V_0^*$, where (H_0^*, V_0^*) is what would be the steady-state of the system if Q_0 was a constant equal to $Q_0(0)$. Under the assumption that $\|Q_0\|_{L^\infty} < +\infty$, it is possible to show the existence of (H_1, V_1) by studying an intermediary family of functions: the family of functions $(H_t^*, V_t^*)_{t\geq 0}$ which, for each t fixed, would be the steady-state of the system if Q_0 was a constant equal to $Q_0(t)$. More details can be found in [90, Proposition 1.1]. From this point we can define the new controller

$$\begin{aligned} H(t, L)V(t, L) &= H_1 V_1(t, L) + k_p(H(t, L) - H_c) + k_I I, \\ \dot{I} &= H(t, L) - H_c, \end{aligned} \tag{6.29}$$

where $H_1V_1(t, L)$ is the additional feedforward components. And the following theorem holds

Theorem 6.10. *There exists $\delta > 0$ such that if $\|\partial_t Q_0\|_{C^3([0,+\infty))} \leq \delta$, and*

$$\begin{aligned} &k_p > 0 \;\; k_I > 0, \\ \text{or} \;\; &k_p < -\frac{gH^*(L) - V^{*2}(L)}{V^*(L)}, \;\; k_I < 0, \end{aligned} \tag{6.30}$$

then the target state (H_1, V_1) of the system (6.22), (6.24), (6.29) is exponentially stable for the H^2 norm.

Since the target state in time-dependent, the definition of exponential stability in this theorem has to change slightly to ensure that it does not depend on the initial time.

Definition 6.2 (Exponential stability)**.** Let $(H_1, V_1) \in C^0([0, T], H^2(0, L)) \cap C^1([0, T], H^1(0, L))$ be a target state of the system (6.22), (6.24), (6.29). This state is said to be exponentially stable for the H^2 norm if there exist $C > 0$, $\eta > 0$ and $\gamma > 0$ such that for any $T > 0$, $t_0 \geq 0$ and any $((H^0, V^0), I_0) \in H^2((0, L); \mathbb{R}^2) \times \mathbb{R}$ with

$$\|H^0 - H_1(t_0, \cdot)\|_{H^2} + \|V^0 - V_1(t_0, \cdot)\|_{H^2} + |I_0| < \eta, \tag{6.31}$$

the system (6.22), (6.24), (6.29) with initial condition $((H^0, V^0), I_0)$ at t_0 has a unique solution $((H, V), I) \in C^0([t_0, t_0 + T], H^2(0, L)) \times C^1([t_0, t_0 + T])$ and

$$\begin{aligned} &\|H(t, \cdot) - H_1(t_0, \cdot)\|_{H^2} + \|V(t, \cdot) - V_1(t_0, \cdot)\|_{H^2} + |I(t)| \\ &\leq Ce^{-\gamma(t-t_0)} \left(\|H^0 - H_1(t_0, \cdot)\|_{H^2} + \|V^0 - V_1(t_0, \cdot)\|_{H^2} + |I_0|\right), \; \forall t \in [t_0, t_0 + T]. \end{aligned} \tag{6.32}$$

Finally it would be interesting to know what can happen if we have no knowledge of the inflow $Q_0(t)$. In this case, it is not possible to compute H_1 and V_1 and to apply the control (6.29). However, it is still possible to treat $\partial_t Q_0$ as a boundary disturbance and to get an ISS estimate

Theorem 6.11. *There exist $\delta > 0$, $C > 0$ $\eta > 0$, and $\gamma > 0$ such that if $\|\partial_t Q_0\|_{C^2([0,+\infty))} \leq \delta$, then for any $T > 0$ and $((H^0, V^0), I_0) \in (H^2(0, L); \mathbb{R}^2) \times \mathbb{R}$ such that*

$$\|H^0 - H^*\|_{H^2} + \|V^0 - H^*\|_{H^2} + |I_0| < \eta, \tag{6.33}$$

the system (6.22)–(6.23) *with initial condition $((H^0, V^0), I_0)$ has a unique solution $((H, V), I) \in C^0([0, T], H^2(0, L)) \times C^1([0, T])$ which satisfies the following ISS inequality*

$$\begin{aligned} &\|H(t,\cdot) - H_0(t,\cdot)\|_{H^2(0,L)} + \|V(t,\cdot) - V_0(t,\cdot)\|_{H^2(0,L)} + |I(t)| \\ &\leq Ce^{-\gamma t}\left(\|H^0 - H^*\|_{H^2} + \|V^0 - V^*\|_{H^2} + I_0 + \int_0^t (|\partial_t Q_0(s)| + |\partial_{tt}^2 Q_0(s)| + |\partial_{ttt}^3 Q_0(s)|)e^{\gamma s} ds\right). \end{aligned} \tag{6.34}$$

6.3. Perspective

This section brings several perspectives and open questions:

- Can the extraction method be successfully applied on a 2×2 system? And on an $n \times n$ system?
- The extraction method works well for a system where the difficulty comes from the PI control. Would it be possible to apply it to the framework of Section 3 with a control of the form (1.3), assuming that necessary and sufficient stability conditions for the linearized system are known? For instance, would it be possible to deal with the example given in [12, Section 5.6]? The difficulty would be to find a way to exploit the information one has on the solution once the limiting part is extracted, in other words to adapt what is done in [47, (61)-(63)] without a PI control.
- Are the sufficient boundary conditions (6.26) for the stabilization of the Saint-Venant equations with PI controller optimal?
- If not, could we get other better conditions on the control parameters k_p and k_I using the extraction method or another method? And what would be some necessary conditions on k_I and k_p?
- Could other methods like the sliding mode be applied in this case (see for instance the promising work of [113])?

7. Conclusion

In this paper we presented cutting-edge tools to stabilize nonlinear 1D hyperbolic systems with boundary controls. We presented how the energy-like Lyapunov functions (or basic quadratic Lyapunov functions) allow us to prove several generic results, and we investigated their limits. We presented how an important class of physical equations –the density-velocity systems– have a particular structure that allows one to always have an energy-like Lyapunov function. We presented a method to stabilize steady-states with a shock in scalar equations and in 2×2 systems on the particular example of Saint-Venant equations. We presented an extraction method applied to scalar systems with a PI control, which allows us to use the spectral information of the linearized system in order to stabilize the nonlinear system. This is motivated by the fact that the stability of the linearized system does not give any information on the stability of the nonlinear system in general, but in many cases both still coincide. Finally, we presented a practical result concerning the stabilization of Saint-Venant equations by PI controls. This framework has been used in practice for the stabilization of navigable rivers and has been studied for many years. This result allows stabilization of the Saint-Venant equations without restriction and works with minimal knowledge of the river. Surprisingly, the only information required is the targeted height of the water. Despite the growing number of results in the last decade, stabilization of nonlinear hyperbolic systems is still a very open field. Even only in one space dimension, many problems still remain unsolved. The absence –so far– of a generic link between the stability of a linearized system and the nonlinear system associated; the wide use in practice of mathematically complicated controllers such as PI controllers; and the natural appearance of

shocks in solutions of hyperbolic systems, are three reasons that make these systems both challenging and very interesting to study. Systems in higher space dimensions, i.e. $2-D$ or $N-D$ systems, remain for the most part inaccessible at present –with a few rare exceptions– but there is good hope that in the future they will be within the reach of the tools and methods of the mathematical theory.

8. Acknowledgments

The author would like to thank Jean-Michel Coron for his advice and constant support. The author would also like to thank Georges Bastin, Peipei Shang, Benedetto Piccoli, Shengquan Xiang, Christophe Zhang and Gabriela Lopez-Ruiz for fruitful discussions. Finally, the author would like to thank the ANR Finite4SoS (No.ANR 15-CE23-0007), the INRIA team CAGE and the French Corps des IPEF.

References

[1] Fabio Ancona, Alberto Bressan, and Giuseppe Maria Coclite. Some results on the boundary control of systems of conservation laws. In *Hyperbolic problems: Theory, numerics, applications. Proceedings of the ninth international conference on hyperbolic problems, Pasadena, CA, USA, March 25–29, 2002*, pages 255–264. Berlin: Springer, 2003.

[2] Karl Johan Åström and Tore Hägglund. *PID controllers: theory, design, and tuning*, volume 2. Instrument society of America Research Triangle Park, NC, 1995.

[3] Karl Johan Åström, Tore Hägglund, and Karl J Astrom. *Advanced PID control*, volume 461. ISA-The Instrumentation, Systems, and Automation Society Research Triangle, 2006.

[4] Karl Johan Åström and Richard M. Murray. *Feedback systems*. Princeton University Press, Princeton, NJ, 2008. An introduction for scientists and engineers.

[5] Jean Auriol, Federico Bribiesca-Argomedo, David Bou Saba, Michael Di Loreto, and Florent Di Meglio. Delay-robust stabilization of a hyperbolic pde–ode system. *Automatica*, 95:494–502, 2018.

[6] Jean Auriol and Florent Di Meglio. Minimum time control of heterodirectional linear coupled hyperbolic PDEs. *Automatica J. IFAC*, 71:300–307, 2016.

[7] George Avalos and Roberto Triggiani. Coupled parabolic-hyperbolic Stokes-Lamé PDE system: limit behaviour of the resolvent operator on the imaginary axis. *Appl. Anal.*, 88(9):1357–1396, 2009.

[8] Abdellahi Bechir Aw and Michel Rascle. Resurrection of "second order" models of traffic flow. *SIAM J. Appl. Math.*, 60(3):916–938, 2000.

[9] Andras Balogh and Miroslav Krstic. Infinite dimensional backstepping-style feedback transformations for a heat equation with an arbitrary level of instability. *European journal of control*, 8(2):165–175, 2002.

[10] Adhémar Barré de Saint-Venant. Théorie du mouvement non permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leur lit. *Comptes Rendus de l'Académie des Sciences*, 53:147–154, 1871.

[11] Georges Bastin and Jean-Michel Coron. On boundary feedback stabilization of non-uniform linear $2\times$ 2 hyperbolic systems over a bounded interval. *Systems & Control Letters*, 60(11):900–906, 2011.

[12] Georges Bastin and Jean-Michel Coron. *Stability and boundary stabilization of 1-d hyperbolic systems*. Springer, 2016.

[13] Georges Bastin and Jean-Michel Coron. A quadratic Lyapunov function for hyperbolic density-velocity systems with nonuniform steady states. *Systems & Control Letters*, 104:66–71, 2017.

[14] Georges Bastin and Jean-Michel Coron. Exponential stability of PI control for Saint-Venant equations with a friction term. *Methods Appl. Anal.*, 26(2):101–112, 2019.

[15] Georges Bastin, Jean-Michel Coron, and Brigitte d'Andréa Novel. On Lyapunov stability of linearised Saint-Venant equations for a sloping channel. *Networks and Heterogeneous Media*, 4(2):177–187, 2009.

[16] Georges Bastin, Jean-Michel Coron, and Amaury Hayat. Input-to-state stability in sup norms for hyperbolic systems with boundary disturbances. *Nonlinear Anal.*, 208:112300, 28, 2021.

[17] Georges Bastin, Jean-Michel Coron, Amaury Hayat, and Peipei Shang. Boundary feedback stabilization of hydraulic jumps. *IFAC J. Syst. Control*, 7:100026, 10, 2019.

[18] Georges Bastin, Jean-Michel Coron, Amaury Hayat, and Peipei Shang. Exponential boundary feedback stabilization of a shock steady state for the inviscid Burgers equation. *Math. Models Methods Appl. Sci.*, 29(2):271–316, 2019.

[19] Georges Bastin, Jean-Michel Coron, and Simona Oana Tamasoiu. Stability of linear density-flow hyperbolic systems under PI boundary control. *Automatica J. IFAC*, 53:37–42, 2015.

[20] Alexandre Bayen, Alexander Keimer, Lukas Pflug, and Tanya Veeravalli. Modeling multi-lane traffic with moving obstacles by nonlocal balance laws. *Preprint*, 09 2020.

[21] Karine Beauchard and Enrique Zuazua. Large time asymptotics for partially dissipative hyperbolic systems. *Arch. Ration. Mech. Anal.*, 199(1):177–227, 2011.

[22] Richard Bellman and Kenneth L. Cooke. *Differential-difference equations*. Academic Press, New York-London, 1963.

[23] Stuart Bennett. *A history of control engineering, 1930-1955*. Number 47. IET, 1993.

[24] Henri Berestycki, Grégoire Nadin, Benoit Perthame, and Lenya Ryzhik. The non-local Fisher–KPP equation: travelling waves and steady states. *Nonlinearity*, 22(12):2813, 2009.

[25] Giulia Bertaglia and Lorenzo Pareschi. Hyperbolic models for the spread of epidemics on networks: kinetic description and numerical methods. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(2):381–407, 2021.
[26] Sébastien Blandin, Xavier Litrico, Maria Laura Delle Monache, Benedetto Piccoli, and Alexandre Bayen. Regularity and Lyapunov stabilization of weak entropy solutions to scalar conservation laws. *IEEE Trans. Automat. Control*, 62(4):1620–1635, 2017.
[27] François Bouchut, Anne Mangeney-Castelnau, Benoît Perthame, and Jean-Pierre Vilotte. A new model of Saint Venant and Savage-Hutter type for gravity driven shallow water flows. *C. R. Math. Acad. Sci. Paris*, 336(6):531–536, 2003.
[28] Dejan M. Bošković, Andras Balogh, and Miroslav Krstić. Backstepping in infinite dimension for a class of parabolic distributed parameter systems. *Math. Control Signals Systems*, 16(1):44–75, 2003.
[29] Alberto Bressan. *Hyperbolic Conservation Laws: An Illustrated Tutorial*, pages 157–245. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.
[30] Alberto Bressan and Giuseppe Maria Coclite. On the boundary control of systems of conservation laws. *SIAM J. Control Optim.*, 41(2):607–622, 2002.
[31] Christopher I. Byrnes and Alberto Isidori. New results and examples in nonlinear feedback stabilization. *Systems Control Lett.*, 12(5):437–442, 1989.
[32] Felipe Castillo, Emmanuel Witrant, Christophe Prieur, and Luc Dugard. Boundary observers for linear and quasi-linear hyperbolic systems with application to flow control. *Automatica*, 49(11):3180–3188, 2013.
[33] Carlos Castro, Francisco Palacios, and Enrique Zuazua. An alternating descent method for the optimal control of the inviscid Burgers equation in the presence of shocks. *Math. Models Methods Appl. Sci.*, 18(3):369–416, 2008.
[34] Carlos Castro and Enrique Zuazua. Concentration and Lack of Observability of Waves in Highly Heterogeneous Media. *Archive for rational mechanics and analysis*, 164(1):39–72, 2002.
[35] Hubert Chanson. *Hydraulics of open channel flow*. Butterworth-Heinemann, 2004.
[36] Gui-Qiang Chen and Philippe G LeFloch. Existence theory for the isentropic euler equations. *Archive for rational mechanics and analysis*, 166(1):81–98, 2003.
[37] Yacine Chitour, Guilherme Mazanti, and Mario Sigalotti. Approximate and exact controllability of linear difference equations. *J. Éc. polytech. Math.*, 7:93–142, 2020.
[38] Jixun Chu, Peipei Shang, and Zhiqiang Wang. Controllability and stabilization of a conservation law modeling a highly re-entrant manufacturing system. *Nonlinear Analysis*, 189:111577, 2019.
[39] Reginald H Clarke. The morning glory: An atmospheric hydraulic jump. *Journal of Applied Meteorology and Climatology*, 11(2):304–311, 1972.
[40] Jean-Michel Coron and Georges Bastin. Dissipative boundary conditions for one-dimensional quasi-linear hyperbolic systems: Lyapunov stability for the C^1-norm. *SIAM Journal on Control and Optimization*, 53(3):1464–1483, 2015.
[41] Jean-Michel Coron, Georges Bastin, and Brigitte d'Andréa Novel. Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. *SIAM Journal on Control and Optimization*, 47(3):1460–1498, 2008.
[42] Jean-Michel Coron and Brigitte d'Andréa Novel. Stabilization of a rotating body beam without damping. *IEEE Trans. Automat. Control*, 43(5):608–618, 1998.
[43] Jean-Michel Coron, Brigitte d'Andréa Novel, and Georges Bastin. A Lyapunov approach to control irrigation canals modeled by Saint-Venant equations. In *CD-Rom Proceedings, Paper F1008-5, ECC99, Karlsruhe, Germany*, pages 3178–3183, 1999.
[44] Jean-Michel Coron, Brigitte d'Andréa Novel, and Georges Bastin. A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws. *IEEE Transactions on Automatic Control*, 52(1):2–11, 2007.
[45] Jean-Michel Coron, Sylvain Ervedoza, Shyam Sundar Ghoshal, Olivier Glass, and Vincent Perrollaz. Dissipative boundary conditions for 2×2 hyperbolic systems of conservation laws for entropy solutions in BV. *J. Differential Equations*, 262(1):1–30, 2017.
[46] Jean-Michel Coron, Ludovick Gagnon, and Morgan Morancey. Rapid stabilization of a linearized bilinear 1-D Schrödinger equation. *J. Math. Pures Appl. (9)*, 115:24–73, 2018.
[47] Jean-Michel Coron and Amaury Hayat. PI controllers for 1-D nonlinear transport equation. *IEEE Trans. Automat. Control*, 64(11):4570–4582, 2019.
[48] Jean-Michel Coron, Amaury Hayat, Shengquan Xiang, and Christophe Zhang. Stabilization of the linearized water tank system. *arXiv preprint arXiv:2103.08293*, 2021.
[49] Jean-Michel Coron, Long Hu, and Guillaume Olive. Finite-time boundary stabilization of general linear hyperbolic balance laws via Fredholm backstepping transformation. *Automatica J. IFAC*, 84:95–100, 2017.
[50] Jean-Michel Coron, Long Hu, Guillaume Olive, and Peipei Shang. Boundary stabilization in finite time of one-dimensional linear hyperbolic balance laws with coefficients depending on time and space. *J. Differential Equations*, 271:1109–1170, 2021.
[51] Jean-Michel Coron, Matthias Kawski, and Zhiqiang Wang. Analysis of a conservation law modeling a highly re-entrant manufacturing system. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4):1337–1359, 2010.
[52] Jean-Michel Coron and Qi Lü. Fredholm transform and local rapid stabilization for a Kuramoto-Sivashinsky equation. *J. Differential Equations*, 259(8):3683–3729, 2015.
[53] Jean-Michel Coron and Hoai-Minh Nguyen. Dissipative boundary conditions for nonlinear 1-D hyperbolic systems: sharp conditions through an approach via time-delay systems. *SIAM J. Math. Anal.*, 47(3):2220–2240, 2015.
[54] Jean-Michel Coron, Ivonne Rivas, and Shengquan Xiang. Local exponential stabilization for a class of korteweg–de vries equations by means of time-varying feedback laws. *Analysis & PDE*, 10(5):1089–1122, 2017.
[55] Constantine M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition,

2010.
[56] Ciro d'Apice, Simone Göttlich, Michael Herty, and Benedetto Piccoli. *Modeling, simulation, and optimization of supply chains: a continuous approach*. SIAM, 2010.
[57] Sergey Dashkovskiy and Andrii Mironchenko. Input-to-state stability of infinite-dimensional control systems. *Math. Control Signals Systems*, 25(1):1–35, 2013.
[58] Sergey Dashkovskiy and Vitalii Slyn'ko. Robust stability of a perturbed nonlinear wave equation. *working paper*, 2020.
[59] J. de Halleux, C. Prieur, J.-M. Coron, B. d'Andréa Novel, and G. Bastin. Boundary feedback control in networks of open channels. *Automatica J. IFAC*, 39(8):1365–1376, 2003.
[60] Jonathan de Halleux, Christophe Prieur, Jean-Michel Coron, Brigitte d'Andréa Novel, and Georges Bastin. Boundary feedback control in networks of open channels. *Automatica. A Journal of IFAC, the International Federation of Automatic Control*, 39(8):1365–1376, 2003.
[61] Riche de Prony. *Nouvelle architecture hydraulique, seconde partie*. Firmin Didot, Paris, 1796.
[62] Belhassen Dehman, Gilles Lebeau, and Enrique Zuazua. Stabilization and control for the subcritical semilinear wave equation. *Ann. Sci. École Norm. Sup. (4)*, 36(4):525–551, 2003.
[63] Lane Desborough and Randy Miller. Increasing customer value of industrial control performance monitoring-honeywell's experience. In *AIChE symposium series*, number 326, pages 169–189. New York; American Institute of Chemical Engineers; 1998, 2002.
[64] Ababacar Diagne, Georges Bastin, and Jean-Michel Coron. Lyapunov exponential stability of 1-d linear hyperbolic systems of balance laws. *Automatica*, 48(1):109–114, 2012.
[65] Markus Dick, Martin Gugat, Michael Herty, Günter Leugering, Sonja Steffensen, and Ke Wang. *Stabilization of Networked Hyperbolic Systems with Boundary Feedback*, pages 487–504. Springer International Publishing, Cham, 2014.
[66] Markus Dick, Martin Gugat, and Günter Leugering. Classical solutions and feedback stabilization for the gas flow in a sequence of pipes. *Netw. Heterog. Media*, 5(4):691–709, 2010.
[67] Markus Dick, Martin Gugat, and Günter Leugering. A strict H^1-Lyapunov function and feedback stabilization for the isothermal Euler equations with friction. *Numer. Algebra Control Optim.*, 1(2):225–244, 2011.
[68] Carlotta Donadello and Vincent Perrollaz. Exact controllability to trajectories for entropy solutions to scalar conservation laws in several space dimensions. *Comptes Rendus Mathématique*, 357(3):263–271, 2019.
[69] Valérie Dos Santos and Christophe Prieur. Boundary control of open channels with numerical and experimental validations. *IEEE transactions on Control systems technology*, 16(6):1252–1264, 2008.
[70] Mathias Dus. BV exponential stability for systems of scalar conservation laws using saturated controls. *SIAM Journal on Control and Optimization*, 59(2):1656–1679, 2021.
[71] Mathias Dus, Francesco Ferrante, and Christophe Prieur. On L^∞ stabilization of diagonal semilinear hyperbolic systems by saturated boundary control. *ESAIM Control Optim. Calc. Var.*, 26:Paper No. 23, 34, 2020.
[72] Mathias Dus, Francesco Ferrante, and Christophe Prieur. Spectral stabilization of linear transport equations with boundary and in-domain couplings. *Preprint*, 2021.
[73] Francesco Ferrante and Christophe Prieur. Boundary Control Design for Conservation Laws in the Presence of Measurement Noise. *working paper*, 2020.
[74] Ludovick Gagnon, Amaury Hayat, Shengquan Xiang, and Christophe Zhang. Fredholm transformation on laplacian and rapid stabilization for the heat equations. *preprint*, 2021.
[75] Ludovick Gagnon, Pierre Lissy, and Swann Marx. A fredholm transformation for the rapid stabilization of a degenerate parabolic equation. *preprint*, 2020.
[76] Mauro Garavello, Paola Goatin, Thibault Liard, and Benedetto Piccoli. A multiscale model for traffic regulation via autonomous vehicles. *J. Differential Equations*, 269(7):6088–6124, 2020.
[77] James Glimm. Solutions in the large for nonlinear hyperbolic systems of equations. *Communications on pure and applied mathematics*, 18(4):697–715, 1965.
[78] James M. Greenberg and Tatsien Li. The effect of boundary damping for the quasilinear wave equation. *J. Differential Equations*, 52(1):66–75, 1984.
[79] Martin Gugat, Markus Dick, and Günter Leugering. Gas flow in fan-shaped networks: classical solutions and feedback stabilization. *SIAM J. Control Optim.*, 49(5):2101–2117, 2011.
[80] Martin Gugat and Stephan Gerster. On the limits of stabilizability for networks of strings. *Systems Control Lett.*, 131:104494, 10, 2019.
[81] Martin Gugat and Michaël Herty. Existence of classical solutions and feedback stabilization for the flow in gas networks. *ESAIM Control Optim. Calc. Var.*, 17(1):28–51, 2011.
[82] Martin Gugat and Michael Herty. Limits of stabilizabilizy for a semilinear model for gas pipeline flow. *Preprint*, 2020.
[83] Martin Gugat and Günter Leugering. Global boundary controllability of the Saint-Venant system for sloped canals with friction. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(1):257–270, 2009.
[84] Martin Gugat, Günter Leugering, Simona Tamasoiu, and Ke Wang. H^2-stabilization of the isothermal Euler equations: a Lyapunov function approach. *Chinese Annals of Mathematics. Series B*, 33(4):479–500, 2012.
[85] Martin Gugat, Vincent Perrollaz, and Lionel Rosier. Boundary stabilization of quasilinear hyperbolic systems of balance laws: exponential decay for small source terms. *J. Evol. Equ.*, 18(3):1471–1500, 2018.
[86] Martin Gugat and Rüdiger Schultz. Boundary feedback stabilization of the isothermal Euler equations with uncertain boundary data. *SIAM J. Control Optim.*, 56(2):1491–1507, 2018.
[87] Martin Gugat and Stefan Ulbrich. The isothermal Euler equations for ideal gas with source term: product solutions, flow reversal and no blow up. *J. Math. Anal. Appl.*, 454(1):439–452, 2017.
[88] Amaury Hayat. On boundary stability of inhomogeneous 2×2 1-D hyperbolic systems for the C^1 norm. *Preprint*, 2017.

[89] Amaury Hayat. Boundary stability of 1-D nonlinear inhomogeneous hyperbolic systems for the C^1 norm. *SIAM J. Control Optim.*, 57(6):3603–3638, 2019.
[90] Amaury Hayat. PI controllers for the general Saint-Venant equations. *Preprint*, January 2019. working paper or preprint.
[91] Amaury Hayat. *Stabilisation de systèmes hyperboliques non-linéaires en dimension un d'espace*. PhD thesis, Sorbonne université, 2019.
[92] Amaury Hayat. Global exponential stability and input-to-state stability of semilinear hyperbolic systems for the L^2 norm. *Systems Control Lett.*, 148:104848, 8, 2021.
[93] Amaury Hayat, Thibault Liard, Francesca Marcellini, and Benedetto Piccoli. A multiscale second order model for the interaction between av and traffic flows: analysis and existence of solutions. *Preprint*, 2021.
[94] Amaury Hayat and Peipei Shang. A quadratic Lyapunov function for Saint-Venant equations with arbitrary friction and space-varying slope. *Automatica J. IFAC*, 100:52–60, 2019.
[95] Amaury Hayat and Peipei Shang. Exponential stability of density-velocity systems with boundary conditions and source term for the H^2 norm. *J. Math. Pures Appl.*, 2021. To appear.
[96] Long Hu, Florent Di Meglio, Rafael Vazquez, and Miroslav Krstic. Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs. *IEEE Trans. Automat. Control*, 61(11):3301–3314, 2016.
[97] Long Hu, Rafael Vazquez, Florent Di Meglio, and Miroslav Krstic. Boundary exponential stabilization of 1-dimensional inhomogeneous quasi-linear hyperbolic systems. *SIAM J. Control Optim.*, 57(2):963–998, 2019.
[98] Iasson Karafyllis and Miroslav Krstic. *Input-to-State Stability for PDEs*. Springer, 2018.
[99] Daniel E Koditschek. Adaptive techniques for mechanical systems. *Proc. 5th. Yale University Conference*, page pp. 259–265, 1987.
[100] Daniel E Koditschek. The application of total energy as a lyapunov function for mechanical control systems. *Contemporary mathematics*, 97:131, 1989.
[101] Miroslav Krstic, Lionel Magnis, and Rafael Vazquez. Nonlinear stabilization of shock-like unstable equilibria in the viscous Burgers PDE. *IEEE Trans. Automat. Control*, 53(7):1678–1683, 2008.
[102] Miroslav Krstic and Andrey Smyshlyaev. Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays. *Systems Control Lett.*, 57(9):750–758, 2008.
[103] Miroslav Krstic and Andrey Smyshlyaev. *Boundary Control of PDEs: A Course on Backstepping Designs*, volume 16 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008.
[104] Peter D. Lax. Hyperbolic systems of conservation laws. II. *Comm. Pure Appl. Math.*, 10:537–566, 1957.
[105] Peter D. Lax. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11.
[106] Hugo Lhachemi and Christophe Prieur. Finite-dimensional observer-based pi regulation control of a reaction-diffusion equation. *arXiv preprint arXiv:2012.05062*, 2020.
[107] Hugo Lhachemi, Christophe Prieur, and Emmanuel Trélat. Neumann trace tracking of a constant reference input for 1-d boundary controlled heat-like equations with delay. In *IFAC World Congress*, 2020.
[108] Hugo Lhachemi, Christophe Prieur, and Emmanuel Trélat. Pi regulation control of a 1-d semilinear wave equation. *arXiv preprint arXiv:2006.10467*, 2020.
[109] Hugo Lhachemi, Christophe Prieur, and Emmanuel Trélat. Pi regulation of a reaction-diffusion equation with delayed boundary control. *IEEE Transactions on Automatic Control*, 2020.
[110] Tatsien Li. *Controllability and observability for quasilinear hyperbolic systems*, volume 3 of *AIMS Series on Applied Mathematics*. American Institute of Mathematical Sciences (AIMS), Springfield, MO; Higher Education Press, Beijing, 2010.
[111] Tatsien Li and Wen Ci Yu. *Boundary value problems for quasilinear hyperbolic systems*. Duke University Mathematics Series, V. Duke University, Mathematics Department, Durham, NC, 1985.
[112] Tong Li. Well-posedness theory of an inhomogeneous traffic flow model. *Discrete & Continuous Dynamical Systems-B*, 2(3):401, 2002.
[113] Thibault Liard, Ismaïla Balogoun, Swann Marx, and Franck Plestan. Boundary sliding mode control of a system of linear hyperbolic equations: a lyapunov approach. *Preprint*, 2020.
[114] Thibault Liard and Benedetto Piccoli. Well-posedness for scalar conservation laws with moving flux constraints. *SIAM J. Appl. Math.*, 79(2):641–667, 2019.
[115] Thibault Liard and Benedetto Piccoli. On entropic solutions to conservation laws coupled with moving bottlenecks. *Communications in mathematical sciences*, 2021.
[116] Mark Lichtner. Spectral mapping theorem for linear hyperbolic systems. *Proc. Amer. Math. Soc.*, 136(6):2091–2101, 2008.
[117] Xavier Litrico, Vincent Fromion, and Jean-Pierre Baume. Tuning of robust distant downstream PI controllers for an irrigation canal pool. II: Implementation issues. *Journal of irrigation and drainage engineering*, 132(4):369–379, 2006.
[118] Xavier Litrico, Vincent Fromion, Jean-Pierre Baume, and Manuel Rijo. Modelling and PI control of an irrigation canal. In *European Control Conference (ECC), 2003*, pages 850–855. IEEE, 2003.
[119] Xavier Litrico, Pierre-Olivier Malaterre, Jean-Pierre Baume, Pierre-Yves Vion, and José Ribot-Bruno. Automatic tuning of pi controllers for an irrigation canal pool. *Journal of irrigation and drainage engineering*, 133(1):27–37, 2007.
[120] Sophie Marbach, Hiroaki Yoshida, and Lydéric Bocquet. Osmotic and diffusio-osmotic flow generation at high solute concentration. I. mechanical approaches. *The Journal of Chemical Physics*, 146(19):194701, 2017.
[121] James Clerk Maxwell. On governors. *Proceedings of the Royal Society of London*, (16):270–283, 1868.
[122] Nicolas Minorsky. Directional stability of automatically steered bodies. *Naval Engineers Journal*, 32(2), 1922.

[123] Andrii Mironchenko. Criteria for input-to-state practical stability. *IEEE Trans. Automat. Control*, 64(1):298–304, 2019.
[124] Andrii Mironchenko, Iasson Karafyllis, and Miroslav Krstic. Monotonicity methods for input-to-state stability of nonlinear parabolic PDEs with boundary disturbances. *SIAM J. Control Optim.*, 57(1):510–532, 2019.
[125] Andrii Mironchenko and Christophe Prieur. Input-to-state stability of infinite-dimensional systems: recent results and open questions. *SIAM Rev.*, 62(3):529–614, 2020.
[126] Andrii Mironchenko and Fabian Wirth. Characterizations of input-to-state stability for infinite-dimensional systems. *IEEE Trans. Automat. Control*, 63(6):1602–1617, 2018.
[127] Aloisio Freiria Neves, Hermano de Souza Ribeiro, and Orlando Lopes. On the spectrum of evolution operators generated by hyperbolic systems. *J. Funct. Anal.*, 67(3):320–344, 1986.
[128] JM Nieto-Villar, R Quintana, and J Rieumont. Entropy production rate as a lyapunov function in chemical systems: Proof. *Physica Scripta*, 68(3):163, 2003.
[129] Vincent Perrollaz. Asymptotic stabilization of entropy solutions to scalar conservation laws through a stationary feedback law. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 30(5):879–915, 2013.
[130] Christophe Prieur and Frédéric Mazenc. ISS-Lyapunov functions for time-varying hyperbolic systems of balance laws. *Math. Control Signals Systems*, 24(1-2):111–134, 2012.
[131] Christophe Prieur, Joseph Winkin, and Georges Bastin. Robust boundary control of systems of conservation laws. *Mathematics of Control, Signals, and Systems*, 20(2):173–197, 2008.
[132] Tie Hu Qin. Global smooth solutions of dissipative boundary value problems for first order quasilinear hyperbolic systems. *Chinese Ann. Math. Ser. B*, 6(3):289–298, 1985. A Chinese summary appears in Chinese Ann. Math. Ser. A **6** (1985), no. 4, 514.
[133] Michael Renardy. On the type of certain C_0-semigroups. *Comm. Partial Differential Equations*, 18(7-8):1299–1307, 1993.
[134] Benjamin Seibold, Morris R Flynn, Aslan R Kasimov, and Rodolfo R Rosales. Constructing set-valued fundamental diagrams from jamiton solutions in second order traffic models. *Networks & Heterogeneous Media*, 8(3):745, 2013.
[135] Peipei Shang. Cauchy problem for multiscale conservation laws: Application to structured cell populations. *Journal of Mathematical Analysis and applications*, 401(2):896–920, 2013.
[136] Andrey Smyshlyaev and Miroslav Krstic. Closed-form boundary state feedbacks for a class of 1-D partial integro-differential equations. *IEEE Trans. Automat. Control*, 49(12):2185–2202, 2004.
[137] Andrey Smyshlyaev, Thomas Meurer, and Miroslav Krstic. Further results on stabilization of shock-like equilibria of the viscous Burgers PDE. *IEEE Trans. Automat. Control*, 55(8):1942–1946, 2010.
[138] Eduardo D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control*, 34(4):435–443, 1989.
[139] Eitan Tadmor and Changhui Tan. Critical thresholds in flocking hydrodynamics with non-local alignment. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 372(2028):20130401, 22, 2014.
[140] Shuxia Tang, Jixun Chu, Peipei Shang, and Jean-Michel Coron. Asymptotic stability of a korteweg–de vries equation with a two-dimensional center manifold. *Advances in Nonlinear Analysis*, 7(4):497–515, 2018.
[141] Ying Tang, Christophe Prieur, and Antoine Girard. Singular perturbation approximation by means of a H^2 Lyapunov function for linear hyperbolic systems. *Systems Control Lett.*, 88:24–31, 2016.
[142] Alexandre Terrand-Jeanne, Vincent Andrieu, Valérie Dos Santos Martins, and Cheng-Zhong Xu. Adding integral action for open-loop exponentially stable semigroups and application to boundary control of PDE systems. *Preprint*, 2018.
[143] Ngoc-Tu Trinh, Vincent Andrieu, and Cheng-Zhong Xu. Design of integral controllers for nonlinear systems governed by scalar hyperbolic partial differential equations. *IEEE Trans. Automat. Control*, 62(9):4527–4536, 2017.
[144] John Tsinias. Sufficient Lyapunov-like conditions for stabilization. *Math. Control Signals Systems*, 2(4):343–357, 1989.
[145] RRP van Nooijen and AG Kolechkina. Water system examples for control education. *IFAC-PapersOnLine*, 53(2):17380–17386, 2020.
[146] Rafael Vazquez, Jean-Michel Coron, Miroslav Krstic, and Georges Bastin. Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping. *50th IEEE Conference on Decision and Control and European Control Conference, Orlando*, pages 1329–1334, 2011.
[147] Ke Wang, Zhiqiang Wang, and Wancong Yao. Boundary feedback stabilization of quasilinear hyperbolic systems with partially dissipative structure. *Systems Control Lett.*, 146:104815, 9, 2020.
[148] Gediyon Y Weldegiyorgis and Mapundi K Banda. Input-to-state stability of non-uniform linear hyperbolic systems of balance laws via boundary feedback control. *Applied Mathematics & Optimization*, pages 1–26, 2020.
[149] Shengquan Xiang. Small-time local stabilization for a korteweg–de vries equation. *Systems & Control Letters*, 111:64–69, 2018.
[150] Shengquan Xiang. Null controllability of a linearized Korteweg–de Vries equation by backstepping approach. *SIAM J. Control Optim.*, 57(2):1493–1515, 2019.
[151] Shengquan Xiang. Quantitative rapid and finite time stabilization of the heat equation. *arXiv preprint arXiv:2010.04696*, 2020.
[152] Cheng-Zhong Xu and Gauthier Sallet. Proportional and integral regulation of irrigation canal systems governed by the st venant equation. *IFAC Proceedings Volumes*, 32(2):2274–2279, 1999.
[153] Cheng-Zhong Xu and Gauthier Sallet. Multivariable boundary PI control and regulation of a fluid flow system. *Math. Control Relat. Fields*, 4(4):501–520, 2014.
[154] Christophe Zhang. Internal rapid stabilization of a 1-D linear transport equation with a scalar feedback. *Preprint*, October 2018. working paper or preprint.
[155] Christophe Zhang. Finite-time internal stabilization of a linear 1-D transport equation. *Systems Control Lett.*, 133:104529, 8, 2019.

[156] Xu Zhang and Enrique Zuazua. Polynomial decay and control of a $1-d$ hyperbolic-parabolic coupled system. *J. Differential Equations*, 204(2):380–438, 2004.
[157] Y.-C. Zhao. *Classical Solutions for Quasilinear Hyperbolic Systems*. PhD thesis, Fudan University, 1986.