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Existence of solutions for Solitons type equations : Derrick's Problem with twice p-laplacian

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Abstract

In this paper we study a class of Lorentz invariant nonlinear field equations in several space dimensions. The fields are characterized by a topological invariant, which we call the charge. We prove the existence of a static solution which minimizes the energy among the configurations with nontrivial charge. In this spirit, a considerable amount of work has been done by V. Benci and collaborators and a model equation proposed in [2] will be the topic of this paper. The main purpose is to obtain soliton-like solutions with twice p-Laplacian.

Key words and phrases: Soliton, variational calculus, splitting lemma.

AMS (MOS) Subject Classifications:35C08,35J35,35J60.

1 Introduction

A soliton is a solution of a field equation whose energy travels as a localized packet and which preserves its form under perturbations. In this respect solitons have a particle-like behavior and they occur in many areas of mathematical physics, such as classical and quantum field theory, nonlinear optics, fluid mechanics, and plasma physics; see [8]. Probably, the simplest equation which has soliton solutions is the sine-Gordon equation,

$$-\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial t^2} + \sin\psi = 0, \quad (1.1)$$

where $\psi = \psi(x, t)$ is a scalar field, x, t are real numbers, representing, respectively, the space and the time variable. Derrick, in a celebrated paper [7], considers the more

realistic three-space-dimension model,

$$-\Delta\psi + \frac{\partial^2\psi}{\partial t^2} + V'(\psi) = 0, \quad (1.2)$$

Δ being the 3-dimensional Laplace operator and V' is the gradient of a nonnegative C^1 real function V . In [7] it is proved by a simple rescaling argument that (1.2) does not possess any nontrivial finite-energy static solution. This fact leads the author to say, “We are thus faced with the disconcerting fact that no equation of type (1.2) has any time-independent solutions which could reasonably be interpreted as elementary particles.” Derrick proposed some possible ways out of this difficulty. The first proposal was to consider models which are the Euler-Lagrange equations of the action functional relative to the functional

$$S = \int \int \mathcal{L} dx dt.$$

The Lorentz invariant Lagrangian density proposed in [7] has the form

$$\mathcal{L}(\psi) = -(|\nabla\psi|^2 - |\psi_t|^2)^{\frac{p}{2}} - V(\psi), \quad p > 3. \quad (1.3)$$

However, Derrick does not continue his analysis and he concludes that a Lagrangian density of type (1.3) leads to a very complicated differential equation. He has been unable to demonstrate either the existence or nonexistence of stable solutions. In this spirit, a considerable amount of work has been done by Benci and collaborators, and a model equation proposed in [2]. The Lorentz invariant Lagrangian density proposed in [2] has the form

$$\rho = |\nabla\psi|^2 - |\psi_t|^2; \quad \alpha(\rho) = a\rho + b|\rho|^{\frac{p}{2}}, \quad p > n,$$

$$\mathcal{L}(\psi, \rho) = -\frac{1}{2}\alpha(\rho) - V(\psi). \quad (1.4)$$

In the case where p is constant, various mathematical results (existence, multiplicity results, asymptotic behavior, . . .), have been obtained for different classes of solution models (see [1–6, 8] and the references therein).

The aim of this paper is to carry out an existence analysis of the finite-energy static solutions in more than one space dimension for a class of Lagrangian densities L which include (1.4) with (r, p) -Laplacian

2 Statement of the Problem

The class of Lagrangian densities we consider generalizes the problem studied in [2], Lagrangian density with variable exponent, in such a way as to include the Lorentz invariant Lagrangian density proposed in [2]. First we introduce some notation. If n, m

are positive integers, and will denote, respectively, the physical space-time (typically $n = 3$) and the internal parameters space. We are interested in the multi-dimensional case, so we assume that $n \geq 2$. A point in \mathbb{R}^{n+1} will be denoted by $X = (x, t)$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The fields we are interested in are maps $\psi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^m$, $\psi = (\psi_1, \dots, \psi_m)$. We set

$$\rho = |\nabla\psi|^2 - |\psi_t|^2,$$

$\nabla\psi$ and ψ_t denoting, respectively, the Jacobian with respect to x and the derivative with respect to t .

We shall consider Lagrangian densities of the form

$$\mathcal{L}(\psi, \rho) = -\frac{1}{2}\alpha(\rho) - V(\psi), \quad (2.1)$$

where the function V is a real function defined in an open subset $\Omega \subset \mathbb{R}^m$ and α is a real function defined by

$$\alpha(\rho) = a\rho|\rho|^{\frac{r}{2}-1} + b|\rho|^{\frac{p}{2}}, \quad a \geq 0, \quad b > 0, \quad 2 \leq r \leq n < p. \quad (2.2)$$

The results of [2] were concerned with the case: $r \equiv 2$. The action functional related to (2.1) is

$$\begin{aligned} S(\psi) &= \int_{\mathbb{R}^{n+1}} \mathcal{L}(\psi, \rho) dx dt \\ &= \int_{\mathbb{R}^{n+1}} -\frac{1}{2}\alpha(\rho) - V(\psi) dx dt. \end{aligned}$$

So the Euler-Lagrange equations are

$$\frac{\partial}{\partial t}(\alpha'\psi_t) - \nabla(\alpha'\nabla\psi) + V'(\psi) = 0, \quad (2.3)$$

where $\nabla(\alpha'\nabla\psi)$ denotes the vector whose j -th component is given by $\text{div}(\alpha'\nabla\psi^j)$, and V' denotes the gradient of V . The equation (2.3) is Lorentz invariant. Static solutions $\psi(x, t) = u(x)$ of (2.3) solve the equation

$$-\nabla(\alpha'\nabla u) + V'(u) = 0. \quad (2.4)$$

Using (2.2) and (2.4) we obtain

$$-a\frac{r}{2}\Delta_r u - b\frac{p}{2}\Delta_p u + V'(u) = 0, \quad (2.5)$$

where

$$\Delta_r u = \nabla(|\nabla u|^{r-2}\nabla u),$$

$$\Delta_p u = \nabla(|\nabla u|^{p-2}\nabla u).$$

Recall that the results of [2] were concerned with the case $r \equiv 2$.

It is easy to verify that, if $u = u(x)$ is a solution of the (2.3) and $v = (\nu, 0, \dots, 0)$ with $|\nu| < 1$, the field

$$\psi_\nu(x, t) = u \left(\frac{x_1 - \nu t}{\sqrt{1 - \nu^2}}, x_2, \dots, x_n \right) \quad (2.6)$$

is solution of (2.3). Notice that the function undergoes a contraction by a factor,

$$\gamma = \frac{1}{\sqrt{1 - \nu^2}},$$

in the direction of the motion; this is a consequence of the fact that (2.3) is Lorentz invariant. Clearly (2.5) are the Euler-Lagrange equations with respect to the energy functional

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2} |\nabla u|^r + \frac{b}{2} |\nabla u|^p + V(u) \right) dx, \quad (2.7)$$

where $m = n + 1$, so the time independent fields u are maps

$$u : \mathbb{R}^n \rightarrow \mathbb{R}^m.$$

For every $\xi \in \mathbb{R}^{n+1}$, we write $\xi = (\xi_0, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^n$. $V : \Omega \rightarrow \mathbb{R}$ where $\Omega = \mathbb{R}^{n+1} \setminus \{\eta\}$, $\eta = (1, 0)$, and V is positive and singular in η . More precisely we assume:

(V₁) $V \in C^1(\Omega, \mathbb{R})$.

(V₂) $V(\xi) \geq V(0) = 0$.

(V₃) V is twice differentiable in 0 and the Hessian matrix $V''(0)$ is nondegenerate.

(V₄) There exist $c, \rho > 0$ such that if $|\xi| < \rho$ then

$$V(\eta + \xi) \geq c|\xi|^{-q}$$

where

$$\frac{1}{q} = \frac{1}{n} - \frac{1}{p}.$$

(V₅) For every $\xi \in \Omega \setminus \{0\}$ we have

$$V(\xi) > 0, \text{ and } \lim_{|\xi| \rightarrow \infty} \inf V(\xi) = v > 0.$$

(V₆) There exist $R > 0$, $|\xi| < R \implies V(\xi) \geq \omega_R |\xi|^r$, $\omega_R > 0$.

Example 2.1. A potential satisfying the assumptions (V₁) – (V₆) is

$$V(\xi) = \omega_0^2 \left(|\xi|^r + \frac{|\xi|^4}{|\xi - \eta|^q} \right).$$

3 Functional Setting

Let $p > n \geq 2$ and, with no loss of generality, we can consider the functional (2.7) with $b = 1$. It will be convenient to introduce the following notation:

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2} |\nabla u|^r + \frac{1}{2} |\nabla u|^p + V(u) \right) dx,$$

and we define the space E_a to be the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ with respect to the norm

$$\|u\|_a = a \|\nabla u\|_{L^r} + \|\nabla u\|_{L^p} + \|u\|_{L^r}, \quad a > 0,$$

$$p > n \geq r \geq 2$$

i.e.,

$$E_a = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_a}$$

$$\|u\|_{L^r} = \left(\sum_{j=1}^{n+1} \|u_j\|_{L^r}^r \right)^{\frac{1}{r}}$$

$$\|\nabla u\|_{L^r} = \left(\sum_{j=1}^{n+1} \|\nabla u_j\|_{L^r}^r \right)^{\frac{1}{r}}$$

and

$$\|\nabla u\|_{L^p} = \left(\sum_{j=1}^{n+1} \|\nabla u_j\|_{L^p}^p \right)^{\frac{1}{p}}.$$

For every $a > 0$, the norms $\|\cdot\|_a$ are equivalent, so we have to study only two cases: $a = 0$, $a > 0$.

Proposition 3.1. *The Banach space E_0 is continuously embedded in $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$, for every $s \in [2, \infty]$.*

Corollary 3.1. *The Banach space E_0 is continuously embedded in $W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1})$.*

Corollary 3.2. *For every $a > 0$, the space E_a can be identified with the Banach space*

$$W = W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1}) \cap W^{1,r}(\mathbb{R}^n, \mathbb{R}^{n+1}),$$

equipped with the usual norm

$$\|u\|_W = \|u\|_{W^{1,r}} + \|u\|_{W^{1,p}}.$$

Recall that η is a singular point of the potential V , so it is reasonable to consider in space E_a , the open subset

$$\Lambda_a = \{u \in E_a : u(x) \neq \eta, \text{ for all } x \in \mathbb{R}^n\},$$

which is open in E_a .

The boundary of Λ_a is given by

$$\partial\Lambda_a = \{u \in E_a : \text{there exist } x \in \mathbb{R}^n \text{ such that } u(x) = \eta\}.$$

We can show that Λ_a has a rich topological structure, more precisely it consists of infinitely many connected components. These components are identified by the topological charge we are going to introduce.

4 Topological charge and connected components of Λ_a

For the sake of simplicity, we consider the function space

$$C = \left\{ u : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1} \setminus \{\eta\} \text{ is continuous and } \lim_{|x| \rightarrow \infty} u(x) = 0 \right\}$$

where $\eta = (1, 0)$. Every function $u \in C$ we write in the form $u(x) = (u_0(x), \tilde{u}(x)) \in \mathbb{R}^{n+1}$ where $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\tilde{u} : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Definition 4.1. For every function $u \in C$ we define the support of u

$$K_u = \{x \in \mathbb{R}^n : u_0(x) > 1\}.$$

Then we define the topological charge of u

$$ch(u) := \begin{cases} \deg(\tilde{u}, K_u, 0) & \text{if } K_u \neq \emptyset, \\ 0 & \text{if } K_u = \emptyset. \end{cases}$$

Such that

$$\deg(\tilde{u}, K_u, 0) = \sum_{x \in \tilde{u}^{-1}(0)} \text{sgn} J_{\tilde{u}}(x),$$

(Brouwer degree) For more information about this subject, see [9];

where $J_{\tilde{u}}$ denotes the determinante of the Jacobian matrix.

We notice that this definition of charge is the same as in [2]. We recall that the topological charge is continuous with respect to the uniform convergence.

Now, for every $q \in \mathbb{Z}$ we set

$$\Lambda_q = \{u \in \Lambda_a : ch(u) = q\}.$$

Since the topological charge is continuous with respect to the uniform convergence

and the continuity of the embeddings E_a in L^∞ (see proposition 3.1) assure that the topological charge is continuous on Λ_a , it follows that Λ_q is open in E_a , since we have also

- $\Lambda_a = \bigcup_{q \in \mathbb{Z}} \Lambda_q$,
- $\Lambda_q \cap \Lambda_p = \emptyset$, $p \neq q$.

We conclude that every Λ_q is a connected component of Λ_a .

We assume that the space dimension is odd then we conclude that for every $q \in \mathbb{Z}$ the component Λ_q is isomorphic to the component Λ_{-q} .

So for every $u \in \Lambda_a$ we can define the charge $ch(u) \in \mathbb{Z}$. Now, we consider the set of a minimizer of f_a in the open set

$$\Lambda^*_q = \{u \in \Lambda_a : ch(u) \neq 0\}.$$

5 Properties of the energy functional

Lemma 5.1. *The functional f_a takes real values and it is continuous on Λ_a .*

Lemma 5.2. *The map $f' : E_a \rightarrow E'_a$ defined by*

$$\begin{aligned} \langle f'_a(u), v \rangle &= \langle -a \frac{r}{2} \Delta_r u - b \frac{p}{2} \Delta_p u + V'(u), v \rangle \\ &= \int_{\mathbb{R}^n} \left(a \frac{r}{2} |\nabla u|^{r-2} (\nabla u | \nabla v) + b \frac{p}{2} |\nabla u|^{p-2} (\nabla u | \nabla v) + V'(u) \cdot v \right) dx \end{aligned}$$

is continuous .

Lemma 5.3. *The functional f_a is coercive in Λ_a ; that is, for every sequence $u_k \subset \Lambda_a$ such that $\|u_k\|_a \rightarrow \infty$, we have $f_a(u_k) \rightarrow \infty$.*

Lemma 5.4. *Let $(u_k) \subset \Lambda_a$ be a weakly converging sequence. If the weak limit belongs to $\partial\Lambda_a$, then*

$$f_a(u_k) \rightarrow \infty \text{ as } k \rightarrow \infty.$$

Corollary 5.1. *For every $b > 0$, there exist $d = d(b)$ such that, for every $u \in \Lambda_a$ we have*

$$f_a(u) \leq b \Rightarrow \min_{x \in \mathbb{R}^n} |u(x) - \eta| \geq d.$$

Lemma 5.5. *The functional f_a is weakly lower semicontinuous in Γ_a .*

Proposition 5.1. *There exists $\Delta_a > 0$ such that, for every $u \in \Lambda_a$ satisfied $\|u\|_{L^\infty} \geq 1$ we have*

$$f_a(u) \geq \Delta_a.$$

6 Existence result

Theorem 6.1. *The minimum points $u \in \Lambda_a$ for the functional f_a are weak solutions of the system (2.5).*

Proposition 6.1. *(Splitting proposition) Let $(u_k) \in \Lambda_a^*$ be a sequence and M be a positive real number such that*

$$f_a(u_k) \leq M.$$

Then there exists $l \in \mathbb{N}$ such that

$$1 \leq l \leq \frac{M}{\Delta_a}$$

where Δ_a was introduced in Proposition 5.1 and there exist $\bar{u}_1, \dots, \bar{u}_l \in \Lambda_a$, $(x_k^1), \dots, (x_k^l) \subset \mathbb{R}^n$ such that, up to a subsequence,

$$u_k(\cdot + x_k^i) \rightarrow \bar{u}_i,$$

$$|x_k^i - x_k^j| \rightarrow \infty, \quad i \neq j,$$

$$\sum_{i=1}^l f_a(\bar{u}_i) \leq \liminf_{k \rightarrow \infty} f_a(u_k),$$

and

$$ch(u_k) = \sum_{i=1}^l ch(\bar{u}_i).$$

The minimum is attained on the set Λ_a , and it is easy to see that $u \equiv 0$ is a trivial solution of the problem. But, of course, we are interested in nontrivial solutions, We consider the following problem

$$I_* = \inf_{u \in \Lambda_a^*} f_a(u), \quad \Lambda_a^* = \{u \in E_a : ch(u) \neq 0\}.$$

The functional is bounded below and the set E_a is not empty. We consider fields u having the form

$$u(x) = \left(\frac{2}{1 + |x|^m}, \frac{1}{1 + |x|^m} x \right). \quad (6.1)$$

Lemma 6.1. *There exists a suitable $m \geq 1$, such that, the field u defined in (6.1) belongs to Λ_a^* .*

And moreover the set Λ_a^* is open in the space E_a ; indeed,

- $\Lambda_a^* = \bigcup_{q \in \mathbb{N}^*} \Lambda_a^q$,
- $\Lambda_a^q \cap \Lambda_a^p = \emptyset$, $p \neq q$.

where Γ_q is a connected component

Theorem 6.2. *Let $a, b > 0$, $p > n \geq r \geq 2$. Then, if V satisfies $(V_1) - (V_6)$, then there exists a weak solution of (2.5) (i.e., a static solution of (2.3)), which is a minimizer of the energy functional (2.7) in the class of maps whose topological charge is different from 0.*

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