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# Existence of solutions for Solitons type equations : Derrick's Problem with twice p-laplacian

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## Abstract

In this paper we study a class of Lorentz invariant nonlinear field equations in several space dimensions. The fields are characterized by a topological invariant, which we call the charge. We prove the existence of a static solution which minimizes the energy among the configurations with nontrivial charge. In this spirit, a considerable amount of work has been done by V. Benci and collaborators and a model equation proposed in [2] will be the topic of this paper. The main purpose is to obtain soliton-like solutions with twice p-Laplacian.

**Key words and phrases:** Soliton, variational calculus, splitting lemma.

**AMS (MOS) Subject Classifications:**35C08,35J35,35J60.

## 1 Introduction

A soliton is a solution of a field equation whose energy travels as a localized packet and which preserves its form under perturbations. In this respect solitons have a particle-like behavior and they occur in many areas of mathematical physics, such as classical and quantum field theory, nonlinear optics, fluid mechanics, and plasma physics; see [8]. Probably, the simplest equation which has soliton solutions is the sine-Gordon equation,

$$-\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial t^2} + \sin \psi = 0, \quad (1.1)$$

where  $\psi = \psi(x, t)$  is a scalar field,  $x, t$  are real numbers, representing, respectively, the space and the time variable. Derrick, in a celebrated paper [7], considers the more

realistic three-space-dimension model,

$$-\Delta\psi + \frac{\partial^2\psi}{\partial t^2} + V'(\psi) = 0, \quad (1.2)$$

$\Delta$  being the 3-dimensional Laplace operator and  $V'$  is the gradient of a nonnegative  $C^1$  real function  $V$ . In [7] it is proved by a simple rescaling argument that (1.2) does not possess any nontrivial finite-energy static solution. This fact leads the author to say, “We are thus faced with the disconcerting fact that no equation of type (1.2) has any time-independent solutions which could reasonably be interpreted as elementary particles.” Derrick proposed some possible ways out of this difficulty. The first proposal was to consider models which are the Euler-Lagrange equations of the action functional relative to the functional

$$S = \int \int \mathcal{L} dx dt.$$

The Lorentz invariant Lagrangian density proposed in [7] has the form

$$\mathcal{L}(\psi) = -(|\nabla\psi|^2 - |\psi_t|^2)^{\frac{p}{2}} - V(\psi), \quad p > 3. \quad (1.3)$$

However, Derrick does not continue his analysis and he concludes that a Lagrangian density of type (1.3) leads to a very complicated differential equation. He has been unable to demonstrate either the existence or nonexistence of stable solutions. In this spirit, a considerable amount of work has been done by Benci and collaborators, and a model equation proposed in [2]. The Lorentz invariant Lagrangian density proposed in [2] has the form

$$\rho = |\nabla\psi|^2 - |\psi_t|^2; \quad \alpha(\rho) = a\rho + b|\rho|^{\frac{p}{2}}, \quad p > n, \\ \mathcal{L}(\psi, \rho) = -\frac{1}{2}\alpha(\rho) - V(\psi). \quad (1.4)$$

In the case where  $p$  is constant, various mathematical results (existence, multiplicity results, asymptotic behavior, ...), have been obtained for different classes of solution models (see [1–6, 8] and the references therein).

The aim of this paper is to carry out an existence analysis of the finite-energy static solutions in more than one space dimension for a class of Lagrangian densities  $L$  which include (1.4) with  $(r, p)$ -Laplacian

## 2 Statement of the Problem

The class of Lagrangian densities we consider generalizes the problem studied in [2], Lagrangian density with variable exponent, in such a way as to include the Lorentz invariant Lagrangian density proposed in [2]. First we introduce some notation. If  $n, m$

are positive integers, and will denote, respectively, the physical space-time (typically  $n = 3$ ) and the internal parameters space. We are interested in the multi-dimensional case, so we assume that  $n \geq 2$ . A point in  $\mathbb{R}^{n+1}$  will be denoted by  $X = (x, t)$ , where  $x \in \mathbb{R}^n$  and  $t \in \mathbb{R}$ . The fields we are interested in are maps  $\psi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^m$ ,  $\psi = (\psi_1, \dots, \psi_m)$ . We set

$$\rho = |\nabla \psi|^2 - |\psi_t|^2,$$

$\nabla \psi$  and  $\psi_t$  denoting, respectively, the Jacobian with respect to  $x$  and the derivative with respect to  $t$ .

We shall consider Lagrangian densities of the form

$$\mathcal{L}(\psi, \rho) = -\frac{1}{2}\alpha(\rho) - V(\psi), \quad (2.1)$$

where the function  $V$  is a real function defined in an open subset  $\Omega \subset \mathbb{R}^m$  and  $\alpha$  is a real function defined by

$$\alpha(\rho) = a\rho|\rho|^{\frac{r}{2}-1} + b|\rho|^{\frac{p}{2}}, \quad a \geq 0, \quad b > 0, \quad 2 \leq r \leq n < p. \quad (2.2)$$

The results of [2] were concerned with the case:  $r \equiv 2$ . The action functional related to (2.1) is

$$\begin{aligned} S(\psi) &= \int_{\mathbb{R}^{n+1}} \mathcal{L}(\psi, \rho) dx dt \\ &= \int_{\mathbb{R}^{n+1}} -\frac{1}{2}\alpha(\rho) - V(\psi) dx dt. \end{aligned}$$

So the Euler-Lagrange equations are

$$\frac{\partial}{\partial t}(\alpha' \psi_t) - \nabla(\alpha' \nabla \psi) + V'(\psi) = 0, \quad (2.3)$$

where  $\nabla(\alpha' \nabla \psi)$  denotes the vector whose  $j$ -th component is given by  $\text{div}(\alpha' \nabla \psi^j)$ , and  $V'$  denotes the gradient of  $V$ . The equation (2.3) is Lorentz invariant. Static solutions  $\psi(x, t) = u(x)$  of (2.3) solve the equation

$$-\nabla(\alpha' \nabla u) + V'(u) = 0. \quad (2.4)$$

Using (2.2) and (2.4) we obtain

$$-a\frac{r}{2}\Delta_r u - b\frac{p}{2}\Delta_p u + V'(u) = 0, \quad (2.5)$$

where

$$\Delta_r u = \nabla(|\nabla u|^{r-2} \nabla u),$$

$$\Delta_p u = \nabla(|\nabla u|^{p-2} \nabla u).$$

Recall that the results of [2] were concerned with the case  $r \equiv 2$ .

It is easy to verify that, if  $u = u(x)$  is a solution of the (2.3) and  $v = (\nu, 0, \dots, 0)$  with  $|\nu| < 1$ , the field

$$\psi_\nu(x, t) = u\left(\frac{x_1 - \nu t}{\sqrt{1 - \nu^2}}, x_2, \dots, x_n\right) \quad (2.6)$$

is solution of (2.3). Notice that the function undergoes a contraction by a factor,

$$\gamma = \frac{1}{\sqrt{1 - \nu^2}},$$

in the direction of the motion; this is a consequence of the fact that (2.3) is Lorentz invariant. Clearly (2.5) are the Euler-Lagrange equations with respect to the energy functional

$$f_a(u) = \int_{\mathbb{R}^n} \left( \frac{a}{2} |\nabla u|^r + \frac{b}{2} |\nabla u|^p + V(u) \right) dx, \quad (2.7)$$

where  $m = n + 1$ , so the time independent fields  $u$  are maps

$$u : \mathbb{R}^n \rightarrow \mathbb{R}^m.$$

For every  $\xi \in \mathbb{R}^{n+1}$ , we write  $\xi = (\xi_0, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^n$ .  $V : \Omega \rightarrow \mathbb{R}$  where  $\Omega = \mathbb{R}^{n+1} \setminus \{\eta\}$ ,  $\eta = (1, 0)$ , and  $V$  is positive and singular in  $\eta$ . More precisely we assume:

(V<sub>1</sub>)  $V \in C^1(\Omega, \mathbb{R})$ .

(V<sub>2</sub>)  $V(\xi) \geq V(0) = 0$ .

(V<sub>3</sub>)  $V$  is twice differentiable in 0 and the Hessian matrix  $V''(0)$  is nondegenerate.

(V<sub>4</sub>) There exist  $c, \rho > 0$  such that if  $|\xi| < \rho$  then

$$V(\eta + \xi) \geq c|\xi|^{-q}$$

where

$$\frac{1}{q} = \frac{1}{n} - \frac{1}{p}.$$

(V<sub>5</sub>) For every  $\xi \in \Omega \setminus \{0\}$  we have

$$V(\xi) > 0, \text{ and } \lim_{|\xi| \rightarrow \infty} \inf V(\xi) = v > 0.$$

(V<sub>6</sub>) There exist  $R > 0$ ,  $|\xi| < R \implies V(\xi) \geq \omega_R |\xi|^r$ ,  $\omega_R > 0$ .

**Example 2.1.** A potential satisfying the assumptions (V<sub>1</sub>) – (V<sub>6</sub>) is

$$V(\xi) = \omega_0^2 \left( |\xi|^r + \frac{|\xi|^4}{|\xi - \eta|^q} \right).$$

### 3 Functional Setting

Let  $p > n \geq 2$  and, with no loss of generality, we can consider the functional (2.7) with  $b = 1$ . It will be convenient to introduce the following notation:

$$f_a(u) = \int_{\mathbb{R}^n} \left( \frac{a}{2} |\nabla u|^r + \frac{1}{2} |\nabla u|^p + V(u) \right) dx,$$

and we define the space  $E_a$  to be the completion of  $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$  with respect to the norm

$$\|u\|_a = a \|\nabla u\|_{L^r} + \|\nabla u\|_{L^p} + \|u\|_{L^r}, \quad a > 0,$$

$$p > n \geq r \geq 2$$

i.e.,

$$E_a = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_a}$$

$$\|u\|_{L^r} = \left( \sum_{j=1}^{n+1} \|u_j\|_{L^r}^r \right)^{\frac{1}{r}}$$

$$\|\nabla u\|_{L^r} = \left( \sum_{j=1}^{n+1} \|\nabla u_j\|_{L^r}^r \right)^{\frac{1}{r}}$$

and

$$\|\nabla u\|_{L^p} = \left( \sum_{j=1}^{n+1} \|\nabla u_j\|_{L^p}^p \right)^{\frac{1}{p}}.$$

For every  $a > 0$ , the norms  $\|\cdot\|_a$  are equivalent, so we have to study only two cases:  $a = 0$ ,  $a > 0$ .

**Proposition 3.1.** *The Banach space  $E_0$  is continuously embedded in  $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$ , for every  $s \in [2, \infty]$ .*

**Corollary 3.1.** *The Banach space  $E_0$  is continuously embedded in  $W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1})$ .*

**Corollary 3.2.** *For every  $a > 0$ , the space  $E_a$  can be identified with the Banach space*

$$W = W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1}) \cap W^{1,r}(\mathbb{R}^n, \mathbb{R}^{n+1}),$$

*equipped with the usual norm*

$$\|u\|_W = \|u\|_{W^{1,r}} + \|u\|_{W^{1,p}}.$$

Recall that  $\eta$  is a singular point of the potential  $V$ , so it is reasonable to consider in space  $E_a$ , the open subset

$$\Lambda_a = \{u \in E_a : u(x) \neq \eta, \text{ for all } x \in \mathbb{R}^n\},$$

which is open in  $E_a$ .

The boundary of  $\Lambda_a$  is given by

$$\partial\Lambda_a = \{u \in E_a : \text{there exist } x \in \mathbb{R}^n \text{ such that } u(x) = \eta\}.$$

We can show that  $\Lambda_a$  has a rich topological structure, more precisely it consists of infinitely many connected components. These components are identified by the topological charge we are going to introduce.

## 4 Topological charge and connected components of $\Lambda_a$

For the sake of simplicity, we consider the function space

$$C = \left\{ u : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1} \setminus \{\eta\} \text{ is continuous and } \lim_{|x| \rightarrow \infty} u(x) = 0 \right\}$$

where  $\eta = (1, 0)$ . Every function  $u \in C$  we write in the form  $u(x) = (u_0(x), \tilde{u}(x)) \in \mathbb{R}^{n+1}$  where  $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\tilde{u} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

**Definition 4.1.** *For every function  $u \in C$  we define the support of  $u$*

$$K_u = \{x \in \mathbb{R}^n : u_0(x) > 1\}.$$

*Then we define the topological charge of  $u$*

$$ch(u) := \begin{cases} \deg(\tilde{u}, K_u, 0) & \text{if } K_u \neq \emptyset, \\ 0 & \text{if } K_u = \emptyset. \end{cases}$$

*Such that*

$$\deg(\tilde{u}, K_u, 0) = \sum_{x \in \tilde{u}^{-1}(0)} \text{sgn} J_{\tilde{u}}(x),$$

*(Brouwer degree) For more information about this subject, see [9];*

*where  $J_{\tilde{u}}$  denotes the determinante of the Jacobian matrix.*

We notice that this definition of charge is the same as in [2]. We recall that the topological charge is continuous with respect to the uniform convergence.

Now, for every  $q \in \mathbb{Z}$  we set

$$\Lambda_q = \{u \in \Lambda_a : ch(u) = q\}.$$

Since the topological charge is continuous with respect to the uniform convergence

and the continuity of the embeddings  $E_a$  in  $L^\infty$  (see proposition 3.1) assure that the topological charge is continuous on  $\Lambda_a$ , it follows that  $\Lambda_q$  is open in  $E_a$ , since we have also

- $\Lambda_a = \bigcup_{q \in \mathbb{Z}} \Lambda_q$ ,
- $\Lambda_q \cap \Lambda_p = \emptyset, \quad p \neq q$ .

We conclude that every  $\Lambda_q$  is a connected component of  $\Lambda_a$ .

We assume that the space dimension is odd then we conclude that for every  $q \in \mathbb{Z}$  the component  $\Lambda_q$  is isomorphic to the component  $\Lambda_{-q}$ .

So for every  $u \in \Lambda_a$  we can define the charge  $ch(u) \in \mathbb{Z}$ . Now, we consider the set of a minimizer of  $f_a$  in the open set

$$\Lambda^*_q = \{u \in \Lambda_a : ch(u) \neq 0\}.$$

## 5 Properties of the energy functional

**Lemma 5.1.** *The functional  $f_a$  takes real values and it is continuous on  $\Lambda_a$ .*

**Lemma 5.2.** *The map  $f' : E_a \rightarrow E'_a$  defined by*

$$\begin{aligned} \langle f'_a(u), v \rangle &= \langle -a \frac{r}{2} \Delta_r u - b \frac{p}{2} \Delta_p u + V'(u), v \rangle \\ &= \int_{\mathbb{R}^n} \left( a \frac{r}{2} |\nabla u|^{r-2} (\nabla u | \nabla v) + b \frac{p}{2} |\nabla u|^{p-2} (\nabla u | \nabla v) + V'(u) \cdot v \right) dx \end{aligned}$$

*is continuous .*

**Lemma 5.3.** *The functional  $f_a$  is coercive in  $\Lambda_a$ ; that is, for every sequence  $u_k \subset \Lambda_a$  such that  $\|u_k\|_a \rightarrow \infty$ , we have  $f_a(u_k) \rightarrow \infty$ .*

**Lemma 5.4.** *Let  $(u_k) \subset \Lambda_a$  be a weakly converging sequence. If the weak limit belongs to  $\partial \Lambda_a$ , then*

$$f_a(u_k) \rightarrow \infty \text{ as } k \rightarrow \infty.$$

**Corollary 5.1.** *For every  $b > 0$ , there exist  $d = d(b)$  such that, for every  $u \in \Lambda_a$  we have*

$$f_a(u) \leq b \Rightarrow \min_{x \in \mathbb{R}^n} |u(x) - \eta| \geq d.$$



**Lemma 5.5.** *The functional  $f_a$  is weakly lower semicontinuous in  $\Gamma_a$ .*

**Proposition 5.1.** *There exists  $\Delta_a > 0$  such that, for every  $u \in \Lambda_a$  satisfied  $\|u\|_{L^\infty} \geq 1$  we have*

$$f_a(u) \geq \Delta_a.$$

## 6 Existence result

**Theorem 6.1.** *The minimum points  $u \in \Lambda_a$  for the functional  $f_a$  are weak solutions of the system (2.5).*

**Proposition 6.1.** *(Splitting proposition) Let  $(u_k) \in \Lambda_a^*$  be a sequence and  $M$  be a positive real number such that*

$$f_a(u_k) \leq M.$$

*Then there exists  $l \in \mathbb{N}$  such that*

$$1 \leq l \leq \frac{M}{\Delta_a}$$

*where  $\Delta_a$  was introduced in Proposition 5.1 and there exist  $\bar{u}_1, \dots, \bar{u}_l \in \Lambda_a$ ,  $(x_k^1), \dots, (x_k^l) \subset \mathbb{R}^n$  such that, up to a subsequence,*

$$u_k(\cdot + x_k^i) \rightarrow \bar{u}_i,$$

$$|x_k^i - x_k^j| \rightarrow \infty, \quad i \neq j,$$

$$\sum_{i=1}^l f_a(\bar{u}_i) \leq \liminf_{k \rightarrow \infty} f_a(u_k),$$

*and*

$$ch(u_k) = \sum_{i=1}^l ch(\bar{u}_i).$$

The minimum is attained on the set  $\Lambda_a$ , and it is easy to see that  $u \equiv 0$  is a trivial solution of the problem. But, of course, we are interested in nontrivial solutions. We consider the following problem

$$I_* = \inf_{u \in \Lambda_a^*} f_a(u), \quad \Lambda_a^* = \{u \in E_a : ch(u) \neq 0\}.$$

The functional is bounded below and the set  $E_a$  is not empty. We consider fields  $u$  having the form

$$u(x) = \left( \frac{2}{1 + |x|^m}, \frac{1}{1 + |x|^m} x \right). \quad (6.1)$$

**Lemma 6.1.** *There exists a suitable  $m \geq 1$ , such that, the field  $u$  defined in (6.1) belongs to  $\Lambda_a^*$ .*

And moreover the set  $\Lambda_a^*$  is open in the space  $E_a$ ; indeed,

- $\Lambda_a^* = \bigcup_{q \in \mathbb{N}^*} \Lambda_a^q$ ,
- $\Lambda_a^q \cap \Lambda_a^p = \emptyset, \quad p \neq q$ .

where  $\Gamma_q$  is a connected component

**Theorem 6.2.** *Let  $a, b > 0, p > n \geq r \geq 2$ . Then, if  $V$  satisfies  $(V_1) - (V_6)$ , then there exists a weak solution of (2.5) (i.e., a static solution of (2.3)), which is a minimizer of the energy functional (2.7) in the class of maps whose topological charge is different from 0.*

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