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Towards Corecursion Without Corecursion in Coq

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Coinduction is an important concept in functional programming. To formally prove properties of corecursive functions one can try to define them in a proof assistant such as Coq [1]. But there are limitations on the functions that can be defined. In particular, corecursive calls must occur directly under a call to a constructor, without any calls to other recursive functions in between. In this paper we show how a partially ordered set endowed with a notion of approximation can be organized as a Complete Partial Order. This makes it possible to define corecursive functions without using Coq's corecursion, as the unique solution of a fixpoint equation, thereby escaping Coq's builtin limitations.

1 Introduction

Coinduction is an important concept in functional programming. Coinductive types are the types of possibly infinite data structures such as the lists of Haskell, and corecursive functions compute values in coinductive types. An example is the Sieve of Eratosthenes that produces the infinite stream of primes.

To formally define corecursive functions one can try to use a proof assistant such as Coq. But then one quickly stumbles on the limitations of Coq on this matter. Indeed the language of this proof assistant is total, meaning that a function must be defined on all its domain (which is not requested for functional programming languages in general). Totality is ensured in Coq by a syntactical check of so-called guardedness. This criterion consists in checking that any corecursive call occurs directly under a call to a constructor of the coinductive type that constitutes the codomain (a.k.a. range) of the function under definition [4]. For example, Rose trees are coinductively defined as the constructor *tree* applied to a *forest*, which is an inductively-defined list of Rose trees. A *mirror* function would be corecursively defined by $mirror\ t = tree(map\ mirror(reverse(forest\ t)))$, but this is rejected by Coq because the corecursive call is not directly under the constructor *tree*, but under *map*, a recursive function defined on lists.

In a previous paper [7] we propose a method for defining such functions by replacing the syntactical guardedness criterion by a semantical proof obligation of productiveness: for each input, an arbitrarily close approximation of the corresponding output is eventually produced. When a functional (i.e., a higher-order function, which constitutes the "blueprint" of a corecursive function under definition) is monotonic and satisfies the productiveness requirement, a corecursive function can be defined as the unique fixpoint of the functional in question. This technique requires that the codomain of the function under definition be organized as a Complete Partial Order (CPO). For some simpler coinductive types such as streams or possibly infinite lists, the CPO can be directly defined using Coq's builtin mechanisms. But for mixed coinductive-inductive types such as Rose trees these mechanisms no longer work, and [7] replaces the native coinductive types of Coq with inductive types, completed with equivalence classes of "ascending" sequences of elements in inductive types, which are the limits of the sequences in question.

However, the construction we ended up with in [7] imposes some not-so-practical conditions. The order had to be *weakly total*, thus excluding the natural prefix order of Rose trees. Although we were able to exhibit a weakly total order, the constructor *tree* is not monotonic w.r.t. this order. As a consequence, functionals that employ this constructor are not monotonic either, which excludes them for being used in

fixpoint equations for the definition of corecursive functions. For example, the *mirror* function cannot be defined in the natural manner, as a fixpoint of the functional $Mirror = \lambda f. \lambda t. tree(map f (reverse(forest t)))$, but in an indirect way that makes it harder to prove that the obtained function is indeed the intended one.

Contributions. In this paper we replace the weak totality condition on orders by a notion of approximation meeting some quite natural conditions. These conditions are easier to satisfy than the (comparatively arbitrary) weak totality. We illustrate with the example of finite trees that the natural (i.e. structural) order on inductive types — quite importantly, this is the least constrained order under which constructors are monotonic — does satisfy the conditions on approximations whereas it did not satisfy weak totality. We then describe the construction of a CPO under the said conditions. This construction is substantially more involved than that in [7], but has to be done only once and can then be instantiated on any inductive type with an adequate notion of approximation in order to generate coinductive types. For Rose trees, the effect of the new construction is that the *tree* constructor is now monotonic w.r.t. their natural *prefix* order. It is now possible to define mixed recursive-corecursive functions in the most natural manner, as the unique solution of the fixpoint equation induced by a monotonic, productive functional of the function.

Outline. Section 2 presents preliminary notions, chief among which an abstract notion of approximations and a number of assumptions over them. We show that the assumptions have a model - finite trees with their natural prefix order and a "cut" function. Some technical developments are also included.

Section 3 defines the completion operation that produces a CPO from a partially ordered set with a least element. Completion is done in several steps: first, one completes an initial set of "finite" elements with new elements that are equivalence classes of ascending sequences of finite elements, thereby providing such sequences with "limits" (least upper bounds) required by CPOs. Second, the order and approximations are extended to the new elements. Third, we show using a "diagonalization" technique that ascending sequences of both old and new elements have limits among the new elements, meaning that the construction of the CPO does not require further completion steps.

Section 4 applies completion to produce the mixed inductive-coinductive type of Rose trees from finite trees. The roadmap for defining the *mirror* recursive-corecursive function as the unique fixpoint of its functional is presented. Section 5 concludes and presents related and future work. This paper will be formalized in Coq, but we keep it language-agnostic and use standard mathematics for better readability.

2 Preliminaries

2.1 Sequences and Complete Partial Orders

Definition 1 Consider a set C and a partial order \leq on C . We denote by $<$ the relation defined by $t < t'$ iff $t \leq t'$ and $t \neq t'$. A sequence $(s_i)_{i \in \mathbb{N}}$ of elements of C is

- increasing whenever for all $i \in \mathbb{N}$, $s_i \leq s_{i+1}$;
- strictly increasing, whenever for all $i \in \mathbb{N}$, $s_i < s_{i+1}$;
- stabilizing to $c \in C$ whenever there exist $m \in \mathbb{N}$ such that for all $i \geq m$, $s_i = c$, and stabilizing whenever it is stabilizing to some $c \in C$;
- ascending whenever it is increasing and non-stabilizing.

Remark. A sequence is ascending iff it is increasing and has a strictly increasing subsequence, and each increasing and stabilizing sequence is stabilizing to a unique value.

Definition 2 A Complete Partial Order (CPO) is a tuple (C, \leq, \perp) , where \leq is a partial order on C , \perp is the least element for \leq , and such that each increasing sequence $(s_n)_{n \in \mathbb{N}}$ over C has a least upper bound.

Hereafter we call least upper bounds *limits* and denote the limit of a sequence $(s_n)_{n \in \mathbb{N}}$ by $\lim[(s_n)_{n \in \mathbb{N}}]$.

An important feature of CPOs is that the order \leq should be interpreted as a definition order. In this sense, \perp is the *least defined* element, and elements that are maximal with respect to the order are interpreted as are "fully defined". Intermediate, non-maximal elements are therefore "partially defined".

2.2 Partially Ordered Sets with Least Element and Approximations

Assumption 1 In the sequel we assume:

1. a partially ordered set (C°, \leq°) with a least element \perp . Elements of C° are often called finite. It is required that no finite element be an upper bound for an ascending sequence of finite elements;
2. for each $N \in \mathbb{N}$, an approximation function $\lceil \cdot \rceil_N : C^\circ \rightarrow C^\circ$. For any $c \in C^\circ$, $\lceil c \rceil_N$ is called the N -th approximation of c , and satisfies the following properties:
 - (a) for all $c \in C^\circ$ and $N \in \mathbb{N}$, $\lceil c \rceil_N \leq^\circ c$;
 - (b) for all $c, c' \in C^\circ$ and $N \in \mathbb{N}$, $c \leq^\circ c'$ implies $\lceil c \rceil_N \leq^\circ \lceil c' \rceil_N$;
 - (c) for all $c \in C^\circ$ and $N, N' \in \mathbb{N}$, $N \leq N'$ implies $\lceil c \rceil_N \leq^\circ \lceil c \rceil_{N'}$;
 - (d) for all $c \in C^\circ$ there exists $N \in \mathbb{N}$ such that $\lceil c \rceil_N = c$;
 - (e) for all $c \in C^\circ$ and $N, N' \in \mathbb{N}$, $N \leq N'$ implies $\lceil \lceil c \rceil_{N'} \rceil_N = \lceil c \rceil_N$;
3. for any $N \in \mathbb{N}$ and any ascending sequence $(s_n)_{n \in \mathbb{N}}$, $(\lceil s_n \rceil_N)_{n \in \mathbb{N}}$ is stabilizing.

Remark. The requirements in Assumption 1 are quite natural. Item 1 holds if finite elements have finitely many finite elements smaller than them, a property which may be expected from finite elements. Item 2.(a) says that approximations do approximate w.r.t. the order, items 2.(b) and 2.(c) says that approximations are monotonic in both arguments; item 2.(d) says that for finite elements approximation can be exact, and item 2.(e) says that approximating first at some order $N' \geq N$ and then approximating at order N amounts to approximating at order N . Finally, item 3 says that mapping approximations to ascending sequences makes them stabilize - the sequence of approximations cannot grow forever.

Example 1 Let C° be the set T of trees inductively defined by the rules $\perp \in T$ and for each list l over T , tree $l \in T$. Let \leq° be the prefix relation on T , inductively defined by $\perp \leq^\circ t$ for each $t \in T$, and, for every pair of lists l, l' over T having the same length, say, m , tree $l \leq^\circ$ tree l' whenever $l[i] \leq^\circ l'[i]$ for each $i < n$ (where $l[i]$ denotes the i -th element of the list l). Let also $\lceil \cdot \rceil_N$ ("cut at height N ") denote the function $\lambda t. \text{if } N = 0 \vee t = \perp \text{ then } \perp \text{ else tree}(\text{map}(\lceil \cdot \rceil_{N-1})(\text{forest } t))$ where $\text{forest}(\text{tree } l) \triangleq l$ and map is the mapping function on lists. We have proved in Coq that Assumption 1 items 1 and 2 hold for these definitions. The code is available at <https://project.inria.fr/wpte2022/>. The third item is not yet proved in Coq, but here is a proof sketch. First, we generalize the third item's statement to increasing sequences:

For any $N \in \mathbb{N}$ and any increasing sequence $(s_n)_{n \in \mathbb{N}}$, $(\lceil s_n \rceil_N)_{n \in \mathbb{N}}$ is stabilizing.

Its proof is by induction on N . For $N = 0$, the sequence to prove stabilizing is $(\lceil s_n \rceil_0)_{n \in \mathbb{N}}$, and this is trivially true because then $\lceil s_n \rceil_0 = \perp$ for all $n \in \mathbb{N}$.

Then, we assume the statement holds for N and prove it for $N + 1$. If the increasing sequence $(s_n)_{n \in \mathbb{N}}$ is the constant sequence with all elements equal to \perp then the statement for $N + 1$ holds directly. Then, assume the increasing sequence $(s_n)_{n \in \mathbb{N}}$ has an element, say, s_i such that $s_i = \text{tree } l$ for some list l of trees. If the list l is empty ($l = []$) then, due to the prefix order relation, it holds that for all $j \geq i$, $s_j = \text{tree}[]$. Then, for all $j \geq i$, $\lceil s_j \rceil_{N+1} = \text{tree}(\text{map}(\lceil \cdot \rceil_N)([])) = \text{tree}[]$ and our stability statement directly

holds in the case $l = []$. Let us now consider the case $l \neq []$. Say the length of the list l is $m > 0$. Then, $s_i = \text{tree}[t_i^1, \dots, t_i^m]$ and, due to the prefix order relation, for all $j \geq i$, $s_j = \text{tree}[t_j^1, \dots, t_j^m]$ for some trees t_j^1, \dots, t_j^m . Also due to the prefix order relation, for all $k = 1, \dots, m$, the sequence $(t_j^k)_{j \geq i}$ is increasing. We can apply the induction hypothesis to all these increasing sequences and get, for each all $k = 1, \dots, m$, an index $i_k \geq i$ and a tree t^k such that for all $j \geq i_k$, $\lceil t_j^k \rceil_N = t^k$. Let I the maximum of all the i_k . Then, for all $j \geq I$, $\lceil s_j \rceil_{N+1} = \text{tree}(\text{map}(\lceil \cdot \rceil_N)[t_j^1, \dots, t_j^m]) = \text{tree}[\lceil t_j^1 \rceil_N, \dots, \lceil t_j^m \rceil_N] = \text{tree}[t^1, \dots, t^m]$, which states the desired stability of the sequence $(\lceil s_j \rceil_{N+1})_{n \in \mathbb{N}}$. This concludes the statement's proof.

Remark. Item 2.(b) in the above Assumption implies that for increasing sequence $(s_n)_{n \in \mathbb{N}}$, the sequence $(\lceil s_n \rceil_N)_{n \in \mathbb{N}}$ is increasing. Hence, for ascending sequences $(s_n)_{n \in \mathbb{N}}$ the sequence $(\lceil s_n \rceil_N)_{n \in \mathbb{N}}$ is increasing and stabilizing (cf. Item 3 of the Assumption) and therefore has a limit.

Lemma 1 Assume two increasing and stabilizing sequences $(s_n)_{n \in \mathbb{N}}, (s'_n)_{n \in \mathbb{N}}$ over C° , such that for all $n \in \mathbb{N}$, $s_n \leq^\circ s'_n$. Then $\lim[(s_n)_{n \in \mathbb{N}}] \leq^\circ \lim[(s'_n)_{n \in \mathbb{N}}]$.

Proof. Per the above observations, $\lim[(s_n)_{n \in \mathbb{N}}]$ is the value at which $(s_n)_{n \in \mathbb{N}}$ stabilizes, hence, there exists $i \in \mathbb{N}$ such that for all $j \geq i$, $s_j = \lim[(s_n)_{n \in \mathbb{N}}]$. Similarly, there exists $i' \in \mathbb{N}$ such that for all $j \geq i'$, $s'_j = \lim[(s'_n)_{n \in \mathbb{N}}]$. Set $i'' := \max i i'$. Then, $\lim[(s_n)_{n \in \mathbb{N}}] = s_{i''}$; by hypothesis of the lemma $s_{i''} \leq^\circ s'_{i''}$ holds; and $s_{i''} = \lim[(s_n)_{n \in \mathbb{N}}]$. By transitivity, $\lim[(s_n)_{n \in \mathbb{N}}] \leq^\circ \lim[(s'_n)_{n \in \mathbb{N}}]$, which proves the lemma.

2.3 An Equivalence Relation on Sequences

Similar sequences eventually reach elements that are pairwise equal up to arbitrary close approximations:

Definition 3 Two sequences $(s_n)_{n \in \mathbb{N}}, (s'_n)_{n \in \mathbb{N}}$ over C° are similar, written $(s_n)_{n \in \mathbb{N}} \sim (s'_n)_{n \in \mathbb{N}}$, whenever for all $N \in \mathbb{N}$ there exists $i \in \mathbb{N}$ such that for all $j \geq i$, $\lceil s_j \rceil_N = \lceil s'_j \rceil_N$.

Lemma 2 The similarity relation \sim is an equivalence relation on sequences over C° .

Proof. Reflexivity and symmetry are immediate. For transitivity, assume $(s_n)_{n \in \mathbb{N}} \sim (s'_n)_{n \in \mathbb{N}}$ and $(s'_n)_{n \in \mathbb{N}} \sim (s''_n)_{n \in \mathbb{N}}$. Choose any $N \in \mathbb{N}$. From $(s_n)_{n \in \mathbb{N}} \sim (s'_n)_{n \in \mathbb{N}}$ we obtain $i \in \mathbb{N}$ such that for all $j \geq i$, $\lceil s_j \rceil_N = \lceil s'_j \rceil_N$, and from $(s'_n)_{n \in \mathbb{N}} \sim (s''_n)_{n \in \mathbb{N}}$ we obtain $i' \in \mathbb{N}$ such that for all $j \geq i'$, $\lceil s'_j \rceil_N = \lceil s''_j \rceil_N$. Setting $i'' := \max i i'$ we get: for all $j \geq i''$, $\lceil s_j \rceil_N = \lceil s'_j \rceil_N = \lceil s''_j \rceil_N$; which proves $(s_n)_{n \in \mathbb{N}} \sim (s''_n)_{n \in \mathbb{N}}$ and establishes transitivity.

Lemma 3 For an ascending sequence $(s_n)_{n \in \mathbb{N}}$, let s be the (unique) value at which $(\lceil s_n \rceil_N)_{n \in \mathbb{N}}$ stabilizes. If $(s_n)_{n \in \mathbb{N}} \sim (s'_n)_{n \in \mathbb{N}}$ then $(\lceil s'_n \rceil_N)_{n \in \mathbb{N}}$ also stabilizes at s .

Proof. From the stabilization hypothesis we obtain $i \in \mathbb{N}$ such that for all $j \geq i$, $\lceil s_j \rceil_N = s$. From $(s_n)_{n \in \mathbb{N}} \sim (s'_n)_{n \in \mathbb{N}}$ we obtain $i' \in \mathbb{N}$ such that for all $j \geq i'$, $\lceil s'_j \rceil_N = \lceil s_j \rceil_N$. Setting $i'' := \max i i'$ we obtain that for all $j \geq i''$, $\lceil s'_j \rceil_N = \lceil s_j \rceil_N = s$. Hence, $(\lceil s'_n \rceil_N)_{n \in \mathbb{N}}$ stabilizes at s ; which proves the lemma.

Lemma 3 helps defining approximations of equivalence classes $\lceil [(s_n)_{n \in \mathbb{N}}] \sim \rceil_N$ of ascending sequences:

Definition 4 For an ascending sequence over C° , $\lceil [(s_n)_{n \in \mathbb{N}}] \sim \rceil_N = \lim[(\lceil s_n \rceil_N)_{n \in \mathbb{N}}]$.

Indeed, Lemma 3 says that the above definition is independent on the chosen representative in the class. This is essential in the rest of the paper. Unless otherwise stated, sequences are assumed to be over C° .

Lemma 4 For ascending sequences $(s_n)_{n \in \mathbb{N}}$ and $(s'_n)_{n \in \mathbb{N}}$ it holds that $(s_n)_{n \in \mathbb{N}} \sim (s'_n)_{n \in \mathbb{N}}$ if and only if for all $N \in \mathbb{N}$, $\lceil [(s_n)_{n \in \mathbb{N}}] \sim \rceil_N = \lceil [(s'_n)_{n \in \mathbb{N}}] \sim \rceil_N$.

Proof. (\Rightarrow): per the above remarks, $\lceil[(s_n)_{n \in \mathbb{N}}]_{\sim}\rceil_N = \lim[\lceil[s_n]_N\rceil_{n \in \mathbb{N}}]$ and $\lceil[(s'_n)_{n \in \mathbb{N}}]_{\sim}\rceil_N = \lim[\lceil[s'_n]_N\rceil_{n \in \mathbb{N}}]$. If $(s_n)_{n \in \mathbb{N}} \sim (s'_n)_{n \in \mathbb{N}}$ we know by Lemma 3 that the value $\lim[\lceil[s_n]_N\rceil_{n \in \mathbb{N}}]$ at which $(\lceil[s_n]_N\rceil_{n \in \mathbb{N}})$ stabilizes equals the value $\lim[\lceil[s'_n]_N\rceil_{n \in \mathbb{N}}]$ at which $(\lceil[s'_n]_N\rceil_{n \in \mathbb{N}})$ stabilizes. Hence the conclusion $\lceil[(s_n)_{n \in \mathbb{N}}]_{\sim}\rceil_N = \lceil[(s'_n)_{n \in \mathbb{N}}]_{\sim}\rceil_N$.

(\Leftarrow): Choose an arbitrary $N \in \mathbb{N}$. Per the above remarks, from the hypothesis $\lceil[(s_n)_{n \in \mathbb{N}}]_{\sim}\rceil_N = \lceil[(s'_n)_{n \in \mathbb{N}}]_{\sim}\rceil_N$ we obtain that the value $\lim[\lceil[s_n]_N\rceil_{n \in \mathbb{N}}]$ at which $(\lceil[s_n]_N\rceil_{n \in \mathbb{N}})$ stabilizes equals the value $\lim[\lceil[s'_n]_N\rceil_{n \in \mathbb{N}}]$ at which $(\lceil[s'_n]_N\rceil_{n \in \mathbb{N}})$ stabilizes. Let i be the least natural number such that for all $j \geq i$, $\lceil[s_j]_N\rceil = \lim[\lceil[s_n]_N\rceil_{n \in \mathbb{N}}]$ and let i' be the least natural number such that for all $j \geq i'$, $\lceil[s'_j]_N\rceil = \lim[\lceil[s'_n]_N\rceil_{n \in \mathbb{N}}]$. Set $i'' := \max i i'$. Hence, for all $j \geq i''$, $\lceil[s_j]_N\rceil = \lceil[s'_j]_N\rceil$ and since N was chosen arbitrarily, by Definition 3, $(s_n)_{n \in \mathbb{N}} \sim (s'_n)_{n \in \mathbb{N}}$; which proves the (\Leftarrow) implication and the lemma.

2.4 An Order Relation on Equivalence Classes of Sequences

Notation. We denote by K the set of equivalence classes of ascending sequences of elements in C° .

Definition 5 For $k, k' \in K$, $k \lesssim k'$ whenever for all $N \in \mathbb{N}$, $\lceil k \rceil_N \leq^\circ \lceil k' \rceil_N$.

Lemma 5 The relation \lesssim is a partial order on K .

Proof. Reflexivity and transitivity of \lesssim easily result from the corresponding properties of \leq° . For anti-symmetry: let $k = \lceil[(s_n)_{n \in \mathbb{N}}]_{\sim}\rceil$ and $k' = \lceil[(s'_n)_{n \in \mathbb{N}}]_{\sim}\rceil$. Then, $k \lesssim k'$ means that for all $N \in \mathbb{N}$, $\lceil[(s_n)_{n \in \mathbb{N}}]_{\sim}\rceil_N \leq \lceil[(s'_n)_{n \in \mathbb{N}}]_{\sim}\rceil_N$, and $k' \lesssim k$ means that for all $N \in \mathbb{N}$, $\lceil[(s'_n)_{n \in \mathbb{N}}]_{\sim}\rceil_N \leq \lceil[(s_n)_{n \in \mathbb{N}}]_{\sim}\rceil_N$. As \leq° is antisymmetric, for all $N \in \mathbb{N}$, $\lceil[(s_n)_{n \in \mathbb{N}}]_{\sim}\rceil_N = \lceil[(s'_n)_{n \in \mathbb{N}}]_{\sim}\rceil_N$, and by Lemma 11, $(s_n)_{n \in \mathbb{N}} \sim (s'_n)_{n \in \mathbb{N}}$, i.e., $\lceil[(s_n)_{n \in \mathbb{N}}]_{\sim}\rceil = \lceil[(s'_n)_{n \in \mathbb{N}}]_{\sim}\rceil$: the lemma is proved.

We give an equivalent characterization of the order \lesssim , expressed as Lemma 7 below. To prove that lemma we need the monotonicity of approximations of equivalence classes:

Lemma 6 If $k \in K$ and $N \leq N'$ then $\lceil k \rceil_N \leq^\circ \lceil k \rceil_{N'}$.

Proof. Let $k = \lceil[(s_n)_{n \in \mathbb{N}}]_{\sim}\rceil$, thus, $\lceil k \rceil_N = \lim[\lceil[s_n]_N\rceil_{n \in \mathbb{N}}]$ and $\lceil k \rceil_{N'} = \lim[\lceil[s_n]_{N'}\rceil_{n \in \mathbb{N}}]$. Using Assumption 1 item 2.(c), for all $n \in \mathbb{N}$, $\lceil[s_n]_N\rceil \leq^\circ \lceil[s_n]_{N'}\rceil$. By Lemma 1 we obtain $\lim[\lceil[s_n]_N\rceil_{n \in \mathbb{N}}] \leq^\circ \lim[\lceil[s_n]_{N'}\rceil_{n \in \mathbb{N}}]$. Hence the conclusion $\lceil k \rceil_N \leq^\circ \lceil k \rceil_{N'}$.

Lemma 7 $k \lesssim k'$ if and only if for all $N \in \mathbb{N}$ there is $N' \in \mathbb{N}$ such that $\lceil k \rceil_N \leq^\circ \lceil k' \rceil_{N'}$.

Proof. (\Rightarrow): this direction is trivial, as by Definition 5 $k \lesssim k'$ means that for all $N \in \mathbb{N}$ there does exist $N' := N$ such that $\lceil k \rceil_N \leq^\circ \lceil k' \rceil_{N'}$.

(\Leftarrow): Choose an arbitrary $N \in \mathbb{N}$. For the corresponding $N' \in \mathbb{N}$ we have two cases:

- either $N' < N$, in which case by Lemma 6, $\lceil k' \rceil_{N'} \leq^\circ \lceil k' \rceil_N$ and by transitivity using hypothesis $\lceil k \rceil_N \leq^\circ \lceil k' \rceil_{N'}$ we obtain $\lceil k \rceil_N \leq^\circ \lceil k' \rceil_N$;
- or $N \geq N'$, and from $\lceil k \rceil_N \leq^\circ \lceil k' \rceil_{N'}$, using Assumption 1 item 2.(b) we obtain $\lceil \lceil k \rceil_N \rceil_{N'} \leq^\circ \lceil \lceil k' \rceil_{N'} \rceil_{N'}$, which, by using item 2.(e), amounts to $\lceil k \rceil_N \leq^\circ \lceil k' \rceil_{N'}$.

Since $N \in \mathbb{N}$ has been arbitrarily chosen and in all cases $\lceil k \rceil_N \leq^\circ \lceil k' \rceil_N$, by Definition 5 we obtain $k \lesssim k'$, which proves the (\Leftarrow) implication and the lemma.

3 Completion

We show how a partially ordered set with a least element $(C^\circ, \leq^\circ, \perp)$ satisfying Assumption 1 can be *completed* in order to obtain a CPO (C, \leq, \perp) . We have already noted that increasing and stabilizing sequences have limits. Hence, limits have to be constructed for the ascending sequences.

Definition 6 *Given a triple $(C^\circ, \leq^\circ, \perp)$ satisfying Assumption 1, the completion operation consists in*

1. *extending the base set C° to a set $C = C^\circ \cup K$ where K is the set of equivalence classes $[(s_n)_{n \in \mathbb{N}}]_{\sim}$ of ascending sequences over C° ;*
2. *defining the limits of each ascending sequence $(s_n)_{n \in \mathbb{N}}$ over C° to be the equivalence class of the sequence: $\text{lim}[(s_n)_{n \in \mathbb{N}}] \triangleq [(s_n)_{n \in \mathbb{N}}]_{\sim}$.*
3. *extending the order \leq° on C° to a relation \leq on C as follows:*
 - (a) *for $c, c' \in C^\circ$, $c \leq c'$ whenever $c \leq^\circ c'$;*
 - (b) *for $c \in C$ and $k \in K$, $c \leq k$ whenever there exists $N \in \mathbb{N}$ such that $c \leq^\circ [k]_N$;*
 - (c) *for $k, k' \in K$, $k \leq k'$ whenever $k \preceq k'$.*

Remark. Definition 6 says nothing about limits of increasing sequences containing elements that are themselves limits (from K). Such sequences are dealt with ahead in the paper and shown to have limits among the new elements. Hence, the CPO construction does not require further completion steps.

A key difference with the earlier completion operation in [7] is that equivalence classes there were all maximal with respect to the order; this was required in order to exclude limit elements (in K) from occurring in ascending sequences. In the new construction, limit elements are not excluded any more from ascending sequences, which enables us to have a nontrivial order on K , in which maximal elements are seen as "fully defined", but in which "partially defined", non-maximal limit elements also exist.

We now show that the relation \leq is an order and that the constructed limits are least upper bounds of the respective sequences. This is then used in the next subsection in order to provide limits to sequences that include both finite elements and equivalence classes, and to prove that those limits are least upper bounds as well, thereby completing the organization of old and new elements and their order as a CPO.

Lemma 8 *In the context of Definition 6, the relation \leq is an order relation on C .*

Proof. The reflexivity of \leq follows from those of \leq° and of \preceq . For antisymmetry: on C° it follows from the antisymmetry of \leq° . On K , it follows from the antisymmetry of \preceq . These are the only situations to consider, as for $c \in C^\circ$ and $k \in K$ having $k \leq c$ is impossible according to Definition 6.

There remains to prove that \leq is transitive. There are only 4 possible cases:

1. $c, c', c'' \in C^\circ$ with $c \leq c'$, $c' \leq c''$: by Definition 6 of completion item 3.(a), here \leq is \leq° and the required $c \leq c''$ results from the transitivity of \leq° ;
2. $c, c' \in C^\circ$ and $k \in K$ with $c \leq c'$ and $c' \leq k$: by Definition 6 item 3.(b), the latter means there exists $N \in \mathbb{N}$ such that $c' \leq^\circ [k]_N$, and since $c \leq c'$, the same N ensures $c \leq^\circ [k]_N$ and thus $c \leq k$;
3. $c \in C^\circ$ and $k, k' \in K$ with $c \leq k$ and $k \leq k'$: by Definition 6 item 3.(b), there exists $N \in \mathbb{N}$ such that $c \leq^\circ [k]_N$, and by Definition 5 and item 3.(c), $[k]_N \leq^\circ [k']_N$, thus, by transitivity, there exists $N \in \mathbb{N}$ such that $c \leq^\circ [k']_N$, meaning that $c \leq k'$;
4. $k, k', k'' \in K$ with $k \leq k'$ and $k' \leq k''$: by Definition 6 item 3.(c), here \leq is \preceq and the required $k \leq k''$ results from the transitivity of \preceq . This concludes the proof of transitivity for \leq and of the lemma as a whole.

Lemma 9 *In the context of Definition 6, the limit $\lim[(s_n)_{n \in \mathbb{N}}]$ of an ascending sequence $(s_n)_{n \in \mathbb{N}}$ over C° is a least upper bound for $(s_n)_{n \in \mathbb{N}}$.*

Proof. By Definition 6, given an ascending sequence $(s_n)_{n \in \mathbb{N}}$, its limit $\lim[(s_n)_{n \in \mathbb{N}}]$ is defined to be the equivalence class $[(s_n)_{n \in \mathbb{N}}]_{\sim}$ of the sequence.

We first prove that $[(s_n)_{n \in \mathbb{N}}]_{\sim}$ is an upper bound for the sequence $(s_n)_{n \in \mathbb{N}}$. That is, we must prove that for all $n \in \mathbb{N}$, $s_n \leq [(s_n)_{n \in \mathbb{N}}]_{\sim}$. Choose an arbitrary $n \in \mathbb{N}$, and choose N such that $\lceil s_n \rceil_N = s_n$. This is made possible by Assumption 1 item 2.(d). We then have $s_n = \lceil s_n \rceil_N \leq^\circ \lim[(\lceil s_n \rceil_N)_{n \in \mathbb{N}}] = [[(s_n)_{n \in \mathbb{N}}]_{\sim}]_N$. Hence, $s_n \leq^\circ [[(s_n)_{n \in \mathbb{N}}]_{\sim}]_N$ and, by Definition 6 item 3.(b), $s_n \leq [(s_n)_{n \in \mathbb{N}}]_{\sim}$. The upper-bound claim of $[(s_n)_{n \in \mathbb{N}}]_{\sim}$ with respect to $(s_n)_{n \in \mathbb{N}}$ is proved.

We now prove that $[(s_n)_{n \in \mathbb{N}}]_{\sim}$ is the least upper bound of $(s_n)_{n \in \mathbb{N}}$. Assume $b \in C$ such that for all $n \in \mathbb{N}$, $s_n \leq b$. There are two cases to consider: $b \in C^\circ$ or $b \in K$.

If $b \in C^\circ$ then for all $n \in \mathbb{N}$, $s_n \leq^\circ b$. But this is impossible by Assumption 1 item 1 since $(s_n)_{n \in \mathbb{N}}$ is ascending.

If $b \in K$, then $b = [(s'_n)_{n \in \mathbb{N}}]_{\sim}$ for some ascending sequence $(s'_n)_{n \in \mathbb{N}}$. From the fact that for all $n \in \mathbb{N}$, $s_n \leq [(s'_n)_{n \in \mathbb{N}}]_{\sim}$, using Assumption 1 item 2.(b) we obtain that for all $n \in \mathbb{N}$, there is $N' \in \mathbb{N}$ such that $s_n \leq^\circ [[(s'_n)_{n \in \mathbb{N}}]_{\sim}]_{N'}$. Choose an arbitrary $N \in \mathbb{N}$. Since for all $n \in \mathbb{N}$, $\lceil s_n \rceil_N \leq^\circ s_n$ we further obtain that for all $n \in \mathbb{N}$ there exists $N' \in \mathbb{N}$ such that $\lceil s_n \rceil_N \leq^\circ [[(s'_n)_{n \in \mathbb{N}}]_{\sim}]_{N'}$. Now, the sequence $(\lceil s_n \rceil_N)_{n \in \mathbb{N}}$ is increasing and stabilizing, and $\lim[(\lceil s_n \rceil_N)_{n \in \mathbb{N}}]$ is an element, say, $\lceil s_k \rceil_N$ of the sequence. We thus obtain that for the arbitrarily chosen $N \in \mathbb{N}$, there exists $N' \in \mathbb{N}$ such that $\lim[(\lceil s_n \rceil_N)_{n \in \mathbb{N}}] \leq^\circ [[(s'_n)_{n \in \mathbb{N}}]_{\sim}]_{N'}$. Moreover, we know that $\lim[(\lceil s_n \rceil_N)_{n \in \mathbb{N}}] = [[(s_n)_{n \in \mathbb{N}}]_{\sim}]_N$. We thus obtain that for all $N \in \mathbb{N}$, there exists $N' \in \mathbb{N}$ such that $[[(s_n)_{n \in \mathbb{N}}]_{\sim}]_N \leq^\circ [[(s'_n)_{n \in \mathbb{N}}]_{\sim}]_{N'}$. By Lemma 7 we obtain $[(s_n)_{n \in \mathbb{N}}]_{\sim} \lesssim [(s'_n)_{n \in \mathbb{N}}]_{\sim}$. Then, by Definition 6 item 3.(c), $[(s_n)_{n \in \mathbb{N}}]_{\sim} \leq [(s'_n)_{n \in \mathbb{N}}]_{\sim} = b$.

Thus, in the only possible case for the upper bound ($b \in K$) we have obtained that $[(s_n)_{n \in \mathbb{N}}]_{\sim}$ is \leq -ordered with b . This concludes the proof of the lemma.

3.1 Completion + Diagonalization = CPO

In this section we show how to "complete" the completion operation in the previous section in order to obtain a CPO. We add what is missing: limits for increasing sequences that include equivalence classes. The main idea is to use a diagonalization technique to extract an ascending sequence of finite elements from an ascending sequence that includes equivalence classes. We need a few technical lemmas. The first lemma adapts Assumption 1 items 2.(a)-(c), which holds elements of C° , to the elements of K .

Lemma 10

1. for all $k \in K$ and $N \in \mathbb{N}$, $\lceil k \rceil_N \leq k$;
2. for all $k, k' \in K$ and $N \in \mathbb{N}$, $k \leq k'$ implies $\lceil k \rceil_N \leq \lceil k' \rceil_N$;
3. for all $k \in K$ and $N, N' \in \mathbb{N}$, $N \leq N'$ implies $\lceil k \rceil_N \leq \lceil k \rceil_{N'}$.

Proof.

1. The approximation $\lceil k \rceil_N$ is an element of C° , hence, by Definition 6 item 3.(b), for $\lceil k \rceil_N \leq k$ it is enough to show that there exists $N' \in \mathbb{N}$ such that $\lceil k \rceil_N \leq \lceil k \rceil_{N'}$. Setting $N' := N$ ensures this;
2. By Definition 6 item 3.(c), $k \leq k'$ is just $k \lesssim k'$, which by Definition 5 is just the conclusion of this item - that for all $N \in \mathbb{N}$, $\lceil k \rceil_N \leq \lceil k' \rceil_N$;
3. Both $\lceil k \rceil_N$ and $\lceil k \rceil_{N'}$ are elements of C° , thus, $\leq = \leq^\circ$, and the conclusion follows by Lemma 6.

Lemma 11 *Assume that for all $N \in \mathbb{N}$ there exists $N' \geq N$ such that $\lceil k \rceil_N = \lceil k' \rceil_{N'}$. Then $k = k'$.*

Proof. $k \lesssim k'$ follows by Lemma 7. There remains to show $k' \lesssim k$. Choose an arbitrary $N \in \mathbb{N}$. For the corresponding $N' \geq N$ from the lemma's hypothesis we have by Lemma 10 item 3, $\lceil k' \rceil_{N'} \leq^\circ \lceil k' \rceil_{N'}$, and since by hypothesis $\lceil k' \rceil_{N'} = \lceil k \rceil_N$, we have obtained that for the arbitrarily chosen $N \in \mathbb{N}$, $\lceil k' \rceil_{N'} \leq^\circ \lceil k \rceil_N$, which by Definition 5 establishes $k' \lesssim k$ and proves the lemma.

Notation. Hereafter, for equivalence classes $k, k' \in K$ we write $k < k'$ for $k \lesssim k'$ and $k \neq k'$.

Lemma 12 *$k < k'$ implies that for all $N \in \mathbb{N}$ there is $N' > N$ such that $\lceil k \rceil_N <^\circ \lceil k' \rceil_{N'}$.*

Proof. Choose an arbitrary $N \in \mathbb{N}$. From Lemma 7 we obtain $N'' \in \mathbb{N}$ such that $\lceil k \rceil_N \leq^\circ \lceil k' \rceil_{N''}$. Let $N' = (\max N N'') + 1$. Then, by Lemma 10 item 3, $\lceil k' \rceil_{N''} \leq^\circ \lceil k' \rceil_{N'}$ and by transitivity of \leq° , $\lceil k \rceil_N \leq^\circ \lceil k' \rceil_{N'}$. There only remains to show that the previous inequality is strict. For, assuming the contrary, we would have that for the arbitrarily chosen $N \in \mathbb{N}$ there exists $N' > N$ such that $\lceil k \rceil_N = \lceil k' \rceil_{N'}$. By Lemma 11 we would have $k = k'$, in contradiction to $k < k'$. Hence, $\lceil k \rceil_N <^\circ \lceil k' \rceil_{N'}$, which proves the lemma.

Notation. We denote by C^ω the set of sequences over a set C .

Next follows the definition of a *diagonalizer* of sequences in $(C^\circ \cup K)^\omega$: a key ingredient for their limits.

Definition 7 *For a increasing sequence $s = (s_n)_{n \in \mathbb{N}} \in (C^\circ \cup K)^\omega$, we define the diagonalizer $\text{diag } s \in \mathbb{N}^\omega$:*

- *base case: $(\text{diag } s)_0 = 0$;*
- *induction step: assume $(\text{diag } s)_{n-1}$ is known. We define $(\text{diag } s)_n$ as follows:*
 - *if $s_n \in C^\circ$ then $(\text{diag } s)_n = 0$;*
 - *if $s_n \in K$, consider the element s_{n-1} :*
 - * *if $s_{n-1} \in C^\circ$: from $s_{n-1} \leq s_n$ and Definition 6 item 3.(b) we can choose $N > 0$ such that $s_{n-1} \leq^\circ \lceil s_n \rceil_N$, and let $(\text{diag } s)_n = N$;*
 - * *if $s_{n-1} \in K$ then:*
 - *either $s_{n-1} = s_n$, and then $(\text{diag } s)_n = (\text{diag } s)_{n-1}$;*
 - *or $s_{n-1} < s_n$. Using Lemma 12, we can choose $N > (\text{diag } s)_{n-1}$ such that $\lceil s_{n-1} \rceil_{(\text{diag } s)_{n-1}} <^\circ \lceil s_n \rceil_N$, and let $(\text{diag } s)_n = N$.*

Lemma 13 *For any increasing sequence $s \in (C^\circ \cup K)^\omega$, the sequence $(\text{diag } s)$ is increasing.*

Proof. For two consecutive elements $(\text{diag } s)_{i-1}$ and $(\text{diag } s)_i$ of $(\text{diag } s)$, we show $(\text{diag } s)_{i-1} \leq (\text{diag } s)_i$:

- *if $s_{i-1} \in C^\circ$, by Definition 7, $(\text{diag } s)_{i-1} = 0 \leq (\text{diag } s)_i$;*
- *If $s_{i-1} \in K$, it holds that $s_i \in K$ since s increasing and no class can be \leq -smaller than a non-class:*
 - *if $s_{i-1} = s_i$ then by Definition 7 $(\text{diag } s)_{i-1} = (\text{diag } s)_i$;*
 - *if $s_{i-1} < s_i$ then by Definition 7 $(\text{diag } s)_{i-1} < (\text{diag } s)_i$; which proves this case and the lemma.*

Lemma 14 *For any ascending sequence $s \in (C^\circ \cup K)^\omega$, the sequence $(\text{diag } s)$ is ascending.*

Proof. Since s is ascending, it is increasing, and by Lemma 13, $(\text{diag } s)$ is increasing. We only need to show that $(\text{diag } s)$ has a strictly increasing subsequence: for all $i \in \mathbb{N}$ there is $j \geq i$ such that $(\text{diag } s)_j > (\text{diag } s)_i$. Consider an arbitrary $i \in \mathbb{N}$.

- *if $s_i \in C^\circ$ then $(\text{diag } s)_i = 0$ and there exist $j > i$ such that $s_j \in K$. By Definition 7, any $j \in \mathbb{N}$ for which $s_j \in K$ satisfies $(\text{diag } s)_j > 0$, hence $(\text{diag } s)_j > (\text{diag } s)_i$;*

- if $s_i \in K$ then, since s is ascending, there exists $j \in \mathbb{N}$ such that $s_j \in K$ and $s_i < s_j$. Pick j minimal with the above property. In this case, the ascending property of s ensures that for all $m \in [i, j)$, $s_i = s_m$ and $s_m < s_j$. By Definition 7, for all $m \in [i, j)$, $(diag\ s)_i = (diag\ s)_m$ and $(diag\ s)_m < (diag\ s)_j$, which proves $(diag\ s)_j > (diag\ s)_i$ in this second case as well and complete the lemma's proof.

We call the *diagonal* of an increasing sequence $s \in (C^\circ \cup K)^\omega$ the sequence $(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}} \in (C^\circ)^\omega$.

Lemma 15 *If $s \in (C^\circ \cup K)^\omega$ is increasing, the sequence $(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}} \in (C^\circ)^\omega$ is increasing as well.*

Proof. Any two consecutive elements $\lceil s_{i-1} \rceil_{(diag\ s)_{i-1}}$ and $\lceil s_i \rceil_{(diag\ s)_i}$ satisfy $\lceil s_{i-1} \rceil_{(diag\ s)_{i-1}} \leq \lceil s_i \rceil_{(diag\ s)_i}$ as both s and $(diag\ s)$ are increasing — the latter, by Lemma 13 — and approximations are monotone by Lemma 10 items 2 and 3.

The next lemma is used in proving the following one. It resembles Lemma 12 but is not quite the same.

Lemma 16 *$k < k'$ implies that for all $N \in \mathbb{N}$, $\lceil k \rceil_N <^\circ \lceil k' \rceil_N$.*

Proof. Choose an arbitrary $N \in \mathbb{N}$. From Lemma 7 we obtain $\lceil k \rceil_N \leq^\circ \lceil k' \rceil_N$. There remains to show that the previous inequality is strict. For, assuming the contrary, we would have that for an the arbitrarily chosen $N \in \mathbb{N}$, $\lceil k \rceil_N = \lceil k' \rceil_N$. Then, $k \lesssim k'$ and $k' \lesssim k$ follow by Definition 7, implying $k = k'$, in contradiction to $k < k'$. Hence, $\lceil k \rceil_N <^\circ \lceil k' \rceil_N$, which proves the lemma.

Lemma 17 *If $s \in (C^\circ \cup K)^\omega$ is ascending and there exists $i \in \mathbb{N}$ such that $s_i \in K$, then $(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}} \in (C^\circ)^\omega$ is ascending.*

Proof. By Lemma 15 the sequence $(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}}$ is increasing. We only need to prove that it has a strictly increasing subsequence. Since s is ascending it has a strictly increasing subsequence $(s_{i_n})_{n \in \mathbb{N}}$. Since $s_i \in K$, we obtain that for all $j \geq i$, $s_j \in K$. Consider the subsequence $(s_{i_n})_{n \in \mathbb{N}, i_n \geq i}$ of $(s_{i_n})_{n \in \mathbb{N}}$. As a subsequence of an strictly increasing subsequence, $(s_{i_n})_{n \in \mathbb{N}, i_n \geq i}$ is strictly increasing, thus, for all $n \in \mathbb{N}$ s.t. $i_n \geq i$, $s_{i_n} < s_{i_{n+1}}$. Fix such an $n \in \mathbb{N}$. Using Lemma 16 with $N := (diag\ s)_{i_n}$ we obtain $\lceil s_{i_n} \rceil_{(diag\ s)_{i_n}} <^\circ \lceil s_{i_{n+1}} \rceil_{(diag\ s)_{i_{n+1}}}$. Since $(diag\ s)$ is increasing, $(diag\ s)_{i_n} \leq (diag\ s)_{i_{n+1}}$ and by Assumption 1 item 2.(c), $\lceil s_{i_{n+1}} \rceil_{(diag\ s)_{i_n}} \leq^\circ \lceil s_{i_{n+1}} \rceil_{(diag\ s)_{i_{n+1}}}$. Overall, $\lceil s_{i_n} \rceil_{(diag\ s)_{i_n}} <^\circ \lceil s_{i_{n+1}} \rceil_{(diag\ s)_{i_{n+1}}}$; and since n was chosen arbitrarily, $(\lceil s_{i_n} \rceil_{(diag\ s)_{i_n}})_{n \in \mathbb{N}, i_n \geq i}$ is an ascending subsequence of $(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}}$, proving the lemma.

The last lemma in this section provides limits to sequences $s \in (C^\circ \cup K)^\omega$ having at least one element in K , as the equivalence sequences of their diagonals, and establishes that the limits are least upper bounds.

Lemma 18 *If $s \in (C^\circ \cup K)^\omega$ is ascending and there exists $i \in \mathbb{N}$ such that $s_i \in K$, then $\llbracket (\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}} \rrbracket \sim$ is the least upper bound for s .*

Proof. We first prove that $\llbracket (\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}} \rrbracket \sim$ is an upper bound for s . Thus, we need to show that for all $j \in \mathbb{N}$, $s_j \leq \llbracket (\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}} \rrbracket \sim$. Since s is increasing, it is enough to show the above inequality for $j \geq i$, i.e., when $s_j \in K$. By Lemma 7 we need to show that for all $N \in \mathbb{N}$, there exists $N' \in \mathbb{N}$ such that $\lceil s_j \rceil_N \leq^\circ \llbracket (\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}} \rrbracket \sim_{N'}$.

Choose an arbitrary $N \in \mathbb{N}$. Since by Lemma 14 $(diag\ s)_{n \in \mathbb{N}}$ is ascending, we can choose $m \in \mathbb{N}$ such that $(diag\ s)_m \geq N$. Let $p = \max\ j\ m$. Thus, by Assumption 1 item 2.(b), $\lceil s_j \rceil_N \leq^\circ \lceil s_p \rceil_N$ and since $(diag\ s)$ is increasing, $(diag\ s)_p \geq N$. Using Assumption 1 item 2.(c), $\lceil s_p \rceil_N \leq^\circ \lceil s_p \rceil_{(diag\ s)_p}$. By transitivity, $\lceil s_j \rceil_N \leq^\circ \lceil s_p \rceil_{(diag\ s)_p}$. But since (by Lemma 17) $(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}}$ is ascending, we know by Lemma 9 that $\llbracket (\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}} \rrbracket \sim$ is an upper bound for $(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}}$. In particular, $\lceil s_p \rceil_{(diag\ s)_p} \leq^\circ$

$[(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}}]_{\sim}$, and by transitivity, $\lceil s_j \rceil_N \leq^\circ [(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}}]_{\sim}$, which by Definition 6 item 3.(b) implies that there exists $N' \in \mathbb{N}$ such that $\lceil s_j \rceil_N \leq^\circ [(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}}]_{\sim} \lceil N' \rceil$; i.e., what remained to prove for the upper-bound claim.

We now prove that $[(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}}]_{\sim}$ is a least upper bound for s . Thus, assuming an upper bound $b \in C^\circ \cup K$ for s , we need to show that $[(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}}]_{\sim} \leq b$. Now, by Lemma 9 $[(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}}]_{\sim}$ is the least upper bound for $(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}}$. Hence in order to show $[(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}}]_{\sim} \leq b$ it is enough to show that b is an upper bound for $(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}}$. Choose an arbitrary $n \in \mathbb{N}$. We have, by Lemma 10 item 1, $\lceil s_n \rceil_{(diag\ s)_n} \leq s_n$ and since b is an upper bound for s , $s_n \leq b$. Hence, for an arbitrary $n \in \mathbb{N}$, $\lceil s_n \rceil_{(diag\ s)_n} \leq b$, which establishes that b is an upper bound for $(\lceil s_n \rceil_{(diag\ s)_n})_{n \in \mathbb{N}}$, which is all that remained to prove in order to complete the least-upper-bound claim of the lemma and the lemma as a whole.

Together, Lemmas 9 and 18 ensure that any ascending sequence over $(C^\circ \cup K)^\omega$ has a limit. For the remaining increasing sequences - the stabilizing ones - the limit is the value at which they stabilize. This completes the transformation of a partial order with a least element satisfying Assumption 1 into a CPO.

4 Rose Trees as CPOs

In this section we focus on the definition of Rose trees, i.e., trees with finite breadth and possibly infinite depth, using the finite trees defined in Example 1 and the CPO construction in the previous section.

Remember from Example 1 that finite trees C° have constructors \perp and $tree\ a\ l$ for finite lists l of finite trees. The prefix order \leq° is such that for all $t \in C^\circ$, $\perp \leq^\circ t$, and for all lists l, l' of trees having the same length m , $tree\ l \leq^\circ tree\ l'$ if and only if for all $i < m$, $l[i] \leq^\circ l'[i]$. Together with their approximations they satisfy Assumption 1 and therefore constitute a CPO $(C^\circ \cup K, \leq, \perp)$ where K is the set of equivalence classes of ascending sequences of finite trees. Such ascending sequences require trees having at least one branch that grows "forever", and their limits (in K) have at least one infinite branch. Ascending sequences including elements of K enable some other finite branches to grow "forever" and, in the limit, to add new infinite branches to the already existing infinite ones. Note, however, that the diagonalization technique employed in the previous section implies that ascending sequences including infinite trees are superfluous, because the same limit can be obtained by their diagonal, a sequence of finite trees only.

Remark. Per the above observations, all sequences considered below are constituted only of finite trees.

Overall, the set K together with the set C° of finite trees constitute the set $C^\circ \cup K$ of Rose trees - with finite breadth and possibly infinite depth. Among Rose trees there are those who do not have \perp as a subtree. Such trees are *maximal* for the \leq order and, since this order is a definition order, the maximal elements are interpreted as "completely defined". There are also elements that have some infinite subtrees and some \perp subtrees. Those are non-maximal for the definition order \leq and hence not fully defined.

4.1 Extending the *tree* Constructor of Finite Trees to Rose Trees

We now define a function *tree* that takes a list of Rose trees (defined above) and produces a Rose tree.

Remark. In the sequel we identify the *equivalence class of an increasing and stabilizing sequence* with the value at which all sequence in the class stabilize, i.e. their common limit. Hence, for increasing and stabilizing sequences, $[(s_n)_{n \in \mathbb{N}}]_{\sim} = \lim[(s_n)_{n \in \mathbb{N}}]$. Since this identity also holds for ascending sequences (cf. Definition 6), it holds for all increasing sequences, which enables us to treat them uniformly.

Consider now a list $l = [t^1, \dots, t^m]$ of elements in $C = C^\circ \cup K$. Thanks to the above identification, each t^i equals the equivalence class of some increasing sequence $(t_n^i)_{n \in \mathbb{N}}$ that converges to it: for all $i = 1, \dots, m$,

$t^i = [(t_n^i)_{n \in \mathbb{N}}]_{\sim} = \lim[(t_n^i)_{n \in \mathbb{N}}]$. Let us consider the sequence of lists $((t_n^1, \dots, t_n^m)_{n \in \mathbb{N}})$. This sequence is "pointwise" increasing: for all $n \in \mathbb{N}$ and $i = 1, \dots, m$, $t_n^i \leq^{\circ} t_{n+1}^i$, and, thanks to the monotony of the \leq° (prefix) order, the sequence $(tree[t_n^1, \dots, t_n^m])_{n \in \mathbb{N}}$ is increasing. This justifies the following definition:

Definition 8 (tree function on Rose Trees) $tree [(t_n^1)_{n \in \mathbb{N}}]_{\sim}, \dots, [(t_n^m)_{n \in \mathbb{N}}]_{\sim} \triangleq \lim[(tree[t_n^1, \dots, t_n^m])_{n \in \mathbb{N}}]$.

We now show that the above definition does not depend on the choice of representatives in the equivalence classes $[(t_n^1)_{n \in \mathbb{N}}]_{\sim}, \dots, [(t_n^m)_{n \in \mathbb{N}}]_{\sim}$. Thus, it is required to prove that for each pair of lists of increasing sequences $[(t_n^1)_{n \in \mathbb{N}}, \dots, (t_n^m)_{n \in \mathbb{N}}]$ and $[(t_n'^1)_{n \in \mathbb{N}}, \dots, (t_n'^m)_{n \in \mathbb{N}}]$ satisfying $(t_n^i)_{n \in \mathbb{N}} \sim (t_n'^i)_{n \in \mathbb{N}}$ for all $i = 1, \dots, m$, it holds that $(tree[t_n^1, \dots, t_n^m])_{n \in \mathbb{N}} \sim (tree[t_n'^1, \dots, t_n'^m])_{n \in \mathbb{N}}$. Fix an arbitrary $N \in \mathbb{N}$. Now, from $(t_n^i)_{n \in \mathbb{N}} \sim (t_n'^i)_{n \in \mathbb{N}}$ for all $i = 1, \dots, m$, we obtain, for each $i = 1, \dots, m$, some $j_i \in \mathbb{N}$ such that for all $j \geq j_i$, $[t_j^i]_N = [t_j'^i]_N$. Setting J to be the maximum of the j_i and by using the definition of approximations for trees, we obtain for all $j \geq J$, $[tree[t_j^1, \dots, t_j^m]]_{N+1} = tree[[t_j^1]_N, \dots, [t_j^m]_N] = tree[[t_j'^1]_N, \dots, [t_j'^m]_N] = [tree[t_j'^1, \dots, t_j'^m]]_{N+1}$. Combined with the fact that for all j (in particular for $j \geq J$), $[tree[t_j^1, \dots, t_j^m]]_0 = \perp = [tree[t_j'^1, \dots, t_j'^m]]_0$, for the arbitrarily chosen $N \in \mathbb{N}$ there is $J \in \mathbb{N}$ such for all $j \geq M$, $[tree[t_j^1, \dots, t_j^m]]_N = ([tree[t_j'^1, \dots, t_j'^m]])_N$. But by definition of \sim this is just $(tree[t_n^1, \dots, t_n^m])_{n \in \mathbb{N}} \sim (tree[t_n'^1, \dots, t_n'^m])_{n \in \mathbb{N}}$, which is what we had to prove in order to show the independence from representatives in Definition 8 of the *tree* function for Rose trees. There remains to prove that the function satisfies the following properties of its restriction to finite trees:

- approximations: $[tree \upharpoonright_0] = \perp$ and $[tree \upharpoonright_{N+1}] = tree(map [\cdot]_N) \upharpoonright$;
- monotonicity : $tree \ l \leq tree \ l'$ iff for some $m \in \mathbb{N}$, $length \ l = length \ l' = m$ and for all $i < m$, $l[i] \leq l'[i]$;
- surjectiveness for nonempty trees: for all $t \in C \setminus \{\perp\}$, there exists l such that $t = tree \ l$;
- injectiveness: for all lists of Rose trees l, l' , $tree \ l = tree \ l'$ implies $l = l'$.

In particular, the last two properties will enable us to define the *forest* accessor for nonempty trees as: *forest* t is the (unique) list l such that $t = tree \ l$.

4.2 Towards Defining Corecursive Functions Without Corecursion

Definition 8 and its properties enables us to define corecursive functions based on some results from [7].

4.2.1 Productive Functionals and Unique Fixpoints

Assume a CPO (C, \leq, \perp) . We extend the order \leq to functions $D \rightarrow C$ by $f_1 \leq f_2$ iff $f_1 x \leq f_2 x$ for all $x \in D$.

Definition 9 A functional $F : (D \rightarrow C) \rightarrow D \rightarrow C$ is increasing if for all $f_1, f_2 : D \rightarrow C$, $f_1 \leq f_2$ implies $F f_1 \leq F f_2$.

Consider a functional $F : (D \rightarrow C) \rightarrow D \rightarrow C$ as above. Let $\perp : D \rightarrow C$ be the constant function such that $\perp x = \perp$, for all $x \in D$, and let $F^n : (D \rightarrow C) \rightarrow D \rightarrow C$ be the functional inductively defined by $F^0 f = f$ and, for all $n \in \mathbb{N}$, $F^{n+1} f = F(F^n f)$.

Definition 10 A functional $F : (D \rightarrow C) \rightarrow D \rightarrow C$ is productive whenever it is increasing and for all $x \in D$, the limit of the (increasing) sequence $(F^n \perp x)_{n \in \mathbb{N}}$ is maximal w.r.t. the order \leq .

Theorem 1 ([7]) If a functional F is productive then $\lim[(F^n \perp)_{n \in \mathbb{N}}]$ is the unique fixpoint of F .

4.2.2 Towards Defining a Mirror Function for Rose Trees

The *mirror* function for Rose trees, which we intend to define via its fixpoint equation, has the following functional: $Mirror = \lambda f. \lambda t. \text{if } t = \perp \text{ then } \perp \text{ else } \text{map } f (\text{reverse } (\text{forest } t))$, where *reverse* is the standard function that reverses lists. For defining the *mirror* function via the equation $\text{mirror} = Mirror \text{ mirror}$ we shall use Theorem 1, which requires us to prove that the functional *Mirror* is productive. We conjecture that this requires us to first define $\text{mirror}^\circ = \lambda t. \text{if } t = \perp \text{ then } \perp \text{ else } \text{map } \text{mirror}^\circ (\text{reverse } (\text{forest } t))$ for finite trees t , and then prove by induction on $n \in \mathbb{N}$ that for all Rose trees t , $Mirror^n \perp t = \text{mirror}^\circ(\lceil t \rceil_n)$. The latter identity is useful because $\lceil t \rceil_n$ has the property that all its \perp subtrees are at the same "distance" from the root of t , which coincides with the tree's height. Hence, for infinite maximal trees t , the sequence $(Mirror^n \perp t)_{n \in \mathbb{N}}$ is ascending and "fair", in the sense that there remain no "undeveloped" \perp subtree.

Hence, in the limit, there is no \perp subtree, which makes the limit maximal and the function itself well-defined: when given as input a completely defined Rose tree it returns a tree of the same nature.

5 Conclusion, Related Work, and Future Work

We have shown a way to build an CPO from a partially ordered set with least element satisfying a certain assumption, and have shown that the assumption holds for finites trees. The resulting CPO of Rose trees enabled us to extend the *tree* constructor of finite trees to a homonymous function on Rose trees and to enable the possibility of defining a mixed recursive-corecursive function - the *mirror* function - as the unique fixpoint of its functional. This is the most natural way to define such a function and earlier attempts at this in [7] failed. The modified completion operation in this paper gets most of the credit for this progress. Although we have focused in this paper on an example — Rose trees and their mirror — we believe that the example is paradigmatic, and that understanding it opens the way to a general approach, perhaps partially automatable, for naturally defining corecursive functions without using corecursion.

Related Work The completion operation has similarities with the classical construction of real numbers based on completing rationals with equivalence classes of Cauchy sequences of rationals. However, Cauchy sequences require metric spaces, with a distance function satisfying certain properties, and it does not seem possible to organize Rose trees and their prefix order as a metric space. The reason is that the distance requires a "weak totality" property that the prefix order does not satisfy. Defining corecursive functions based on Cauchy sequences is mentioned in [5]. Using Banach's fixpoint theorem, corecursive functions are defined as unique fixpoints of *eventually contracting* functionals. However, their work is not concerned with extending the definition of corecursive functions beyond the guarded.

Corecursion is present in several major proof assistants. In Coq, corecursive function definitions have to satisfy a guardedness-by-constructors criterion. In some cases, a function that is not guarded can be transformed into an equivalent, guarded function [2]. Their idea is to use an ad-hoc predicate stating that the definition under study is, in some sense, productive. However, they do not handle the case where corecursive calls are guarded by non-constructor functions, such as our mirror function for Rose trees.

Agda [9] is a proof assistant with an underlying type theory close to that of Coq. It also offers support for corecursive function definition. In the core tool there is a guardedness checker similar to that of Coq, but more liberal as it allows, e.g., the definition of two mutually dependent functions, one of which is recursive and the other one corecursive. This enables it to accept the definition of the mirror function on Rose trees, which Coq does not accept. Extensions to Agda with sized types [8] provide users with a uniform, automatic way of handling termination and productiveness. The implementation of sized types

is currently unsound (cf. <https://github.com/agda/agda/issues/3026>).

Isabelle/HOL [10] is also major proof assistant which supports corecursive functions. It accept function that go beyond guarded corecursion [3], provided the functions are *friendly* (a friendly function needs to destruct at most one constructor of input to produce one constructor of output). Unguarded corecursive calls are also accepted, provided they eventually produce a constructor of output. Like in our case, the user needs to prove the conditions ensuring the well-formedness of corecursive functions.

Future Work The next steps are to prove that relevant properties of the *tree* constructor of finite trees are also satisfied by the homonymous function on Rose trees, and to exploit these properties together with results from [7] in order to define corecursive functions as the unique solution of a fixpoint equation. The presented and planned developments will be implemented in Coq. Another goal is to build a tool on top of Coq for corecursive function definition, where proof obligations for productiveness of a given function are automatically generated by the tool for the user to prove. The present paper could also be formalized in other proof assistants: an interesting target is Lean [6], a dependently-typed language and proof assistant that to our best knowledge does not offer support for coinductive types and corecursive function definition; implementing the present generic CPO construction in Lean represents one manner to endow it with coinductive/corecursive features. Interestingly, Lean includes the feature of *quotient types* that would naturally encode equivalence classes, which are extensively used in the present work.

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