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Lagrange dynamics of lumped-mass multibody models of overhead cranes in 2D and 3D operational spaces

Bernard Brogliato*

February 3, 2023

Abstract

In this report the multibody Lagrange dynamics of overhead cranes is analyzed in some particular cases, for 2-dimensional and 3-dimensional operational spaces. Cables are modeled with lumped-mass chains.

1 Lagrange Dynamics in Finite Dimensions

Let a mechanical system with n degrees of freedom be given, let $q \in \mathbb{R}^n$ be the vector of its generalised coordinates. The Lagrange equations are given as:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q \quad (1)$$

where $L(q, \dot{q})$ is the Lagrangian function, $L(q, \dot{q}) = T(q, \dot{q}) - U(q)$, $T(q, \dot{q})$ is the system's kinetic energy, $U(q)$ is its potential energy, Q is the vector of generalized forces (that perform work on the coordinate q). It is assumed that q is minimal, which means that there are no redundant coordinates, and (1) describes the system's dynamics without the need to add any equality (bilateral) holonomic constraints. Usually it is much easier to write the dynamics with redundant coordinates, and adding bilateral constraints that represent the joints. As is well-known [3], the Lagrange dynamics can be rewritten as:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = Q \quad (2)$$

where: $M(q) = M^\top(q) \succ 0$ is the mass matrix, or inertia matrix (it may be just $\succcurlyeq 0$), $C(q, \dot{q})\dot{q}$ contains centrifugal and Coriolis nonlinear generalised forces, $g(q) = \frac{\partial U}{\partial q}$ is the vector of forces that derive from a potential (gravity, elasticity, *etc*). The next property is also well-known (since at least the 1980s in the Automatic Control and the Robotics scientific communities):

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Property 1 The matrix $C(q, \dot{q})$, when written with the so-called Christoffel's symbols¹, verifies

$$\dot{M}(q, \dot{q}) = C(q, \dot{q}) + C^\top(q, \dot{q}) \Leftrightarrow \dot{M}(q, \dot{q}) - 2C(q, \dot{q}) \text{ is skew-symmetric,} \quad (3)$$

where $\dot{M}(q, \dot{q})$ is the matrix obtained from $M(q)$ by differentiating each one of its entries with respect to time.

Property (3) is a key property for the design of passivity-based controllers [3], it implies passivity but it is not implied by passivity. Another fundamental property is that the system (1), seen as an input/output operator $Q \mapsto \dot{q}$, and under a boundedness condition on the potential energy, is passive, *i.e.*, $V(q(t), \dot{q}(t)) - V(q(0), \dot{q}(0)) \leq \int_0^t Q^\top(s) \dot{q}(s) ds$ along the trajectories of the system, and where $V(q, \dot{q}) = T(q, \dot{q}) + U(q)$ is the total mechanical energy.

Remark 1 In this manuscript we focus mainly on the calculation of the mass matrix $M(q)$. The Coriolis/centrifugal forces can be deduced from it using the classical Christoffel's symbols as

$$C(q, \dot{q}) = \left(\sum_{k=1}^n \Gamma_{ijk} \dot{q}_k \right)_{ij}, \quad (4)$$

where [3, Lemma 6.16]

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial m_{ij}}{\partial q_k} + \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{kj}}{\partial q_i} \right). \quad (5)$$

Then we have $\dot{m}_{ij}(q, \dot{q}) = c_{ij}(q, \dot{q}) + c_{ji}(q, \dot{q})$, *i.e.*, (3) holds. In spite of the fact that such calculations may not be straightforward in general, they are doable and this is what we shall use in the sequel. Notice that each symbol requires the calculation of three partial derivatives, and each component $c_{ij}(q, \dot{q})$ requires the calculation of n symbols Γ_{ijk} . The $n \times n$ matrix $C(q, \dot{q})$ thus requires the calculation of $3n^3$ partial derivatives of the form $\frac{\partial m_{ij}}{\partial q_k}$.

2 Overhead Cranes with Rigid Links and Varying Lengths in 2D Operating Space

Let us consider a system as in Figures 1 or 2. This is an overhead crane with a finite-dimensional approximation of a cable, made of N rigid links (a series kinematic chain with pivot joints, $N = 3$ in the figure). Such multibody cable model is sometimes used [5]. The basic assumptions are:

1. Each rigid link is assumed here to possess a varying length $l_i(t)$. Indeed links may possess some longitudinal stiffness (see section 7), or be assumed perfectly rigid but with some winch mechanism (it could be assumed that at least l_1 is acted upon by a control input, realised with some winding devices at A_0).

¹Indeed there is a general not a unique way to write the matrix $C(q, \dot{q})$ in $C(q, \dot{q})\dot{q}$, just like $axy + bx^2$ can be written as $(ax, bx) \begin{pmatrix} y \\ x \end{pmatrix}$ or as $(ay, bx) \begin{pmatrix} x \\ x \end{pmatrix}$.

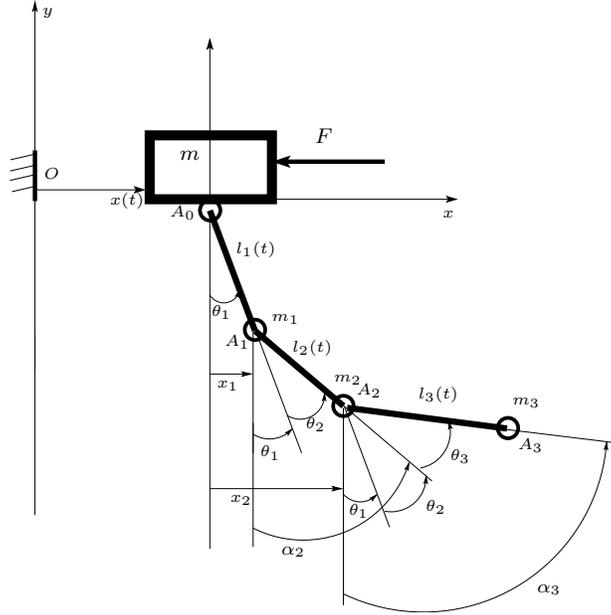


Figure 1: Overhead crane with pendulum-like cable (controlled links' lengths).

2. Links are massless, the masses are lumped at the joints, that one may name nodes (points A_i , $1 \leq i \leq N$).
3. The trolley has mass $m > 0$ and is acted upon by a force $F(t)$, that may be a control input.
4. The payload has mass $m_N > 0$, each node has a mass $m_i \geq 0$.
5. Since cables, contrarily to beams, usually have negligible bending stiffness, there is no angular springs at the joints, or these stiffnesses are very small.
6. Sagging (static) effects are neglected, contrarily to cable models for cable-cars in which they are a prominent effect.
7. Sometimes it may be justified to assume that the payload mass m_N is much bigger than the nodes A_i masses (said otherwise, the cable's mass is much smaller than the payload mass), so that the latter may be taken equal to zero in the dynamics [7, 11]. This is particularly true for cranes carrying very large payloads of several tenths of tons.

Let us derive kinetic energies and potential energy. The Lagrange equations cannot be written explicitly in the general case (*i.e.*, $N \geq 2$) since the many couplings make it quite cumbersome to write expressions for the Coriolis and centrifugal nonlinearities. This may be the reason why such pendulum-like models are limited to $N = 1$ and $N = 2$ in the Automatic Control literature. However some properties can be deduced from energies. In the general case with varying lengths, the system has $2N + 1$ degrees of freedom, and the chosen generalised coordinates vector are $q = (x, \theta_1, \dots, \theta_N, l_1, \dots, l_N)^\top \in \mathbb{R}^{2N+1}$ or $z = (x, \alpha_1, \dots, \alpha_N, l_1, \dots, l_N)^\top \in \mathbb{R}^{2N+1}$.

3 Kinetic energies

- Kinetic energy of the trolley:

$$T_{trol} = \frac{1}{2}m\dot{x}^2. \quad (6)$$

so that

$$\frac{d}{dt} \frac{\partial T_{trol}}{\partial \dot{x}} = m\ddot{x}. \quad (7)$$

- Translational kinetic energy of each point-mass m_i : let the translational velocity of point A_i be denoted as $V_{A_i} \in \mathbb{R}^3$ in the (O, x, y, z) Galilean frame. For instance, $OA_1 = \begin{pmatrix} x + l_1 \sin(\theta_1) \\ -l_1 \cos(\theta_1) \\ 0 \end{pmatrix}$ where $OA_0 = (x(t), 0, 0)^\top$. Thus

$$\begin{aligned} V_{A_1} &= \begin{pmatrix} \dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1) \\ l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1) \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix} + l_1 \dot{\theta}_1 \begin{pmatrix} \cos(\theta_1) \\ \sin(\theta_1) \\ 0 \end{pmatrix} + \dot{l}_1 \begin{pmatrix} \sin(\theta_1) \\ -\cos(\theta_1) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix} + l_1 \dot{\alpha}_1 \begin{pmatrix} \cos(\alpha_1) \\ \sin(\alpha_1) \\ 0 \end{pmatrix} + \dot{l}_1 \begin{pmatrix} \sin(\alpha_1) \\ -\cos(\alpha_1) \\ 0 \end{pmatrix}, \end{aligned} \quad (8)$$

and $OA_2 = \begin{pmatrix} x + l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \\ -l_1 \cos(\theta_1) - l_2 \cos(\theta_1 + \theta_2) \\ 0 \end{pmatrix} = \begin{pmatrix} x + l_1 \sin(\alpha_1) + l_2 \sin(\alpha_2) \\ -l_1 \cos(\alpha_1) - l_2 \cos(\alpha_2) \\ 0 \end{pmatrix}$, so that:

$$\begin{aligned} V_{A_2} &= \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix} + l_1 \dot{\theta}_1 \begin{pmatrix} \cos(\theta_1) \\ \sin(\theta_1) \\ 0 \end{pmatrix} + \dot{l}_1 \begin{pmatrix} \sin(\theta_1) \\ -\cos(\theta_1) \\ 0 \end{pmatrix} + l_2 (\dot{\theta}_1 + \dot{\theta}_2) \begin{pmatrix} \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \\ 0 \end{pmatrix} \\ &\quad + \dot{l}_2 \begin{pmatrix} \sin(\theta_1 + \theta_2) \\ -\cos(\theta_1 + \theta_2) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix} + l_1 \dot{\alpha}_1 \begin{pmatrix} \cos(\alpha_1) \\ \sin(\alpha_1) \\ 0 \end{pmatrix} + \dot{l}_1 \begin{pmatrix} \sin(\alpha_1) \\ -\cos(\alpha_1) \\ 0 \end{pmatrix} + l_2 \dot{\alpha}_2 \begin{pmatrix} \cos(\alpha_2) \\ \sin(\alpha_2) \\ 0 \end{pmatrix} \\ &\quad + \dot{l}_2 \begin{pmatrix} \sin(\alpha_2) \\ -\cos(\alpha_2) \\ 0 \end{pmatrix} \end{aligned} \quad (9)$$

Similarly:

$$\begin{aligned}
V_{A_3} &= \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix} + l_1 \dot{\theta}_1 \begin{pmatrix} \cos(\theta_1) \\ \sin(\theta_1) \\ 0 \end{pmatrix} + \dot{l}_1 \begin{pmatrix} \sin(\theta_1) \\ -\cos(\theta_1) \\ 0 \end{pmatrix} + l_2 (\dot{\theta}_1 + \dot{\theta}_2) \begin{pmatrix} \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \\ 0 \end{pmatrix} \\
&+ \dot{l}_2 \begin{pmatrix} \sin(\theta_1 + \theta_2) \\ -\cos(\theta_1 + \theta_2) \\ 0 \end{pmatrix} + l_3 (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) \begin{pmatrix} \cos(\theta_1 + \theta_2 + \theta_3) \\ \sin(\theta_1 + \theta_2 + \theta_3) \\ 0 \end{pmatrix} + \dot{l}_3 \begin{pmatrix} \sin(\theta_1 + \theta_2 + \theta_3) \\ -\cos(\theta_1 + \theta_2 + \theta_3) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix} + l_1 \dot{\alpha}_1 \begin{pmatrix} \cos(\alpha_1) \\ \sin(\alpha_1) \\ 0 \end{pmatrix} + \dot{l}_1 \begin{pmatrix} \sin(\alpha_1) \\ -\cos(\alpha_1) \\ 0 \end{pmatrix} + l_2 \dot{\alpha}_2 \begin{pmatrix} \cos(\alpha_2) \\ \sin(\alpha_2) \\ 0 \end{pmatrix} \\
&+ \dot{l}_2 \begin{pmatrix} \sin(\alpha_2) \\ -\cos(\alpha_2) \\ 0 \end{pmatrix} + l_3 \dot{\alpha}_3 \begin{pmatrix} \cos(\alpha_3) \\ \sin(\alpha_3) \\ 0 \end{pmatrix} + \dot{l}_3 \begin{pmatrix} \sin(\alpha_3) \\ -\cos(\alpha_3) \\ 0 \end{pmatrix}
\end{aligned} \tag{10}$$

Clearly there is a recursive structure for linear velocities of joints nodes A_i :

$$\begin{aligned}
V_{A_i} &= \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sum_{k=1}^i l_k \left(\sum_{j=1}^k \dot{\theta}_j \right) \cos \left(\sum_{j=1}^k \theta_j \right) \\ \sum_{k=1}^i l_k \left(\sum_{j=1}^k \dot{\theta}_j \right) \sin \left(\sum_{j=1}^k \theta_j \right) \\ 0 \end{pmatrix} + \begin{pmatrix} \sum_{k=1}^i \dot{l}_k \sin \left(\sum_{j=1}^k \theta_j \right) \\ -\sum_{k=1}^i \dot{l}_k \cos \left(\sum_{j=1}^k \theta_j \right) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) \\ \sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) \\ 0 \end{pmatrix} + \begin{pmatrix} \sum_{k=1}^i \dot{l}_k \sin(\alpha_k) \\ -\sum_{k=1}^i \dot{l}_k \cos(\alpha_k) \\ 0 \end{pmatrix}
\end{aligned} \tag{11}$$

The translational kinetic energy of each point mass A_i is $T_{V_{A_i}} = \frac{1}{2} m_i V_{A_i}^\top V_{A_i}$:

$$\begin{aligned}
T_{V_{A_i}} &= \frac{1}{2} m_i \left(\dot{x} + \sum_{k=1}^i l_k \left(\sum_{j=1}^k \dot{\theta}_j \right) \cos \left(\sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left(\sum_{j=1}^k \theta_j \right) \right)^2 \\
&+ \frac{1}{2} m_i \left(\sum_{k=1}^i l_k \left(\sum_{j=1}^k \dot{\theta}_j \right) \sin \left(\sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left(\sum_{j=1}^k \theta_j \right) \right)^2 \\
&= \frac{1}{2} m_i \left(\dot{x} + \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right)^2 \\
&+ \frac{1}{2} m_i \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right)^2
\end{aligned} \tag{12}$$

and the total translational kinetic energy of all mass-points is:

$$\begin{aligned}
T_{trans} &= \frac{1}{2} \sum_{i=1}^N m_i \left(\dot{x} + \sum_{k=1}^i l_k \left(\sum_{j=1}^k \dot{\theta}_j \right) \cos \left(\sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left(\sum_{j=1}^k \theta_j \right) \right)^2 \\
&+ \frac{1}{2} \sum_{i=1}^N m_i \left(\sum_{k=1}^i l_k \left(\sum_{j=1}^k \dot{\theta}_j \right) \sin \left(\sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left(\sum_{j=1}^k \theta_j \right) \right)^2 \\
&= \frac{1}{2} \sum_{i=1}^N m_i \left(\dot{x} + \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right)^2 \\
&+ \frac{1}{2} \sum_{i=1}^N m_i \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right)^2
\end{aligned} \tag{13}$$

Therefore the total kinetic energy of the system is

$$T(q, \dot{q}) = T_{trol} + T_{trans} = T(z, \dot{z}). \tag{14}$$

The coordinate z may be in general easier to work with, since it yields simpler expression of the kinetic energy.

4 Lagrange Dynamics of Overhead Cranes in 2D Operating Spaces

In this manuscript the mass matrix is denoted as:

$$M(z) = \begin{pmatrix} M_{x\alpha}(z) & M_{x\alpha l}(z) \\ M_{l\alpha}(z) & M_{ll}(z) \end{pmatrix} \in \mathbb{R}^{(1+2N) \times (1+2N)}, \tag{15}$$

with $M_{x\alpha}(z) \in \mathbb{R}^{(1+N) \times (1+N)}$, $M_{ll}(z) \in \mathbb{R}^{N \times N}$, $M_{x\alpha l}(z) = M_{l\alpha}^\top(z) \in \mathbb{R}^{(1+N) \times N}$. It is also possible to go a step further with:

$$M_{x\alpha}(z) = \begin{pmatrix} M_{xx}(z) & \bar{M}_{x\alpha}(z) \\ \bar{M}_{x\alpha}^\top(z) & M_{\alpha\alpha}(z) \end{pmatrix} \tag{16}$$

with $M_{xx}(z) \in \mathbb{R}$, $\bar{M}_{x\alpha}(z) \in \mathbb{R}^{1 \times N}$, $M_{\alpha\alpha}(z) \in \mathbb{R}^{N \times N}$.

This expression of the mass matrix corresponds to the choice of the generalized coordinates as $z = (x, \alpha_1, \dots, \alpha_N, l_1, \dots, l_N)^\top \in \mathbb{R}^{2N+1}$. Other choices can be made, like those splitting the coordinates into actuated q_a and nonactuated ones q_{na} , which is a classical way of doing in the Control literature [16].

4.1 Calculation of $\frac{d}{dt} \frac{\partial T}{\partial \dot{z}}$

This is the term that allows us to derive the mass matrix.

4.1.1 Calculation of $\frac{d}{dt} \frac{\partial T}{\partial \dot{x}}$

First we compute $\frac{\partial T}{\partial \dot{x}} = m\dot{x} + \frac{\partial T_{trans}}{\partial \dot{x}}$:

$$\begin{aligned} \frac{\partial T_{trans}}{\partial \dot{x}} &= \frac{1}{2} \frac{\partial}{\partial \dot{x}} \sum_{i=1}^N m_i \left(\dot{x} + \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right)^2 \\ &= \frac{1}{2} \sum_{i=1}^N m_i \frac{\partial}{\partial \dot{x}} \left(\dot{x} + \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right)^2 \\ &= \sum_{i=1}^N m_i \left(\dot{x} + \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) \end{aligned} \quad (17)$$

Therefore:

$$\begin{aligned} \frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{x}} &= \frac{d}{dt} \sum_{i=1}^N m_i \left(\dot{x} + \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) \\ &= \sum_{i=1}^N m_i \left(\ddot{x} + \sum_{k=1}^i l_k \ddot{\alpha}_k \cos(\alpha_k) + \ddot{l}_k \sin(\alpha_k) \right) \\ &\quad + \sum_{i=1}^N m_i \left(\sum_{k=1}^i \dot{l}_k \dot{\alpha}_k \cos(\alpha_k) - l_k \dot{\alpha}_k^2 \sin(\alpha_k) + \dot{l}_k \dot{\alpha}_k \cos(\alpha_k) \right) \end{aligned} \quad (18)$$

The first row and first column (by symmetry) of $M(z)$ are deduced as follows:

$$m_{11}(z) = m + \sum_{i=1}^N m_i \quad (19)$$

and for all $2 \leq k \leq N+1$ (components which multiply $\ddot{\alpha}_k$):

$$m_{1k}(z) = m_{k1}(z) = \left(\sum_{i=k-1}^N m_i \right) l_{k-1} \cos(\alpha_{k-1}) \quad (20)$$

and for all $N+2 \leq k \leq 2N+1$ (components which multiply \ddot{l}_k):

$$m_{1k}(z) = m_{k1}(z) = \left(\sum_{i=k-3}^N m_i \right) \sin(\alpha_{k-3}) \quad (21)$$

Thus this determines the submatrices $M_{xx}(z)$ ($= m_{11}$) and $\bar{M}_{x\alpha}(z)$ (in (20)). The N components in (21) are the first row of the matrix $M_{x\alpha l}(z)$.

4.1.2 Calculation of $\frac{d}{dt} \frac{\partial T}{\partial \dot{\alpha}_n}$, $1 \leq n \leq N$

This corresponds to the row $n+1$ of $M(z)$. The row and column $n+1$ are calculated as follows. First let us compute $\frac{\partial T}{\partial \dot{\alpha}_n}$, $1 \leq n \leq N$:

$$\begin{aligned}
\frac{\partial T}{\partial \dot{\alpha}_n} &= \sum_{i=1}^N m_i \left(\dot{x} + \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) \frac{\partial T}{\partial \dot{\alpha}_n} \left(\sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) \\
&+ \sum_{i=1}^N m_i \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right) \frac{\partial T}{\partial \dot{\alpha}_n} \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right) \\
&= \sum_{i=n}^N m_i \left(\dot{x} + \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) \frac{\partial T}{\partial \dot{\alpha}_n} \left(\sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) \\
&+ \sum_{i=n}^N m_i \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right) \frac{\partial T}{\partial \dot{\alpha}_n} \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right) \\
&= \sum_{i=n}^N m_i \left(\dot{x} + \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) \frac{\partial T}{\partial \dot{\alpha}_n} \left(\sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) \right) \\
&+ \sum_{i=n}^N m_i \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right) \frac{\partial T}{\partial \dot{\alpha}_n} \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) \right) \\
&= \sum_{i=n}^N m_i \left(\dot{x} + \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) l_n \cos(\alpha_n) \\
&+ \sum_{i=n}^N m_i \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right) l_n \sin(\alpha_n) \\
&= \sum_{i=n}^N m_i \dot{x} l_n \cos(\alpha_n) + \sum_{i=n}^N m_i \sum_{k=1}^i \left(l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) l_n \cos(\alpha_n) \\
&+ \left(l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right) l_n \sin(\alpha_n) \\
&= \sum_{i=n}^N m_i \dot{x} l_n \cos(\alpha_n) + \sum_{i=n}^N m_i \sum_{k=1}^i \left(l_k l_n \dot{\alpha}_k \cos(\alpha_k - \alpha_n) + \dot{l}_k l_n \sin(\alpha_k - \alpha_n) \right)
\end{aligned} \tag{22}$$

It is deduced the components of the first column for $1 \leq n \leq N$ (first components of the rows 2 to $N + 1$ of $M(z)$, factors of \dot{x} in (22)):

$$m_{(n+1)1}(z) = m_{1(n+1)}(z) = \left(\sum_{i=n}^N m_i \right) l_n \cos(\alpha_n). \tag{23}$$

Let us compute now the components 2 to $N + 1$ of the row $n + 1$, *i.e.*, for $2 \leq j \leq N + 1$. The factor of $\dot{\alpha}_1$ in (22) is:

$$m_{(n+1)2}(z) = \left(\sum_{i=n}^N m_i \right) l_1 l_n \cos(\alpha_1 - \alpha_n) \tag{24}$$

The factor of $\dot{\alpha}_2$ in (22) is:

$$m_{(n+1)3}(z) = \left(\sum_{i=\max(2,n)}^n m_i \right) l_n l_2 \cos(\alpha_n - \alpha_2) \tag{25}$$

The factor of $\dot{\alpha}_3$ in (22) is:

$$m_{(n+1)4}(z) = \left(\sum_{i=\max(3,n)}^N m_i \right) l_n l_3 \cos(\alpha_n - \alpha_3) \tag{26}$$

Thus the factor of $\dot{\alpha}_j$ in (22), for $1 \leq j \leq N$, is equal to:

$$m_{(n+1)(j+1)}(z) = \left(\sum_{i=\max(j,n)}^N m_i \right) l_n l_j \cos(\alpha_n - \alpha_j) = m_{(j+1)(n+1)}(z) \quad (27)$$

In particular when $j = n$ we get (for $1 \leq n \leq N$):

$$m_{(n+1)(n+1)}(z) = \left(\sum_{i=n}^N m_i \right) l_n^2 \quad (28)$$

These are the diagonal terms of $M(z)$, factors of $\ddot{\alpha}_n$ in the Lagrange dynamics.

\rightsquigarrow Therefore from symmetry we have for the moment calculated the principal submatrix of $M(z)$ of size $(N+1) \times (N+1)$.

Let us calculate the factors of \dot{l}_j , $1 \leq j \leq N$, in (22), which will furnish the last N components of the rows 2 to $N+1$. The factor of \dot{l}_1 in the row $n+1$ is:

$$m_{(n+1)(N+2)}(z) = \left(\sum_{i=n}^N m_i \right) l_n \sin(\alpha_1 - \alpha_n) \quad (29)$$

The factor of \dot{l}_2 in the row $n+1$ is:

$$m_{(n+1)(N+3)}(z) = \left(\sum_{i=\max(n,2)}^N m_i \right) l_n \sin(\alpha_2 - \alpha_n) \quad (30)$$

The factor of \dot{l}_3 in the row $n+1$ is:

$$m_{(n+1)(N+4)}(z) = \left(\sum_{i=\max(n,3)}^N m_i \right) l_n \sin(\alpha_3 - \alpha_n) \quad (31)$$

Thus the factor of \dot{l}_j in the row $n+1$ is:

$$m_{(n+1)(N+j+1)}(z) = \left(\sum_{i=\max(n,j)}^N m_i \right) l_n \sin(\alpha_j - \alpha_n) \quad (32)$$

In particular for $n = j$ we get $m_{(n+1)(N+n+1)}(z) = 0$.

\rightsquigarrow Therefore the full characterization of the rows 2 to $N+1$ of $M(z)$ is obtained. By symmetry the full characterization of the columns 2 to $N+1$ of $M(z)$ is obtained.

The only submatrix which remains to be calculated is $M_{ll}(z)$.

4.1.3 Calculation of $\frac{d}{dt} \frac{\partial T}{\partial \dot{l}_n}$, $1 \leq n \leq N$

Let us now calculate the factors of \dot{l}_j in (22), to determine the components of the principal submatrix $M_{ll}(z)$. Let us compute first $\frac{\partial T}{\partial \dot{l}_n}$ using (13):

$$\begin{aligned}
\frac{\partial T}{\partial \dot{l}_n} &= \sum_{i=1}^N m_i \left(\dot{x} + \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) \frac{\partial}{\partial \dot{l}_n} \left(\dot{x} + \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) \\
&+ \sum_{i=1}^N m_i \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right) \frac{\partial}{\partial \dot{l}_n} \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right) \\
&= \sum_{i=1}^N m_i \left(\dot{x} + \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) \frac{\partial}{\partial \dot{l}_n} \left(\sum_{k=1}^i \dot{l}_k \sin(\alpha_k) \right) \\
&+ \sum_{i=1}^N m_i \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right) \frac{\partial}{\partial \dot{l}_n} \left(\sum_{k=1}^i -\dot{l}_k \cos(\alpha_k) \right) \\
&= \sum_{i=n}^N m_i \left(\dot{x} + \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) \sin(\alpha_n) \\
&- \sum_{i=n}^N m_i \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right) \cos(\alpha_n) \\
&= \sum_{i=n}^N m_i \dot{x} \sin(\alpha_n) + \sum_{i=n}^N m_i \left(\sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) \sin(\alpha_n) \\
&- \sum_{i=n}^N m_i \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right) \cos(\alpha_n) \\
&= \sum_{i=n}^N m_i \dot{x} \sin(\alpha_n) + \sum_{i=n}^N m_i \left(\left(\sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) \sin(\alpha_n) \right. \\
&- \left. \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right) \cos(\alpha_n) \right) \\
&= \sum_{i=n}^N m_i \dot{x} \sin(\alpha_n) + \sum_{i=n}^N m_i \left(\sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) \sin(\alpha_n) + \dot{l}_k \sin(\alpha_k) \sin(\alpha_n) \right. \\
&- \left. l_k \dot{\alpha}_k \sin(\alpha_k) \cos(\alpha_n) + \dot{l}_k \cos(\alpha_k) \cos(\alpha_n) \right) \\
&= \sum_{i=n}^N m_i \dot{x} \sin(\alpha_n) + \sum_{i=n}^N m_i \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_n - \alpha_k) + \dot{l}_k \cos(\alpha_n - \alpha_k) \right)
\end{aligned} \tag{33}$$

This allows us to calculate the components of the row $N + 1 + n$. Therefore the factor of \ddot{x} in this row is:

$$m_{(N+1+n)1}(z) = \left(\sum_{i=n}^N m_i \right) \sin(\alpha_n) \tag{34}$$

The factor of $\ddot{\alpha}_j$, $1 \leq j \leq N$, in this row is:

$$m_{(N+1+n)(j+1)}(z) = m_{(j+1)(N+1+n)}(z) = \left(\sum_{i=\max(n,j)}^N m_i \right) l_j \sin(\alpha_n - \alpha_j) \quad (35)$$

In particular we see that $m_{(N+1+n)(n+1)}(z) = m_{(n+1)(N+1+n)}(z) = 0$. The factor of \ddot{l}_j , $1 \leq j \leq N$, in this row is:

$$m_{(N+1+n)(N+j+1)}(z) = m_{(N+j+1)(N+1+n)}(z) = \left(\sum_{i=\max(j,n)}^N m_i \right) \cos(\alpha_n - \alpha_j) \quad (36)$$

These are the components of the submatrix $M_{ll}(z)$.

4.2 Characterization of $M(z)$ and its submatrices

We have the following:

- $M_{xx}(z) = m_{11}(z) = m + \sum_{i=1}^N m_i$
- The components of $M_{ll}(z)$ are

$$m_{(N+1+n)(N+j+1)}(z) = m_{(N+j+1)(N+1+n)}(z) = \left(\sum_{i=\max(j,n)}^N m_i \right) \cos(\alpha_n - \alpha_j)$$

$$1 \leq j \leq N, 1 \leq n \leq N.$$

- The components of $M_{\alpha\alpha}(z)$ are $m_{(n+1)(j+1)}(z) = \left(\sum_{i=\max(j,n)}^N m_i \right) l_n l_j \cos(\alpha_n - \alpha_j) = m_{(j+1)(n+1)}(z)$, $1 \leq j \leq N$, $1 \leq n \leq N$.
- The components of $\bar{M}_{x\alpha}(z)$ are: for all $2 \leq k \leq N+1$ (components which multiply $\ddot{\alpha}_k$): $m_{1k}(z) = m_{k1}(z) = \left(\sum_{i=k-1}^N m_i \right) l_{k-1} \cos(\alpha_{k-1})$.
- The components of $M_{x\alpha l}$ are: for all $N+2 \leq k \leq 2N+1$, $m_{1k}(z) = m_{k1}(z) = \left(\sum_{i=k-3}^N m_i \right) \sin(\alpha_{k-3})$; and for $1 \leq n \leq N$, $1 \leq j \leq N$: $m_{(n+1)(N+j+1)}(z) = \left(\sum_{i=\max(n,j)}^N m_i \right) l_n \sin(\alpha_j - \alpha_n)$.

Some properties can be deduced:

1. The diagonal components of $M(z)$ are always all constant positive.
2. The components of $M_{x\alpha l}$ are small for small angles (neighborhood of the cable's vertical posture) with $\sin(\alpha_k) \approx \alpha_k$. Hence the mass matrix possesses an almost-diagonal structure for small angles, in the limit: $M(z) = \begin{pmatrix} M_{x\alpha}(z) & 0 \\ 0 & M_{ll}(z) \end{pmatrix} \in \mathbb{R}^{(1+2N) \times (1+2N)}$.
When the links are aligned (equal angles α_i) the inertial couplings between the angles and the lengths vanish, with $\sin(\alpha_j - \alpha_n) \approx \alpha_j - \alpha_n$.

3. The components of $M_{\alpha\alpha}(z)$ and of $M_{ll}(z)$ are constant positive for small angles.
4. Assume that $M(z) = M(z)^\top \succ 0$ for all z . Then from the Schur Complement Lemma [3, Theorem A.65] [2, Proposition 10.2.5], it follows that $M_{\theta\theta}(z) \succ 0$ and $M_{ll}(z) \succ 0$.
5. In case of large payload mass (*i.e.*, $m_N \gg m_i$ for all $1 \leq i \leq N-1$), the mass matrix becomes ill-conditioned, since $M_{ll}(z)$ loses its rank around the vertical position when $m_i = 0$, $1 \leq i \leq N-1$, and has low rank when in addition the zero order approximation for small angles is made.

The proof of items 1 to 4 are easy consequences of the above developments. The proof of item 5 is as follows. The components of $M_{ll}(z)$ are $m_{(N+1+n)(N+j+1)}(z) = \left(\sum_{i=\max(j,n)}^N m_i \right) \cos(\alpha_n - \alpha_j)$, $1 \leq j \leq N$, $1 \leq n \leq N$. The zero-order approximation implies $\cos(\alpha_n - \alpha_j) \approx 1$, hence $m_{(N+1+n)(N+j+1)}(z) = \sum_{i=\max(j,n)}^N m_i$. In the limit where $m_i = 0$ for all $1 \leq i \leq N-1$, then all components are equal to m_N and M_{ll} has rank 1.

4.3 Potential Energy

If no elastic torsional joint stiffness is modeled, this reduces to the gravity potential energy of each link, that of the trolley being constant chosen equal to zero.

$$\begin{aligned} U_g(\alpha, l) &= - \sum_{i=1}^N m_i g \sum_{k=1}^i l_k \cos(\alpha_k) \\ &= - \sum_{k=1}^N g l_k \cos(\alpha_k) \sum_{i=k}^N m_i \end{aligned} \quad (37)$$

Thus for $1 \leq n \leq N$:

$$\begin{aligned} \frac{\partial U_g}{\partial \alpha_n} &= -g \frac{\partial}{\partial \alpha_n} \sum_{k=1}^N l_k \cos(\alpha_k) \sum_{i=k}^N m_i \\ &= g l_n \sin(\alpha_n) \sum_{i=n}^N m_i \end{aligned} \quad (38)$$

$$\begin{aligned} \frac{\partial U_g}{\partial l_n} &= -g \frac{\partial}{\partial l_n} \sum_{k=1}^N l_k \cos(\alpha_k) \sum_{i=k}^N m_i \\ &= -g \cos(\alpha_n) \sum_{i=n}^N m_i \end{aligned} \quad (39)$$

From the Lagrange dynamics structure, the generalized forces that derive from the potential are equal to $-\frac{\partial U_g}{\partial z}$.

Joint flexibility: If joint flexibility is added at joints A_i , $1 \leq i \leq N$, then the additional potential energy is $U_{flex}(\theta) = \frac{1}{2} \sum_{j=0}^{N-1} \kappa_j \theta_{j+1}^2 = \frac{1}{2} \theta^\top \mathcal{K} \theta$, where $\kappa_j \geq 0$ is the angular stiffness at joint A_j and $\mathcal{K} = \text{diag}(\kappa_i)$, $0 \leq i \leq N-1$. Thus:

$$\frac{\partial U_{flex}(\theta)}{\partial \theta_i} = \kappa_{i-1} \theta_i, \quad 1 \leq i \leq N. \quad (40)$$

This introduces no coupling between the coordinates, contrarily to what occurs in flexible joint manipulators [12, 4, 19]. In the coordinate angles $\alpha_i = \sum_{j=1}^i \theta_j$, $1 \leq i \leq N$, $\alpha = J\theta$ with J in (129), $\theta = J^{-1}\alpha$ with J^{-1} in (130), hence $U_{flex}(\alpha) = \frac{1}{2} \theta^\top \mathcal{K} \theta = \frac{1}{2} \alpha^\top J^{-\top} \mathcal{K} J^{-1} \alpha$. In

view of (130): $J^{-1}\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 - \alpha_1 \\ \alpha_3 - \alpha_2 \\ \vdots \\ \alpha_N - \alpha_{N-1} \end{pmatrix}$, hence $U_{flex}(\alpha) = \frac{1}{2}\kappa_0\alpha_1^2 + \sum_{i=1}^{N-1} \kappa_i(\alpha_{i+1} - \alpha_i)^2$. It is

inferred that

$$\frac{\partial U_{flex}(\alpha)}{\partial \alpha_1} = \kappa_0\alpha_1 - \kappa_1(\alpha_2 - \alpha_1) = (\kappa_0 + \kappa_1)\alpha_1 - \kappa_1\alpha_2, \quad (41)$$

for $2 \leq j \leq N - 1$:

$$\frac{\partial U_{flex}(\alpha)}{\partial \alpha_j} = \kappa_{j-1}(\alpha_j - \alpha_{j-1}) - \kappa_j(\alpha_{j+1} - \alpha_j) = (\kappa_{j-1} + \kappa_j)\alpha_j - \kappa_j\alpha_{j+1} - \kappa_{j-1}\alpha_{j-1}, \quad (42)$$

and

$$\frac{\partial U_{flex}(\alpha)}{\partial \alpha_N} = \kappa_{N-1}(\alpha_N - \alpha_{N-1}), \quad (43)$$

so that

$$\frac{\partial U_{flex}(\alpha)}{\partial \alpha} = \begin{pmatrix} \kappa_0 + \kappa_1 & -\kappa_1 & 0 & \dots & & & 0 \\ -\kappa_1 & \kappa_1 + \kappa_2 & -\kappa_2 & 0 & \dots & & 0 \\ 0 & -\kappa_2 & \kappa_2 + \kappa_3 & -\kappa_3 & 0 & \dots & 0 \\ \vdots & & & & & & \vdots \\ \vdots & & & & & & \vdots \\ 0 & \dots & & & 0 & -\kappa_{N-2} & \kappa_{N-2} + \kappa_{N-1} & -\kappa_{N-1} \\ 0 & \dots & & & & 0 & -\kappa_{N-1} & \kappa_{N-1} \end{pmatrix} \quad (44)$$

This time the joint flexibility introduces a triangular structure in the torques that derive from the elasticity potential, since the row $1 + n$ of the Lagrange dynamics, $1 \leq n \leq N$, corresponding to $M_{(1+n)\bullet}(q_\alpha)$ and $\ddot{\alpha}_n$, contains the flexibility torque $(-\kappa_{n-1}\alpha_{n-1} + (\kappa_{n-1} + \kappa_n)\alpha_n) - \kappa_n\alpha_{n+1}$. One can think of using a backstepping-like control design using the fictitious input α_{n+1} . However, the *vis-à-vis* terms in the submatrix $M_{\alpha\alpha}(q_\alpha)$ always contain strong inertial couplings between $\ddot{\alpha}_n$ and the other angular accelerations. Therefore the global triangular structure of Spong's model for flexible-joint manipulators [12, 4] does not exist in such overhead crane systems.

4.4 Recapitulation

The Lagrange dynamics of the overhead crane is given by:

$$\frac{d}{dt} \frac{\partial(T_{trol} + T_{trans} - U_g)}{\partial \dot{z}} - \frac{\partial(T_{trol} + T_{trans} - U_g - U_{flex})}{\partial z} = Q, \quad (45)$$

where Q_i works on z_i . More precisely:

$$\begin{cases} \frac{d}{dt} \frac{\partial(T_{trol} + T_{trans})}{\partial \dot{x}} = Q_x = F \\ \frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{\alpha}} - \frac{\partial(T_{trans} - U_g - U_{flex})}{\partial \alpha} = Q_\alpha = 0 \\ \frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{l}} - \frac{\partial(T_{trans} - U_g - U_{flex})}{\partial l} = Q_l = (F_1, 0, \dots, 0)^\top \end{cases} \quad (46)$$

It may be useful to introduce viscous friction (Rayleigh dissipation [3, Definition 6.12]) at the joints, and some joint stiffness as well. This may allow to avoid chaotic behaviours of conservative pendula. Such dynamical behaviour is unlikely to occur in real cables where a small amount of dissipation and of bending stiffness is always present.

5 The Case $N = 2$

This may be of interest in practice for cranes with two cables related by a hook. It could be of interest in such a case to investigate energy-based controllers, which have been derived in case $N = 1$ [18]. For clarity let us rewrite the above energies for $N = 2$, starting from scratch, then checking that the computed values of the mass matrix match with the general expressions in section 4. In this section we go a step further by calculating the whole Lagrange dynamics. We have:

$$T_{trol} = \frac{1}{2}m\dot{x}^2. \quad (47)$$

This kinetic energy concerns only the first component m_{11} of the mass matrix.

$$\begin{aligned} T_{trans} &= \frac{1}{2} \sum_{i=1}^2 m_i \left(\dot{x} + \sum_{k=1}^i l_k \left(\sum_{j=1}^k \dot{\theta}_j \right) \cos \left(\sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left(\sum_{j=1}^k \theta_j \right) \right)^2 \\ &+ \frac{1}{2} \sum_{i=1}^2 m_i \left(\sum_{k=1}^i l_k \left(\sum_{j=1}^k \dot{\theta}_j \right) \sin \left(\sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left(\sum_{j=1}^k \theta_j \right) \right)^2 \\ &= \frac{1}{2} \sum_{i=1}^2 m_i \left(\dot{x} + \sum_{k=1}^i l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right)^2 \\ &+ \frac{1}{2} \sum_{i=1}^2 m_i \left(\sum_{k=1}^i l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right)^2 \end{aligned} \quad (48)$$

Let us split T_{trans} into three parts:

$$\begin{aligned} T_{trans,1} &= \frac{1}{2}m_1(\dot{x} + l_1\dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1))^2 + \frac{1}{2}m_1(l_1\dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1))^2 \\ &= \frac{1}{2}m_1(\dot{x} + l_1\dot{\alpha}_1 \cos(\alpha_1) + \dot{l}_1 \sin(\alpha_1))^2 + \frac{1}{2}m_1(l_1\dot{\alpha}_1 \sin(\alpha_1) - \dot{l}_1 \cos(\alpha_1))^2 \end{aligned} \quad (49)$$

and

$$\begin{aligned} T_{trans,2} &= \frac{1}{2}m_2 \left(\dot{x} + \sum_{k=1}^2 l_k \left(\sum_{j=1}^k \dot{\theta}_j \right) \cos \left(\sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left(\sum_{j=1}^k \theta_j \right) \right)^2 \\ &= \frac{1}{2}m_2 \left(\dot{x} + \sum_{k=1}^2 l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right)^2 \end{aligned} \quad (50)$$

and

$$\begin{aligned} T_{trans,3} &= \frac{1}{2}m_2 \left(\sum_{k=1}^2 l_k \left(\sum_{j=1}^k \dot{\theta}_j \right) \sin \left(\sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left(\sum_{j=1}^k \theta_j \right) \right)^2 \\ &= \frac{1}{2}m_2 \left(\sum_{k=1}^2 l_k \dot{\alpha}_k \sin(\alpha_k) - \dot{l}_k \cos(\alpha_k) \right)^2. \end{aligned} \quad (51)$$

The potential energy due to gravity is:

$$\begin{aligned}
U_g(\alpha_1, \alpha_2, l_1, l_2) &= -\sum_{i=1}^2 m_i g \sum_{k=1}^i l_k \cos(\sum_{j=1}^k \theta_j) = -m_1 g l_1 \cos(\theta_1) \\
&\quad -m_2 g (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
&= -\sum_{i=1}^2 m_i g \sum_{k=1}^i l_k \cos(\alpha_k) = -m_1 g l_1 \cos(\alpha_1) \\
&\quad -m_2 g (l_1 \cos(\alpha_1) + l_2 \cos(\alpha_2)).
\end{aligned} \tag{52}$$

Thus: $\frac{\partial U_g}{\partial x} = 0$, $\frac{\partial U_g}{\partial \alpha_1} = (m_1 + m_2) g l_1 \sin(\alpha_1)$, $\frac{\partial U_g}{\partial \alpha_2} = m_2 g \sin(\alpha_2)$, $\frac{\partial U_g}{\partial l_1} = -(m_1 + m_2) g \cos(\alpha_1)$, $\frac{\partial U_g}{\partial l_2} = -m_2 g \cos(\alpha_2)$.

Let us calculate the 5×5 inertia matrix. As above we denote $q = (x, \theta_1, \theta_2, l_1, l_2)^\top$, and also $z = (x, \alpha_1, \alpha_2, l_1, l_2)^\top$.

5.1 Calculation of $\frac{d}{dt} \frac{\partial T_{trans,1}}{\partial \dot{q}}$ and $\frac{d}{dt} \frac{\partial T_{trans,1}}{\partial \dot{z}}$

We have:

$$\frac{\partial T_{trans,1}}{\partial \dot{x}} = m_1 (\dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1)) \tag{53}$$

We infer that the first row of the mass matrix $M(q)$ contains the terms:

$$\begin{aligned}
m_{11}(q) &: m_1 \\
m_{12}(q) &: m_1 l_1 \cos(\theta_1) \\
m_{13}(q) &: 0 \\
m_{14}(q) &: m_1 \sin(\theta_1) \\
m_{15}(q) &: 0
\end{aligned} \tag{54}$$

while the first row of the mass matrix $M(z)$ contains the terms:

$$\begin{aligned}
m_{11}(z) &: m_1 \\
m_{12}(z) &: m_1 l_1 \cos(\alpha_1) \\
m_{13}(z) &: 0 \\
m_{14}(z) &: m_1 \sin(\alpha_1) \\
m_{15}(z) &: 0
\end{aligned} \tag{55}$$

$$\begin{aligned}
\frac{\partial T_{trans,1}}{\partial \dot{\theta}_1} &= m_1 (\dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1)) l_1 \cos(\theta_1) + m_1 (l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1)) l_1 \sin(\theta_1) \\
&= m_1 l_1 \cos(\theta_1) \dot{x} + m_1 l_1^2 \dot{\theta}_1
\end{aligned} \tag{56}$$

We infer that the second row of the mass matrix $M(q)$ contains the terms:

$$\begin{aligned}
m_{21}(q) &: m_1 l_1 \cos(\theta_1) \\
m_{22}(q) &: m_1 l_1^2 \\
m_{23}(q) &: 0 \\
m_{24}(q) &: 0 \\
m_{25}(q) &: 0
\end{aligned} \tag{57}$$

while the second row of the mass matrix $M(z)$ contains the terms:

$$\begin{aligned}
m_{21}(z) &: m_1 l_1 \cos(\alpha_1) \\
m_{22}(z) &: m_1 l_1^2 \\
m_{23}(z) &: 0 \\
m_{24}(z) &: 0 \\
m_{25}(z) &: 0
\end{aligned} \tag{58}$$

$$\frac{\partial T_{trans,1}}{\partial \theta_2} = 0 \tag{59}$$

This does not contribute to the third row of $M(q)$.

$$\begin{aligned}
\frac{\partial T_{trans,1}}{\partial \dot{l}_1} &= m_1(\dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1)) \sin(\theta_1) - m_1(l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1)) \cos(\theta_1) \\
&= m_1 \sin(\theta_1) \dot{x} + m_1 \dot{l}_1
\end{aligned} \tag{60}$$

We infer that the fourth row of the mass matrix $M(q)$ contains the terms:

$$\begin{aligned}
m_{41}(q) &: m_1 \sin(\theta_1) \\
m_{42}(q) &: 0 \\
m_{43}(q) &: 0 \\
m_{44}(q) &: m_1 \\
m_{45}(q) &: 0
\end{aligned} \tag{61}$$

while the fourth row of the mass matrix $M(z)$ contains the terms:

$$\begin{aligned}
m_{41}(z) &: m_1 \sin(\alpha_1) \\
m_{42}(z) &: 0 \\
m_{43}(z) &: 0 \\
m_{44}(z) &: m_1 \\
m_{45}(z) &: 0
\end{aligned} \tag{62}$$

$$\frac{\partial T_{trans,1}}{\partial \dot{l}_2} = 0 \tag{63}$$

This does not contribute to fifth row of the mass matrices $M(q)$ and $M(z)$.

$$\frac{\partial T_{trans,1}}{\partial \dot{\alpha}_1} = \frac{\partial T_{trans,1}}{\partial \theta_1} = m_1 l_1 \cos(\alpha_1) \dot{x} + m_1 l_1^2 \dot{\alpha}_1 \tag{64}$$

This contributes to the second row of $M(z)$ as:

$$\begin{aligned}
m_{21}(z) &: m_1 l_1 \cos(\alpha_1) \\
m_{22}(z) &: m_1 l_1^2 \\
m_{23}(z) &: 0 \\
m_{24}(z) &: 0 \\
m_{25}(z) &: 0
\end{aligned} \tag{65}$$

$$\frac{\partial T_{trans,1}}{\partial \dot{\alpha}_2} = 0 \tag{66}$$

This does not contribute to the third row of $M(z)$.

5.2 Calculation of $\frac{d}{dt} \frac{\partial T_{trans,2}}{\partial \dot{q}}$ and $\frac{d}{dt} \frac{\partial T_{trans,2}}{\partial \dot{z}}$

We have in the q -coordinate:

$$\begin{aligned}
\frac{\partial T_{trans,2}}{\partial \dot{x}} &= m_2 \left(\dot{x} + \sum_{k=1}^2 l_k \left(\sum_{j=1}^k \dot{\theta}_j \right) \cos(\sum_{j=1}^k \theta_j) + \dot{l}_k \sin(\sum_{j=1}^k \theta_j) \right) \\
&= m_2 \left(\dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1) + l_2 (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2) + \dot{l}_2 \sin(\theta_1 + \theta_2) \right) \\
&= m_2 \dot{x} + m_2 (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \dot{\theta}_1 + m_2 l_2 \cos(\theta_1 + \theta_2) \dot{\theta}_2 \\
&\quad + m_2 \sin(\theta_1) \dot{l}_1 + m_2 \sin(\theta_1 + \theta_2) \dot{l}_2
\end{aligned} \tag{67}$$

and we also have in the z -coordinate:

$$\begin{aligned}
\frac{\partial T_{trans,2}}{\partial \dot{x}} &= m_2 \left(\dot{x} + \sum_{k=1}^2 l_k \dot{\alpha}_k \cos(\alpha_k) + \dot{l}_k \sin(\alpha_k) \right) \\
&= m_2 \left(\dot{x} + l_1 \dot{\alpha}_1 \cos(\alpha_1) + \dot{l}_1 \sin(\alpha_1) + l_2 \dot{\alpha}_2 \cos(\alpha_2) + \dot{l}_2 \sin(\alpha_2) \right) \\
&= m_2 \dot{x} + m_2 l_1 \cos(\alpha_1) \dot{\alpha}_1 + m_2 l_2 \cos(\alpha_2) \dot{\alpha}_2 + m_2 \sin(\alpha_1) \dot{l}_1 + m_2 \sin(\alpha_2) \dot{l}_2
\end{aligned} \tag{68}$$

We infer that the first row of the mass matrix $M(q)$ contains the terms:

$$\begin{aligned}
m_{11}(q) &: m_2 \\
m_{12}(q) &: m_2 (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
m_{13}(q) &: m_2 l_2 \cos(\theta_1 + \theta_2) \\
m_{14}(q) &: m_2 \sin(\theta_1) \\
m_{15}(q) &: m_2 \sin(\theta_1 + \theta_2)
\end{aligned} \tag{69}$$

while the first row of the mass matrix $M(z)$ contains the terms:

$$\begin{aligned}
m_{11}(z) &: m_2 \\
m_{12}(z) &: m_2 l_1 \cos(\alpha_1) \\
m_{13}(z) &: m_2 l_2 \cos(\alpha_2) \\
m_{14}(z) &: m_2 \sin(\alpha_1) \\
m_{15}(z) &: m_2 \sin(\alpha_2)
\end{aligned} \tag{70}$$

Now let us pass to the next coordinate in q -coordinate (second row of the mass matrix):

$$\begin{aligned}
\frac{\partial T_{trans,2}}{\partial \dot{\theta}_1} &= m_2 \left(\dot{x} + l_1 \cos(\theta_1) \dot{\theta}_1 + \sin(\theta_1) \dot{l}_1 + l_2 (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2) + \dot{l}_2 \sin(\theta_1 + \theta_2) \right) \\
&\quad \times (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2))
\end{aligned} \tag{71}$$

and to the next coordinate in z -coordinate:

$$\frac{\partial T_{trans,2}}{\partial \dot{\alpha}_1} = m_2 \left(\dot{x} + l_1 \dot{\alpha}_1 \cos(\alpha_1) + \dot{l}_1 \sin(\alpha_1) + l_2 \dot{\alpha}_2 \cos(\alpha_2) + \dot{l}_2 \sin(\alpha_2) \right) l_1 \cos(\alpha_1) \tag{72}$$

From (71) it can be deduced terms that belong to the second row of the mass matrix $M(q)$:

$$\begin{aligned}
m_{21}(q) &: m_2 (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
m_{22}(q) &: m_2 (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2))^2 \\
m_{23}(q) &: m_2 l_2 \cos(\theta_1 + \theta_2) (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
m_{24}(q) &: m_2 \sin(\theta_1) (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
m_{25}(q) &: m_2 \sin(\theta_1 + \theta_2) (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2))
\end{aligned} \tag{73}$$

and from (72) terms that belong to the second row of the mass matrix $M(z)$:

$$\begin{aligned}
m_{21}(z) &: m_2 l_1 \cos(\alpha_1) \\
m_{22}(z) &: m_2 l_1^2 \cos^2(\alpha_1) \\
m_{23}(z) &: m_2 l_2 l_1 \cos(\alpha_1) \cos(\alpha_2) \\
m_{24}(z) &: m_2 l_1 \cos(\alpha_1) \sin(\alpha_1) \\
m_{25}(z) &: m_2 l_1 \cos(\alpha_1) \sin(\alpha_2)
\end{aligned} \tag{74}$$

Let us pass to the third row in q -coordinate:

$$\begin{aligned}
\frac{\partial T_{trans,2}}{\partial \dot{\theta}_2} &= m_2 \left(\dot{x} + l_1 \cos(\theta_1) \dot{\theta}_1 + \sin(\theta_1) \dot{l}_1 + l_2 (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2) + \dot{l}_2 \sin(\theta_1 + \theta_2) \right) \\
&\quad \times l_2 \cos(\theta_1 + \theta_2)
\end{aligned} \tag{75}$$

and in z -coordinate:

$$\frac{\partial T_{trans,2}}{\partial \dot{\alpha}_2} = m_2 \left(\dot{x} + l_1 \dot{\alpha}_1 \cos(\alpha_1) + \dot{l}_1 \sin(\alpha_1) + l_2 \dot{\alpha}_2 \cos(\alpha_2) + \dot{l}_2 \sin(\alpha_2) \right) l_2 \cos(\alpha_2) \tag{76}$$

From (75) it can be deduced terms that belong to the third row of the mass matrix $M(q)$

:

$$\begin{aligned}
m_{31}(q) &: m_2 l_2 \cos(\theta_1 + \theta_2) \\
m_{32}(q) &: m_2 l_1 l_2 \cos(\theta_1) \cos(\theta_1 + \theta_2) + m_2 l_2^2 \cos^2(\theta_1 + \theta_2) \\
m_{33}(q) &: m_2 l_2^2 \cos^2(\theta_1 + \theta_2) \\
m_{34}(q) &: m_2 l_2 \sin(\theta_1) \cos(\theta_1 + \theta_2) \\
m_{35}(q) &: m_2 l_2 \sin(\theta_1 + \theta_2) \cos(\theta_1 + \theta_2)
\end{aligned} \tag{77}$$

From (76) it can be deduced terms that belong to the third row of the mass matrix $M(z)$:

$$\begin{aligned}
m_{31}(z) &: m_2 l_2 \cos(\alpha_2) \\
m_{32}(z) &: m_2 l_1 l_2 \cos(\alpha_2) \cos(\alpha_1) \\
m_{33}(z) &: m_2 l_2^2 \cos^2(\alpha_2) \\
m_{34}(z) &: m_2 l_2 \sin(\alpha_1) \cos(\alpha_2) \\
m_{35}(z) &: m_2 l_2 \sin(\alpha_2) \cos(\alpha_2)
\end{aligned} \tag{78}$$

Now terms of the fourth row of the mass matrix $M(q)$ are obtained from:

$$\begin{aligned}
\frac{\partial T_{trans,2}}{\partial \dot{l}_1} &= m_2 \left(\dot{x} + \sum_{k=1}^2 l_k \left(\sum_{j=1}^k \dot{\theta}_j \right) \cos \left(\sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left(\sum_{j=1}^k \theta_j \right) \right) \sin(\theta_1) \\
&= m_2 \left(\dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1) + l_2 (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2) + \dot{l}_2 \sin(\theta_1 + \theta_2) \right) \sin(\theta_1)
\end{aligned} \tag{79}$$

and terms of the fourth row of the mass matrix $M(z)$ are obtained from:

$$\frac{\partial T_{trans,2}}{\partial \dot{l}_1} = m_2 \left(\dot{x} + l_1 \dot{\alpha}_1 \cos(\alpha_1) + \dot{l}_1 \sin(\alpha_1) + l_2 \dot{\alpha}_2 \cos(\alpha_2) + \dot{l}_2 \sin(\alpha_2) \right) \sin(\alpha_1) \tag{80}$$

From (79) it can be deduced terms that belong to the fourth row of $M(q)$:

$$\begin{aligned}
m_{41}(q) &: m_2 \sin(\theta_1) \\
m_{42}(q) &: m_2 l_1 \cos(\theta_1) \sin(\theta_1) + m_2 l_2 \cos(\theta_1 + \theta_2) \sin(\theta_1) \\
m_{43}(q) &: m_2 l_2 \cos(\theta_1 + \theta_2) \sin(\theta_1) \\
m_{44}(q) &: m_2 \sin^2(\theta_1) \\
m_{45}(q) &: m_2 \sin(\theta_1) \sin(\theta_1 + \theta_2)
\end{aligned} \tag{81}$$

and from (80) it can be deduced terms that belong to the fourth row of $M(q)$:

$$\begin{aligned}
m_{41}(z) &: m_2 \sin(\alpha_1) \\
m_{42}(z) &: m_2 l_1 \cos(\alpha_1) \sin(\alpha_1) \\
m_{43}(z) &: m_2 l_2 \cos(\alpha_2) \sin(\alpha_1) \\
m_{44}(z) &: m_2 \sin^2(\alpha_1) \\
m_{45}(z) &: m_2 \sin(\alpha_2) \sin(\alpha_1)
\end{aligned} \tag{82}$$

Let us end with terms of the fifth row of the mass matrix $M(q)$:

$$\begin{aligned}
\frac{\partial T_{trans,2}}{\partial l_2} &= m_2 \left(\dot{x} + \sum_{k=1}^2 l_k \left(\sum_{j=1}^k \dot{\theta}_j \right) \cos \left(\sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left(\sum_{j=1}^k \theta_j \right) \right) \sin(\theta_1) \\
&= m_2 \left(\dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1) + l_2 (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2) + \dot{l}_2 \sin(\theta_1 + \theta_2) \right) \sin(\theta_1 + \theta_2)
\end{aligned} \tag{83}$$

and the fifth row of the mass matrix $M(z)$:

$$\frac{\partial T_{trans,2}}{\partial l_2} = m_2 \left(\dot{x} + l_1 \dot{\alpha}_1 \cos(\alpha_1) + \dot{l}_1 \sin(\alpha_1) + l_2 \dot{\alpha}_2 \cos(\alpha_2) + \dot{l}_2 \sin(\alpha_2) \right) \sin(\alpha_2) \tag{84}$$

From (83) it can be deduced terms that belong to the fifth row of the mass matrix $M(q)$:

$$\begin{aligned}
m_{51}(q) &: m_2 \sin(\theta_1 + \theta_2) \\
m_{52}(q) &: m_2 l_1 \cos(\theta_1) \sin(\theta_1 + \theta_2) + m_2 l_2 \cos(\theta_1 + \theta_2) \sin(\theta_1 + \theta_2) \\
m_{53}(q) &: m_2 l_2 \cos(\theta_1 + \theta_2) \sin(\theta_1 + \theta_2) \\
m_{54}(q) &: m_2 \sin(\theta_1) \sin(\theta_1 + \theta_2) \\
m_{55}(q) &: m_2 \sin^2(\theta_1 + \theta_2)
\end{aligned} \tag{85}$$

and from (84) it can be deduced terms that belong to the fifth row of the mass matrix $M(z)$:

$$\begin{aligned}
m_{51}(z) &: m_2 \sin(\alpha_2) \\
m_{52}(z) &: m_2 l_1 \cos(\alpha_1) \sin(\alpha_2) \\
m_{53}(z) &: m_2 l_2 \cos(\alpha_2) \sin(\alpha_2) \\
m_{54}(z) &: m_2 \sin(\alpha_1) \sin(\alpha_2) \\
m_{55}(z) &: m_2 \sin^2(\alpha_2)
\end{aligned} \tag{86}$$

5.3 Calculation of $\frac{d}{dt} \frac{\partial T_{trans,3}}{\partial \dot{q}}$ and $\frac{d}{dt} \frac{\partial T_{trans,3}}{\partial \dot{z}}$

Recall that

$$\begin{aligned}
T_{trans,3} &= \frac{1}{2} m_2 \left(\sum_{k=1}^2 l_k \left(\sum_{j=1}^k \dot{\theta}_j \right) \sin \left(\sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left(\sum_{j=1}^k \theta_j \right) \right)^2 \\
&= \frac{1}{2} m_2 \left(l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1) + l_2 (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2) - \dot{l}_2 \cos(\theta_1 + \theta_2) \right)^2
\end{aligned} \tag{87}$$

$$T_{trans,3} = \frac{1}{2} m_2 \left(l_1 \dot{\alpha}_1 \sin(\alpha_1) - \dot{l}_1 \cos(\alpha_1) + l_2 \dot{\alpha}_2 \sin(\alpha_2) - \dot{l}_2 \cos(\alpha_2) \right)^2 \tag{88}$$

Hence:

$$\frac{\partial T_{trans,3}}{\partial \dot{x}} = 0. \tag{89}$$

This does not contribute to the first row of $M(q)$ and $M(z)$. Concerning the second row:

$$\begin{aligned} \frac{\partial T_{trans,3}}{\partial \dot{\theta}_1} &= m_2 \left(l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1) + l_2 (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2) - \dot{l}_2 \cos(\theta_1 + \theta_2) \right) \\ &\times (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \end{aligned} \quad (90)$$

and

$$\frac{\partial T_{trans,3}}{\partial \dot{\alpha}_1} = m_2 \left(l_1 \dot{\alpha}_1 \sin(\alpha_1) - \dot{l}_1 \cos(\alpha_1) + l_2 \dot{\alpha}_2 \sin(\alpha_2) - \dot{l}_2 \cos(\alpha_2) \right) l_1 \sin(\alpha_1) \quad (91)$$

From (90) the following terms are in the second row of $M(q)$:

$$\begin{aligned} m_{21}(q) &: 0 \\ m_{22}(q) &: m_2 (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2))^2 \\ m_{23}(q) &: m_2 l_2 \sin(\theta_1 + \theta_2) (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \\ m_{24}(q) &: -m_2 \cos(\theta_1) (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \\ m_{25}(q) &: -m_2 \cos(\theta_1 + \theta_2) (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \end{aligned} \quad (92)$$

From (91) the following terms are in the second row of $M(z)$:

$$\begin{aligned} m_{21}(z) &: 0 \\ m_{22}(z) &: m_2 l_1^2 \sin^2(\alpha_1) \\ m_{23}(z) &: m_2 l_1 l_2 \sin(\alpha_1) \sin(\alpha_2) \\ m_{24}(z) &: -m_2 l_1 \cos(\alpha_1) \sin(\alpha_1) \\ m_{25}(z) &: -m_2 l_1 \cos(\alpha_2) \sin(\alpha_1) \end{aligned} \quad (93)$$

Let us pass to the third rows:

$$\begin{aligned} \frac{\partial T_{trans,3}}{\partial \dot{\theta}_2} &= m_2 \left(l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1) + l_2 (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2) - \dot{l}_2 \cos(\theta_1 + \theta_2) \right) \\ &\times l_2 \sin(\theta_1 + \theta_2) \end{aligned} \quad (94)$$

and

$$\frac{\partial T_{trans,3}}{\partial \dot{\alpha}_2} = m_2 \left(l_1 \dot{\alpha}_1 \sin(\alpha_1) - \dot{l}_1 \cos(\alpha_1) + l_2 \dot{\alpha}_2 \sin(\alpha_2) - \dot{l}_2 \cos(\alpha_2) \right) l_2 \sin(\alpha_2) \quad (95)$$

From (94) the following terms are in the third row of $M(q)$:

$$\begin{aligned} m_{31}(q) &: 0 \\ m_{32}(q) &: m_2 (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) l_2 \sin(\theta_1 + \theta_2) \\ m_{33}(q) &: m_2 l_2^2 \sin^2(\theta_1 + \theta_2) \\ m_{34}(q) &: -m_2 l_2 \cos(\theta_1) \sin(\theta_1 + \theta_2) \\ m_{35}(q) &: -m_2 l_2 \cos(\theta_1 + \theta_2) \sin(\theta_1 + \theta_2) \end{aligned} \quad (96)$$

From (95) the following terms are in the third row of $M(z)$:

$$\begin{aligned} m_{31}(z) &: 0 \\ m_{32}(z) &: m_2 l_1 l_2 \sin(\alpha_1) \sin(\alpha_2) \\ m_{33}(z) &: m_2 l_2^2 \sin^2(\alpha_2) \\ m_{34}(z) &: -m_2 l_2 \cos(\alpha_1) \sin(\alpha_2) \\ m_{35}(z) &: -m_2 l_2 \cos(\alpha_2) \sin(\alpha_2) \end{aligned} \quad (97)$$

Let us pass to the fourth rows:

$$\begin{aligned} \frac{\partial T_{trans,3}}{\partial \dot{l}_1} &= -m_2 \left(l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1) + l_2 (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2) - \dot{l}_2 \cos(\theta_1 + \theta_2) \right) \\ &\quad \times \cos(\theta_1) \end{aligned} \quad (98)$$

and

$$\frac{\partial T_{trans,3}}{\partial \dot{l}_1} = -m_2 \left(l_1 \dot{\alpha}_1 \sin(\alpha_1) - \dot{l}_1 \cos(\alpha_1) + l_2 \dot{\alpha}_2 \sin(\alpha_2) - \dot{l}_2 \cos(\alpha_2) \right) \cos(\alpha_1) \quad (99)$$

From (98) the following terms are in the fourth row of $M(q)$:

$$\begin{aligned} m_{41}(q) &: 0 \\ m_{42}(q) &: -m_2(l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \cos(\theta_1) \\ m_{43}(q) &: -m_2 l_2 \sin(\theta_1 + \theta_2) \cos(\theta_1) \\ m_{44}(q) &: m_2 \cos^2(\theta_1) \\ m_{45}(q) &: m_2 \cos(\theta_1 + \theta_2) \cos(\theta_1) \end{aligned} \quad (100)$$

From (99) the following terms are in the fourth row of $M(z)$:

$$\begin{aligned} m_{41}(z) &: 0 \\ m_{42}(z) &: -m_2 l_1 \sin(\alpha_1) \cos(\alpha_1) \\ m_{43}(z) &: -m_2 l_2 \sin(\alpha_2) \cos(\alpha_1) \\ m_{44}(z) &: m_2 \cos^2(\alpha_1) \\ m_{45}(z) &: m_2 \cos(\alpha_2) \cos(\alpha_1) \end{aligned} \quad (101)$$

Let us pass to the fifth rows:

$$\begin{aligned} \frac{\partial T_{trans,3}}{\partial \dot{l}_2} &= -m_2 \left(l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1) + l_2 (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2) - \dot{l}_2 \cos(\theta_1 + \theta_2) \right) \\ &\quad \times \cos(\theta_1 + \theta_2) \end{aligned} \quad (102)$$

and

$$\frac{\partial T_{trans,3}}{\partial \dot{l}_2} = -m_2 \left(l_1 \dot{\alpha}_1 \sin(\alpha_1) - \dot{l}_1 \cos(\alpha_1) + l_2 \dot{\alpha}_2 \sin(\alpha_2) - \dot{l}_2 \cos(\alpha_2) \right) \cos(\alpha_2) \quad (103)$$

From (102) the following terms are in the fourth row of $M(q)$:

$$\begin{aligned} m_{51}(q) &: 0 \\ m_{52}(q) &: -m_2(l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \cos(\theta_1 + \theta_2) \\ m_{53}(q) &: -m_2 l_2 \sin(\theta_1 + \theta_2) \cos(\theta_1 + \theta_2) \\ m_{54}(q) &: m_2 \cos(\theta_1) \cos(\theta_1 + \theta_2) \\ m_{55}(q) &: m_2 \cos^2(\theta_1 + \theta_2) \end{aligned} \quad (104)$$

From (103) the following terms are in the fourth row of $M(z)$:

$$\begin{aligned} m_{51}(z) &: 0 \\ m_{52}(z) &: -m_2 l_1 \sin(\alpha_1) \cos(\alpha_2) \\ m_{53}(z) &: -m_2 l_2 \sin(\alpha_2) \cos(\alpha_2) \\ m_{54}(z) &: m_2 \cos(\alpha_1) \cos(\alpha_2) \\ m_{55}(z) &: m_2 \cos^2(\alpha_2) \end{aligned} \quad (105)$$

5.4 Recapitulation: Calculation of $M(q)$

Let us gather the above results to obtain $M(q)$:

$$\begin{aligned}
m_{11}(q) &= m + m_1 + m_2 \\
m_{12}(q) &= m_1 l_1 \cos(\theta_1) + m_2(l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
&= (m_1 + m_2)l_1 \cos(\theta_1) + m_2 l_2 \cos(\theta_1 + \theta_2) \\
m_{13}(q) &= m_2 l_2 \cos(\theta_1 + \theta_2) \\
m_{14}(q) &= m_1 \sin(\theta_1) + m_2 \sin(\theta_1) = (m_1 + m_2) \sin(\theta_1) \\
m_{15}(q) &= m_2 \sin(\theta_1 + \theta_2)
\end{aligned} \tag{106}$$

$$\begin{aligned}
m_{21}(q) &= m_1 l_1 \cos(\theta_1) + m_2(l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
&= (m_1 + m_2)l_1 \cos(\theta_1) + m_2 l_2 \cos(\theta_1 + \theta_2) \\
m_{22}(q) &= m_1 l_1^2 + m_2(l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2))^2 + m_2(l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2))^2 \\
m_{23}(q) &= m_2 l_2 \cos(\theta_1 + \theta_2)(l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) + m_2 l_2 \sin(\theta_1 + \theta_2)(l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \\
&= m_2 l_1 l_2 \cos(\theta_2) + m_2 l_2^2 \\
m_{24}(q) &= m_2 \sin(\theta_1)(l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) - m_2 \cos(\theta_1)(l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \\
&= -m_2 l_2 \sin(\theta_2) \\
m_{25}(q) &= m_2 \sin(\theta_1 + \theta_2)(l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) - m_2 \cos(\theta_1 + \theta_2)(l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \\
&= m_2 l_1 \sin(\theta_2)
\end{aligned} \tag{107}$$

$$\begin{aligned}
m_{31}(q) &= m_2 l_2 \cos(\theta_1 + \theta_2) \\
m_{32}(q) &= m_2 l_1 l_2 \cos(\theta_1) \cos(\theta_1 + \theta_2) + m_2 l_2^2 \cos^2(\theta_1 + \theta_2) + m_2(l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2))l_2 \sin(\theta_1 + \theta_2) \\
&= m_2 l_2^2 + m_2 l_1 l_2 \cos(\theta_2) \\
m_{33}(q) &= m_2 l_2^2 \cos^2(\theta_1 + \theta_2) + m_2 l_2^2 \sin^2(\theta_1 + \theta_2) = m_2 l_2^2 \\
m_{34}(q) &= m_2 l_2 \sin(\theta_1) \cos(\theta_1 + \theta_2) - m_2 l_2 \cos(\theta_1) \sin(\theta_1 + \theta_2) \\
&= -m_2 l_2 \sin(\theta_2) \\
m_{35}(q) &= m_2 l_2 \sin(\theta_1 + \theta_2) \cos(\theta_1 + \theta_2) - m_2 l_2 \cos(\theta_1 + \theta_2) \sin(\theta_1 + \theta_2) = 0
\end{aligned} \tag{108}$$

$$\begin{aligned}
m_{41}(q) &= m_1 \sin(\theta_1) + m_2 \sin(\theta_1) = (m_1 + m_2) \sin(\theta_1) \\
m_{42}(q) &= m_2 l_1 \cos(\theta_1) \sin(\theta_1) + m_2 l_2 \cos(\theta_1 + \theta_2) \sin(\theta_1) - m_2(l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \cos(\theta_1) \\
&= -m_2 l_2 \sin(\theta_2) \\
m_{43}(q) &= m_2 l_2 \cos(\theta_1 + \theta_2) \sin(\theta_1) - m_2 l_2 \sin(\theta_1 + \theta_2) \cos(\theta_1) \\
&= -m_2 l_2 \sin(\theta_2) \\
m_{44}(q) &= m_1 + m_2 \sin^2(\theta_1) + m_2 \cos^2(\theta_1) = m_1 + m_2 \\
m_{45}(q) &= m_2 \sin(\theta_1) \sin(\theta_1 + \theta_2) + m_2 \cos(\theta_1) \cos(\theta_1 + \theta_2) \\
&= m_2 \cos(\theta_2)
\end{aligned} \tag{109}$$

$$\begin{aligned}
m_{51}(q) &= m_2 \sin(\theta_1 + \theta_2) \\
m_{52}(q) &= m_2 l_1 \cos(\theta_1) \sin(\theta_1 + \theta_2) + m_2 l_2 \cos(\theta_1 + \theta_2) \sin(\theta_1 + \theta_2) \\
&\quad - m_2 (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \cos(\theta_1 + \theta_2) \\
&= m_2 l_1 \cos(\theta_1) \sin(\theta_1 + \theta_2) - m_2 l_1 \sin(\theta_1) \cos(\theta_1 + \theta_2) \\
&= m_2 l_1 \sin(\theta_2) \\
m_{53}(q) &= m_2 l_2 \cos(\theta_1 + \theta_2) \sin(\theta_1 + \theta_2) - m_2 l_2 \cos(\theta_1 + \theta_2) \sin(\theta_1 + \theta_2) = 0 \\
m_{54}(q) &= m_2 \sin(\theta_1) \sin(\theta_1 + \theta_2) + m_2 \cos(\theta_1) \cos(\theta_1 + \theta_2) = m_2 \cos(\theta_2) \\
m_{55}(q) &= m_2 \cos^2(\theta_1 + \theta_2) + m_2 \sin^2(\theta_1 + \theta_2) = m_2
\end{aligned} \tag{110}$$

and to obtain $M(z)$:

$$\begin{aligned}
m_{11}(z) &= m + m_1 + m_2 \\
m_{12}(z) &= m_1 l_1 \cos(\alpha_1) + m_2 l_1 \cos(\alpha_1) = (m_1 + m_2) l_1 \cos(\alpha_1) \\
m_{13}(z) &= m_2 l_2 \cos(\alpha_2) \\
m_{14}(z) &= m_1 \sin(\alpha_1) + m_2 \sin(\alpha_1) = (m_1 + m_2) \sin(\alpha_1) \\
m_{15}(z) &= m_2 \sin(\alpha_2)
\end{aligned} \tag{111}$$

$$\begin{aligned}
m_{21}(z) &= m_1 l_1 \cos(\alpha_1) + m_2 l_1 \cos(\alpha_1) = (m_1 + m_2) l_1 \cos(\alpha_1) \\
m_{22}(z) &= m_1 l_1^2 + m_2 l_1^2 \cos^2(\alpha_1) + m_2 l_1^2 \sin^2(\alpha_1) = (m_1 + m_2) l_1^2 \\
m_{23}(z) &= m_2 l_1 l_2 \cos(\alpha_1) \cos(\alpha_2) + m_2 l_1 l_2 \sin(\alpha_1) \sin(\alpha_2) \\
&= m_2 l_1 l_2 \cos(\alpha_1 - \alpha_2) \\
m_{24}(z) &= m_2 l_1 \cos(\alpha_1) \sin(\alpha_1) - m_2 l_1 \cos(\alpha_1) \sin(\alpha_1) = 0 \\
m_{25}(z) &= m_2 l_1 \cos(\alpha_1) \sin(\alpha_2) - m_2 l_1 \cos(\alpha_2) \sin(\alpha_1) \\
&= m_2 l_1 \sin(\alpha_2 - \alpha_1)
\end{aligned} \tag{112}$$

$$\begin{aligned}
m_{31}(z) &= m_2 l_2 \cos(\alpha_2) \\
m_{32}(z) &= m_2 l_1 l_2 \cos(\alpha_2) \cos(\alpha_1) + m_2 l_1 l_2 \sin(\alpha_2) \sin(\alpha_1) \\
&= m_2 l_1 l_2 \cos(\alpha_1 - \alpha_2) \\
m_{33}(z) &= m_2 l_2^2 \cos^2(\alpha_2) + m_2 l_2^2 \sin^2(\alpha_2) = m_2 l_2^2 \\
m_{34}(z) &= m_2 l_2 \sin(\alpha_1) \cos(\alpha_2) - m_2 l_2 \cos(\alpha_1) \sin(\alpha_2) \\
&= m_2 l_2 \sin(\alpha_1 - \alpha_2) \\
m_{35}(z) &= m_2 l_2 \sin(\alpha_2) \cos(\alpha_2) - m_2 l_2 \sin(\alpha_2) \cos(\alpha_2) = 0
\end{aligned} \tag{113}$$

$$\begin{aligned}
m_{41}(z) &= m_1 \sin(\alpha_1) + m_2 \sin(\alpha_1) = (m_1 + m_2) \sin(\alpha_1) \\
m_{42}(z) &= m_2 l_1 \cos(\alpha_1) \sin(\alpha_1) - m_2 l_1 \cos(\alpha_1) \sin(\alpha_1) = 0 \\
m_{43}(z) &= m_2 l_2 \cos(\alpha_2) \sin(\alpha_1) - m_2 l_2 \cos(\alpha_1) \sin(\alpha_2) = m_2 l_2 \sin(\alpha_1 - \alpha_2) \\
m_{44}(z) &= m_1 + m_2 \sin^2(\alpha_1) + m_2 \cos^2(\alpha_1) = m_1 + m_2 \\
m_{45}(z) &= m_2 \sin(\alpha_2) \sin(\alpha_1) + m_2 \cos(\alpha_1) \cos(\alpha_2) = m_2 \cos(\alpha_1 - \alpha_2)
\end{aligned} \tag{114}$$

$$\begin{aligned}
m_{51}(z) &= m_2 \sin(\alpha_2) \\
m_{52}(z) &= m_2 l_1 \cos(\alpha_1) \sin(\alpha_2) - m_2 l_1 \sin(\alpha_1) \cos(\alpha_2) = m_2 l_1 \sin(\alpha_2 - \alpha_1) \\
m_{53}(z) &= m_2 l_2 \cos(\alpha_2) \sin(\alpha_2) - m_2 l_2 \cos(\alpha_2) \sin(\alpha_2) = 0 \\
m_{54}(z) &= m_2 \sin(\alpha_1) \sin(\alpha_2) + m_2 \cos(\alpha_1) \cos(\alpha_2) = m_2 \cos(\alpha_1 - \alpha_2) \\
m_{55}(z) &= m_2 \sin^2(\alpha_2) + m_2 \cos^2(\alpha_2) = m_2
\end{aligned} \tag{115}$$

5.4.1 $m_2 \gg m_1$ and small angles

This occurs if m_1 (the hook's mass) is much smaller than m_2 (the tool plus payload mass). Then $m_1 = 0$ can be taken in $M(z)$ and $M(q)$. It follows that the submatrix $M_{ll}(\theta)$ satisfies $\det(M_{ll}(\theta)) = m_2 \sin^2(\theta_2)$ and $\det(M_{ll}(\alpha)) = m_2 \sin^2(\alpha_1 - \alpha_2)$. If the small angles assumption is made ($\sin(\alpha_i) \approx \alpha_i$, $\cos(\alpha_i) \approx 1$), then $\det(M_{ll}(\theta)) = m_2 \theta_2^2$ and $\det(M_{ll}(\alpha)) = m_2 (\alpha_1 - \alpha_2)^2$. In the limit there is a decoupling between the (x, θ_1, θ_2) dynamics and the (l_1, l_2) dynamics in the mass matrix which has a block diagonal structure.

5.5 Coriolis/centrifugal forces

Let us calculate them in the z -coordinates. These generalized forces are equal to $\frac{d}{dt}(M(z))\dot{z} - \frac{1}{2} \frac{\partial}{\partial z}(\dot{z}^\top M(z) \dot{z})$.

5.5.1 Calculation of $\frac{d}{dt}(M(z))\dot{z} \in \mathbb{R}^5$

1. $-(m_1 + m_2)l_1 \sin(\alpha_1) \dot{\alpha}_1^2 - m_2 l_2 \sin(\alpha_2) \dot{\alpha}_2^2 + 2(m_1 + m_2) \cos(\alpha_1) \dot{\alpha}_1 \dot{l}_1 + 2m_2 \cos(\alpha_2) \dot{\alpha}_2 \dot{l}_2$
2. $-(m_1 + m_2)l_1 \sin(\alpha_1) \dot{\alpha}_1 \dot{x} - m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2) (\dot{\alpha}_1 - \dot{\alpha}_2) \dot{\alpha}_2 + m_2 l_1 \cos(\alpha_2 - \alpha_1) (\dot{\alpha}_2 - \dot{\alpha}_1) \dot{l}_2 + (m_1 + m_2) \dot{l}_1 \dot{x} \cos(\alpha_1) + 2(m_1 + m_2) l_1 \dot{l}_1 \dot{\alpha}_1 + m_2 l_1 \dot{l}_2 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2) + m_2 l_2 \dot{l}_1 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2) + m_2 \dot{l}_1 \dot{l}_2 \sin(\alpha_2 - \alpha_1)$
3. $m_2 \dot{l}_2 \dot{x} \cos(\alpha_2) - m_2 l_2 \sin(\alpha_2) \dot{\alpha}_2 \dot{x} - m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2) (\dot{\alpha}_1 - \dot{\alpha}_2) \dot{\alpha}_1 + m_2 l_2 \cos(\alpha_1 - \alpha_2) (\dot{\alpha}_1 - \dot{\alpha}_2) \dot{l}_1 + m_2 \dot{l}_1 \dot{\alpha}_1 l_2 \cos(\alpha_1 - \alpha_2) + m_2 l_1 \dot{l}_2 \dot{\alpha}_1 \cos(\alpha_1 - \alpha_2) + 2m_2 l_2 \dot{l}_2 \dot{\alpha}_2 + m_2 \dot{l}_2 \dot{l}_1 \sin(\alpha_1 - \alpha_2)$
4. $(m_1 + m_2) \cos(\alpha_1) \dot{\alpha}_1 \dot{x} + m_2 l_2 \cos(\alpha_1 - \alpha_2) (\dot{\alpha}_1 - \dot{\alpha}_2) \dot{\alpha}_2 - m_2 \sin(\alpha_1 - \alpha_2) (\dot{\alpha}_1 - \dot{\alpha}_2) \dot{l}_2 + m_2 \dot{l}_2 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2)$
5. $m_2 \cos(\alpha_2) \dot{\alpha}_2 \dot{x} + m_2 l_1 \cos(\alpha_2 - \alpha_1) (\dot{\alpha}_2 - \dot{\alpha}_1) \dot{\alpha}_1 - m_2 \sin(\alpha_1 - \alpha_2) (\dot{\alpha}_1 - \dot{\alpha}_2) \dot{l}_1 + m_2 \sin(\alpha_2 - \alpha_1) \dot{l}_1 \dot{\alpha}_1$

5.5.2 Calculation of $\frac{1}{2} \frac{\partial}{\partial z}(\dot{z}^\top M(z) \dot{z}) \in \mathbb{R}^5$

$$\dot{z}^\top M(z) \dot{z} = \sum_{k=1}^5 \dot{z}_k \sum_{j=1}^5 m_{kj}(z) \dot{z}_j \quad (116)$$

so that

$$\begin{aligned}
\dot{z}^\top M(z) \dot{z} &= \sum_{k=1}^5 \dot{z}_k \sum_{j=1}^5 m_{1j}(z) \dot{z}_j \\
&= \dot{x} \sum_{j=1}^5 m_{1j}(z) \dot{z}_j + \dot{\alpha}_1 \sum_{j=1}^5 m_{2j}(z) \dot{z}_j + \dot{\alpha}_2 \sum_{j=1}^5 m_{3j}(z) \dot{z}_j \\
&\quad + \dot{l}_1 \sum_{j=1}^5 m_{4j}(z) \dot{z}_j + \dot{l}_2 \sum_{j=1}^5 m_{5j}(z) \dot{z}_j \\
&= \dot{x}^2(m + m_1 + m_2) + \dot{x}\dot{\alpha}_1(m_1 + m_2)l_1 \cos(\alpha_1) + \dot{x}\dot{\alpha}_2 m_2 l_2 \cos(\alpha_2) + \dot{x}\dot{l}_1(m_1 + m_2) \sin(\alpha_1) \\
&\quad + \dot{x}\dot{l}_2 m_2 \sin(\alpha_2) \\
&\quad + \dot{\alpha}_1 \dot{x}(m_1 + m_2)l_1 \cos(\alpha_1) + \dot{\alpha}_1^2(m_1 + m_2)l_1^2 + \dot{\alpha}_1 \dot{\alpha}_2 m_2 l_1 l_2 \cos(\alpha_1 - \alpha_2) \\
&\quad + \dot{\alpha}_1 \dot{l}_2 m_2 l_1 \sin(\alpha_2 - \alpha_1) \\
&\quad + \dot{\alpha}_2 \dot{x} m_2 l_2 \cos(\alpha_2) + \dot{\alpha}_2 \dot{\alpha}_1 m_2 l_1 l_2 \cos(\alpha_1 - \alpha_2) + \dot{\alpha}_2^2 m_2 l_2^2 + \dot{\alpha}_2 \dot{l}_1 m_2 l_2 \sin(\alpha_1 - \alpha_2) \\
&\quad + \dot{l}_1 \dot{x}(m_1 + m_2) \sin(\alpha_1) + \dot{l}_1 \dot{\alpha}_2 m_2 l_2 \sin(\alpha_1 - \alpha_2) + \dot{l}_1^2(m_1 + m_2) + \dot{l}_1 \dot{l}_2 m_2 \cos(\alpha_1 - \alpha_2) \\
&\quad + \dot{l}_2 \dot{x} m_2 \sin(\alpha_2) + \dot{l}_2 \dot{\alpha}_1 m_2 l_1 \sin(\alpha_2 - \alpha_1) + \dot{l}_2 \dot{l}_1 m_2 \cos(\alpha_1 - \alpha_2) + \dot{l}_2^2 m_2.
\end{aligned} \tag{117}$$

Therefore:

$$\frac{\partial}{\partial x}(\dot{z}^\top M(z) \dot{z}) = 0 \tag{118}$$

$$\begin{aligned}
\frac{\partial}{\partial \alpha_1}(\dot{z}^\top M(z) \dot{z}) &= -2\dot{x}\dot{\alpha}_1(m_1 + m_2)l_1 \sin(\alpha_1) + 2\dot{x}\dot{l}_1(m_1 + m_2) \cos(\alpha_1) - 2\dot{\alpha}_1 \dot{\alpha}_2 m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2) \\
&\quad - 2\dot{\alpha}_1 \dot{l}_2 m_2 l_1 \cos(\alpha_2 - \alpha_1) + 2\dot{l}_1 \dot{\alpha}_2 m_2 l_2 \cos(\alpha_1 - \alpha_2) - 2\dot{l}_1 \dot{l}_2 m_2 \sin(\alpha_1 - \alpha_2)
\end{aligned} \tag{119}$$

$$\begin{aligned}
\frac{\partial}{\partial \alpha_2}(\dot{z}^\top M(z) \dot{z}) &= -2\dot{x}\dot{\alpha}_2 m_2 l_2 \sin(\alpha_2) + 2\dot{x}\dot{l}_2 m_2 \cos(\alpha_2) + 2\dot{\alpha}_2 \dot{\alpha}_1 m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2) \\
&\quad + 2\dot{\alpha}_1 \dot{l}_2 m_2 l_1 \cos(\alpha_2 - \alpha_1) - 2\dot{\alpha}_2 \dot{l}_1 m_2 l_2 \cos(\alpha_1 - \alpha_2) + 2\dot{l}_1 \dot{l}_2 m_2 \sin(\alpha_1 - \alpha_2)
\end{aligned} \tag{120}$$

$$\begin{aligned}
\frac{\partial}{\partial l_1}(\dot{z}^\top M(z) \dot{z}) &= 2\dot{x}\dot{\alpha}_1(m_1 + m_2) \cos(\alpha_1) + 2\dot{\alpha}_1 \dot{\alpha}_2 m_2 l_2 \cos(\alpha_1 - \alpha_2) \\
&\quad + 2\dot{l}_2 \dot{\alpha}_1 m_2 \sin(\alpha_2 - \alpha_1) + 2\dot{\alpha}_1^2(m_1 + m_2)l_1
\end{aligned} \tag{121}$$

$$\begin{aligned}
\frac{\partial}{\partial l_2}(\dot{z}^\top M(z) \dot{z}) &= 2\dot{x}\dot{\alpha}_2 m_2 \cos(\alpha_2) + 2\dot{\alpha}_2 \dot{\alpha}_1 m_2 l_1 \cos(\alpha_1 - \alpha_2) + 2\dot{l}_1 \dot{\alpha}_2 m_2 \sin(\alpha_1 - \alpha_2) + 2\dot{\alpha}_2^2 m_2 l_2
\end{aligned} \tag{122}$$

5.6 Calculation of $C(z, \dot{z}) \dot{z} \in \mathbb{R}^5$

1. First component: $-(m_1 + m_2)l_1 \sin(\alpha_1)\dot{\alpha}_1^2 - m_2 l_2 \sin(\alpha_2)\dot{\alpha}_2^2 + 2(m_1 + m_2) \cos(\alpha_1)\dot{\alpha}_1 \dot{l}_1 + 2m_2 \cos(\alpha_2)\dot{\alpha}_2 \dot{l}_2$. So the first row of $C(z, \dot{z})$ is

$$\begin{aligned}
C_{1\bullet}(z, \dot{z}) &= (0, -(m_1 + m_2)l_1 \sin(\alpha_1)\dot{\alpha}_1 + (m_1 + m_2) \cos(\alpha_1)\dot{l}_1, -m_2 l_2 \sin(\alpha_2)\dot{\alpha}_2 + m_2 \cos(\alpha_2)\dot{l}_2, \\
&\quad (m_1 + m_2) \cos(\alpha_1)\dot{\alpha}_1, m_2 \cos(\alpha_2)\dot{\alpha}_2)
\end{aligned} \tag{123}$$

2. Second component: $-(m_1 + m_2)l_1 \sin(\alpha_1)\dot{\alpha}_1 \dot{x} - m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2)(\dot{\alpha}_1 - \dot{\alpha}_2)\dot{\alpha}_2 + m_2 l_1 \cos(\alpha_2 - \alpha_1)(\dot{\alpha}_2 - \dot{\alpha}_1)\dot{l}_2 + (m_1 + m_2)\dot{l}_1 \dot{x} \cos(\alpha_1) + 2(m_1 + m_2)l_1 \dot{l}_1 \dot{\alpha}_1 + m_2 l_1 \dot{l}_2 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2) + m_2 l_2 \dot{l}_1 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2) + m_2 \dot{l}_1 \dot{l}_2 \sin(\alpha_2 - \alpha_1) + \dot{x}\dot{\alpha}_1(m_1 + m_2)l_1 \sin(\alpha_1) - \dot{x}\dot{l}_1(m_1 + m_2) \cos(\alpha_1) +$

$\dot{\alpha}_1 \dot{\alpha}_2 m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2) + \dot{\alpha}_1 \dot{l}_2 m_2 l_1 \cos(\alpha_2 - \alpha_1) - \dot{l}_1 \dot{\alpha}_2 m_2 l_2 \cos(\alpha_1 - \alpha_2) + \dot{l}_1 \dot{l}_2 m_2 \sin(\alpha_1 - \alpha_2)$, after simplifications: $m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2) \dot{\alpha}_2^2 + 2m_2 l_1 \cos(\alpha_2 - \alpha_1) \dot{\alpha}_2 \dot{l}_2 + 2(m_1 + m_2) l_1 \dot{l}_1 \dot{\alpha}_1 + m_2 l_1 l_2 \sin(\alpha_2 - \alpha_1) + l_1 \dot{l}_2 m_2 \sin(\alpha_1 - \alpha_2) = m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2) \dot{\alpha}_2^2 + 2m_2 l_1 \cos(\alpha_2 - \alpha_1) \dot{\alpha}_2 \dot{l}_2 + 2(m_1 + m_2) l_1 \dot{l}_1 \dot{\alpha}_1$

so the second row of $C(z, \dot{z})$ can be chosen as

$$C_{2\bullet}(z, \dot{z}) = (0, (m_1 + m_2) l_1 \dot{l}_1, m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2) \dot{\alpha}_2 + m_2 l_1 \cos(\alpha_2 - \alpha_1) \dot{l}_2, (m_1 + m_2) l_1 \dot{\alpha}_1, m_2 l_1 \cos(\alpha_2 - \alpha_1) \dot{\alpha}_2) \quad (124)$$

3. Third component: $m_2 \dot{l}_2 \dot{x} \cos(\alpha_2) - m_2 l_2 \sin(\alpha_2) \dot{\alpha}_2 \dot{x} - m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2) (\dot{\alpha}_1 - \dot{\alpha}_2) \dot{\alpha}_1 + m_2 \dot{l}_2 \cos(\alpha_1 - \alpha_2) (\dot{\alpha}_1 - \dot{\alpha}_2) \dot{l}_1 + m_2 \dot{l}_1 \dot{\alpha}_1 l_2 \cos(\alpha_1 - \alpha_2) + m_2 l_1 \dot{l}_2 \dot{\alpha}_1 \cos(\alpha_1 - \alpha_2) + 2m_2 l_2 l_2 \dot{\alpha}_2 + m_2 \dot{l}_2 \dot{l}_1 \sin(\alpha_1 - \alpha_2) + \dot{x} \dot{\alpha}_2 m_2 l_2 \sin(\alpha_2) - \dot{x} \dot{l}_2 m_2 \cos(\alpha_2) - \dot{\alpha}_2 \dot{\alpha}_1 m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2) - \dot{\alpha}_1 \dot{l}_2 m_2 l_1 \cos(\alpha_2 - \alpha_1) + \dot{\alpha}_2 \dot{l}_1 m_2 l_2 \cos(\alpha_1 - \alpha_2) - \dot{l}_1 \dot{l}_2 m_2 \sin(\alpha_1 - \alpha_2)$, after simplifications: $-m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2) \dot{\alpha}_1^2 - m_2 l_2 \cos(\alpha_1 - \alpha_2) \dot{\alpha}_2 \dot{l}_1 + 2m_2 \dot{l}_1 \dot{\alpha}_1 l_2 \cos(\alpha_1 - \alpha_2) + 2m_2 l_2 l_2 \dot{\alpha}_2 + m_2 \dot{l}_2 \dot{l}_1 \sin(\alpha_1 - \alpha_2) + \dot{\alpha}_2 \dot{l}_1 m_2 l_2 \cos(\alpha_1 - \alpha_2) - \dot{l}_1 \dot{l}_2 m_2 \sin(\alpha_1 - \alpha_2) = -m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2) \dot{\alpha}_1^2 + 2m_2 \dot{l}_1 \dot{\alpha}_1 l_2 \cos(\alpha_1 - \alpha_2) + 2m_2 l_2 l_2 \dot{\alpha}_2$

so the third row of $C(z, \dot{z})$ can be chosen as

$$C_{3\bullet}(z, \dot{z}) = (0, -m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2) \dot{\alpha}_1 + m_2 \dot{l}_1 l_2 \cos(\alpha_1 - \alpha_2), m_2 l_2 l_2, +m_2 \dot{\alpha}_1 l_2 \cos(\alpha_1 - \alpha_2), m_2 l_2 \dot{\alpha}_2) \quad (125)$$

4. Fourth component: $(m_1 + m_2) \cos(\alpha_1) \dot{\alpha}_1 \dot{x} + m_2 l_2 \cos(\alpha_1 - \alpha_2) (\dot{\alpha}_1 - \dot{\alpha}_2) \dot{\alpha}_2 - m_2 \sin(\alpha_1 - \alpha_2) (\dot{\alpha}_1 - \dot{\alpha}_2) \dot{l}_2 + m_2 \dot{l}_2 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - \dot{x} \dot{\alpha}_1 (m_1 + m_2) \cos(\alpha_1) - \dot{\alpha}_1 \dot{\alpha}_2 m_2 l_2 \cos(\alpha_1 - \alpha_2) - \dot{l}_2 \dot{\alpha}_1 m_2 \sin(\alpha_2 - \alpha_1) - \dot{\alpha}_1^2 (m_1 + m_2) l_1$, after simplifications: $-m_2 l_2 \cos(\alpha_1 - \alpha_2) \dot{\alpha}_2^2 + 2m_2 \dot{l}_2 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - \dot{\alpha}_1^2 (m_1 + m_2) l_1$

so the fourth row of $C(z, \dot{z})$ can be chosen as

$$C_{4\bullet}(z, \dot{z}) = (0, -\dot{\alpha}_1 (m_1 + m_2) l_1, -m_2 l_2 \cos(\alpha_1 - \alpha_2) \dot{\alpha}_2 + m_2 \dot{l}_2 \sin(\alpha_1 - \alpha_2), 0, m_2 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2)) \quad (126)$$

5. Fifth component: $m_2 \cos(\alpha_2) \dot{\alpha}_2 \dot{x} + m_2 l_1 \cos(\alpha_2 - \alpha_1) (\dot{\alpha}_2 - \dot{\alpha}_1) \dot{\alpha}_1 - m_2 \sin(\alpha_1 - \alpha_2) (\dot{\alpha}_1 - \dot{\alpha}_2) \dot{l}_1 + m_2 \sin(\alpha_2 - \alpha_1) \dot{l}_1 \dot{\alpha}_1 - \dot{x} \dot{\alpha}_2 m_2 \cos(\alpha_2) - \dot{\alpha}_2 \dot{\alpha}_1 m_2 l_1 \cos(\alpha_1 - \alpha_2) - \dot{l}_1 \dot{\alpha}_2 m_2 \sin(\alpha_1 - \alpha_2) - \dot{\alpha}_2^2 m_2 l_2$, after simplifications: $-m_2 l_1 \cos(\alpha_2 - \alpha_1) \dot{\alpha}_1^2 - 2m_2 \sin(\alpha_1 - \alpha_2) \dot{\alpha}_1 \dot{l}_1 - \dot{\alpha}_2^2 m_2 l_2$

so the fifth row of $C(z, \dot{z})$ can be chosen as

$$C_{5\bullet}(z, \dot{z}) = (0, -m_2 l_1 \cos(\alpha_2 - \alpha_1) \dot{\alpha}_1 + m_2 \sin(\alpha_2 - \alpha_1) \dot{l}_1, -\dot{\alpha}_2 m_2 l_2, -m_2 \sin(\alpha_1 - \alpha_2) \dot{\alpha}_1, 0) \quad (127)$$

↪ It is noteworthy that the above choice for $C(z, \dot{z})$ corresponds to the Christoffel's symbols in (5) with $\frac{d}{dt}(M(z)) = C(z, \dot{z}) + C^\top(z, \dot{z})$.

↪ This choice is important because this form of the matrix $C(z, \dot{z})$ is useful in passivity-based controllers requiring the well-known skew-symmetry property [3].

5.7 Forces that derive from gravity potential energy

Using (52) we obtain:

$$\begin{aligned}
 \frac{\partial U_g}{\partial x} &= 0, & \frac{\partial U_g}{\partial \alpha_1} &= (m_1 + m_2)gl_1 \sin(\alpha_1), \\
 \frac{\partial U_g}{\partial \alpha_2} &= m_2g \sin(\alpha_2), & \frac{\partial U_g}{\partial l_1} &= -(m_1 + m_2)g \cos(\alpha_1), \\
 \frac{\partial U_g}{\partial l_2} &= -m_2g \cos(\alpha_2)
 \end{aligned} \tag{128}$$

The corresponding generalized forces satisfy $F_g(z) = -\frac{\partial U_g}{\partial z}$.

6 Mass Matrix Structure

6.1 Other Generalized Coordinates: Angles α_i , $1 \leq i \leq N$

Most of the authors work with the angles $\alpha_i = \sum_{j=1}^i \theta_j$, with generalized coordinate $q_\alpha = (x, \alpha_1, \dots, \alpha_N, l_1, \dots, l_N)^\top$, see Figure 1. The transformation matrix $q_\alpha = Tq$ is given by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & I_N \end{pmatrix}, J \in \mathbb{R}^{N \times N}, \text{ with :}$$

$$J = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & & & & \vdots \\ 1 & 1 & 1 & \dots & 1 & 1 & 0 \\ 1 & 1 & 1 & \dots & & 1 & 1 \end{pmatrix} \tag{129}$$

The matrix J is full-rank Toeplitz [1, Definition 3.1.3], and its inverse is [1, Fact 3.18.11]:

$$J^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 0 & 0 & \dots & & 0 \\ 0 & -1 & 1 & 0 & \dots & & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & \dots & 0 & -1 & 1 & 0 \\ 0 & 0 & \dots & & 0 & -1 & 1 \end{pmatrix} \tag{130}$$

Then

$$\begin{aligned}
M_\alpha(q_\alpha) &= T^{-\top} M(q) T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & J^{-\top} & 0 \\ 0 & 0 & I_N \end{pmatrix} M(q) \begin{pmatrix} 1 & 0 & 0 \\ 0 & J^{-1} & 0 \\ 0 & 0 & I_N \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & J^{-\top} & 0 \\ 0 & 0 & I_N \end{pmatrix} \begin{pmatrix} M_{x\theta}(q) & 0 \\ 0 & M_{ll} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & J^{-1} & 0 \\ 0 & 0 & I_N \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & J^{-\top} \end{pmatrix} M_{x\theta}(q) \begin{pmatrix} 1 & 0 \\ 0 & J^{-1} \end{pmatrix} & 0 \\ 0 & M_{ll} \end{pmatrix} = \begin{pmatrix} M_{x\alpha}(q_\alpha) & 0 \\ 0 & M_{ll} \end{pmatrix}.
\end{aligned} \tag{131}$$

Remark 2 (Differences with flexible-joint manipulators) *The mass matrix in coordinates q or q_α , has a block-diagonal structure, with small off-diagonal blocks in the neighborhood of the cable's vertical posture. But, this does not seem to put the system's dynamics in an amenable form for control design, as it is for instance the case for Spong's model of flexible joint manipulators. One reason is the number and location of the inputs in (46): the underactuated structure of overhead cranes and of flexible-joint manipulators, totally differ one from each other. It is noteworthy that Spong's model for flexible-joint manipulators, is obtained by neglecting the kinetic energy of actuators' shafts, which is a justified assumption in many applications, and allows one to get rid of some inertial couplings. Doing a similar "triangularization" assumption in $M_{\theta\theta}(q)$ or $M_{\alpha\alpha}(z)$ is not possible. In this sense the overhead crane dynamics is closer to the complete flexible-joint manipulators dynamics where off-diagonal blocks are present in the mass matrix [19]. It is known that the backstepping method no longer applies in this case.*

6.2 Horizontal Positions of the Mass-Points

Let us analyse the system when the coordinates are the horizontal coordinates of each point-mass A_i : $x_i = x + \sum_{j=1}^i l_j \sin(\sum_{k=1}^j \theta_k)$, and the angles θ_i , see Figure 1. The small angle and large payload mass assumptions are made to ease the calculations of the transformation (which otherwise involves trigonometric functions): $x_i = x + \sum_{j=1}^i l_j (\sum_{k=1}^j \theta_k)$. It is noteworthy that this type of coordinates together with the small angles assumptions, are used in some PDEs approaches (wave, or string equations) [9, 8]. The new generalized coordinate is $q_x = (x, \theta_1, \dots, \theta_N, x_1, \dots, x_N)^\top$, and the transformation from q to q_x is nonlinear: $q_x = Q(q)$, hence $\dot{q}_x = \frac{\partial Q}{\partial q}(q) \dot{q}$ and $\ddot{q}_x = \frac{\partial^2 Q}{\partial q^2}(q) \ddot{q} + \frac{d}{dt} \left(\frac{\partial Q}{\partial q} \right) (q, \dot{q}) \dot{q}$. In the new coordinates $M_x(q_x) = \frac{\partial Q^{-\top}}{\partial q} M(q) \frac{\partial Q^{-1}}{\partial q}$. We have:

$$\frac{\partial Q}{\partial q}(q) = \begin{pmatrix} I_{N+1} & 0 \\ 1 & \\ \vdots & Q_L \quad Q_\theta \\ 1 & \end{pmatrix}, \tag{132}$$

with:

$$Q_L = \begin{pmatrix} l_1 & 0 & \dots & & 0 \\ l_1 + l_2 & l_2 & 0 & \dots & 0 \\ l_1 + l_2 + l_3 & l_2 + l_3 & l_3 & 0 & \dots & 0 \\ \vdots & & & & \vdots & \\ \vdots & & & & l_{N-1} & 0 \\ \sum_{k=1}^N l_k & \sum_{k=2}^N l_k & \dots & l_{N-1} + l_N & l_N \end{pmatrix} \quad (133)$$

and

$$Q_\theta = \begin{pmatrix} \theta_1 & 0 & \dots & 0 \\ \theta_1 & \theta_1 + \theta_2 & 0 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & 0 \\ \theta_1 & \theta_1 + \theta_2 & \dots & \sum_{k=1}^N \theta_k \end{pmatrix} \quad (134)$$

Clearly Q_θ is not invertible at $\theta_1 = 0$, *etc*, its determinant being equal to $\theta_1(\theta_1 + \theta_2) \dots (\sum_{k=1}^N \theta_k)$, and neither is $\frac{\partial Q}{\partial q}(q)$ at these angles, because the lengths l_k cannot be obtained from q_x at those coordinates values. From the matrix inversion Lemma:

$$\frac{\partial Q^{-1}}{\partial q}(q) = \begin{pmatrix} I_{N+1} & 0 \\ -Q_\theta^{-1} \bar{Q}_L & Q_\theta^{-1} \end{pmatrix} \quad (135)$$

with $\bar{Q}_L = \begin{pmatrix} 1 \\ \vdots \\ Q_L \\ 1 \end{pmatrix}$, and $(\dot{x}_1, \dots, \dot{x}_N)^\top = \bar{Q}_L \dot{\theta} + Q_\theta \dot{L}$, $L = (l_1, \dots, l_N)^\top$. Therefore:

$$M_x(q_x) = \begin{pmatrix} I_{N+1} & -(Q_\theta^{-1} \bar{Q}_L)^\top \\ 0 & Q_\theta^{-\top} \end{pmatrix} \begin{pmatrix} M_{x\theta}(q) & M_{x\theta l}(q) \\ M_{lx\theta}(q) & M_{ll}(q) \end{pmatrix} \begin{pmatrix} I_{N+1} & 0 \\ -Q_\theta^{-1} \bar{Q}_L & Q_\theta^{-1} \end{pmatrix} \quad (136)$$

In order to analyze the dynamics using the positions x_i let us try to start directly with the kinetic energy, doing some basic assumptions as done in the PDEs approaches [9, 8], like constant total length, that we may translate here by constant length, and pure horizontal motion of the nodes. In this case $q = (x, x_1, x_2, \dots, x_N)^\top$, $V_{A_i} = \dot{x} + \dot{x}_i$, and the total kinetic energy is $T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \sum_{i=1}^N m_i \dot{x}_i^2$. In this case the mass matrix is linear invariant with $M(q) = \text{diag}(m, m_1, \dots, m_N)$, and the Lagrange dynamics is easily obtained.

7 Links with Linear Longitudinal Elasticity (2D Operating Space)

Flexibility at the joints have already been studied in section 4.3. Now the same system is considered as in Figure 1, with elastic links to approximate extensible cables. This is depicted in Figure 2. Each link has a longitudinal linear elasticity with stiffness $k_i > 0$, $1 \leq i \leq N$. The same framework as in the foregoing sections is adopted, but the potential energy is augmented with terms $\frac{1}{2} k_i (l_i - l_{i,r})^2$ (assuming that springs are at rest for $l_i = l_{i,r}$). This

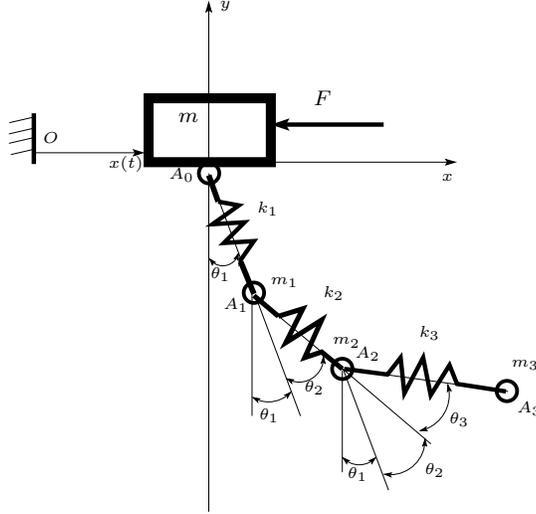


Figure 2: Overhead crane with pendulum-like cable (elastic links).

model is close in spirit to the lumped-mass models developed in [6, 11], but nonlinearities are considered here. It is also easy to add some viscous friction (linear spring-dashpot model) $c_i \dot{l}_i$, which is some kind of Rayleigh dissipation [3]. Remind that $l = (l_1, l_2, \dots, l_N)^\top$, one has:

$$\frac{\partial U_{elas}}{\partial l} = (k_1(l_1 - l_{1,r}), \dots, k_N(l_N - l_{N,r}))^\top \quad (137)$$

Thus (46) is changed to:

$$\left\{ \begin{array}{l} (a) \quad \frac{d}{dt} \frac{\partial(T_{trol} + T_{trans})}{\partial \dot{x}} = Q_x = F \\ (b) \quad \frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{\theta}} - \frac{\partial(T_{trans} - U_g - U_{flex})}{\partial \theta} + C_\theta \dot{\theta} = Q_\theta = 0 \\ (c) \quad \frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{l}} - \frac{\partial(T_{trans} - U_g - U_{flex} - U_{elas})}{\partial l} + C_l \dot{l} = Q_l = (F_1, 0, \dots, 0)^\top \end{array} \right. \quad (138)$$

with $C_l = \text{diag}(c_i)$, the form of $C_\theta \succcurlyeq 0$ depends on the choice of the coordinates.

Remark 3 (Linear chains) *The overhead crane dynamics and control may be seen as an extension of the linear chains dynamics and control [14, 15]: it is a nonlinear chain that is much more complex than its linear counterpart, where (138) (a) and (b) are absent.*

Remark 4 (Influence of flexibilities) *The flexibility potential energy due to links' elasticity, does not fundamentally modify the system's structure for control, as it does for instance for flexible joint kinematic chains with triangular structure, rendering the system suitable for tracking control via backstepping and passivity-based inputs, as proposed for the first time in [12, 4]. It is noteworthy also that the triangular structure for flexible-joint chains is obtained at the price of neglecting the kinetic energy of some parts, since otherwise the mass matrix is "dense" [19]. As said above, joint flexibilities in the joints A_i is not a reasonable modelling assumption for most cables, since cables usually possess a very low bending stiffness.*

Moreover this does not seem to bring anything useful to the control problem (at least with the chosen q).

From the above the Lagrange equations are (which are also given in [18, Equations (6) (7) (8)] and [17, Equations (1) (2) (3)]):

$$\left\{ \begin{array}{l} \text{(a) } (m + m_1)\ddot{x} + m_1 \left(l_1 \cos(\theta_1)\ddot{\theta}_1 + \sin(\theta_1)\ddot{l}_1 + 2\dot{l}_1\dot{\theta}_1 \cos(\theta_1) - l_1\dot{\theta}_1^2 \sin(\theta_1) \right) = F \\ \text{(b) } m_1 l_1 \cos(\theta_1)\ddot{x} + m_1 l_1^2 \ddot{\theta}_1 + 2m_1 \dot{l}_1 l_1 \dot{\theta}_1 + m_1 g l_1 \sin(\theta_1) = 0 \\ \text{(c) } m_1 \sin(\theta_1)\ddot{x} + m_1 \ddot{l}_1 - m_1 g \cos(\theta_1) - m_1 l_1 \dot{\theta}_1^2 = F_1, \end{array} \right. \quad (139)$$

where F_1 is the force exerted on the ‘‘cable’’ by some winding device mounted at joint A_0 . Property (3) can be checked. Here $Q = (F, 0, F_1)^\top$ and passivity holds with supply rate $w(Q, \dot{q}) = F\dot{x} + F_1\dot{l}_1$. The mass matrix for $q = (x, \theta_1, l_1)^\top$ is:

$$M(q) = \begin{pmatrix} m + m_1 & m_1 l_1 \cos(\theta_1) & m_1 \sin(\theta_1) \\ m_1 l_1 \cos(\theta_1) & m_1 l_1^2 & 0 \\ m_1 \sin(\theta_1) & 0 & m_1 \end{pmatrix} = \begin{pmatrix} M_{x\theta_1} & M_{x\theta_1 l_1} \\ M_{x\theta_1 l_1}^\top & M_{ll} \end{pmatrix}. \quad (140)$$

with $M_{ll} = m_1$. Following [16], if the coordinate is rearranged as $q_a = (x, l_1)^\top$ and $q_{na} = \theta_1$, then:

$$M(q) = \begin{pmatrix} m + m_1 & m_1 \sin(\theta_1) & m_1 l_1 \cos(\theta_1) \\ m_1 \sin(\theta_1) & m_1 & 0 \\ m_1 l_1 \cos(\theta_1) & 0 & m_1 l_1^2 \end{pmatrix} = \begin{pmatrix} M_{xl_1} & M_{xl_1 \theta_1} \\ M_{xl_1 \theta_1}^\top & M_{\theta_1} \end{pmatrix}, \quad (141)$$

with $M_{xl_1 \theta_1} = m_1 l_1 \cos(\theta_1)$, $M_{\theta_1} = m_1 l_1^2$.

Remark 5 *If instead of lumped-mass model we consider that the links are rigid bodies with center of mass G_i situated inside the segments $A_i A_{i+1}$, and with moments of inertia J_i , then the above developments have to be modified accordingly. Their kinetic energies are the sum of the translational gravity centers energy $T_{G_i} = \frac{1}{2} m_i V_{G_i}^\top V_{G_i}$, plus the rotational kinetic energies $\frac{1}{2} J_i \dot{\theta}_i^2$.*

8 Single-pendulum in 3D operational space and varying links’ lengths

The angles notation are the classical ones, see [13, Figure 1] and [10, Fig. 1] and Figure 3. The velocity of the payload mass at the link’s tip A_1 in the Galilean frame of reference is given by:

$$V_{A_1} = \begin{pmatrix} \dot{x} + \dot{l}_1 \sin(\theta_1) \cos(\theta_2) + l_1 \dot{\theta}_1 \cos(\theta_1) \cos(\theta_2) - l_1 \sin(\theta_1) \sin(\theta_2) \dot{\theta}_2 \\ \dot{y} + \dot{l}_1 \sin(\theta_2) + l_1 \dot{\theta}_2 \cos(\theta_2) \\ -\dot{l}_1 \cos(\theta_1) \cos(\theta_2) + l_1 \dot{\theta}_1 \sin(\theta_1) \cos(\theta_2) + l_1 \dot{\theta}_2 \cos(\theta_1) \sin(\theta_2) \end{pmatrix} \quad (142)$$

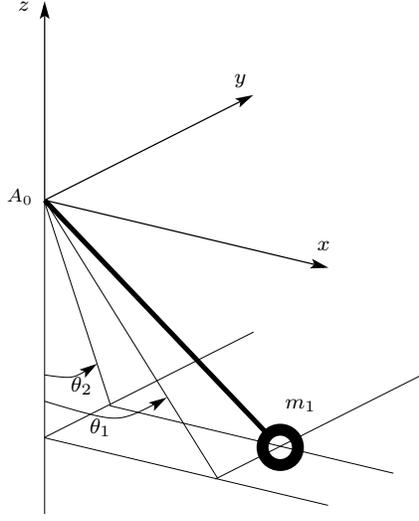


Figure 3: 3D single-pendulum.

Let us now deduce the system's kinetic energy which is the sum of the kinetic energy of the cart and the kinetic energy of the point mass $\frac{1}{2}m_1 V_{A_1}^\top V_{A_1}$, from which $M(q)$ in (150) is derived, as follows:

$$\begin{aligned}
T(q, \dot{q}) &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m_1(\dot{y} + \dot{l}_1 \sin(\theta_2) + l_1 \dot{\theta}_2 \cos(\theta_2))^2 \\
&+ \frac{1}{2}m_1(\dot{x} + \dot{l}_1 \sin(\theta_1) \cos(\theta_2) + l_1 \dot{\theta}_1 \cos(\theta_1) \cos(\theta_2) - l_1 \sin(\theta_1) \sin(\theta_2) \dot{\theta}_2)^2 \quad (143) \\
&+ \frac{1}{2}m_1(-\dot{l}_1 \cos(\theta_1) \cos(\theta_2) + l_1 \dot{\theta}_1 \sin(\theta_1) \cos(\theta_2) + l_1 \dot{\theta}_2 \cos(\theta_1) \sin(\theta_2))^2
\end{aligned}$$

The mass matrix is obtained by examining $\frac{\partial T}{\partial \dot{q}}$, $q = (x, y, \theta_1, \theta_2, l_1)^\top$, which yields:

$$\begin{aligned}
M(q) &= \\
&\begin{pmatrix}
(m + m_1) & 0 & m_1 l_1 \cos(\theta_1) \cos(\theta_2) & -m_1 l_1 \sin(\theta_1) \sin(\theta_2) & m_1 \sin(\theta_1) \cos(\theta_2) \\
0 & (m + m_1) & 0 & m_1 l_1 \cos(\theta_1) & m_1 \sin(\theta_2) \\
m_1 l_1 \cos(\theta_1) \cos(\theta_2) & 0 & m_1 l_1^2 \cos^2(\theta_2) & 0 & 0 \\
-m_1 l_1 \sin(\theta_1) \sin(\theta_2) & m_1 l_1 \cos(\theta_2) & 0 & m_1 l_1^2 & 0 \\
m_1 \sin(\theta_1) \cos(\theta_2) & m_1 \sin(\theta_2) & 0 & 0 & m_1
\end{pmatrix} \quad (144)
\end{aligned}$$

The nonlinear inertial forces matrix in (150) which satisfies the skew-symmetry property

$\frac{d}{dt}(M(q)) = C(q, \dot{q}) + C^\top(q, \dot{q})$ is given by (row by row):

$$C_{1\bullet}(q, \dot{q}) = \begin{pmatrix} 0, 0, -m_1 l_1 \sin(\theta_1) \cos(\theta_2) \dot{\theta}_1 - m_1 l_1 \cos(\theta_1) \sin(\theta_2) \dot{\theta}_2 + m_1 \cos(\theta_1) \cos(\theta_2) \dot{l}_1, \\ -m_1 l_1 \sin(\theta_1) \cos(\theta_2) \dot{\theta}_2 - m_1 l_1 \sin(\theta_1) \sin(\theta_2) \dot{l}_1 - m_1 l_1 \cos(\theta_1) \sin(\theta_2) \dot{\theta}_1, \\ -m_1 \sin(\theta_1) \sin(\theta_2) \dot{\theta}_2 + m_1 \cos(\theta_1) \cos(\theta_2) \dot{\theta}_1 \end{pmatrix} \quad (145)$$

$$C_{2\bullet}(q, \dot{q}) = (0, 0, 0, -m_1 l_1 \sin(\theta_2) \dot{\theta}_2 + m_1 \cos(\theta_2) \dot{l}_1, m_1 \cos(\theta_2) \dot{\theta}_2) \quad (146)$$

$$C_{3\bullet}(q, \dot{q}) = \begin{pmatrix} 0, 0, m_1 l_1 \cos^2(\theta_2) \dot{l}_1 - m_1 l_1^2 \cos(\theta_2) \sin(\theta_2) \dot{\theta}_2, -m_1 l_1^2 \sin(\theta_2) \cos(\theta_2) \dot{\theta}_1, \\ m_1 l_1 \cos^2(\theta_2) \dot{\theta}_1 \end{pmatrix} \quad (147)$$

$$C_{4\bullet}(q, \dot{q}) = (0, 0, m_1 l_1^2 \cos(\theta_2) \sin(\theta_2) \dot{\theta}_1, m_1 l_1 \dot{l}_1, m_1 l_1 \dot{\theta}_2) \quad (148)$$

$$C_{5\bullet}(q, \dot{q}) = (0, 0, -m_1 l_1 \cos^2(\theta_2) \dot{\theta}_1, -m_1 l_1 \dot{\theta}_2, 0) \quad (149)$$

As above the coefficients can be obtained from the Christoffel's symbols, or by computing $C(q, \dot{q})\dot{q} = (\frac{d}{dt}M(q))\dot{q} - \frac{1}{2}\frac{\partial}{\partial q} \dot{q}^\top M(q)\dot{q}$ and rearranging the terms in a suitable way so that $\dot{m}_{ij} = c_{ij} + c_{ji}$ for all $1 \leq i, j \leq 5$. The single-pendulum 3D dynamics take the form:

$$\left\{ \begin{array}{l} (m + m_1)\ddot{x} + m_1 l_1 \cos(\theta_1) \cos(\theta_2) \ddot{\theta}_1 - m_1 l_1 \sin(\theta_1) \sin(\theta_2) \ddot{\theta}_2 + m_1 \sin(\theta_1) \cos(\theta_2) \ddot{l}_1 + \\ \quad + 2m_1 \cos(\theta_1) \cos(\theta_2) \dot{l}_1 \dot{\theta}_1 - 2m_1 \sin(\theta_1) \sin(\theta_2) \dot{l}_1 \dot{\theta}_1 - m_1 l_1 \sin(\theta_1) \cos(\theta_2) \dot{\theta}_1^2 \\ \quad - m_1 l_1 \sin(\theta_1) \cos(\theta_2) \dot{\theta}_2^2 - 2m_1 l_1 \cos(\theta_1) \sin(\theta_2) \dot{\theta}_1 \dot{\theta}_2 = F_1 \\ \\ (m + m_1)\ddot{y} + m_1 l_1 \cos(\theta_1) \ddot{\theta}_2 + m_1 \sin(\theta_2) \ddot{l}_1 + 2m_1 \cos(\theta_2) \dot{l}_1 \dot{\theta}_2 - m_1 l_1 \sin(\theta_2) \dot{\theta}_2^2 = F_2 \\ \\ m_1 l_1 \cos(\theta_1) \cos(\theta_2) \ddot{x} + m_1 l_1^2 \cos^2(\theta_2) \ddot{\theta}_1 + 2m_1 l_1 \cos^2(\theta_2) \dot{l}_1 \dot{\theta}_1 - 2m_1 l_1^2 \sin(\theta_2) \cos(\theta_2) \dot{\theta}_1 \dot{\theta}_2 \\ \quad + m_1 g l_1 \sin(\theta_1) \cos(\theta_2) = 0 \\ \\ -m_1 l_1 \sin(\theta_1) \sin(\theta_2) \ddot{x} + m_1 l_1 \cos(\theta_2) \ddot{y} + m_1 l_1^2 \ddot{\theta}_2 + m_1 l_1^2 \cos(\theta_2) \sin(\theta_2) \dot{\theta}_1^2 + 2m_1 l_1 \dot{l}_1 \dot{\theta}_2 \\ \quad + m_1 g l_1 \cos(\theta_1) \sin(\theta_2) = 0 \\ \\ m_1 \sin(\theta_1) \cos(\theta_2) \ddot{x} + m_1 \sin(\theta_2) \ddot{y} + m_1 \ddot{l}_1 - m_1 l_1 \cos^2(\theta_2) \dot{\theta}_1^2 - m_1 l_1 \dot{\theta}_2^2 + m_1 g \cos(\theta_1) \cos(\theta_2) \\ = F_l \end{array} \right. \quad (150)$$

where $q = (x, y, \theta_1, \theta_2, l_1)^\top$, F_1 , F_2 and F_l are as above.

9 Double-pendulum in 3D operational space with varying links' lengths and constant masses

The dynamics of this OC have been derived in [13, Equations (1)-(6)] and [10, Equations (1)-(6)] in the case of constant links' lengths. The angles notations follow [13, Figure 1] and [10, Fig. 1]. When the links' lengths vary, the joints' centers A_1 and A_2 positions in a fixed

Galilean frame of reference are given by:

$$A_0A_1 = \begin{pmatrix} l_1 \sin(\theta_1) \cos(\theta_2) \\ l_1 \sin(\theta_2) \\ -l_1 \cos(\theta_1) \cos(\theta_2) \end{pmatrix} \quad (151)$$

where A_0 is the attachment joint between the cable and the cart, and

$$A_1A_2 = \begin{pmatrix} l_2 \sin(\theta_3) \cos(\theta_4) \\ l_2 \sin(\theta_4) \\ -l_2 \cos(\theta_3) \cos(\theta_4) \end{pmatrix} \quad (152)$$

so that

$$OA_2 = \begin{pmatrix} x + l_1 \sin(\theta_1) \cos(\theta_2) + l_2 \sin(\theta_3) \cos(\theta_4) \\ y + l_1 \sin(\theta_2) + l_2 \sin(\theta_4) \\ -l_1 \cos(\theta_1) \cos(\theta_2) - l_2 \cos(\theta_3) \cos(\theta_4) \end{pmatrix} \quad (153)$$

The spherical joints' velocities at points A_1 and A_2 are given by:

$$V_{A_1} = \begin{pmatrix} \dot{x} + \dot{l}_1 \sin(\theta_1) \cos(\theta_2) + l_1 \dot{\theta}_1 \cos(\theta_1) \cos(\theta_2) - l_1 \sin(\theta_1) \sin(\theta_2) \dot{\theta}_2 \\ \dot{y} + \dot{l}_1 \sin(\theta_2) + l_1 \dot{\theta}_2 \cos(\theta_2) \\ -\dot{l}_1 \cos(\theta_1) \cos(\theta_2) + l_1 \dot{\theta}_1 \sin(\theta_1) \cos(\theta_2) + l_1 \dot{\theta}_2 \cos(\theta_1) \sin(\theta_2) \end{pmatrix} \quad (154)$$

and

$$V_{A_2} = \begin{pmatrix} \dot{x} + \dot{l}_1 \sin(\theta_1) \cos(\theta_2) + l_1 \dot{\theta}_1 \cos(\theta_1) \cos(\theta_2) - l_1 \sin(\theta_1) \sin(\theta_2) \dot{\theta}_2 + \dot{l}_2 \sin(\theta_3) \cos(\theta_4) \\ \quad + l_2 \dot{\theta}_3 \cos(\theta_3) \cos(\theta_4) - l_2 \dot{\theta}_4 \sin(\theta_3) \sin(\theta_4) \\ \dot{y} + \dot{l}_1 \sin(\theta_2) + l_1 \dot{\theta}_2 \cos(\theta_2) + \dot{l}_2 \sin(\theta_4) + l_2 \dot{\theta}_4 \cos(\theta_4) \\ -\dot{l}_1 \cos(\theta_1) \cos(\theta_2) + l_1 \dot{\theta}_1 \sin(\theta_1) \cos(\theta_2) + l_1 \dot{\theta}_2 \cos(\theta_1) \sin(\theta_2) - \dot{l}_2 \cos(\theta_3) \cos(\theta_4) \\ \quad + l_2 \dot{\theta}_3 \sin(\theta_3) \cos(\theta_4) + l_2 \dot{\theta}_4 \cos(\theta_3) \sin(\theta_4) \end{pmatrix} \quad (155)$$

Thus the overhead crane system's total kinetic energy is obtained as:

$$\begin{aligned}
T(q, \dot{q}) = & \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m_1[\dot{y} + \dot{l}_1 \sin(\theta_2) + l_1\dot{\theta}_2 \cos(\theta_2)]^2 \\
& + \frac{1}{2}m_1[\dot{x} + \dot{l}_1 \sin(\theta_1) \cos(\theta_2) + l_1\dot{\theta}_1 \cos(\theta_1) \cos(\theta_2) - l_1 \sin(\theta_1) \sin(\theta_2)\dot{\theta}_2]^2 \\
& + \frac{1}{2}m_1[-\dot{l}_1 \cos(\theta_1) \cos(\theta_2) + l_1\dot{\theta}_1 \sin(\theta_1) \cos(\theta_2) + l_1\dot{\theta}_2 \cos(\theta_1) \sin(\theta_2)]^2 \\
& + \frac{1}{2}m_2[\dot{x} + \dot{l}_1 \sin(\theta_1) \cos(\theta_2) + l_1\dot{\theta}_1 \cos(\theta_1) \cos(\theta_2) - l_1\dot{\theta}_2 \sin(\theta_1) \sin(\theta_2) \\
& + \dot{l}_2 \sin(\theta_3) \cos(\theta_4) + l_2\dot{\theta}_3 \cos(\theta_3) \cos(\theta_4) - l_2\dot{\theta}_4 \sin(\theta_4) \sin(\theta_4)]^2 \\
& + \frac{1}{2}m_2[\dot{y} + \dot{l}_1 \sin(\theta_2) + l_1\dot{\theta}_2 \cos(\theta_2) + \dot{l}_2 \sin(\theta_4) + l_2\dot{\theta}_4 \cos(\theta_4)]^2 \\
& + \frac{1}{2}m_2[-\dot{l}_1 \cos(\theta_1) \cos(\theta_2) + l_1\dot{\theta}_1 \sin(\theta_1) \cos(\theta_2) + l_1\dot{\theta}_2 \cos(\theta_1) \sin(\theta_2) \\
& - \dot{l}_2 \cos(\theta_3) \cos(\theta_4) + l_2\dot{\theta}_3 \sin(\theta_3) \cos(\theta_4) + l_2\dot{\theta}_4 \cos(\theta_3) \sin(\theta_4)]^2
\end{aligned} \tag{156}$$

The mass matrix is deduced by examining the term $\frac{\partial T}{\partial \dot{q}}$, and it is calculated:

$$\begin{aligned}
m_{11}(q) &= m + m_1 + m_2, \quad m_{12}(q) = 0, \quad m_{13}(q) = (m_1 + m_2)l_1 \cos(\theta_1) \cos(\theta_2), \\
m_{14}(q) &= -(m_1 + m_2)l_1 \sin(\theta_1) \sin(\theta_2), \quad m_{15}(q) = m_2l_2 \cos(\theta_3) \cos(\theta_4), \\
m_{16}(q) &= -m_2l_2 \sin(\theta_3) \sin(\theta_4), \quad m_{17}(q) = (m_1 + m_2) \sin(\theta_1) \cos(\theta_2), \\
m_{18}(q) &= m_2 \sin(\theta_3) \cos(\theta_4)
\end{aligned} \tag{157}$$

$$\begin{aligned}
m_{21}(q) &= 0, \quad m_{22}(q) = m + m_1 + m_2, \quad m_{23}(q) = 0, \quad m_{24}(q) = (m_1 + m_2)l_1 \cos(\theta_2), \\
m_{25}(q) &= 0, \quad m_{26}(q) = m_2l_2 \cos(\theta_4), \quad m_{27}(q) = (m_1 + m_2) \sin(\theta_2), \quad m_{28}(q) = m_2 \sin(\theta_4)
\end{aligned} \tag{158}$$

$$\begin{aligned}
m_{31}(q) &= m_{13}(q), \quad m_{32}(q) = m_{23}(q) = 0, \quad m_{33}(q) = (m_1 + m_2)l_1 \cos^2(\theta_2), \\
m_{34}(q) &= 0, \quad m_{35}(q) = m_2l_1l_2 \cos(\theta_2) \cos(\theta_4) \cos(\theta_1 - \theta_3), \\
m_{36}(q) &= m_2l_1l_2 \sin(\theta_2) \cos(\theta_2) \sin(\theta_1 - \theta_3), \quad m_{37}(q) = 0, \\
m_{38}(q) &= m_2l_1 \cos(\theta_4) \cos(\theta_2) \sin(\theta_3 - \theta_1)
\end{aligned} \tag{159}$$

$$\begin{aligned}
m_{41}(q) &= m_{14}(q), \quad m_{42}(q) = m_{24}(q), \quad m_{43}(q) = m_{34}(q) = 0, \quad m_{44}(q) = (m_1 + m_2)l_1^2, \\
m_{45}(q) &= m_2l_1l_2 \sin(\theta_2) \cos(\theta_4) \sin(\theta_3 - \theta_1), \quad m_{46}(q) = m_1l_1l_2 \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) \sin(\theta_4) \\
& + m_2l_1l_2[\cos(\theta_2) \cos(\theta_4) + \cos(\theta_1) \sin(\theta_1) \cos(\theta_3) \sin(\theta_2)], \quad m_{47}(q) = 0, \\
m_{48}(q) &= m_2l_1[\cos(\theta_2) \sin(\theta_4) - \cos(\theta_4) \sin(\theta_2) \cos(\theta_1 - \theta_3)]
\end{aligned} \tag{160}$$

$$\begin{aligned}
m_{51}(q) &= m_{15}(q), \quad m_{52}(q) = m_{25}(q), \quad m_{53}(q) = m_{35}(q), \quad m_{54}(q) = m_{45}(q), \\
m_{55}(q) &= m_2l_2^2 \cos^2(\theta_4), \quad m_{56}(q) = 0, \quad m_{57}(q) = m_2l_2 \cos(\theta_2) \cos(\theta_4) \sin(\theta_1 - \theta_3), \\
m_{58}(q) &= 0
\end{aligned} \tag{161}$$

$$\begin{aligned}
m_{61}(q) &= m_{16}(q), \quad m_{62}(q) = m_{26}(q), \quad m_{63}(q) = m_{36}(q), \quad m_{64}(q) = m_{46}(q), \quad m_{65}(q) = m_{56}(q), \\
m_{66}(q) &= m_2 l_2^2, \quad m_{67}(q) = m_2 l_2 [\cos(\theta_4) \sin(\theta_2) - \sin(\theta_4) \cos(\theta_2) \cos(\theta_1 - \theta_3)], \\
m_{68}(q) &= 0
\end{aligned} \tag{162}$$

$$\begin{aligned}
m_{71}(q) &= m_{17}(q), \quad m_{72}(q) = m_{27}(q), \quad m_{73}(q) = m_{37}(q), \quad m_{74}(q) = m_{47}(q), \\
m_{75}(q) &= m_{57}(q), \quad m_{76}(q) = m_{67}(q), \quad m_{77}(q) = m_1 + m_2, \\
m_{78}(q) &= m_2 [\sin(\theta_2) \sin(\theta_4) + \cos(\theta_2) \cos(\theta_4) \cos(\theta_1 - \theta_3)]
\end{aligned} \tag{163}$$

$$\begin{aligned}
m_{81}(q) &= m_{18}(q), \quad m_{82}(q) = m_{28}(q), \quad m_{83}(q) = m_{38}(q), \quad m_{84}(q) = m_{48}(q), \quad m_{85}(q) = m_{58}(q), \\
m_{86}(q) &= m_{68}(q), \quad m_{87}(q) = m_{78}(q), \quad m_{88}(q) = m_2
\end{aligned} \tag{164}$$

Then the Christoffel's symbols can be used to compute the Coriolis/centrifugal nonlinear inertial generalized forces, as $C(q, \dot{q}) = (\sum_{k=1}^8 \Gamma_{ijk} \dot{q}_k)_{ij}$, where $\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial m_{ij}}{\partial q_k} + \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{kj}}{\partial q_i} \right)$ [3, Lemma 6.16]. The matrix $C(q, \dot{q})$ has 64 components, and each component ij is calculated with 8 symbols Γ_{ijk} , each symbol involves the calculation of 3 partial derivatives: the overall matrix' calculation therefore involves $64 \times 8 \times 3 = 1536$ terms of the form $\frac{\partial m_{ij}}{\partial q_k}$:

$$\begin{aligned}
C_{1\bullet}(q, \dot{q}) &= (0, 0, c_{13} = (m_1 + m_2)[-l_1 \sin(\theta_1) \cos(\theta_2) \dot{\theta}_1 - l_1 \cos(\theta_1) \sin(\theta_2) \dot{\theta}_2 + \cos(\theta_1) \cos(\theta_2) \dot{l}_1], \\
&c_{14} = -(m_1 + m_2)[l_1 \cos(\theta_1) \sin(\theta_2) \dot{\theta}_1 + \sin(\theta_1) \sin(\theta_2) \dot{l}_1 + l_1 \sin(\theta_1) \cos(\theta_2) \dot{\theta}_2], \\
&c_{15} = m_2[-l_2 \sin(\theta_3) \cos(\theta_4) \dot{\theta}_3 - l_2 \cos(\theta_3) \sin(\theta_4) \dot{\theta}_4 + \cos(\theta_3) \cos(\theta_4) \dot{l}_2], \\
&c_{16} = -m_2[l_2 \cos(\theta_3) \sin(\theta_4) \dot{\theta}_3 + l_2 \sin(\theta_3) \cos(\theta_4) \dot{\theta}_4 + \sin(\theta_3) \sin(\theta_4) \dot{l}_2], \\
&c_{17} = (m_1 + m_2)[\cos(\theta_1) \cos(\theta_2) \dot{\theta}_1 - \sin(\theta_1) \sin(\theta_2) \dot{\theta}_2], \quad c_{18} = m_2[\cos(\theta_3) \cos(\theta_4) \dot{\theta}_3 \\
&- \sin(\theta_3) \sin(\theta_4) \dot{\theta}_4]
\end{aligned} \tag{165}$$

$$\begin{aligned}
C_{2\bullet}(q, \dot{q}) &= (0, 0, 0, c_{24} = (m_1 + m_2)[-l_1 \sin(\theta_2) \dot{\theta}_2 + \cos(\theta_2) \dot{l}_1], \quad 0, \quad c_{26} = -m_2 l_2 \sin(\theta_4) \dot{\theta}_4 \\
&+ m_2 \cos(\theta_4) \dot{l}_2, \quad c_{27} = (m_1 + m_2) \cos(\theta_2) \dot{\theta}_2, \quad c_{28} = m_2 \cos(\theta_4) \dot{\theta}_4
\end{aligned} \tag{166}$$

$$\begin{aligned}
C_{3\bullet}(q, \dot{q}) &= (0, 0, c_{33} = -(m_1 + m_2) l_1 \cos(\theta_2) \sin(\theta_2) \dot{\theta}_2 + \frac{1}{2} (m_1 + m_2) \cos^2(\theta_2) \dot{l}_1, \\
&c_{34} = -(m_1 + m_2) l_1 \cos(\theta_2) \sin(\theta_2) \dot{\theta}_2, \quad c_{35} = m_2 l_1 l_2 \cos(\theta_2) \cos(\theta_2) \sin(\theta_1 - \theta_3) \dot{\theta}_3 \\
&- m_2 l_1 l_2 \cos(\theta_2) \sin(\theta_4) \cos(\theta_1 - \theta_3) \dot{\theta}_4 + m_2 l_1 \cos(\theta_2) \cos(\theta_4) \cos(\theta_1 - \theta_3) \dot{l}_2, \\
&c_{36} = \frac{1}{2} (m_2 - m_1) l_1 l_2 \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) \sin(\theta_4) \dot{\theta}_2 \\
&- m_2 l_1 l_2 \cos(\theta_2) \sin(\theta_4) \cos(\theta_1 - \theta_3) \dot{\theta}_3 + m_2 l_1 l_2 \cos(\theta_4) \cos(\theta_2) \sin(\theta_1 - \theta_3) \dot{\theta}_4 \\
&+ m_2 l_1 \sin(\theta_4) \cos(\theta_2) \sin(\theta_1 - \theta_3) \dot{l}_2, \quad c_{37} = \frac{1}{2} (m_1 + m_2) \cos^2(\theta_2) \dot{\theta}_1, \\
&c_{38} = m_2 l_1 \cos(\theta_2) \cos(\theta_4) \cos(\theta_1 - \theta_3) \dot{\theta}_3 + m_2 l_1 \sin(\theta_4) \cos(\theta_2) \sin(\theta_1 - \theta_3) \dot{\theta}_4
\end{aligned} \tag{167}$$

$$\begin{aligned}
C_{4\bullet}(q, \dot{q}) = & (0, 0, c_{43} = (m_1 + m_2)l_1 \cos(\theta_2) \sin(\theta_2) \dot{\theta}_2, c_{44} = m_{(1+m_2)}l_1 \dot{l}_1, \\
& c_{45} = m_2 l_1 l_2 \sin(\theta_{1y}) \cos(\theta_{2y}) \cos(\theta_1 - \theta_3) \dot{\theta}_3 - m_2 l_1 l_2 \sin(\theta_2) \sin(\theta_4) \sin(\theta_3) \cos(\theta_1) \dot{\theta}_4 \\
& + \frac{m_1+m_2}{2} l_1 l_2 \sin(\theta_1) \sin(\theta_2) \cos(\theta_3) \sin(\theta_4) \dot{\theta}_4 + m_2 l_1 \sin(\theta_2) \cos(\theta_4) \sin(\theta_3 - \theta_1) \dot{l}_2, \\
& c_{46} = \frac{m_1+m_2}{2} l_1 l_2 \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) \sin(\theta_4) \dot{\theta}_1 + \frac{1}{2} m_1 l_1 l_2 \sin(\theta_1) \sin(\theta_2) \cos(\theta_3) \sin(\theta_4) \dot{\theta}_3 \\
& - \frac{1}{2} m_2 l_1 l_2 \sin(\theta_2) \sin(\theta_4) \sin(\theta_3 - \theta_1) \dot{\theta}_3 + m_1 l_1 l_2 \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) \cos(\theta_4) \dot{\theta}_4 \\
& - m_2 l_1 l_2 \cos(\theta_2) \sin(\theta_4) \dot{\theta}_4 + m_2 l_1 l_2 \cos(\theta_1) \sin(\theta_2) \cos(\theta_3) \cos(\theta_4) \dot{\theta}_4 \\
& + \frac{m_1-m_2}{2} l_2 \sin(\theta_1) \sin(\theta_2) \sin(\theta) \sin(\theta_4) \dot{l}_1 + \frac{m_1+m_2}{2} l_1 \sin(\theta_4) \sin(\theta_2) \sin(\theta_1) \sin(\theta_3) \dot{l}_2 \\
& + m_2 l_1 \cos(\theta_2) \cos(\theta_4) \dot{l}_2 + m_2 l_1 \cos(\theta_1) \sin(\theta_2) \cos(\theta_3) \sin(\theta_4) \dot{l}_2, c_{47} = (m_1 + m_2)l_1 \dot{\theta}_2 \\
& + \frac{m_1-m_2}{2} l_2 \sin(\theta_4) \sin(\theta_2) \sin(\theta_1) \sin(\theta_3) \dot{\theta}_4, c_{48} = m_2 l_1 \cos(\theta_2) \cos(\theta_4) \dot{\theta}_4 \\
& + m_2 l_1 \sin(\theta_4) \sin(\theta_2) \cos(\theta_1) \cos(\theta_3) \dot{\theta}_4 + \frac{m_1+m_2}{2} l_1 \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) \sin(\theta_4) \dot{\theta}_4)
\end{aligned} \tag{168}$$

$$\begin{aligned}
C_{5\bullet}(q, \dot{q}) = & (0, 0, c_{53} = -m_2 l_1 l_2 \cos(\theta_2) \cos(\theta_4) \sin(\theta_1 - \theta_3) \dot{\theta}_1 - m_2 l_1 l_2 \sin(\theta_2) \cos(\theta_4) \cos(\theta_1 - \theta_3) \dot{\theta}_2 \\
& + m_2 l_2 \cos(\theta_2) \cos(\theta_4) \cos(\theta_1 - \theta_3) \dot{l}_1, \\
& c_{54} = -m_2 l_1 l_2 \sin(\theta_2) \cos(\theta_4) \cos(\theta_1 - \theta_3) \dot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_2) \cos(\theta_4) \sin(\theta_3 - \theta_1) \dot{\theta}_2 \\
& + \frac{m_2-m_1}{2} l_1 l_2 \sin(\theta_1) \sin(\theta_2) \sin(\theta_4) \cos(\theta_3) \dot{\theta}_4 + m_2 l_2 \sin(\theta_2) \cos(\theta_4) \sin(\theta_3 - \theta_1) \dot{l}_1, \\
& c_{55} = -m_2 l_2^2 \cos(\theta_4) \sin(\theta_4) \dot{\theta}_4 + m_2 l_2 \cos^2(\theta_4) \dot{l}_2, \\
& c_{56} = \frac{m_2-m_1}{2} l_1 l_2 \sin(\theta_2) \sin(\theta_4) \cos(\theta_3) \sin(\theta_1) \dot{\theta}_2 - m_2 l_2^2 \cos(\theta_4) \sin(\theta_4) \dot{\theta}_3, \\
& c_{57} = m_2 l_2 \cos(\theta_2) \cos(\theta_4) \cos(\theta_1 - \theta_3) \dot{\theta}_1 - m_2 l_2 \sin(\theta_2) \cos(\theta_4) \sin(\theta_1 - \theta_3) \dot{\theta}_2, \\
& c_{58} = -m_2 l_2 \cos^2(\theta_4) \dot{\theta}_3)
\end{aligned} \tag{169}$$

$$\begin{aligned}
C_{6\bullet}(q, \dot{q}) = & (0, 0, c_{63} = m_2 l_1 l_2 \sin(\theta_4) \cos(\theta_2) \cos(\theta_1 - \theta_3) \dot{\theta}_1 - m_2 l_1 l_2 \sin(\theta_4) \sin(\theta_2) \sin(\theta_1) \cos(\theta_3) \dot{\theta}_2 \\
& + \frac{1}{2} (m_1 + m_2) l_1 l_2 \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) \sin(\theta_4) \dot{\theta}_2 + m_2 l_2 \sin(\theta_4) \cos(\theta_2) \sin(\theta_1 - \theta_3) \dot{l}_1, \\
& c_{64} = -m_2 l_1 l_2 \sin(\theta_1) \sin(\theta_2) \cos(\theta_3) \sin(\theta_4) \dot{\theta}_1 + \frac{m_1+m_2}{2} l_1 l_2 \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) \sin(\theta_4) \dot{\theta}_1 \\
& + m_1 l_1 l_2 \sin(\theta_1) \cos(\theta_2) \sin(\theta_3) \sin(\theta_4) \dot{\theta}_2 - m_2 l_1 l_2 \sin(\theta_2) \cos(\theta_4) \dot{\theta}_2 \\
& + m_2 l_1 l_2 \cos(\theta_1) \cos(\theta_2) \cos(\theta_3) \sin(\theta_4) \dot{\theta}_2 + \frac{m_1-m_2}{2} l_1 l_2 \sin(\theta_2) \sin(\theta_4) \cos(\theta_3) \sin(\theta_1) \dot{\theta}_3 \\
& + m_2 l_2 \cos(\theta_1) \sin(\theta_2) \cos(\theta_3) \sin(\theta_4) \dot{l}_1 + m_2 l_2 \cos(\theta_2) \cos(\theta_4) \dot{l}_1 \\
& + \frac{m_1+m_2}{2} l_2 \sin(\theta_4) \sin(\theta_2) \sin(\theta_1) \sin(\theta_3) \dot{l}_1 + \frac{m_1-m_2}{2} l_1 \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) \sin(\theta_4) \dot{l}_2, \\
& c_{65} = \frac{m_1-m_2}{2} l_1 l_2 \sin(\theta_2) \sin(\theta_4) \cos(\theta_3) \sin(\theta_1) \dot{\theta}_2 + m_2 l_2^2 \cos(\theta_4) \sin(\theta_4) \dot{\theta}_3, \\
& c_{66} = m_2 l_2 \dot{l}_2, c_{67} = m_2 l_2 \cos(\theta_2) \sin(\theta_4) \sin(\theta_1 - \theta_3) \dot{\theta}_1 + m_2 l_2 \cos(\theta_4) \cos(\theta_2) \dot{\theta}_2 \\
& + m_2 l_2 \cos(\theta_1) \sin(\theta_2) \cos(\theta_3) \sin(\theta_4) \dot{\theta}_2 + \frac{m_1+m_2}{2} l_2 \sin(\theta_4) \sin(\theta_2) \sin(\theta_1) \sin(\theta_3) \dot{\theta}_2 \\
& c_{67} = m_2 l_2 \cos(\theta_{1y}) \sin(\theta_4) \sin(\theta_1 - \theta_3) \dot{\theta}_1 + m_2 l_2 \cos(\theta_4) \cos(\theta) \dot{\theta}_2 \\
& + m_2 l_2 \cos(\theta_1) \sin(\theta_2) \cos(\theta_3) \sin(\theta_4) \dot{\theta}_2 + \frac{m_1+m_2}{2} l_2 \sin(\theta_4) \sin(\theta_2) \sin(\theta_1) \sin(\theta_3) \dot{\theta}, \\
& c_{68} = m_2 l_2 \dot{\theta}_4 + \frac{m_1-m_2}{2} l_1 \sin(\theta_4) \sin(\theta_2) \sin(\theta_1) \sin(\theta_3) \dot{\theta}_2)
\end{aligned} \tag{170}$$

$$\begin{aligned}
C_{7\bullet}(q, \dot{q}) = & (0, 0, c_{73} = -\frac{1}{2}(m_1 + m_2) \cos^2(\theta_2) \dot{\theta}_1, c_{74} = -(m_1 + m_2) l_1 \dot{\theta}_2 \\
& - \frac{m_1 - m_2}{2} l_2 \sin(\theta_4) \sin(\theta_2) \sin(\theta_1) \sin(\theta_3) \dot{\theta}_4, \\
c_{75} = & -m_2 l_2 \cos(\theta_2) \cos(\theta_4) \cos(\theta_1 - \theta_3) \dot{\theta}_3 - m_2 l_2 \cos(\theta_2) \sin(\theta_4) \sin(\theta_1 - \theta_3) \dot{\theta}_4 \\
& + m_2 \cos(\theta_2) \cos(\theta_4) \sin(\theta_1 - \theta_4) \dot{l}_2, c_{76} = \frac{m_2 - m_1}{2} l_2 \sin(\theta_4) \sin(\theta_2) \sin(\theta_1) \sin(\theta_3) \dot{\theta}_2 \\
& - m_2 l_2 \cos(\theta_2) \sin(\theta_4) \sin(\theta_1 - \theta_3) \dot{\theta}_3 - m_2 l_2 \sin(\theta_4) \sin(\theta_2) \dot{\theta}_4 \\
& - m_2 l_2 \cos(\theta_4) \cos(\theta_2) \cos(\theta_1 - \theta_3) \dot{\theta}_4 + m_2 \cos(\theta_4) \sin(\theta_2) \dot{l}_2 \\
& - m_2 \sin(\theta_4) \cos(\theta_2) \cos(\theta_1 - \theta_3) \dot{l}_2, c_{77} = 0, c_{78} = m_2 \cos(\theta_2) \cos(\theta_4) \sin(\theta_1 - \theta_3) \dot{\theta}_3 \\
& - m_2 \cos(\theta_2) \sin(\theta_4) \cos(\theta_1 - \theta_3) \dot{\theta}_4)
\end{aligned} \tag{171}$$

$$\begin{aligned}
C_{8\bullet}(q, \dot{q}) = & (0, 0, c_{83} = -m_2 l_1 \cos(\theta_4) \cos(\theta_2) \cos(\theta_1 - \theta_3) \dot{\theta}_1 - m_2 l_1 \cos(\theta_4) \sin(\theta_2) \sin(\theta_3 - \theta_1) \dot{\theta}_2 \\
& + m_2 \cos(\theta_4) \cos(\theta_2) \sin(\theta_3 - \theta_1) \dot{l}_1, c_{84} = m_2 l_1 \cos(\theta_4) \sin(\theta_2) \sin(\theta_1 - \theta_3) \dot{\theta}_1 \\
& - m_2 l_1 \sin(\theta_2) \sin(\theta_4) \dot{\theta}_2 - m_2 l_1 \cos(\theta_2) \cos(\theta_2) \cos(\theta_1 - \theta_3) \dot{\theta}_2 + \frac{m_2 - m_1}{2} l_1 \sin(\theta_1) \sin(\theta_2) \\
& \sin(\theta_{2x}) \sin(\theta_2) \dot{\theta}_4 + m_2 \cos(\theta_2) \sin(\theta_4) \dot{l}_1 - m_2 \cos(\theta_4) \sin(\theta_2) \cos(\theta_1 - \theta_3) \dot{l}_1, \\
c_{85} = & m_2 l_2 \cos^2(\theta_4) \dot{\theta}_3, c_{86} = \frac{m_2 - m_1}{2} l_1 \sin(\theta_4) \sin(\theta_2) \sin(\theta_3) \sin(\theta_1) \dot{\theta}_2 - m_2 l_2 \dot{\theta}_4, \\
c_{87} = & -m_2 \cos(\theta_2) \cos(\theta_4) \sin(\theta_1 - \theta_3) \dot{\theta}_1 + m_2 \cos(\theta_2) \sin(\theta_4) \dot{\theta}_2 \\
& - m_2 \cos(\theta_4) \sin(\theta_4) \cos(\theta_1 - \theta_3) \dot{\theta}_2, c_{88} = 0)
\end{aligned} \tag{172}$$

It can be checked that each term satisfies $\dot{m}_{ij} = c_{ij} + c_{ji}$. The Lagrange dynamics are then complete by adding forces that derive from a potential (gravity, elasticity) and nonconservative forces (like torque inputs).

Remark 6 *It would be good to extend these calculations to the case where the payload mass point is replaced by a rigid body modeling a container or else. Then three orientation angles have to be added in the generalized coordinates, the Euler or the Bryan angles could be suitable. However the system then has 11 degrees of freedom and the above calculations, especially those for $C(q, \dot{q})$, become very lengthy and cumbersome. The first step would be to calculate the kinetic energy, then the 11×11 mass matrix, which are two reasonable steps. Some formal calculation tool for the Christoffel's symbols (5) seem mandatory to avoid calculating by hand $3 \times 11^3 = 3993$ partial derivatives like $\frac{\partial m_{ij}}{\partial q_k}$.*

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