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On a Central Limit Theorem and a Confidence Interval for Randomized Quasi-Monte Carlo Methods

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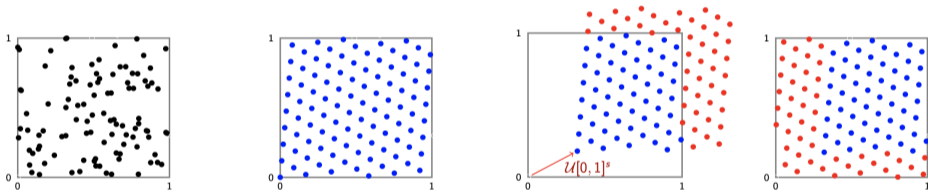
Stochastic Numerics and Statistical Learning workshop
May 2022

Motivation

- Estimate μ and provide **computable measure of error**, where

$$\mu = \int_{[0,1]^s} h(u) du = \mathbb{E}[h(U)], \quad U \sim \mathcal{U}[0,1]^s$$

- Monte Carlo (MC), Quasi-Monte Carlo (QMC), **Randomized QMC (RQMC)**



- Central limit theorem (CLT) not proven for many RQMC settings.**
- Goal:** General conditions for RQMC to ensure
 - CLT
 - Asymptotically valid confidence interval (AVCI)**

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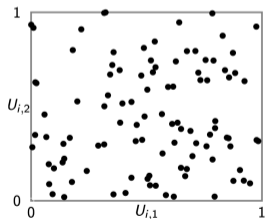
- 1 Review: MC, QMC, RQMC
- 2 RQMC CLT and AVCI
- 3 Corollaries Ensuring CLT and AVCI: Tradeoffs
- 4 Concluding Remarks

Review: Monte Carlo (MC)

- MC: **random sampling** to estimate $\mu = \mathbb{E}[h(U)]$

$$\hat{\mu}_n^{\text{MC}} = \frac{1}{n} \sum_{i=1}^n h(U_i)$$

- U_1, U_2, \dots, U_n i.i.d. $\mathcal{U}[0, 1]^s$



Review: MC — Error Estimation Easy, But Slow Convergence

- MC estimator: $\hat{\mu}_n^{\text{MC}} = \frac{1}{n} \sum_{i=1}^n h(U_i)$
- **CLT**: If $\psi^2 \equiv \text{Var}[h(U)] \in (0, \infty)$, then [Billingsley 1995]

$$\sqrt{\frac{n}{\psi^2}} \left[\hat{\mu}_n^{\text{MC}} - \mu \right] \Rightarrow \mathcal{N}(0, 1) \quad \text{as } n \rightarrow \infty$$

- Approximate $100\gamma\%$ confidence interval (CI) for μ :

$$I_{n,\gamma}^{\text{MC}} \equiv \left[\hat{\mu}_n^{\text{MC}} \pm z_\gamma \frac{\hat{\psi}_n}{\sqrt{n}} \right]$$

▶ $\hat{\psi}_n^2 = \frac{1}{n-1} \sum_{i=1}^n [h(U_i) - \hat{\mu}_n^{\text{MC}}]^2$ and $\Phi(z_\gamma) = 1 - (1 - \gamma)/2$.

- **Asymptotically valid CI (AVCI)**:

$$\mathbb{P}(\mu \in I_{n,\gamma}^{\text{MC}}) \rightarrow \gamma, \quad \text{as } n \rightarrow \infty$$

- Root mean-squared error: $\text{RMSE}[\hat{\mu}_n^{\text{MC}}] = \frac{\psi}{\sqrt{n}}$

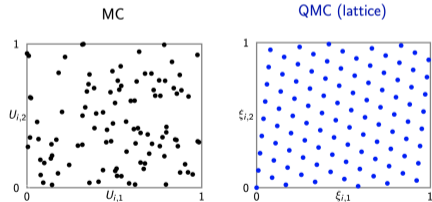
Review: Quasi-Monte Carlo (QMC)

- QMC: **deterministic points** to estimate $\mu = \mathbb{E}[h(U)]$

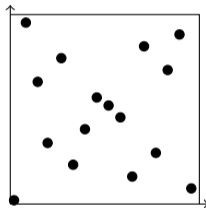
$$\hat{\mu}_n^Q = \frac{1}{n} \sum_{i=1}^n h(\xi_i)$$

- **Low-discrepancy sequence** $\Xi = (\xi_i : i = 1, 2, \dots)$

- ▶ Ξ is **deterministic** and evenly fill $[0, 1]^s$
- ▶ **lattices** (e.g., Korobov, ...),

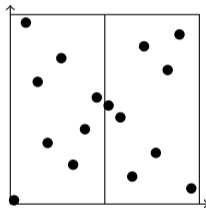


- **Digital nets/sequences** (e.g., Sobol', Faure, ...)



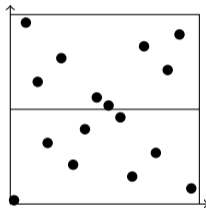
- Is this 16 points sequence well-distributed?

- **Digital nets/sequences** (e.g., Sobol', Faure, ...)



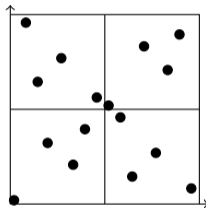
- If we split $[0, 1]^2$, we get the right number of points in the rectangles.

- **Digital nets/sequences** (e.g., Sobol', Faure, ...)



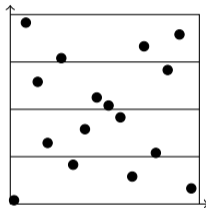
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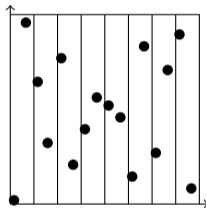
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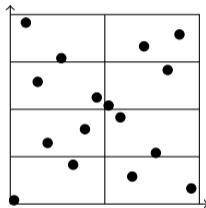
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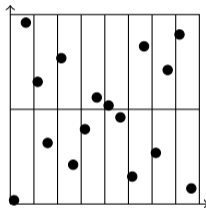
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- **Digital nets/sequences** (e.g., Sobol', Faure, ...)



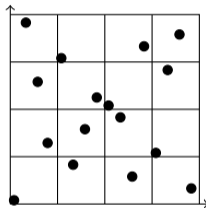
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- **Digital nets/sequences** (e.g., Sobol', Faure, ...)



- If we split $[0, 1]^2$, we get the right number of points in the rectangles.

Review: QMC — Fast Convergence

- QMC: **deterministic points** to estimate $\mu = \mathbb{E}[h(U)]$

$$\hat{\mu}_n^Q = \frac{1}{n} \sum_{i=1}^n h(\xi_i), \quad \Xi = (\xi_i : i = 1, 2, \dots)$$

- **Koksma-Hlawka (K-H) inequality** [Niederreiter 1992]: for each $n > 1$,

$$|\hat{\mu}_n^Q - \mu| \leq V_{\text{HK}}(h) D_n^*(\Xi)$$

- ▶ **Hardy-Krause variation** $V_{\text{HK}}(h) \in [0, \infty]$: “roughness” of h
- ▶ **Star-discrepancy** $D_n^*(\Xi) \in [0, 1]$: how unevenly first n points of Ξ fill $[0, 1]^s$

$$D_n^*(\Xi) = O(n^{-1}(\ln n)^s) \approx O(n^{-1}), \quad n \rightarrow \infty.$$

- ▶ If $V_{\text{HK}}(h) < \infty$ (BVHK), then K-H bound shrinks at faster rate than MC rate $\Theta(n^{-1/2})$

$$|\hat{\mu}_n^Q - \mu| \approx O(n^{-1}).$$

- ★ BVHK: “bounded variation in sense of Hardy and Krause”

Review: QMC — But Error Estimation Difficult

Two components: $V_{\text{HK}}(h)$ and $D_n^*(\Xi)$.

- Variation

- ▶ Difficult to compute: sum of $2^s - 1$ variations in the sense of Vitali (each difficult to estimate).
- ▶ $V_{\text{HK}}(h) = \infty$ often, even for simple functions such as

$$h(x_1, x_2, x_3) = \min(x_1 + x_2 + x_3, 1)$$

for which QMC converges (faster than for MC).

- Discrepancy

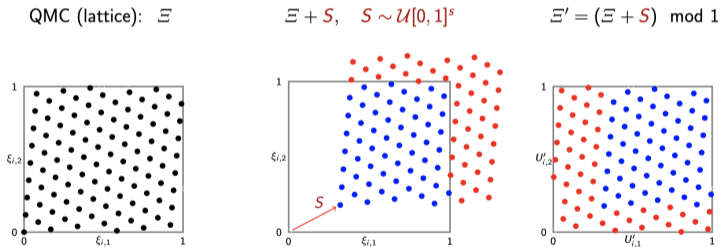
- ▶ Expression $D_n^*(\Xi) = \sup_{x \in [0,1]^s} \left| \frac{\sum_{i=1}^n \mathbb{1}_{\prod_{k=1}^s [0, x_k]}(\xi_i)}{n} - \prod_{k=1}^s x_k \right|$.
- ▶ We only have bounds with asymptotic usefulness, not for practice use.
- ▶ Ex: bounds for the $(0, s)$ -Niederreiter sequence (while we know $D_n^*(\Xi) < 1!$)

$s = 5$	n	10^3	10^4	10^6	10^9	10^{12}	10^{16}
	Bound	1.50	$3.34 \cdot 10^{-1}$	$2.14 \cdot 10^{-2}$	$1.21 \cdot 10^{-4}$	$5.83 \cdot 10^{-7}$	$1.92 \cdot 10^{-10}$
$s = 25$	n	10^3	10^9	10^{15}	10^{21}	10^{35}	
	Bound	$2.88 \cdot 10^5$	$1.11 \cdot 10^6$	$2.46 \cdot 10^5$	$5.22 \cdot 10^3$	$1.62 \cdot 10^{-3}$	

- Other bounds exist, in general with the same issues.

Review: Randomized Quasi-Monte Carlo (RQMC)

- i.i.d. randomizations of $\Xi = (\xi_i : i \geq 1)$, each yielding $\Xi' = (U'_i : i \geq 1)$
 - ▶ Each $U'_i \sim \mathcal{U}[0, 1]^s$
 - ▶ Ξ' retains low-discrepancy properties of Ξ
- **Lattice:** random shift [Cranley & Patterson 1976]



- **Digital net:** nested scrambling [Owen 1995], digital shift [L'Ecuyer & Lemieux 2002], ...

Review: Randomized Quasi-Monte Carlo (RQMC)

- RQMC computation budget of n evaluations of h (as for MC)
 - ▶ **allocation** (m_n, r_n) with $m_n \times r_n \approx n$
 - ▶ $r_n = \#$ i.i.d. randomizations
 - ▶ $m_n = \#$ points used from j th randomized sequence $\Xi'_j = (U'_{i,j} : i \geq 1)$, $j = 1, 2, \dots, r_n$
- RQMC: $r_n \geq 2$ i.i.d. **randomizations** to estimate $\mu = \mathbb{E}[h(U)]$

$$\hat{\mu}_{m_n, r_n}^{\text{RQ}} = \frac{1}{r_n} \sum_{j=1}^{r_n} X_{n,j}, \quad \text{where} \quad X_{n,j} = \frac{1}{m_n} \sum_{i=1}^{m_n} h(U'_{i,j})$$

- ▶ $X_{n,1}, X_{n,2}, \dots, X_{n,r_n}$ i.i.d.: estimate $\sigma_{m_n}^2 \equiv \text{Var}[X_{n,1}]$ by

$$\hat{\sigma}_{m_n, r_n}^2 = \frac{1}{r_n - 1} \sum_{j=1}^{r_n} (X_{n,j} - \hat{\mu}_{m_n, r_n}^{\text{RQ}})^2.$$

- Approx γ -level CI for μ

$$I_{m_n, r_n, \gamma}^{\text{RQ}} \equiv \left[\hat{\mu}_{m_n, r_n}^{\text{RQ}} \pm z_\gamma \frac{\hat{\sigma}_{m_n, r_n}}{\sqrt{r_n}} \right]$$

- ▶ $X_{n,1}, X_{n,2}, \dots, X_{n,r_n}$ i.i.d., but distn of each $X_{n,j}$ depends on n : **Triangular array.**

How to choose RQMC Allocation (m_n, r_n) with $m_n \times r_n \approx n$?

- **Heuristic:** For given budget n , choose r_n small and $m_n \approx n/r_n$ large to exploit QMC.
 - ▶ CI: $I_{m_n, r_n, \gamma}^{\text{RQ}} \equiv \left[\hat{\mu}_{m_n, r_n}^{\text{RQ}} \pm z_\gamma \frac{\hat{\sigma}_{m_n, r_n}}{\sqrt{r_n}} \right]$
 - ▶ $r_n = \#$ i.i.d. randomizations
 - ▶ $m_n = \#$ points used from each randomized sequence
- But heuristic lacks rigorous justification.

How to choose RQMC Allocation (m_n, r_n) with $m_n \times r_n \approx n$?

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 - ▶ $r_n = \#$ i.i.d. randomizations
 - ▶ $m_n = \#$ points used from each randomized sequence
- But heuristic lacks rigorous justification.

Question and Goal

- Is this heuristic theoretically valid?
- **Goal:** Obtain sufficient conditions to ensure CLT and AVCI (as $n \approx m_n \times r_n \rightarrow \infty$).

How to choose RQMC Allocation (m_n, r_n) with $m_n \times r_n \approx n$?

- AVCI relies on **CLT**: **not established for many RQMC settings**.
 - ▶ Nested scrambling of digital nets: CLT as $m_n = n \rightarrow \infty$, **fixed** $r_n = 1$ [Loh 2003]
 - ▶ Randomly shifted lattices: **no** CLT as $m_n = n/r_n \rightarrow \infty$, **fixed** $r_n \geq 1$ [L'Ecuyer et al. 2010]

How to choose RQMC Allocation (m_n, r_n) with $m_n \times r_n \approx n$?

- AVCI relies on **CLT**: **not established for many RQMC settings**.
 - ▶ Nested scrambling of digital nets: CLT as $m_n = n \rightarrow \infty$, **fixed** $r_n = 1$ [Loh 2003]
 - ▶ Randomly shifted lattices: **no** CLT as $m_n = n/r_n \rightarrow \infty$, **fixed** $r_n \geq 1$ [L'Ecuyer et al. 2010]
- **Assumption 1**. On
 - ▶ **Assumption 1.A**: $m_n r_n \leq n$ for each $n \geq 1$, with $m_n r_n / n \rightarrow 1$ and $r_n \rightarrow \infty$ as $n \rightarrow \infty$.
 - ▶ **Assumption 1.B**: “Simple allocation”: $(m_n, r_n) = (n^c, n^{1-c})$ for constant $c \in (0, 1)$.
 - ▶ **Main Issue**: How to choose c ?
 - ▶ More general allocation (m_n, r_n) : $r_n \rightarrow \infty$ with $m_n \times r_n \approx n$ as $n \rightarrow \infty$.
- **Assumption 2**. $\sigma_{m_n}^2 \equiv \text{Var}[X_{n,1}] > 0$ for all n large enough.
- Results based on CLT for triangular arrays. For a triangular array, the r_n variables within each row n are independent, but variables in different rows may be dependent. Here:
 $X_{n,1}, X_{n,2}, \dots, X_{n,r_n}$ are i.i.d., each with some distribution F_n .

Theorem 1

If Assumptions 1 and 2 hold, then RQMC estimator $\hat{\mu}_{m_n, r_n}^{RQ}$ satisfies **CLT**

$$\sqrt{\frac{r_n}{\sigma_{m_n}^2}} [\hat{\mu}_{m_n, r_n}^{RQ} - \mu] \Rightarrow \mathcal{N}(0, 1), \quad \text{as } n \rightarrow \infty$$

under either

Lindeberg condition:

$$\frac{\mathbb{E} \left[(X_{n,1} - \mu)^2; |X_{n,1} - \mu| > t \sqrt{r_n \sigma_{m_n}^2} \right]}{\mathbb{E} \left[(X_{n,1} - \mu)^2 \right]} \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad \forall t > 0;$$

or

Lyapounov condition:

$$\frac{\mathbb{E} \left[|X_{n,1} - \mu|^{2+b'} \right]}{r_n^{b'/2} \sigma_{m_n}^{2+b'}} \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad \text{for some } b' > 0.$$

- $\sigma_{m_n}^2 = \mathbb{E}[(X_{n,1} - \mu)^2]$ = variance of estimator $X_{n,1}$ from single randomization of m_n points.

RQMC Asymptotically Valid CI (AVCI)

- Recall **Lyapounov condition**:

$$\frac{\mathbb{E} \left[|X_{n,1} - \mu|^{2+b'} \right]}{r_n^{b'/2} \sigma_{m_n}^{2+b'}} \rightarrow 0, \quad \text{as } n \rightarrow \infty, \text{ for some } b' > 0.$$

- $\hat{\sigma}_{m_n, r_n}^2 = \frac{1}{r_n - 1} \sum_{j=1}^{r_n} (X_{n,j} - \hat{\mu}_{m_n, r_n}^{\text{RQ}})^2$ is unbiased estimator of $\sigma_{m_n}^2 = \text{Var}[X_{n,1}]$.
- Approx. γ -level CI for μ

$$I_{m_n, r_n, \gamma}^{\text{RQ}} = \left[\hat{\mu}_{m_n, r_n}^{\text{RQ}} \pm z_\gamma \frac{\hat{\sigma}_{m_n, r_n}}{\sqrt{r_n}} \right]$$

Theorem 2

If Assumptions 1 and 2 hold, along with Lyapounov condition for $b' = 2$, then **CLT**

$$\sqrt{\frac{r_n}{\hat{\sigma}_{m_n, r_n}^2}} [\hat{\mu}_{m_n, r_n}^{\text{RQ}} - \mu] \Rightarrow \mathcal{N}(0, 1), \quad \text{as } n \rightarrow \infty$$

and **AVCI**

$$P(\mu \in I_{m_n, r_n, \gamma}^{\text{RQ}}) \rightarrow \gamma, \quad \text{as } n \rightarrow \infty.$$

Ensuring CLT (with $\sigma_m^2 > 0$)

Instead of condition (1), impose alternative conditions on integrand h

- **Assumption 3.A:** $V_{\text{HK}}(h) < \infty$ (BVHK)
- **Assumption 3.B:** h is bounded
- **Assumption 3.C:** $\mathbb{E}[|h(U) - \mu|^{2+b}] < \infty$ for some $b > 0$, where $U \sim \mathcal{U}[0, 1]^s$.

Proposition 1

- Under Assumption 3.A, if $\exists \lambda \in (0, 1)$ $r_n^{1-\lambda} \left(\frac{m_n \sigma_{m_n}}{(\ln m_n)^s} \right)^2 \rightarrow \infty$ as $n \rightarrow \infty$,
- Under Assumption 3.B, if $s_n^2 = r_n \sigma_{m_n}^2 \rightarrow \infty$ as $n \rightarrow \infty$,
- Under Assumption 3.C, if $r_n^{b/(2+b)} \sigma_{m_n}^2 \rightarrow \infty$ as $n \rightarrow \infty$,

then the CLT holds

Proposition 2

Assumption 3.A \implies 3.B \implies 3.C, leading reversely strict conditions in the above proposition.

Ensuring AVCI (with $\sigma_m^2 > 0$)

Instead of condition (1), impose alternative conditions on integrand h

- **Assumption 3.A:** $V_{\text{HK}}(h) < \infty$ (BVHK)
- **Assumption 3.B:** h is bounded
- **Assumption 3.C:** $\mathbb{E}[|h(U) - \mu|^{2+b}] < \infty$ for some $b > 0$, where $U \sim \mathcal{U}[0, 1]^s$.

Under Assumption 1.A:

Proposition 3

- Under Assumption 3.A, if $r_n \left(\frac{m_n \sigma_{m_n}}{(\ln m_n)^s} \right)^4 \rightarrow \infty$ as $n \rightarrow \infty$,
- Under Assumption 3.C, if $\frac{\mathbb{E}[|X_{m,1} - \mu|^{2+b'}]}{\sigma_m^{2+b'}} \leq k_1$ with $b' = 2$,

then AVCI holds

Corollaries Ensuring CLT or AVCI under Assumption 1.B

Assumption

For estimator $X_{n,1}$ from **single** randomization of m_n points,

$$\sigma_{m_n} \equiv \sqrt{\text{Var}[X_{n,1}]} \approx \Theta(m_n^{-\alpha_*}) \quad \text{as } m_n \rightarrow \infty, \quad \text{where } \alpha_* \equiv - \lim_{m_n \rightarrow \infty} \frac{\ln(\sigma_{m_n})}{\ln(m_n)} > \frac{1}{2}$$

- $\alpha_* \geq 1$ when $V_{\text{HK}}(h) < \infty$ (BVHK).
- Under Assumption 1.B [$(m_n, r_n) = (n^c, n^{1-c}), c \in (0, 1)$],

$$\text{RMSE} [\hat{\mu}_{m_n, r_n}^{\text{RQ}}] = \frac{\sigma_{m_n}}{\sqrt{r_n}} \approx \Theta(n^{-v(\alpha_*, c)}) \quad \text{as } n \rightarrow \infty, \quad \text{with } v(\alpha_*, c) \equiv c \left[\alpha_* - \frac{1}{2} \right] + \frac{1}{2}.$$

Corollaries Ensuring CLT or AVCI under Assumption 1.B

Assumption

For estimator $X_{n,1}$ from **single** randomization of m_n points,

$$\sigma_{m_n} \equiv \sqrt{\text{Var}[X_{n,1}]} \approx \Theta(m_n^{-\alpha_*}) \quad \text{as } m_n \rightarrow \infty, \quad \text{where } \alpha_* \equiv -\lim_{m_n \rightarrow \infty} \frac{\ln(\sigma_{m_n})}{\ln(m_n)} > \frac{1}{2}$$

- $\alpha_* \geq 1$ when $V_{\text{HK}}(h) < \infty$ (BVHK).
- Under Assumption 1.B [$(m_n, r_n) = (n^c, n^{1-c}), c \in (0, 1)$],

$$\text{RMSE} [\hat{\mu}_{m_n, r_n}^{\text{RQ}}] = \frac{\sigma_{m_n}}{\sqrt{r_n}} \approx \Theta(n^{-\nu(\alpha_*, c)}) \quad \text{as } n \rightarrow \infty, \quad \text{with } \nu(\alpha_*, c) \equiv c \left[\alpha_* - \frac{1}{2} \right] + \frac{1}{2}.$$

- Corollary $k = 1, 2, \dots, 6$: ensure CLT or AVCI under constraint $c < c_k(\alpha_*)$
 - ▶ $c_k(\alpha_*) \in (0, 1]$, sometimes $c_k(\alpha_*) = 1$.
 - ▶ Optimal RMSE: take $c < c_k(\alpha_*)$ with $c \approx c_k(\alpha_*)$

$$\text{RMSE} [\hat{\mu}_{m_n, r_n}^{\text{RQ}}] \approx \Theta(n^{-\nu_k(\alpha_*)}) \quad \text{as } n \rightarrow \infty, \quad \text{with } \nu_k(\alpha_*, c) \equiv c_k(\alpha_*) \left[\alpha_* - \frac{1}{2} \right] + \frac{1}{2} > \frac{1}{2}$$

\implies **RQMC better than MC.**

Corollaries Ensuring CLT or AVCI

Corollary 1

Suppose that Assumptions 1 and 2 hold, and $\exists b' > 0$ and $k_1 \in (0, \infty)$ such that

$$\frac{\mathbb{E}\left[|X_{n,1}-\mu|^{2+b'}\right]}{\sigma_{m_n}^{2+b'}} \leq k_1 \quad \forall m_n \text{ sufficiently large.} \quad (1)$$

Then **CLT** holds for allocation $(m_n, r_n) = (n^c, n^{1-c})$ with any

$$c < 1 \equiv c_1(\alpha_*),$$

and optimal $RMSE \approx \Theta(n^{-v_1(\alpha_*)})$ as $n \rightarrow \infty$ with

$$v_1(\alpha_*) \equiv \alpha_*.$$

If (1) holds for $b' = 2$, then **AVCI** holds for $c < c_1(\alpha_*)$, and RMSE rate exponent is $v_1(\alpha_*)$.

- CLT/AVCI with $r_n \rightarrow \infty$ since $c < 1$.
- RMSE rate same as $\sigma_{m_n} \approx \Theta(m_n^{-\alpha_*})$ for **single** randomization of $m_n = n$ points.

Corollaries Ensuring CLT or AVCI: Tradeoffs

Impose again alternative conditions on integrand h

- **Assumption 3.A:** $V_{\text{HK}}(h) < \infty$ (BVHK)
- **Assumption 3.B:** h is bounded
- **Assumption 3.C:** $\mathbb{E}[|h(U) - \mu|^{2+b}] < \infty$ for some $b > 0$, where $U \sim \mathcal{U}[0, 1]^s$.

Proposition 4

- *Assumption 3.A \implies 3.B \implies 3.C,
leading to successively smaller $c_k(\alpha_*)$ for Corollaries k (next slide).*
- *Under Assumption 3.x, for $c_k(\alpha_*)$ ensuring CLT and $c_{k'}(\alpha_*)$ ensuring AVCI,*

$$c_k(\alpha_*) \geq c_{k'}(\alpha_*) \quad (\text{often } >).$$

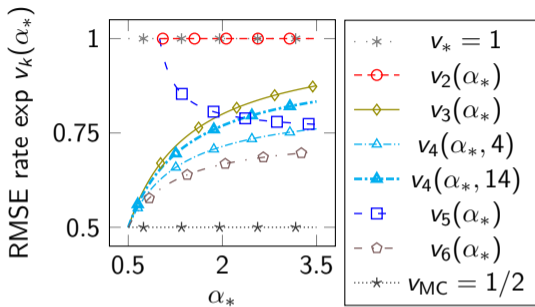
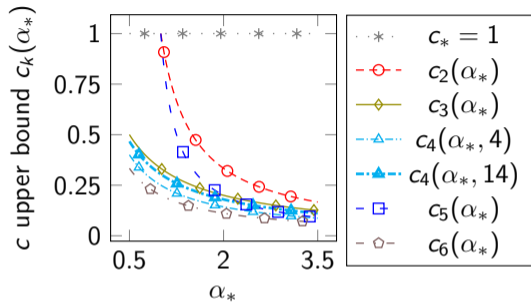
- **Assumption 1:** $(m_n, r_n) = (n^c, n^{1-c})$, $c \in (0, 1)$
- Corollary k : $c < c_k(\alpha_*)$
- $\sigma_{m_n} \approx \Theta(m_n^{-\alpha_*})$, $\alpha_* > 1/2$

Corollaries CLT or AVCI: Tradeoffs

Cor. k	Ensures	Assumption on h	c upper bd $c_k(\alpha_*)$	RMSE rate exp $v_k(\alpha_*)$
2	CLT	3.A (BVHK)	$\frac{1}{2\alpha_* - 1} >$	$1 >$
3	CLT	3.B (h bdd)	$\frac{1}{2\alpha_* + 1} >$	$\frac{2\alpha_*}{2\alpha_* + 1} >$
4	CLT	3.C ($b > 0$)	$\frac{1}{2\alpha_*(1+\frac{2}{b})+1} \in (0, \frac{1}{2})$	$\frac{2\alpha_*(1+\frac{1}{b})}{2\alpha_*(1+\frac{2}{b})+1} > \frac{1}{2}$
5	AVCI	3.A (BVHK)	$\frac{1}{4\alpha_* - 3} >$	$\frac{3\alpha_* - 2}{4\alpha_* - 3} >$
6	AVCI	3.C ($b = 2$)	$\frac{1}{4\alpha_* + 1} \in (0, \frac{1}{3})$	$\frac{3\alpha_*}{4\alpha_* + 1} > \frac{1}{2}$

- 3.A \implies 3.B \implies 3.C
 - ▶ **Assumption 3.A:** $V_{\text{HK}}(h) < \infty$ (BVHK: $\implies \alpha_* \geq 1$)
 - ▶ **Assumption 3.B:** h is bounded.
 - ▶ **Assumption 3.C:** $\mathbb{E}[|h(U) - \mu|^{2+b}] < \infty$ for some $b > 0$, where $U \sim \mathcal{U}[0, 1]^s$.
- Comparisons for fixed $\alpha_* > 1/2$
 - ▶ $(m_n, r_n) = (n^c, n^{1-c})$, $c < c_k(\alpha_*)$, $\text{opt RMSE} \approx \Theta(n^{-v_k(\alpha_*)})$.

Conditions Ensuring CLT or AVCI: Tradeoffs



- All $c_k(\alpha_*) \downarrow$ as $\alpha_* \uparrow$
 - ▶ Corollary k : $c < c_k(\alpha_*)$ in $(m_n, r_n) = (n^c, n^{1-c})$.
 - ▶ $\sigma_{m_n} \approx \Theta(m_n^{-\alpha_*})$, $\alpha_* > 1/2$ (≥ 1 BVHK)
- Most $v_k(\alpha_*) \uparrow$ as $\alpha_* \uparrow$
 - ▶ Optimal RMSE $\approx \Theta(n^{-v_k(\alpha_*)})$, $n \rightarrow \infty$
 - ▶ Larger α_* usually yields better RQMC performance.

Concluding Remarks

- Sufficient conditions for RQMC CLT and AVCI for $\mu = \mathbb{E}[h(U)]$
 - ▶ Corollary k : $c < c_k(\alpha_*)$ for allocation $(m_n, r_n) = (n^c, n^{1-c})$.
 - ▶ Tradeoffs: Weaker conditions on integrand h
→ stronger restrictions on c .
 - ▶ AVCI imposes (often strictly) stronger restrictions on c than CLT.
 - ▶ RMSE: RQMC outperforms MC.
- Current work
 - ▶ Estimate $c_k(\alpha_*)$.
 - ▶ Weaker conditions.
 - ▶ CLT and AVCI for biased estimators, e.g., quantiles.
 - ▶ Alternative CIs: Bootstrap- t , etc.

Thank you!