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# Stability and String Stability of Car-following Models with Reaction-time Delay

April 6, 2020

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## Abstract

We investigate the transfer function emanating from the linearization of a car-following model, when taking into account a driver reaction time. This leads to stability conditions, which are explicitly given. We also show how this reaction time can introduce a *partial string instability*. This is a preliminary step of a forthcoming work about traffic stabilization via connected automated vehicles.

## 1 Introduction

It has been known for some time, and even demonstrated in the field [16, 15], that vehicles on a highway can suffer from stop-and-go traffic conditions without any external intervention. To explain this phenomenon, traffic evolution is studied at a microscopic level and further expanded as a series of interconnected systems named car-following models. These refer to the way drivers maintain a spacing gap towards the preceding vehicle. In other words, car-following models predict the motion of a human-driven vehicle in a vehicular stream, by studying the dynamics of the inter-vehicle gap.

Conceptualized for the first time by Pipes [12] and Reuschel [1], several car-following models have been proposed in the literature to simulate mathematically the driving behavior at a microscopic level. The review of car-following models in [19] highlights five groups: Gazis-Herman-Rothery (GHR, also known as GM model), collision-avoidance models, linear models, psycho-physical or action-point models and fuzzy logic-based models.

Over the years, the aforementioned models served as a basis for more complex and accurate models. The Gipps' car-following model [4], has been widely

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used, both in research and practice for its accuracy. Later, Treiber *et al.* [22] introduced the *Intelligent Driver Model* (IDM) which gained a lot of attention, becoming the most popular car-following model for human drivers. In [21], IDM was extended to take into account driver's adaptation effect using memory functions. More recently, IDM+ has been proposed for traffic flow study [14].

Several studies added that reaction-time delays of the human driver should be considered to represent more accurately upstream wave propagation in a traffic lane. For instance, stop-and-go traffic jams have been shown to be connected to these delays in [6, 8, 7], demonstrating that an unexpected deceleration of the front vehicle is followed by an even stronger ego-deceleration to avoid rear-end collision. This amplification of vehicle states along a string can lead to a traffic jam. An analysis of such a delayed car-following model on a circle can be found in [9].

Several authors [3, 2, 23] have suggested that it is possible to use a small proportion of automated vehicles to regulate the traffic. These studies are set in a traffic regime which exhibits string instability, which means in terms of transfer function that any excitation of a frequency below a certain limit is amplified. We are interested here in a slightly different setting, where reaction time is taken into account for human drivers. In this case, we show that the system may be unstable under some conditions, but also that there is an interesting regime where string instability only concerns frequencies in a particular band.

## 1.1 Motivation

This paper is intended as a foundation of a larger work on traffic stabilization by means of a fleet of cooperative automated vehicles. However, contrary to the work in [3], our approach is based on a car-following model with reaction-time delay, rather than on a first order fluid model. The continuation of these studies will concern shockwave analysis and adequate traffic-stabilizing control strategies.

We start from a classical *car-following model* on a line with identical vehicles, in which we introduce a reaction time  $\tau$  for each driver. For simplicity of presentation, we follow the notation and the development of [23]. Let  $x_j(t)$  be the position of vehicle  $j$  along the road at time  $t$ ,  $v_j(t)$  its speed and  $a_j(t)$  its acceleration. We assume that vehicle  $j$  obeys a car-following model of the form

$$\begin{aligned} a_j(t + \tau) &= f(x_{j-1}(t) - x_j(t), v_{j-1}(t) - v_j(t), v_j(t)) \\ &\stackrel{\text{def}}{=} f(\Delta x_j(t), \Delta v_j(t), v_j(t)). \end{aligned} \quad (1.1)$$

The acceleration at time  $t$  is thus a function of the following variables, delayed

by  $\tau$ : the distance to the vehicle in front, their relative speed and the speed of vehicle  $j$  itself. A typical example of such a car-following model is the Intelligent Driver Model (IDM) [20], which we will use in our examples.

The relation at equilibrium between the heading  $\Delta x^*$  and the speed  $v^*$  is given by the equation

$$0 = f(\Delta x^*, 0, v^*). \quad (1.2)$$

The linearization of (1.1) at point  $(\Delta x^*, 0, v^*)$  yields

$$a_j(t + \tau) = k_{dx}(\Delta x_j(t) - \Delta x^*) + k_{dv}\Delta v_j(t) + k_v(v_j(t) - v^*), \quad (1.3)$$

where the parameters  $k_{dv}, k_{dx}, k_v$  are positive and stand for the following partial derivatives at equilibrium

$$k_{dx} = \frac{\partial f}{\partial \Delta x}, \quad k_{dv} = \frac{\partial f}{\partial \Delta v}, \quad k_v = -\frac{\partial f}{\partial v}.$$

Equation (1.3) defines a second-order linear differential equation for the positions of the vehicles. For all  $\Re(s) \geq 0$ , let  $V_j(s)$  denote the Laplace transform of  $v_j(t) - v^*$ . Then the fundamental transfer function takes the form

$$U(s) \stackrel{\text{def}}{=} \frac{V_j(s)}{V_{j-1}(s)} = \frac{k_{dv}s + k_{dx}}{s^2 e^{s\tau} + (k_{dv} + k_v)s + k_{dx}}, \quad \Re(s) \geq 0, \quad (1.4)$$

Note that, unlike in [23],  $U(s)$  is no more a rational but a transcendental function, because of the term  $e^{sT}$ . As we shall see in the next sections, the analysis and the behavior of the system become more intricate.

## 2 Stability region

The first basic question about the dynamics of (1.3) concerns its stability.

**Definition 2.1** (Stability). The system is *stable* if and only if  $U(s)$  has no pole in the region  $\Re(s) \geq 0$ .

The physical meaning of this property is that a finite perturbation of  $x_i$  should never have an unbounded effect. Note that stability always holds when  $\tau = 0$  as in [2, 23].

Let us analyze the poles of  $U(s)$  in the region  $\Re(s) \geq 0$ . For this purpose, it will be convenient to introduce the scaling

$$z = s\tau, \quad \alpha = \tau^2 k_{dx}, \quad \beta = \tau k_{dv}, \quad \gamma = \tau k_v, \quad \delta = \beta + \gamma. \quad (2.1)$$

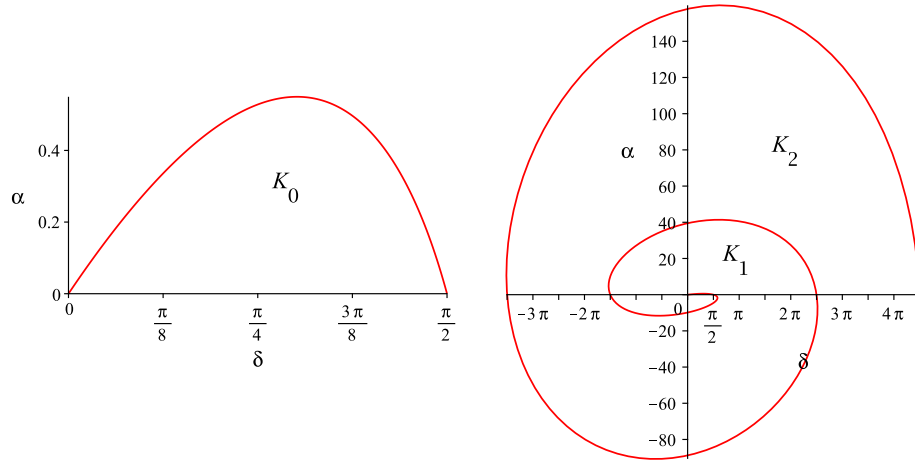


Fig. 2.1: Left: stability region in terms of the parameters  $\alpha$  and  $\delta$ . Right: the spiral (2.3) represented for  $y \in [0, \frac{9\pi}{2}]$ ; note that the stability region  $K_0$  of the left panel is barely visible in the display of the right panel.

Then (1.4) can be rewritten as

$$U(z) = \frac{\beta z + \alpha}{D(z)}, \quad \text{where } D(z) = z^2 e^z + \delta z + \alpha. \quad (2.2)$$

We state now necessary and sufficient conditions for the system to be stable. They are nontrivial due the presence of the reaction time  $\tau$ .

**Theorem 2.2.** *In the  $(\delta, \alpha)$  plane, let  $\mathcal{L}$  be the curve shown in Figure 2.1 and represented in the parametric form*

$$\begin{cases} \delta = y \sin y, \\ \alpha = y^2 \cos y. \end{cases} \quad (2.3)$$

*Then the system is stable if and only if the couple of parameters  $(\alpha, \delta)$  belongs to the finite domain  $K_0$ , bounded by the curve  $\mathcal{L}$  and the open segment  $]0, \pi/2[$  on the  $\delta$ -axis.*

*In the positive quarter plane  $(\delta, \alpha)$ , define  $K_j$ , the  $j$ -th positive sector of  $\mathcal{L}$ , as the domain corresponding in (2.3) to  $y \in [0, \pi/2] + 2j\pi, j \geq 0$ , see Figure 2.1. Then, when  $(\delta, \alpha) \in K_j$ , the function  $D(z)$  has exactly  $2j$  conjugate zeros with positive real parts.*

*In addition, the set of equations (2.3) is equivalent to the algebraic differential*

equation

$$\frac{d\alpha}{d\delta} = \frac{4\alpha - \delta(\delta^2 + \sqrt{\delta^4 + 4\alpha^2})}{2(\alpha + \delta)}, \quad (2.4)$$

subject to the initial conditions  $\alpha(0) = 0$  with  $\alpha'(0) = 1$ .

A clear consequence of this result is that, when the parameters  $(\alpha, \delta)$  do not belong to  $K_0$ , the system is *unstable*.

*Proof.* We want to find conditions ensuring the existence of zeros for the function  $D$  in the right half-plane  $\Re(z) \geq 0$ . The main ingredient will be a proper use of Rouché's theorem, together with an argument of continuity with respect to the parameters  $(\delta, \alpha)$ .

First consider the imaginary axis  $\{z = iy, y \in \mathbb{R}\}$ , where  $D$  may have two possible zeros at the points  $iy_0$  and  $-iy_0$ , which are the real solutions of the system

$$\begin{cases} y^2 \cos y = \alpha, \\ y \sin y = \delta, \end{cases}$$

which coincides exactly with (2.3). When  $y_0$  exists, it is immediate to check that, necessarily,

$$y_0 = \left( \frac{\delta^2 + \sqrt{\delta^4 + 4\alpha^2}}{2} \right)^{\frac{1}{2}} > \delta. \quad (2.5)$$

Viewing in (2.3) the variables  $y$  and  $\alpha$  as functions of  $\delta$ , one gets by differentiation

$$\begin{cases} \frac{dy_0}{d\delta} (2y_0 \cos y_0 - y_0^2 \sin y_0) = \frac{d\alpha}{d\delta}, \\ \frac{dy_0}{d\delta} (\sin y_0 + y_0 \cos y_0) = 1, \end{cases}$$

so that

$$\frac{d\alpha}{d\delta} = \frac{2\alpha - \delta y_0^2}{\alpha + \delta},$$

which is exactly equivalent to (2.4) by using (2.5).

It is important to observe that the differential equation (2.4) has the classical form

$$\frac{d\alpha}{d\delta} = F(\delta, \alpha), \quad \forall (\delta, \alpha) \in \Delta \in \mathbb{R}^{+2},$$

where  $\Delta$  is a finite domain including the point  $(0, 0)$ , where  $F(\delta, \alpha)$  is not continuous, and nor does satisfy a classical Lipschitz condition with respect to  $\alpha$ .

On the other hand, whenever system (2.3) holds, we have the immediate following results.

1.

$$\alpha'(0) = \lim_{\delta \rightarrow 0} \frac{\alpha}{\delta} = 1.$$

2. If  $\alpha = 0$ , then either  $\delta = 0$  or  $\delta = \frac{\pi}{2} + 2k\pi$ , for any integer  $k \geq 0$ .3. If  $\delta = 0$ , then  $\alpha = (2\ell\pi)^2$ , for any integer  $\ell \geq 0$ .

The curve  $\mathcal{L}$ , which is parametrized by (2.3), looks like an Archimedean spiral, the spires of which are dilated along the  $\alpha$ -axis in the  $(\delta, \alpha)$ -plane.

In the course of the proof, we shall use the so-called Rouché's theorem, which we recall for the sake of completeness.

**Theorem 2.3** (Rouché, see e.g. [18]). *For any two complex-valued functions  $f$  and  $g$  holomorphic inside some region  $\mathcal{H}$  with simple closed contour  $\partial\mathcal{H}$ , if  $|g(z)| < |f(z)|$  on  $\partial\mathcal{H}$ , then  $f$  and  $f + g$  have the same number of zeros inside  $\mathcal{H}$ , where each zero is counted with its multiplicity.*

**Proof of stability.** The baseline of reasoning will be, *starting with convenient small values for  $(\alpha, \delta)$ , to apply Theorem 2.3 to the functions  $f(z) = z^2e^z + \delta z$  and  $g(z) = \alpha$  (constant) on a properly chosen contour, and then to proceed by continuity with respect to the parameters  $(\delta, \alpha)$ .*

In the right half-plane, consider the closed contour  $C(\delta, R)$  consisting of the two semicircles centered at the origin of respective radii  $R$  and  $\delta$ , plus the two segments of the imaginary axis  $[i\delta, iR]$  and  $[-iR, -i\delta]$  respectively. Henceforth,  $R$  will stand for a large positive number.

Fix  $0 < \alpha < \delta < \varepsilon$ , where  $\varepsilon$  is an arbitrary small number.

**Lemma 2.4.** *For all  $(\alpha, \delta)$  in the region  $0 < \alpha < \delta^2 < 1/4$ , the following inequality holds.*

$$|z^2e^z + \delta z| > \alpha, \quad \forall z \in C(\delta, R). \quad (2.6)$$

*Proof.* Inequality (2.6) is evidently satisfied for  $z$  on the semicircle of radius  $R$ . The situation for the remaining components of the contour is twofold.

(i)  $z \in [i\delta, iR] \cup [-iR, i\delta]$ . Setting  $z = iy$ ,  $y$  real,

$$|ze^z + \delta|^2 = y^2 - 2\delta y \sin y + \delta^2 \geq (1 - 2\delta)y^2 + \delta^2 > \delta^2$$

(ii)  $z$  belongs to the semicircle of radius  $\delta$ , that is  $z = \delta e^{i\theta}$ ,  $-\pi/2 \leq \theta \leq \pi/2$ . Setting  $Y \stackrel{\text{def}}{=} e^{\delta \cos \theta} \geq 1$ , we get directly

$$|ze^z + \delta|^2 = \delta^2 [Y^2 + 2Y \cos(\theta + \delta \sin \theta) + 1]. \quad (2.7)$$

By the monotony of the sine and cosine functions, we have

$$\cos(\theta + \delta \sin \theta) > \cos(\pi/2 + \delta) = -\sin \delta, \quad \forall \theta \in [0, \pi/2],$$

and (2.7) yields

$$|ze^z + \delta|^2 > 2\delta^2(1 - \sin \delta).$$

Finally, for  $0 < \alpha < \delta^2 < 1/4$ , (2.6) holds everywhere on  $C(\delta, R)$ , and the lemma is proved.  $\blacksquare$

In a second step, we shall show in addition that  $D(z)$  has no zeros in the closed domain bounded by the half-disk  $z = \delta e^{i\theta}$ ,  $\theta \in [-\pi/2, \pi/2]$  in the right half-plane.

**Lemma 2.5.** *For all  $\alpha > \delta^2(e^{2\delta} - 1)^{1/2}$ , the equation*

$$z^2 e^z + \delta z + \alpha = 0$$

*has no roots in the half-disk  $z = \rho e^{i\theta}$ ,  $0 \leq \rho \leq \delta$  and  $\theta \in [-\pi/2, \pi/2]$ .*

*Proof.* By Rouché's theorem 2.3, it suffices to establish on the contour of this half-disk the inequality

$$|\delta z + \alpha| > |z^2 e^z|. \quad (2.8)$$

Letting  $z = x + iy$ , we analyze separately the two components of the domain.

(i) On the diameter of the half-disk,  $x = 0$  and

$$|\delta z + \alpha|^2 - |z^2 e^z|^2 = \delta^2 y^2 + \alpha^2 - y^4 > 0, \quad 0 \leq |y| \leq \delta,$$

which shows (2.8).

(ii)  $z = \delta e^{i\theta}$ ,  $\theta \in [-\pi/2, \pi/2]$ . Here,

$$|\delta z + \alpha|^2 = \delta^4 + 2\alpha\delta x + \alpha^2, \quad \text{and} \quad |z^2 e^z|^2 = \delta^4 e^{2x}.$$

Now, as  $x \geq 0$ , (2.8) will clearly be satisfied by choosing for instance

$$\alpha^2 > \delta^4(e^{2\delta} - 1),$$

and the proof of the lemma is terminated.  $\blacksquare$

Putting together the results of Lemmas 2.4 and 2.5, we can conclude that  $D(z)$  has no zeros in the right half-plane for the parameter region

$$\left\{ \delta^2(e^{2\delta} - 1)^{1/2} < \alpha < \delta^2 < \frac{(\log 2)^2}{4} \right\},$$



which lies clearly in the interior of the domain  $K_0$ .

On the other hand, by the principle of the argument (see e.g. [10]), the number of zeros of  $D(z)$  can be expressed by means of a Cauchy type integral of the form

$$\frac{1}{2i\pi} \int_{\mathcal{U}} \frac{D'(t)}{D(t)} dt,$$

$\mathcal{U}$  being the boundary the semicircle of radius  $R$  in the right half-plane. *This integral takes integer values and is continuous with respect to  $(\delta, \alpha)$ , as long as  $D(z)$  does not vanish, which would in particular imply  $|z^2 e^z| = |\alpha + \delta z|$ .* Hence in the interior of  $K_0$ ,  $D(z)$  has no zeros and the stability part of the theorem is proved.

**The zeros of  $D(z)$  for the parameter domain  $K_j, j \leq 1$ .** In the parameter space, take a point  $P = (\delta, \alpha) \geq (0, 0)$  located inside  $K_j$  and let  $P$  move to the boundary of  $K_j$ , to enter  $K_{j+1}$ , so that system (2.3) holds, which means that two zeros of  $D(z)$  appear on the imaginary axis in the  $z$ -plane. These conjugate zeros come from the left half-plane and all what we have to show is that their real part becomes strictly positive when  $P$  switches from  $K_j$  to  $K_{j+1}$ . We deal with the case  $j = 0$ , the argument being the same for all  $j$ .

Let  $P$  reach the boundary of  $K_0$  to enter  $K_1$  via the point  $(\pi/2, 0)$ , and suppose for a while that  $\alpha$  is an arbitrary positive function of  $\delta$ , locally differentiable around  $\delta = \pi/2$ , with a derivative  $q \stackrel{\text{def}}{=} \alpha'(\pi/2) > -\pi^2/4$  (the value of the slope of the curve  $\mathcal{L}$  at  $\delta = \pi/2$ ) to ensure that the point  $P = (\pi/2 + \varepsilon, \varepsilon)$  belongs to  $K_1$  for  $\varepsilon$  sufficiently small.

Then the possible zeros of  $D(z)$  are locally differentiable with respect to  $\delta$  around  $\delta = \pi/2$ , whence, by taking the derivative of  $D(z) = 0$  and writing  $z'(\delta) = \frac{dz}{d\delta}$ ,

$$z'(\delta)[ze^z(z+2) + \delta] + z + \alpha'(\delta) = 0. \quad (2.9)$$

Instantiating  $\delta = \pi/2$  in (2.9), we have  $z = \pm i\pi/2$ . Choosing for example  $z = i\pi/2$ , we obtain

$$z'(\pi/2) = \frac{q + i\pi/2}{\pi(1 + i\pi/4)} = \frac{q + \pi^2/4 + i\pi(1 - q)/4}{\pi(1 + \pi^2/4)},$$

and the real part of  $z'(\pi/2)$  is positive as soon as  $q + \pi^2/4 > 0$ . Thus, when the point  $P$  enters  $K_1$ , two conjugate zeros of  $D(z)$  come into the right complex half-plane.

A similar phenomenon occurs when  $P$  passes from  $K_j$  to  $K_{j+1}$ , creating two additional zeros of  $D(z)$  in the right.

The proof of the theorem is terminated. ■

**Remark 2.6.** We always assumed  $\alpha > 0$ . For the sake of completeness, let us briefly analyze the case  $\alpha = 0$ , although it is of little physical interest, since it implies  $k_{dx}$ .

The solution in  $z$  of the equation

$$w = ze^z$$

is the classical special LambertW-function. It has an infinite number of branches, exactly one of them being analytic around the origin and given by

$$z(w) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1} w^n}{n!},$$

where the series is convergent for  $|w| < e^{-1}$ . In particular,  $\text{LambertW}(-\delta)$  does not vanish in the interior of the disk of radius  $e^{-1}$ . Moreover,

$$\Re[\text{LambertW}(-\delta)] < 0, \quad 0 < \delta < \pi/2.$$

### 3 Partial string stability

Even if the system is stable, another phenomenon may occur, when, for some  $y \in \mathbb{R}$ ,  $|U(iy)| > 1$ : indeed, in this case, since

$$V_j(iy) = U^n(iy)V_{j-n}(iy), \quad \forall 1 \leq j \leq n,$$

it appears that small perturbations could be amplified as they propagate in upstream direction. A metric has been introduced in the literature that relates on how systems' states energy is propagated element-wise in a string of systems. It is referred as *string stability* [17, 13].

**Definition 3.1** (String stability). The system is called *string stable* if and only if  $|U(iy)| \leq 1$  for all  $y \in \mathbb{R}$ .

String stability is easy to investigate when the reaction time is  $\tau = 0$ : either it holds, or  $|U(iy)| \geq 1$  in a neighborhood of 0. When  $\tau > 0$ , there is another possible situation, that we introduce as *partial string stability*.

**Definition 3.2** (Partial string stability). The system is *partially string stable* if and only if there exists  $y_c > 0$  such that  $|U(iy)| \leq 1$  for all  $|y| < y_c$ .

Partial string stability covers cases where  $|U(iy)|$  can be greater than 1 for some  $|y| > y_c$ . Accordingly, the next step will be to analyze the modulus of  $U(z)$  on the imaginary axis, especially its position with respect to the value 1. As the reader will note, finding exact necessary and sufficient conditions for string stability in the parameter domain is a thorny problem, which we intend to tackle hereafter.

### 3.1 Necessary conditions for partial string stability

In the next theorem, we give the maximum interval containing all possible frequencies  $y$  satisfying  $F(y) \stackrel{\text{def}}{=} |U(iy)| = 1$ .

**Theorem 3.3.** *The function  $F(y)$ ,  $y \in \mathbb{R}$ , enjoys the following properties.*

(i) *The system is string stable or partially string stable if and only if*

$$2\alpha < \gamma(2\beta + \gamma) \stackrel{\text{def}}{=} \mu, \quad (3.1)$$

and

$$\frac{1}{2} < \delta < \frac{\pi}{2}. \quad (3.2)$$

(ii) *Assume condition (3.1) holds. Then any critical  $y_c$  such that  $F(y_c) = 1$  satisfies the inequalities*

$$\sqrt{\beta^2 + \delta^2 - 2\sqrt{\beta^2\delta^2 + \alpha^2}} < y_c < \sqrt{\beta^2 + \delta^2 + 2\sqrt{\beta^2\delta^2 + \alpha^2}}. \quad (3.3)$$

One can check that condition (3.1) does not depend on  $\tau > 0$  and is the same as condition (10) in [2], where  $\tau = 0$ .

*Proof.* An easy algebra yields

$$F(0) = 1, \quad F'(0) = 0, \quad F''(0) = \frac{2(2\alpha - \mu)}{\alpha}.$$

On the other hand,

$$\frac{1 - F(y)}{y^2} = \frac{G(y)}{|D(iy)|^2}, \quad \text{where } G(y) \stackrel{\text{def}}{=} y^2 - 2(\delta y \sin y + \alpha \cos y) + \mu \quad (3.4)$$

and now (3.1) becomes immediate, since here

$$\lim_{y \rightarrow 0} \frac{1 - F(y)}{y^2} = -\frac{F''(0)}{2}.$$

On the other hand, since

$$G(y) = y(y - 2\delta \sin y) + \mu - 2\alpha > y^2(1 - 2\delta) + \mu - 2\alpha, \quad \forall y > 0,$$

(3.4) may have a positive root only if  $\delta > 1/2$ . In addition, as illustrated on Figure 2.1, equation (2.4) entails  $\delta < \pi/2$ , which proves (3.2) and part (i) of the theorem.

As for part (ii), we note first

$$G(0) = \mu - 2\alpha, \quad G'(0) = 0, \quad G''(0) = 2(1 + \alpha - 2\delta).$$

One could think that the curvature of  $G$  at the origin (i.e. the sign of  $G''(0)$ ) plays a decisive role in the way  $F(y)$  reaches and exceeds 1. Actually it will emerge in the sequel that this is not the case.

It is worth illustrating on a toy example the existence of a set of positive parameters  $\alpha, \beta, \gamma$ , for which  $F(y)$  becomes strictly larger than 1 for some  $y$ . Taking for instance  $y = \pi/4$ , one can check that the quantity

$$G(\pi/4) = \gamma^2 + \left(2\beta - \frac{\pi\sqrt{2}}{4}\right) \gamma + \frac{\pi^2}{4} - \sqrt{2} \left(\frac{\beta\pi}{4} + \alpha\right)$$

can be rendered negative for some  $\gamma_0$ , as soon as  $\beta > \frac{\pi}{\sqrt{2}}$ .

The problem of the existence of  $y_c$  in (3.3) can be formulated in terms of the intricate following puzzle.

*For given parameters  $\alpha, \beta, \gamma$ , find the conditions ensuring the existence of  $y_c$ , the smallest positive root of  $G(y) = 0$ , where  $G(y)$  was defined by equation (3.4).*

To this end, let us introduce the family of parametrized trinomials  $H(y, z)$ , where  $z$  stands for a positive real parameter, and

$$H(y, z) = y^2 - 2\delta y \sin z - 2\alpha \cos z + \mu. \quad (3.5)$$

Setting  $t = \tan(z/2)$  and using the classical formulas

$$\sin z = \frac{2t}{1+t^2}, \quad \cos z = \frac{1-t^2}{1+t^2},$$

one checks easily that the equation  $H(y, z) = 0$  corresponds in a one-to-one way to the function  $K(y, t) = 0$ , where

$$\begin{aligned} K(y, t) &= (y^2 + 2\alpha + \mu)t^2 - 4\delta ty + y^2 - 2\alpha + \mu \\ &= (1+t^2)y^2 - 4\delta ty + (\mu + 2\alpha)t^2 + \mu - 2\alpha. \end{aligned} \quad (3.6)$$

In the sequel  $K(y, t)$  will be viewed, *ad libitum*, either as a trinomial in  $y$  or a trinomial in  $t$ .

Hence,  $G(y) \equiv H(y, y) = 0$  is plainly equivalent to the system

$$\begin{cases} K(y, t) = 0, \\ t = \tan(y/2). \end{cases} \quad (3.7)$$

Since  $G(y)$  is an even function of  $y$ , we shall only look for its positive roots, remarking that if  $(y, t)$  satisfies (3.7), then  $y > 0 \Rightarrow t > 0$ . This means that

we can restrict ourselves to the roots of  $G$  of the form  $y = y_1 + 2k\pi$  with  $0 < y_1 < \pi$  and  $k \geq 0$ .

In the sequel, at our convenience, we shall use the variables  $Y \stackrel{\text{def}}{=} y^2$  and  $T \stackrel{\text{def}}{=} t^2$ . Also, according to (3.1), it is worth recalling that  $\mu = \delta^2 - \beta^2$ .

The reduced discriminant of  $K(y, t) = 0$ , considered as a quadratic equation in  $t$ , is equal to

$$A(Y) \stackrel{\text{def}}{=} -Y^2 + 2(2\delta^2 - \mu)Y + 4\alpha^2 - \mu^2, \quad (3.8)$$

which has two real positive roots  $Y_+, Y_-$ , where

$$Y_- = \beta^2 + \delta^2 - 2\sqrt{\beta^2\delta^2 + \alpha^2}, \quad Y_+ = \beta^2 + \delta^2 + 2\sqrt{\beta^2\delta^2 + \alpha^2}. \quad (3.9)$$

Quite similarly, the reduced discriminant of  $K(y, t) = 0$ , considered now as a quadratic equation in  $y$ , is equal to

$$B(T) \stackrel{\text{def}}{=} -(2\alpha + \mu)T^2 + 2(2\delta^2 - \mu)T + 2\alpha - \mu, \quad (3.10)$$

which has two real positive roots  $T_+, T_-$ , where

$$T_- = \frac{Y_-}{2\alpha + \mu}, \quad T_+ = \frac{Y_+}{2\alpha + \mu}.$$

Hence, any possible positive solution  $(y, t)$  of (3.7) necessarily satisfies the inequalities

$$\sqrt{Y_-} \leq y \leq \sqrt{Y_+}, \quad \sqrt{T_-} \leq t \leq \sqrt{T_+}, \quad (3.11)$$

since in this case  $A(Y)$  and  $B(T)$ , respectively given by (3.8) and (3.10), are positive, which proves in particular (3.3).  $\blacksquare$

**Remark 3.4.** On the other hand, by using the elementary inequalities

$$y - y^3/6 \leq \sin y \leq y, \quad 1 - y^2/2 \leq \cos y \leq 1 - y^2/2 + y^4/24,$$

we get the following easy bounds, valid for all  $y > 0$ ,

$$\frac{\delta}{3}y^4 \geq G(y) - (1 + \alpha - 2\delta)y^2 + 2\alpha - \mu \geq -\frac{\alpha}{12}y^4,$$

which can be helpful to find sufficient conditions (in terms of concrete parameters) for the existence of real zeros of  $G$ .

### 3.2 About a surface in the parameter domain ensuring partial string stability

From now on, we assume that the system is stable and that condition (3.1) holds. To characterize the exact parameter regions ensuring the existence of positive roots for equation (3.4), we propose two directions, both of them having a geometrical flavour.

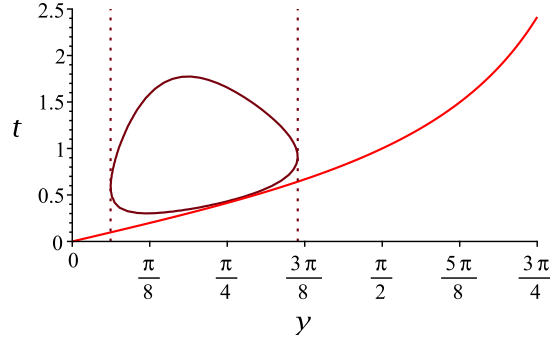


Fig. 3.1: Representation of  $K(x, t) = 0$  and  $t = \tan(y/2)$  for arbitrary parameter values  $\alpha = 0.074$ ,  $\beta = 0.45$  and  $\gamma = 0.235$ , chosen so that the curves are tangent. The vertical dotted lines mark the bounds (3.3).

**Intersection of a quartic curve with a tangent curve.** Finding the exact necessary and sufficient conditions for string stability is tantamount to locate the closed curve  $\{K(y, t) = 0\}$  with respect to the periodic curve  $\{t = \tan(y/2)\}$ , in the region defined by (3.11) in the  $(y, t)$ -plane. In particular, one must specify their possible tangency points, obtained by means of the additional relation

$$\frac{\partial K(y, t)}{\partial t} + \frac{d \tan(y/2)}{dy} = 0,$$

which, by using the identity

$$\frac{d \tan(y/2)}{dy} = \frac{1 + t^2}{2},$$

yields the algebraic equation

$$t(1 + t^2)y^2 + 2(1 - \delta)(t^2 + 1)y + (2\alpha + \mu)t^3 + (2\alpha + \mu - 4\delta)t = 0. \quad (3.12)$$

The system made by (3.7) and (3.12) contains three equations with only two unknowns, and therefore there exists a relationship between the parameters  $(\alpha, \delta, \mu)$ , actually a *surface* allowing to discriminate between existence and non existence of solutions. However, the related algebra as well as numerical implementations are rather tedious, and this question remains a pending issue.

**Intersection of a parabola with a linear-sinusoidal curve.** Here the idea is to analyze the *first possible tangency point* in the  $(y, z)$ -plane of the two curves (see Figure 3.2)

$$\begin{cases} z = y^2 + \mu, & (\text{parabola}) \\ z = 2(\delta y \sin y + \alpha \cos y). \end{cases} \quad (3.13)$$

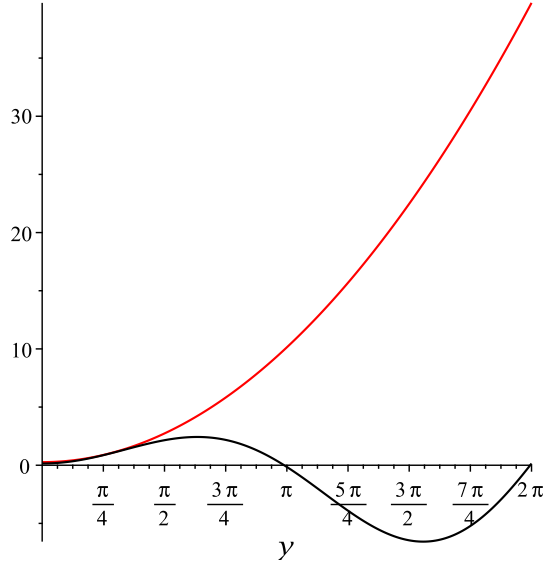


Fig. 3.2: The two functions defined in (3.13), for same parameter values as in Figure 3.1.

By identifying the normal vectors at the contact point of the two curves, we immediately obtain the following system

$$\begin{cases} \delta y \cos y + (\delta - \alpha) \sin y = y, \\ 2(\alpha \cos y + \delta y \sin y) = y^2 + \mu, \end{cases} \quad (3.14)$$

which in turn gives

$$\begin{cases} \cos y = \frac{(\alpha + \delta)y^2 + \mu(\alpha - \delta)}{2[\delta^2 y^2 + \alpha(\alpha - \delta)]}, \\ \sin y = \frac{\delta y^3 + (\mu\delta - 2\alpha)y}{2[\delta^2 y^2 + \alpha(\alpha - \delta)]}. \end{cases} \quad (3.15)$$

System (3.15) produces the following third degree equation in  $Y = y^2$ , namely

$$\begin{aligned} \delta^2 Y^3 + [(\alpha - \delta)^2 + 2\delta^2(\mu - 2\delta^2)]Y^2 \\ + [2(\mu - 4\delta^2 + 2)\alpha^2 + 4\alpha\delta(2\delta^2 - \mu) + \delta^2\mu(\mu - 2)]Y \\ + (\delta - \alpha)^2(\mu^2 - 4\alpha^2) = 0, \end{aligned} \quad (3.16)$$

the convenient root of which (in a sense to be made precise below) will be denoted by  $Y_m = y_m^2$ . The discriminant of (3.16) is equal to

$$\Delta = 16 W (\alpha^2 + \delta^4 - \delta^2\mu)(\alpha^2 + \alpha\delta - \delta^2\mu)^2, \quad (3.17)$$

where  $W$  is the following polynomial of degree 2 with respect to  $\mu$

$$W = \delta^2(\delta^2 - 1)\mu^2 - [12\delta^4 - 10\alpha\delta^3 + (2\alpha^2 - 11)\delta^2 + 8\alpha\delta - \alpha^2]\mu + 16\delta^6 - 16\alpha\delta^5 - 8\delta^4 - 4\alpha\delta^3 + (25\alpha^2 + 1)\delta^2 - (10\alpha^2 + 6)\alpha\delta + \alpha^2(\alpha^2 + 1).$$

Since  $\mu = \delta^2 - \beta^2$ , the two explicit factors in (3.17) are nonnegative, so that  $\Delta$  and  $W$  have the same sign.

From condition (3.1), the product of the roots in (3.16) is negative and there always exists a negative root which is of course not admissible.

- If  $W < 0$ , (3.16) has no admissible solution.
- If  $W \geq 0$ , (3.16) has three real roots. Then, either they are all negative and there is no admissible solution, or two of them are positive and the issue is to select the correct one.

It might be useful to note that the discriminant of the quadratic polynomial  $W$  takes the simple form

$$\Delta_W = (\alpha - \delta)(4\delta^3 - 5\delta + \alpha)^3,$$

noting that, for all  $(\alpha, \delta) \in K_0$ , we have  $\delta \geq \alpha$  by (2.4).

Clearly, for  $(\alpha, \delta) \in K_0$  and  $\delta$  satisfying (3.2),  $W = 0$  represents a *necessary separating surface*  $S_1$  in the parameter domain  $(\alpha, \delta, \mu)$ : on one side of  $S_1$ ,  $G$  defined in (3.4) may have real zeros, while it has none on the other side.

To proceed further, we fix  $\delta$  and consider  $\mu$  and  $y_m$  as functions of the variable  $\alpha$ . Then, upon combining the first equation of (3.14) with (3.15), we get

$$\begin{aligned} \frac{dy_m}{d\alpha} &= \frac{\sin y_m}{(2\delta - \alpha) \cos y_m - \delta y_m \sin y_m - 1} \\ &= \frac{-y_m(\delta y_m^2 + \mu\delta - 2\alpha)}{\delta^2 y_m^4 + (\mu\delta^2 + \alpha^2 - 3\alpha\delta)y_m^2 + ((\mu + 2)\alpha - 2\delta\mu)(\alpha - \delta)}. \end{aligned} \quad (3.18)$$

Similarly, after some algebra (!), the second equation of (3.14), allied with (3.18), gives

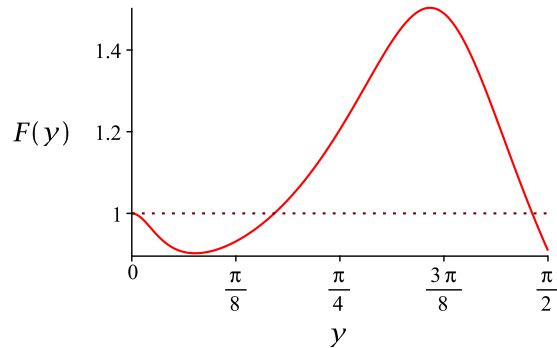
$$\begin{aligned} \frac{d\mu}{d\alpha} &= 2[y_m(\delta \cos y_m - 1) + (\delta - \alpha) \sin y_m] \frac{dy_m}{d\alpha} + 2 \cos y_m \\ &= \frac{(\alpha + \delta)Y_m + \mu(\alpha - \delta)}{\delta^2 Y_m + \alpha(\alpha - \delta)}, \end{aligned} \quad (3.19)$$

which nicely simplifies to a pleasant homographic function of  $Y_m$ . Thus the function  $\mu(\alpha)$  satisfies the nonlinear algebraic differential equation (3.19),



Parameter	value
Desired velocity $v_0$	33 m/s
Safe time headway $T$	1.5 s
Maximum acceleration $a$	1.5 m/s <sup>2</sup>
Desired deceleration $b$	1.5 m/s <sup>2</sup>
Acceleration exponent $\delta$	4
Jam distance $s_0$	2 m
Reaction time $\tau$	1.5 s

Tab. 1: Parameters for the IDM model

Fig. 4.1: Representation of the function  $F(y) = |U(iy)|$  for the Intelligent Driver Model with parameters given in Table 1 and  $v^* = 25$  m/s.

which has unique solution  $S_2(\alpha, \delta, \mu) = 0$ . To select the convenient domains delimited by the intersection of the surfaces  $S_1$  and  $S_2$ , one can proceed by continuity with respect to parameters, starting for instance from a condition of the form  $\mu = 2\alpha + \varepsilon$ , for  $\varepsilon$  sufficiently small. Currently, we leave the details of this kind of bootstrapping as an open question.

#### 4 Application to the Intelligent Driver Model

Although our results are usable with any local linearization of a car-following model of the form (1.1), it will be convenient to use the Intelligent Driver Model (IDM, see [20]), whose parameter values are shown in Table 1. As far as the reaction time is concerned, several studies suggest that the choice of  $\tau = 1.5$  is quite reasonable, although the actual value depends on the circumstances [5, 11].

In order to see what happens on a particular example, let us pick an equilibrium speed  $v^* = 25$  m/s. It is easy to solve explicitly the equilibrium equation (1.2), and find that  $\Delta x^* = 48.23$  m and that the parameters of the

linear system are respectively  $\alpha = 0.0975$ ,  $\beta = 0.6366$  and  $\gamma = 0.2332$ . Then the module  $F$  of the transfer function is less than 1 until  $y_m = 0.5379$  and above 1 between  $y_m$  and 1.5116 (see Figure 4.1).

This example shows that, beyond the string stable and string unstable cases highlighted in [2, 23], taking into account the reaction time can lead to an hybrid situation which is string stable for values of  $y$  below some threshold.

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