Subpredictor Approach for Event-Triggered Control of Discrete-Time Systems with Input Delays

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Abstract— **We propose a new output event-triggered control design for linear discrete-time systems with constant arbitrarily long input delays, using delay compensating subpredictors. We prove input-to-state stability of the closed loop system, using framers and the theory of positive systems. A novel feature of our approach is our use of matrices of absolute values, instead of Euclidean norms, in our discrete-time event triggers for our delay compensating control design. We illustrate our approach using a model of the BlueROV2 marine vehicle, where our new event triggers lead to a smaller number of control recomputation times as compared with standard event triggers that were based on Euclidean norms, without sacrificing on settling times or on other performance metrics.**

I. Introduction

This work continues our development (which we started in [18], [20], [21], and [22]) of event-triggered control methods for discrete- and continuous-time systems using positive systems and interval observers. While [18] provided event-triggered subpredictor-based approaches to compensate for input delays in continuous-time systems, and [20] and [21] covered time-varying continuous-time event-triggered systems with uncertain vector fields or state delays, and [22] covered undelayed discrete-time linear systems with outputs, here we solve a complementary problem of compensating for input delays in event-triggered discrete-time linear systems.

As in [18], [20], [21], and [22], a key innovation is our use of triggers which are based on matrices of absolute values instead of standard Euclidean norms. As in [22], such triggers can lead to less conservative lower bounds on the intersample times as compared with the corresponding triggers that would arise from using standard Euclidean norms; see Section V. This is advantageous for applications that call for taking communication constraints into account, by reducing the number of time instants when control values are changed.

The need to reduce the number of times when control values are changed has led to a large literature on event-triggered regimes that address the computational challenges that are created by controls whose values are changed unnecessarily often; see, e.g., the survey [9]. Other notable and significant works include [1], [5], [10], [14], [15], [23], and [27]. Designing event-triggered control mechanisms entails the co-design of feedback controls and event triggering rules that indicate which events call for changing control values. In both discrete- and continuous-time cases, these events are usually specified as times when a measurement enters a specified region of the measurement space that is defined in terms of the usual Euclidean norm. This differs from standard zero-order hold controls whose control recomputation times are independent of the measurements from the systems.

Another main ingredient of our approach is positive systems, which are dynamics for which the nonnegative orthant is positively invariant. The literature on positive systems provided new control methods to help overcome technical challenges of using standard Lyapunov functionals for time delay systems. Positive systems have been used in conjunction with interval observers (as defined, e.g., in [7] and [25]), which yield intervals containing values of unknown states when the inequalities are viewed as being componentwise; see [16]. Dynamical systems theory based on interval observers and positive systems has led to notable contributions in aerospace engineering, mathematical biology, and other applications.

By providing a new design for event-triggered control for linear discrete-time systems with outputs and arbitrarily long constant input delays, we believe that this work is the first to use positive systems and interval observers to prove input-to-state stability (or ISS) for discrete-time systems with outputs whose delays are compensated by subpredictors. In particular, our work is novel relative to notable works on predictors that did not involve event triggering, such as [2], [3], [4], and [19]. For discrete-time systems, we do not have to rule out the possibility of Zeno's phenomenon. Hence, it may be worth discretizing a continuous time system and to next use event triggered control, instead of using event triggered control for continuous-time systems. Our method uses the structure of our dynamics to provide an alternative to small-gain methods (e.g., from [11] and [12]) and so can lead to less frequent control computations.

After presenting our definitions and notation and new theorem in Section II, followed by our main lemmas in Section III and the proof of our theorem in Section IV, Section V illustrates how discretized continuous time systems are amenable to our new theorem. Section V also applies our work to a model of the BlueROV2 underwater vehicle, where our input delay compensation results complement our results for this dynamics in undelayed cases in [22]. The example

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illustrates how our new event triggers can lead to fewer control recomputation times, compared with standard event triggers that were based on the Euclidean norm, without sacrificing the settling times or other performance metrics. Since the BlueROV2 is widely used for the study of corals and for other environmental surveys, this illustrates the value of our approach in a significant example from ecological robotics, where there is strong motivation for reducing the number of control recomputation times. We close in Section VI with a summary and our ideas for follow up research.

II. Definitions, Notation, and Main Result

We use standard notation, which we simplify when no confusion would arise. The dimensions of our Euclidean spaces are arbitrary, unless we indicate otherwise. We omit arguments of functions when no confusion would arise. We set $\mathbb{Z}_0 = \{0, 1, 2, ...\}$. For a matrix $G = [g_{ij}] \in \mathbb{R}^{r\times s}$, we set $|G| = [|g_{ij}|]$, so the entries of $|G|$ are the absolute values of the corresponding entries g_{ij} of the matrix G. We use $||\cdot||$ to denote the usual Euclidean norm. By G^+, we denote the matrix whose entries are $\max\{0, g_{ij}\}$ and $G^- = G^+ - G$. A square matrix is called Schur stable provided its spectral radius is in $[0, 1)$. For matrices $D = [d_{ij}]$ and $E = [e_{ij}]$ of the same size, we write $D < E$ (resp., $D \le E$) provided $d_{ij} < e_{ij}$ (resp., $d_{ij} \le e_{ij}$) for all i and j. We also use $D \nleq E$ to mean that there is a pair (i, j) such that $d_{ij} > e_{ij}$. We adopt similar notation for vectors. We call a matrix S positive (resp., nonnegative) provided $0 < S$ (resp., $0 \le S$), where 0 is the zero matrix. We use standard notions of ISS for discrete-time systems [13], and I is the identity matrix.

We consider the system with outputs

$$\left\{\begin{array}{rcl} x_{k+1} &=& Ax_k + Bu_{k-r} + \delta_k \\ y_k &=& Cx_k \end{array}\right. \tag{1}$$

with $k \in \mathbb{Z}_0$, x valued in $\mathbb{R}^n$, the control u (that we will specify in our theorem) valued in $\mathbb{R}^p$, the output y valued in $\mathbb{R}^q$, the unknown sequence $\delta_k \in \mathbb{R}^n$ representing a disturbance, and the constant integer $r \ge 1$ representing an input delay that will arise in the measurements that enter the control u. A key feature of our work is that we allow r to be any positive integer. This arbitrarily long input delay is compensated for by our chain of subpredictors. Throughout this work, we make these three assumptions:

Assumption 1: There is a matrix $K \in \mathbb{R}^{p\times n}$ such that

$$H = A + BK \tag{2}$$

is nonnegative. There is a matrix $\Gamma \in \mathbb{R}^{n\times n}$ such that

$$H + |BK|\Gamma \tag{3}$$

is Schur stable and such that $\Gamma > 0$. □

Assumption 2: There is a matrix $L \in \mathbb{R}^{n\times q}$ such that $A + LC$ is Schur stable. □

Assumption 3: The matrix A is invertible. □

See Remark 1 for the motivation for Assumptions 1-3 and the ease with which we can often check that they are satisfied. To specify u, fix matrices K, L, and Γ satisfying Assumptions 1-2 and a matrix $R \in \mathbb{R}^{n\times n}$ such that $A+R$ is Schur stable (e.g., $R = -A$). We then use the r subpredictors

$$\left\{\begin{array}{rcl} z_{1,k+1} &=& Az_{1,k} + Bu_{k-r+1} + ALCA^{-1}z_{1,k} \\ && -ALy_k - ALCA^{-1}Bu_{k-r} \\ z_{2,k+1} &=& Az_{2,k} + Bu_{k-r+2} \\ && +R(z_{2,k} - Az_{1,k} - Bu_{k-r+1}) \\ &\vdots& \\ z_{r,k+1} &=& Az_{r,k} + Bu_k \\ && +R(z_{r,k} - Az_{r-1,k} - Bu_{k-1}) \end{array}\right. \tag{4}$$

where each $z_{i,k}$ is valued in $\mathbb{R}^n$. We also use the control law

$$u_{k-r} = Kz_{r,\sigma(k)-r} \tag{5}$$

with K in Assumption 1 and $\sigma : \mathbb{Z}_0 \to \mathbb{Z}_0$ defined by

$$\begin{array}{rcl} \sigma(0) &=& 0 \\ \sigma(j+1) &=& j+1 \quad \text{if} \\ && |z_{r,j+1-r} - z_{r,\sigma(j)-r}| \nleq \Gamma|z_{r,j+1-r}| \\ \sigma(j+1) &=& \sigma(j) \quad \text{if} \\ && |z_{r,j+1-r} - z_{r,\sigma(j)-r}| \le \Gamma|z_{r,j+1-r}|. \end{array} \tag{6}$$

See Remark 3 for the motivation for our triggering rule (6). Then the corresponding closed-loop system is

$$x_{k+1} = Ax_k + BKz_{r,\sigma(k)-r} + \delta_k. \tag{7}$$

In terms of the notation

$$\begin{array}{c} e_{i,k} = z_{i,k} - z_{i-1,k+1} \text{ for } i = 1, \dots, r, \\ E_k = [e_{1,k},\ \dots, e_{r,k}]^\top, \text{ and } z_{0,k} = x_k, \end{array} \tag{8}$$

where the $z_{i,k}$'s are the states of (4), our main result is:

Theorem 1: Let (1) be such that Assumptions 1-3 are satisfied. Choose any matrices K, L, and Γ that satisfy our assumptions, and any matrix R such that $A + R$ is Schur stable. Then we can find real constants $\bar{c}_i > 0$ for $i = 1, 2, 3$ such that all solutions of the closed-loop system (7) with the control (5)-(6) defined by the subpredictors (4) satisfy

$$\begin{array}{rcl} ||x_k|| &\le& \bar{c}_1 e^{-\bar{c}_2(k-k_0)}\left(||x_{k_0}|| + ||E_{k_0-r}||\right) \\ && \bar{c}_3 \displaystyle\sup_{q\in\{k_0-r,\dots,k-1\}} ||\delta_q|| \end{array} \tag{9}$$

for all $k_0 > r$ and all integers $k \ge k_0$.

Remark 1: Assumptions 1-3 are easily checked in many cases. For instance, Assumption 1 is not very restrictive. Indeed, when a pair (A_a, B_a) is controllable, we can choose a matrix K_a such that $A_a+B_aK_a$ is Schur stable, having distinct real eigenvalues on the interval $(0, 1)$. Then, there is an invertible matrix $P \in \mathbb{R}^{n\times n}$ such that $P(A_a+B_aK_a)P^{-1}$ is Schur stable and nonnegative (by diagonalizing $A_a+B_aK_a$). Consequently, if we consider $Z_{k+1} = A_aZ_k + B_au + \delta_{a,k}$, then the change of coordinates $X_k = PZ_k$ gives

$$X_{k+1} = AX_k + Bu + P\delta_{a,k}, \tag{10}$$

with $A = PA_aP^{-1}$ and $B = PB_a$, and (10) satisfies Assumption 1, because the choice $K = K_aP^{-1}$ gives

$$\begin{array}{rcl} A + BK &=& PA_aP^{-1} + PB_aK_aP^{-1} \\ &=& P(A_a + B_aK_a)P^{-1} \end{array} \tag{11}$$

which is both Schur stable and nonnegative, and because when the matrix H is Schur stable, there always exists a positive matrix $\Gamma > 0$ such that $H + |BK|\Gamma$ is Schur stable, by choosing Γ so its entries are small enough.

Assumption 2 is satisfied when the pair (A, C) is observable, which is a standard observability condition. Assumption 3 is a technical assumption, and holds in the following generic sense. The invertibility of A can be viewed as the requirement that a real analytic function defined on $\mathbb{R}^{n^2}$ takes a nonzero value, namely, the determinant of A viewed as a function of the n^2 entries of A. It follows from a standard analytic continuation argument that the set of all invertible $n \times n$ matrices form an open and dense subset of $\mathbb{R}^{n\times n}$. Hence, invertibility of A is generic, in the same sense of genericity that controllable pairs (and so also observable pairs) form a generic set; see, e.g., [26, Section 3.3]. □

Remark 2: By contrast with the subsequential observer of [17], (4) depends on y_k, and not on y_{k+1}. This is the reason why we have to assume that A is invertible. □

Remark 3: Since our focus is on the state of the dynamics (1), we did not include the actuator state on the left side of (9). Our choice of the triggering rule (6) was made because (as we show in our proof of our theorem) it ensures that

$$|z_{r,\sigma(k)-r} - z_{r,k-r}| \leq \Gamma|z_{r,k-r}| \tag{12}$$

holds for all integers $k \geq 0$. The preceding inequality plays an essential role in our stability analysis for our closed-loop system. It contrasts with the trigger requirement

$$||z_{r,\sigma(k)-r} - z_{r,k-r}|| \leq \sigma_* ||z_{r,k-r}|| \tag{13}$$

that would arise from using the usual Euclidean norm $||\cdot||$ and a constant σ_*. In fact, the largest $\sigma_* > 0$ such that all pairs $(z_{r,\sigma(k)-r}, z_{r,k-r})$ satisfying (13) also satisfy (12) is $\sigma_* = \min_{ij} \Gamma_{ij}$, i.e., the smallest entry of $\Gamma = [\Gamma_{ij}]$. This property of the minimum entry of Γ was shown in our study [22] of the undelayed case, and the same reasoning applies in the delayed case that we consider here. In Section V-B, we illustrate how using our new condition (12) can lead to fewer event triggers on given time horizons as compared to the number of event triggers that we would have required if we had instead used the traditional trigger (13) with the corresponding least conservative choice $\sigma_* = \min_{ij} \Gamma_{ij}$. □

III. Key Lemmas to Prove Theorem 1

Our first lemma is:

Lemma 1: With the preceding assumptions and notation, we can construct matrices Ω and Λ such that the equality

$$E_{a+b} = \Omega^b E_a + \sum_{p=0}^{b-1} \Omega^{b-p-1} \Lambda \begin{bmatrix} \delta_{a+p} \\ \delta_{a+p+1} \end{bmatrix} \tag{14}$$

is satisfied for all integers $a > 0$ and $b > 0$. □

Proof: First, let us observe that

$$\begin{aligned} ALy_k + ALCA^{-1}Bu_{k-r} &= ALC(x_k + A^{-1}Bu_{k-r}) \\ &= F(x_{k+1} - \delta_k), \end{aligned} \tag{15}$$

where $F = ALCA^{-1}$. Here and in the sequel, all equalities are for all $k \in \mathbb{Z}_0$, unless otherwise indicated. It follows that $z_{1,k+1} = Az_{1,k} + Bu_{k-r+1} + Fz_{1,k} - Fx_{k+1} + F\delta_k$, which can be rewritten as

$$z_{1,k+1} = Az_{1,k} + Bu_{k-r+1} + Fe_{1,k} + F\delta_k. \tag{16}$$

Next, consider the z_2-subsystem of (4). Since (16) gives $-Az_{1,k} - Bu_{k-r+1} = -z_{1,k+1} + Fe_{1,k} + F\delta_k$, we have

$$z_{2,k+1} = Az_{2,k} + Bu_{k-r+2} + Re_{2,k} + RFe_{1,k} + RF\delta_k. \tag{17}$$

Similarly, we can prove (by induction on a) that

$$\begin{aligned} z_{a,k+1} \;=\; & Az_{a,k} + Bu_{k-r+a} + Re_{a,k} \\ & + R^2 e_{a-1,k} + ... + R^{a-1} e_{2,k} \\ & + R^{a-1} Fe_{1,k} + R^{a-1} F\delta_k \end{aligned} \tag{18}$$

for $a = 2$ to r. Hence, from (16), we deduce that

$$\begin{aligned} e_{1,k+1} &= Az_{1,k} + Bu_{k-r+1} + Fe_{1,k} + F\delta_k \\ &\quad - [Ax_{k+1} + Bu_{k-r+1} + \delta_{k+1}] \\ &= (A + F)\, e_{1,k} + F\delta_k - \delta_{k+1} \\ &= Me_{1,k} + F\delta_k - \delta_{k+1}, \end{aligned} \tag{19}$$

where $M = A(A + LC)A^{-1}$.

Therefore, from (16)-(17), we deduce that

$$\begin{aligned} e_{2,k+1} &= A(z_{2,k} - z_{1,k+1}) + Re_{2,k} + RFe_{1,k} \\ &\quad + RF\delta_k - Fe_{1,k+1} - F\delta_{k+1} \\ &= (A + R)e_{2,k} + RFe_{1,k} - Fe_{1,k+1} \\ &\quad + RF\delta_k - F\delta_{k+1} \\ &= (A + R)e_{2,k} + (RF - FM)e_{1,k} \\ &\quad + (RF - F^2)\delta_k. \end{aligned} \tag{20}$$

Then (18) gives

$$\begin{aligned} e_{a,k+1} &= z_{a,k+1} - z_{a-1,k+2} \\ &= (A + R)e_{a,k} + R^2 e_{a-1,k} + ... \\ &\quad + R^{a-1} e_{2,k} + R^{a-1} Fe_{1,k} + R^{a-1} F\delta_k \\ &\quad - \big(Re_{a-1,k+1} + R^2 e_{a-2,k+1} + ... \\ &\quad + R^{a-2} e_{2,k+1} + R^{a-2} Fe_{1,k+1} \\ &\quad + R^{a-2} F\delta_{k+1}\big) \end{aligned} \tag{21}$$

for $a = 3$ to r, and so also

$$\begin{aligned} e_{a,k+1} &= (A + R)e_{a,k} + R^2 e_{a-1,k} + ... \\ &\quad + R^{a-1} e_{2,k} + R^{a-1} Fe_{1,k} - (Re_{a-1,k+1} \\ &\quad + R^2 e_{a-2,k+1} + ... + R^{a-2} e_{2,k+1}\big) \\ &\quad - R^{a-2} F(Me_{1,k} + F\delta_k - \delta_{k+1}) \\ &\quad + R^{a-1} F\delta_k - R^{a-2} F\delta_{k+1} \end{aligned} \tag{22}$$

for $a = 3$ to r, where the last equality is a consequence of the last equality in (19). We deduce that

$$\begin{aligned} e_{a,k+1} &= (A + R)e_{a,k} + R^2 e_{a-1,k} + ... \\ &\quad + R^{a-1} e_{2,k} + R^{a-1} Fe_{1,k} \\ &\quad - \big(Re_{a-1,k+1} + R^2 e_{a-2,k+1} + ... \\ &\quad + R^{a-2} e_{2,k+1}\big) \\ &\quad - R^{a-2} FMe_{1,k} + (R^{a-1} - R^{a-2} F)F\delta_k. \end{aligned} \tag{23}$$

It now follows from (19), (20) and (23) that there is a matrix

Ω of the form

$$\Omega = \begin{bmatrix} M & 0 & \dots & \dots & 0 \\ \star & A+R & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ \star & \dots & \dots & \star & A+R \end{bmatrix}. \tag{24}$$

and a matrix Λ such that

$$E_{k+1} = \Omega E_k + \Lambda \begin{bmatrix} \delta_k \\ \delta_{k+1} \end{bmatrix} \tag{25}$$

with E_j defined in (8). Therefore, for all integers a and b such that $a > 0$ and $b > 0$, we can argue by induction on b to show that the equality (14) is satisfied. ∎

Using our definitions (8) of the $e_{i,k}$'s and the formulas (8) for the E_k's, we can also prove the following:

Lemma 2: With the preceding notation with Ω and Λ satisfying the requirements from Lemma 1, we have

$$z_{r,k} = x_{k+r} + \xi_k, \text{ where } \xi_k = \sum_{j=0}^{r-1} e_{r-j,k+j} \tag{26}$$

for all $k \geq 0$. Also, if we let $V_i \in \mathbb{R}^{n\times(rn)}$ denote the matrix $[V_{(i,1)}, \dots, V_{(i,r)}]$ where $V_{(i,j)} = 0 \in \mathbb{R}^{n\times n}$ if $j \neq i$ and $V_{(i,i)} = I \in \mathbb{R}^{n\times n}$, and if we choose

$$W = \sum_{j=0}^{r-1} V_{r-j}\Omega^j, \tag{27}$$

then the following holds for all $k \geq 1$:

$$\begin{aligned} \xi_k &= WE_k \\ &+ \sum_{j=1}^{r-1} V_{r-j} \sum_{p=0}^{j-1} \Omega^{j-p-1}\Lambda \begin{bmatrix} \delta_{k+p} \\ \delta_{k+p+1} \end{bmatrix} \end{aligned} \tag{28}$$

if $r > 1$ and

$$\xi_k = V_1 E_k \tag{29}$$

if $r = 1$. □

Proof: Condition (26) follows from a telescoping sum argument. Also, we can rewrite the ξ_k formula in (26) as

$$\xi_k = \sum_{j=0}^{r-1} V_{r-j}E_{k+j}. \tag{30}$$

It follows from (14) and our choice (27) of W that

$$\begin{aligned} \xi_k &= V_r E_k + \sum_{j=1}^{r-1} V_{r-j}\Bigg(\Omega^j E_k \\ &\quad + \sum_{p=0}^{j-1} \Omega^{j-p-1}\Lambda \begin{bmatrix} \delta_{k+p} \\ \delta_{k+p+1} \end{bmatrix}\Bigg) \\ &= WE_k + \sum_{j=1}^{r-1} V_{r-j} \sum_{p=0}^{j-1} \Omega^{j-p-1}\Lambda \begin{bmatrix} \delta_{k+p} \\ \delta_{k+p+1} \end{bmatrix} \end{aligned} \tag{31}$$

if $r > 1$, and that (29) holds if $r = 1$. ∎

IV. Proof of Theorem

We now use the preceding two lemmas to prove Theorem 1. The proof has three parts. In the first part, we provide a new representation of the closed-loop system, to facilitate our analysis. In the second part, we build a framer for the state of the closed-loop system. In the final part, we use the framer to provide a stability analysis to prove the ISS result.

First Part: New Representation of Closed-Loop System. The system (7) can be rewritten as

$$\begin{aligned} x_{k+1} &= Ax_k + BKz_{r,k-r} \\ &\quad + BK(z_{r,\sigma(k)-r} - z_{r,k-r}) + \delta_k. \end{aligned} \tag{32}$$

From the equality (26), we deduce that

$$\begin{aligned} x_{k+1} &= Ax_k + BK(x_k + \xi_{k-r}) \\ &\quad + BK(z_{r,\sigma(k)-r} - z_{r,k-r}) + \delta_k \end{aligned} \tag{33}$$

for all $k \geq r$. Then, with H defined in (2), the representation

$$\begin{aligned} x_{k+1} &= Hx_k + BK(z_{r,\sigma(k)-r} - z_{r,k-r}) \\ &\quad + BK\xi_{k-r} + \delta_k \end{aligned} \tag{34}$$

is obtained.

Second Part: Framer Construction. With a view to the stability analysis of the closed-loop system, and using our notation from Section II, we introduce the dynamic extension

$$\left\{ \begin{aligned} \overline{x}_{k+1} &= H\overline{x}_k + (BK)^+(z_{r,\sigma(k)-r} - z_{r,k-r})^+ \\ &\quad +(BK)^-(z_{r,\sigma(k)-r} - z_{r,k-r})^- \\ &\quad +(BK\xi_{k-r})^+ + \delta_k^+ \\ \underline{x}_{k+1} &= H\underline{x}_k - (BK)^+(z_{r,\sigma(k)-r} - z_{r,k-r})^- \\ &\quad -(BK)^-(z_{r,\sigma(k)-r} - z_{r,k-r})^+ \\ &\quad -(BK\xi_{k-r})^- - \delta_k^- \end{aligned} \right. \tag{35}$$

with $k \geq r$. Let $k_0 > r$. We have

$$\begin{aligned} x_{k+1} &= Hx_k + (BK)^+(z_{r,\sigma(k)-r} - z_{r,k-r})^+ \\ &\quad +(BK)^-(z_{r,\sigma(k)-r} - z_{r,k-r})^- \\ &\quad -(BK)^+(z_{r,\sigma(k)-r} - z_{r,k-r})^- \\ &\quad -(BK)^-(z_{r,\sigma(k)-r} - z_{r,k-r})^+ \\ &\quad +(BK\xi_{k-r})^+ - (BK\xi_{k-r})^- + \delta_k^+ - \delta_k^-. \end{aligned} \tag{36}$$

It follows that

$$\begin{aligned} \overline{x}_{k+1} - x_{k+1} &= H(\overline{x}_k - x_k) \\ &\quad +(BK)^+(z_{r,\sigma(k)-r} - z_{r,k-r})^- \\ &\quad +(BK)^-(z_{r,\sigma(k)-r} - z_{r,k-r})^+ \\ &\quad +(BK\xi_{k-r})^- + \delta_k^- \\ x_{k+1} - \underline{x}_{k+1} &= H(x_k - \underline{x}_k) \\ &\quad +(BK)^+(z_{r,\sigma(k)-r} - z_{r,k-r})^+ \\ &\quad +(BK)^-(z_{r,\sigma(k)-r} - z_{r,k-r})^- \\ &\quad +(BK\xi_{k-r})^+ + \delta_k^+. \end{aligned} \tag{37}$$

Since the matrix H is nonnegative, using (37), one can prove by induction that if $\overline{x}_{k_0} - x_{k_0}$ and $x_{k_0} - \underline{x}_{k_0} \geq 0$, then $\overline{x}_k - x_k \geq 0$ and $x_k - \underline{x}_k \geq 0$ for all $k \geq k_0$. Thus

$$\underline{x}_k \leq x_k \leq \overline{x}_k \text{ for all } k \geq k_0. \tag{38}$$

Similarly, if $\overline{x}_{k_0} \geq 0$ and $\underline{x}_{k_0} \leq 0$ then

$$\overline{x}_k \geq 0 \text{ and } \underline{x}_k \leq 0 \tag{39}$$

for all $k \geq k_0$, again by induction. It follows that (35) satisfies the requirements to be a framer for (34). The framer (35) is inspired by those constructed, e.g., in [6].

Third Part: Stability Analysis. We use the interval observer (35) to establish stability properties for (34). Let us introduce

$$\tilde{x}_k = \overline{x}_k - \underline{x}_k. \tag{40}$$

We deduce from (38) and (39) that

$$|x_k| \leq \tilde{x}_k \tag{41}$$

for all $k \geq k_0$. On the other hand, (35) gives

$$\begin{array}{rcl} \tilde{x}_{k+1} & = & H\tilde{x}_k + |BK||z_{r,\sigma(k)-r} - z_{r,k-r}| \\ & & + |BK\xi_{k-r}| + |\delta_k|. \end{array} \tag{42}$$

Also, by (39), we have $\tilde{x}_k \geq 0$ for all $k \geq k_0$. Also, we deduce from the definition of σ in (6) that

$$\tilde{x}_{k+1} \leq H\tilde{x}_k + |BK|\Gamma|z_{r,k-r}| + |BK\xi_{k-r}| + |\delta_k| \tag{43}$$

for all $k \geq r$. To see why, notice that if $|z_{r,\sigma(k)-r} - z_{r,k-r}| \nleq \Gamma|z_{r,k-r}|$, then $\sigma(k) \neq k$, so the definition of σ gives $\sigma(k) = \sigma(k-1)$, so $|z_{r,\sigma(k-1)-r} - z_{r,k-r}| \nleq \Gamma|z_{r,k-r}|$, so (6) with the choice $j = k-1$ gives $\sigma(k) = k$, which is a contradiction.

Let us take $k_0 > r$. Again using (26), we obtain

$$\begin{array}{rcl} \tilde{x}_{k+1} & \leq & H\tilde{x}_k + |BK|\Gamma|x_k + \xi_{k-r}| \\ & & +|BK\xi_{k-r}| + |\delta_k| \\ & \leq & H\tilde{x}_k + |BK|\Gamma|x_k| + |BK|\Gamma|\xi_{k-r}| \\ & & +|BK\xi_{k-r}| + |\delta_k| \end{array} \tag{44}$$

for all $k \geq k_0$, by (43). From (41), we deduce that

$$\begin{array}{rcl} \tilde{x}_{k+1} & \leq & (H + |BK|\Gamma)\tilde{x}_k + |BK|\Gamma|\xi_{k-r}| \\ & & +|BK\xi_{k-r}| + |\delta_k| \\ & \leq & \kappa_1\tilde{x}_k + \kappa_2|\xi_{k-r}| + |\delta_k|, \end{array} \tag{45}$$

where $\kappa_1 = H + |BK|\Gamma$ and $\kappa_2 = |BK|(\Gamma + I)$. On the other hand, from (28), it follows that

$$\begin{array}{rcl} \xi_{k-r} & = & WE_{k-r} \\ & & + \sum\limits_{j=1}^{r-1} V_{r-j} \sum\limits_{p=0}^{j-1} \Omega^{j-p-1}\Lambda \left[\begin{array}{c} \delta_{k-r+p} \\ \delta_{k-r+p+1} \end{array} \right] \end{array} \tag{46}$$

when $k \geq k_0$ and $r > 1$. Hence,

$$\begin{array}{rcl} \tilde{x}_{k+1} & \leq & \kappa_1\tilde{x}_k + \kappa_2|W||E_{k-r}| + |\delta_k| \\ & & + \kappa_2 \left| \sum\limits_{j=1}^{r-1} V_{r-j} \sum\limits_{p=0}^{j-1} \Omega^{j-p-1}\Lambda \left[\begin{array}{c} \delta_{k-r+p} \\ \delta_{k-r+p+1} \end{array} \right] \right| \end{array} \tag{47}$$

when $r > 1$. Thus, when $k \geq k_0$, we have:

$$\begin{array}{l} \tilde{x}_{k+1} \leq \kappa_1\tilde{x}_k + \kappa_2|W||E_{k-r}| + |\delta_k| \\ \qquad + \kappa_2 \left| \sum\limits_{j=1}^{r-1} V_{r-j} \sum\limits_{p=0}^{j-1} \Omega^{j-p-1}\Lambda \left[\begin{array}{c} \delta_{k-r+p} \\ \delta_{k-r+p+1} \end{array} \right] \right| \\ E_{k-r+1} = \Omega E_{k-r} + \Lambda \left[\begin{array}{c} \delta_{k-r} \\ \delta_{k-r+1} \end{array} \right]. \end{array} \tag{48}$$

These equalities can be written in the compact form

$$\left[\begin{array}{c} \tilde{x}_{k+1} \\ E_{k-r+1} \end{array} \right] \leq \Upsilon \left[\begin{array}{c} \tilde{x}_k \\ E_{k-r} \end{array} \right] + \Theta(\delta_{k-r}, ..., \delta_k), \tag{49}$$

where $\Upsilon = \left[\begin{array}{cc} \kappa_1 & \kappa_2|W| \\ 0 & \Omega \end{array} \right]$ and $\Theta(\delta_{k-r}, ..., \delta_k) =$

$$\left[\begin{array}{c} \kappa_2 \left| \sum\limits_{j=1}^{r-1} V_{r-j} \sum\limits_{p=0}^{j-1} \Omega^{j-p-1}\Lambda \left[\begin{array}{c} \delta_{k-r+p} \\ \delta_{k-r+p+1} \end{array} \right] \right| + |\delta_k| \\ \Lambda \left[\begin{array}{c} \delta_{k-r} \\ \delta_{k-r+1} \end{array} \right] \end{array} \right] \tag{50}$$

when $k > r > 1$. Hence, when $k > l > r$, we have

$$\begin{array}{l} \left[\begin{array}{c} \tilde{x}_k \\ E_{k-r} \end{array} \right] \leq \Upsilon^{k-l} \left[\begin{array}{c} \tilde{x}_l \\ E_{l-r} \end{array} \right] \\ \qquad + \sum\limits_{p=0}^{k-l-1} \Upsilon^{k-l-p-1}\Theta(\delta_{l-r+p}, ..., \delta_{l+p}), \end{array} \tag{51}$$

by an induction argument that is similar to the one that gave (14). Assumption 1 ensures that κ_1 is Schur stable. The matrix Ω is Schur stable too because $A + R$ and $A + LC$ are Schur stable (by Assumption 2).

It follows that the matrix Υ is Schur stable. Therefore there are constants $\beta_i > 0$ such that $||\Upsilon^\ell|| \leq \beta_1 e^{-\beta_2 \ell}$ for all integers $\ell \geq 0$. Then, we deduce from (51) that

$$\begin{array}{l} \left|\left|\left[\begin{array}{c} \tilde{x}_k \\ E_{k-r} \end{array} \right]\right|\right| \leq \beta_1 e^{-\beta_2(k-l)} \left|\left|\left[\begin{array}{c} \tilde{x}_l \\ E_{l-r} \end{array} \right]\right|\right| \\ + \sum\limits_{p=0}^{k-l-1} \beta_1 e^{-\beta_2(k-l-p-1)} ||\Theta(\delta_{l-r+p}, ..., \delta_{l+p})||. \end{array} \tag{52}$$

We deduce there is a constant $\beta_3 > 0$ such that

$$\begin{array}{l} \left|\left|\left[\begin{array}{c} \tilde{x}_k \\ E_{k-r} \end{array} \right]\right|\right| \leq \beta_1 e^{-\beta_2(k-l)} \left|\left|\left[\begin{array}{c} \tilde{x}_l \\ E_{l-r} \end{array} \right]\right|\right| \\ +\beta_3 \sum\limits_{p=0}^{k-l-1} e^{-\beta_2(k-l-p)} \sup\limits_{q\in\{l-r,...,k-1\}} ||\delta_q||. \end{array} \tag{53}$$

Thus,

$$\begin{array}{l} \left|\left|\left[\begin{array}{c} \tilde{x}_k \\ E_{k-r} \end{array} \right]\right|\right| \leq \beta_1 e^{-\beta_2(k-l)} \left|\left|\left[\begin{array}{c} \tilde{x}_l \\ E_{l-r} \end{array} \right]\right|\right| \\ +\beta_3 \sup\limits_{q\in\{l-r,...,k-1\}} ||\delta_q||. \end{array} \tag{54}$$

From (41), it follows that

$$\begin{array}{l} \sqrt{||x_k||^2 + ||E_{k-r}||^2} \\ \leq \beta_1 e^{-\beta_2(k-l)} \sqrt{||\tilde{x}_l||^2 + ||E_{l-r}||^2} \\ +\beta_3 \sup\limits_{q\in\{l-r,...,k-1\}} ||\delta_q||. \end{array} \tag{55}$$

This, (41), and the subadditivity of the square root allow us to conclude when $r > 1$, since we can assume that $\bar{x}_k \leq 2|x_k|$ and $\bar{x}_k \geq -2|x_k|$ for all $k \leq k_0$ to get $||\tilde{x}_{k_0}|| \leq 4||x_{k_0}||$. The case $r = 1$ is similar. This concludes the proof.

V. Illustrations

A. Discretization of Continuous Time Systems

Starting from a continuous time system of the form

$$\dot{x}(t) = Fx(t) + Gu(t-\tau) + \gamma(t) \tag{56}$$

where x is valued in $\mathbb{R}^n$, u is valued in $\mathbb{R}^p$, the components of γ are piecewise continuous bounded functions representing uncertainty, and $\tau > 0$ is a constant representing a delay, we can often obtain a discretized system that is amenable to Theorem 1. To see how, we select constants r and $\nu > 0$ such that $\tau = r\nu$, and introduce the sequence $s_i = i\nu$ for all $i \in \mathbb{Z}_0$; our choice $\tau = r\nu$ is motivated by the fact that we found it to be useful for modeling our marine robotic dynamics below. We restrict our attention to piecewise constant feedbacks that are defined by $u(t-\tau) = u(s_{i-r})$ if $s_i \le t < s_{i+1}$. Then we can apply the method of variation of parameters to $\dot{x}(t) = Fx(t) + Gu(s_{i-r}) + \gamma(t)$ on the interval $[s_i, s_{i+1})$ to obtain

$$\begin{aligned} x(s_{i+1}) \;=\; & e^{\nu F}x(s_i) + \textstyle\int_{s_i}^{s_{i+1}} e^{F(s_{i+1}-\ell)}\mathrm{d}\ell G u(s_{i-r}) \\ & + \textstyle\int_{s_i}^{s_{i+1}} e^{F(s_{i+1}-\ell)}\gamma(\ell)\mathrm{d}\ell \end{aligned} \tag{57}$$

for all $i \in \mathbb{Z}_0$. Thus, we obtain the discrete-time system

$$\begin{aligned} & x(s_{i+1}) = A_\nu x(s_i) + B_\nu u(s_{i-r}) + \Delta_i \text{ where} \\ & A_\nu = e^{\nu F},\ B_\nu = \textstyle\int_0^\nu e^{F\ell}\mathrm{d}\ell G, \text{ and} \\ & \Delta_i = \textstyle\int_{s_i}^{s_{i+1}} e^{F(s_{i+1}-\ell)}\gamma(\ell)\mathrm{d}\ell. \end{aligned} \tag{58}$$

We can then apply Theorem 1 to (58), provided (i) (F, G) is a controllable pair, (ii) the coefficient matrix C of the output is such that (A_ν, C) is observable, (iii) there are no eigenvalues λ and μ of A_ν such that $\nu(\lambda-\mu)$ is a nonzero integer multiple of 2π, and (iv) A_ν is invertible, possibly after a change of coordinates as in Remark 1. This follows from the Kalman-Ho-Narendra conditions, e.g., from [26, p.102]. That way, we get a stabilizing event triggered control law for the discrete-time system (58), and this feedback also ensures ISS of the system (56). As we illustrate in the next section, our use of event triggers that are based on vectors of absolute values instead of standard Euclidean norms can reduce the number of control computations without increasing oscillations or settling times, and thereby offer computational advantages.

B. BlueROV2 Marine Vehicle

We apply Theorem 1 to underwater robotic vehicles that we studied in the undelayed case in [22]. The dynamics are for the control of the depth and pitch degrees-of-freedom (or DOF) of the BlueROV2 vehicle, which is widely used to study corals and other ecosystems, but similar reasoning applies to similar underwater vehicles. Following [22], we assume that the vehicle has a Doppler Velocity Logger (or DVL) for estimating the vehicle's velocity. When close to the sea floor, the DVL typically experiences bottom lock, making it impractical to continuously change the control values, and producing an input delay. Therefore, build a control for the depth plane, using a more practical event-triggered delay-compensating sample data subpredictor approach.

As noted in [24, Equation (9.28)], after linearization and assuming that the vehicle is neutrally buoyant, we obtain the following linearized dynamics in the depth plane:

$$\begin{aligned} & (m - X_{\dot{w}(t)})\dot{w}(t) - (mx_g + Z_{\dot{q}})\dot{q}(t) \\ & -Z_w w(t) - (mU + z_q)q(t) = Z_{\gamma_s}u_Z \\ & \text{and } (mx_g + M_{\dot{w}}(t))\dot{w}(t) + (I_{yy} - M_{\dot{q}})\dot{q}(t) \\ & -M_w w(t) + (mx_gU - M_q)q(t) - M_\theta\theta = M_{\gamma_s}u_M \end{aligned} \tag{59}$$

whose parameters were obtained experimentally computed and reported in [24]. We assume that the nominal surge velocity is $U = 0.1$m/s. This produces a two state system, whose states are the depth and pitch velocity $x = [w,\ q]^\top$, and the control inputs u_Z and u_M are the force and moment required to produce motion of the vehicle. Using the parameter values and controller from [24], the system (59) takes the form $\dot{x}(t) = Fx(t) + Gu$ with

$$F = \begin{bmatrix} -0.17742 & -0.3027 \\ 0.5394 & -1.4685 \end{bmatrix} \text{ and } G = \begin{bmatrix} -0.2063 \\ -0.7629 \end{bmatrix} \tag{60}$$

and so is amenable to the method from Section V-A. Hence, we assume that the control is piecewise constant with a constant sample rate $\bar{s}$, to convert the dynamics into

$$\begin{aligned} & Z_{k+1} = A_a Z_k + B_a u, \text{ where} \\ & A_a = e^{\bar{s}F} \text{ and } B_a = \textstyle\int_0^{\bar{s}} e^{F\ell}\mathrm{d}\ell G. \end{aligned} \tag{61}$$

This conversion to a discrete time system is strongly motivated by the fact that when implementing robotic controllers using the Robot Operating System (ROS) for any robot, the implementation must be done in discrete time.

We next show how to satisfy our Assumptions 1-3 of Theorem 1, after a change of coordinates, by finding the required matrices Γ, K, and L, where $C = [1,\ 1]$. Following our analysis of the undelayed case in [22], we choose $\bar{s} = 0.5$, and a matrix K_a such that $H_a = A_a + B_a K_a$ has the eigenvalues 0.25 and 0.5 to obtain the required Schur stability condition on H_a, by using the command `StateFeedbackGains` in the Mathematica program. We can then diagonalize H_a to obtain a new matrix $P(A_a + B_a K_a)P^{-1} = H$ that is both Schur stable and nonnegative. It follows that with the choices $A = PA_aP^{-1}$, $B = PB_a$, $L = -[1.5, 0]^\top$, and all entries of Γ being small enough positive constants, the requirements of Theorem 1 all hold.

We found that the event triggers from Theorem 1 produced fewer control recomputation times t_i, as compared with the corresponding event-trigger $||z_{r,k-r} - z_{r,\sigma(k)-r}|| \le \sigma_*||z_{r,k-r}||$ with σ_* being the smallest entry of Γ; see Remark 3 above. We illustrate this in Fig. 1, which shows our MATLAB simulations for the depth-pitch controller of the AUV using the delay-compensating positive system sub-predictor approach from Theorem 1 with $r = 3$. Our figure shows results for different initial states. Our simulations used the K and L from the preceding paragraphs, and

$$\Gamma = \begin{bmatrix} 0.015 & 0.045 \\ 0.15 & 0.15 \end{bmatrix}, \tag{62}$$

which satisfied our requirements from Assumption 1 with the preceding choices of K and L. For our simulations on the time horizon of $[0,\ 15]$ seconds, the event trigger from Theorem 1 produced an average of 28 trigger times t_i. In Fig. 2, the behavior of the BlueROV2 is presented when the sampling rate is instead $\bar{s} = 0.05$ seconds, and Γ is

$$\Gamma = \begin{bmatrix} 0.015 & 0.045 \\ 1.95 & 1.95 \end{bmatrix} \tag{63}$$

with K L as before and with the larger delay value $r = 5$. In this case, the event triggered 16 times for our simulation. By comparison, using the Euclidean norm in the event trigger with the same initial conditions, over the same time horizon and with $\sigma_* = 0.015$ being the smallest element of Γ, the event triggered 40 times on average in our simulations. Hence, our event trigger from our theorem produced a reduction of 60% in the number of triggering times in this case. Both event triggers produced similar settling times. Therefore, this illustrates the value of our approach for decreasing the numbers of event triggers, without adversely affecting the control performance, and with larger delays.

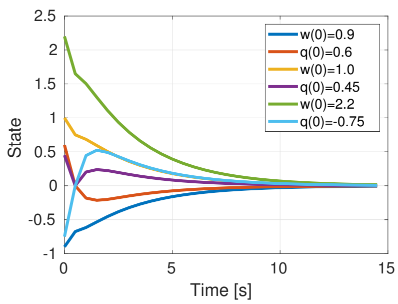

Fig. 1. Simulation for depth and pitch control of BlueROV2 with sampling rate $\bar{s} = 0.5$ and delay $r = 3$.

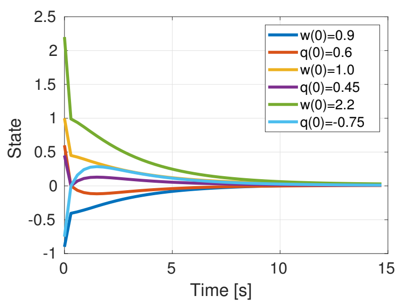

Fig. 2. Simulation for depth and pitch control of BlueROV2 with sampling rate $\bar{s} = 0.05$ and delay $r = 5$

VI. Conclusions

We proposed a new triggered control design for discrete-time systems with input delays and outputs. Relative to previous methods, key novel features were our use of sub-predictors, which made it possible to achieve an ISS-like result under an arbitrarily long constant input delay, and our event triggers that were based on matrices of absolute values instead of Euclidean norms. We hope to develop analogs for nonconstant delays, which could entail generalizing [19] (which did not allow event-triggering) to have the choice of the subpredictor in the control depend on the value of the delay, instead of choosing the last subpredictor in the control. Extensions to time-varying systems are also expected.

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