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# Lagrange dynamics of lumped-mass multibody models of overhead cranes in 2D operational space

Bernard Brogliato\*

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## Abstract

In this report the Lagrange dynamics of overhead cranes is analyzed, in the particular 2D (two-dimensional) operating space.

## 1 Generalities on Modeling Aspects

Models can be classified according to different criteria: linear *vs* nonlinear, finite-dimensional *vs* infinite-dimensional (PDEs), control-design oriented *vs* numerical-simulation oriented, 2D operating space *vs* 3D operating space, *etc.* In this report we shall focus on 2D operational space models, and we shall provide a detailed study of a lumped-mass Lagrangian model of the cable. In spite of the fact that several toolboxes allow to derive automatically (or numerically) the dynamics of such multibody systems, it is of interest to obtain a deeper understanding of their structure and properties, to pave the way for control design and stability analysis.

### 1.1 Models for Control

For control-design purposes two main classes have been used so far:

1. Finite-dimensional models with very few degrees-of-freedom (dof), hence tractable for control design and analysis: trolley + multibody-like pendulum, where the pendulum is a series kinematic chain made of rigid links with joints (revolute or spherical), in a multibody system framework. In the Automatic Control literature, the pendulum has one, or two dof, rarely more. They may be nonlinear (Coriolis and centrifugal torques, gravity torques) or linearized around the vertical posture of the cable (in which case

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linear modes can be computed easily and used for control tuning purposes). It seems that few subclasses have emerged in the Automatic Control literature (this list is not exhaustive):

- (a) 2D operating space,
- (b) 3D operating space,
- (c) Single-cable cranes,
- (d) Multiple-cable cranes,
- (e) Fixed-length links,
- (f) Variable-length links, with or without control of the length coordinate,
- (g) With or without stiffnesses (at joints, or inside the links) which correspond to various potential functions.

It has to be stressed that not all of these may be suitable for a multibody-system modeling approach.

2. Infinite-dimensional models (which correspond to a coupling ODE+PDE): trolley + flexible cable. The cable may be represented by a linear wave equation, valid in a neighborhood of the cable's vertical posture (*i.e.*, small angles and small transversal displacement) [6, 7, 8]. These are strongly related to finite-element models, which are necessary for the control implementation and for simulation. However, while the continuous-time studies seem to be restricted to linear systems (PDE), many FEM models yield nonlinear systems, though not Lagrangian dynamics [23].

It is noteworthy that the models for control, are either nonlinear and with few dof (typically less than 5), or linear and infinite dimensional. The first class is the most used in the Automatic Control literature, by far, see, *e.g.*, [9, 24, 25, 26, 27, 28] to cite very few references.

*As made clear later, and as a known fact, controlling overhead cranes efficiently, necessarily implies to take into account and use the inertial and dynamical couplings between the state variables, because of their strong underactuation. It is therefore quite useful to characterize rigorously these couplings.*

## 1.2 Models for Simulation

For the sake of numerical simulation (in order to settle numerical benchmark tests for robustness analysis), one finds:

1. Finite-element-like approximations of PDEs: Rayleigh-Ritz approach, the ALE-ANCF approach [13, 12], see also [23, 17]. These yield multibody-like dynamics, however not Lagrange dynamics but rather some kind of quasi-Lagrange systems. Very few works

have used FEM models to design feedback control for overhead cranes, among these we find [11, 10, 30, 15, 14] for bridge’s cables with significant sag, or [6, 8] for the FEM of the string model. The FEM dynamics derived in [23] looks promising for control design, as it combines both the advantages of multibody systems models with global nonlinearities (large rotations) and of FEM dynamics which are local in nature.

2. Finite-dimensional multibody models of trolley+pendulum with a very large number of pendulum dof (thus *a priori* untractable for control design).

The boundaries between these two big classes is, of course, never frozen, as new controller design may emerge that use more complex cable’s models.

### 1.3 Disturbances, Uncertainties, Neglected Dynamics

An unwanted effect may be considered as a disturbance, or as being part of the model for control design. In addition to cable flexibilities, other effects considered as disturbances can be incorporated in any of the above for control design or in the numerical benchmarks:

1. Disturbances:
  - (a) Coulomb-like friction between the trolley and the rails,
  - (b) mechanical play (static or dynamic backlash),
  - (c) measurement noise (sensors),
  - (d) uncertain parameters (inertial parameters),
2. Large parameters variations (*e.g.*, switching between heavy-payload and payload-free subtasks),
3. Neglected dynamics:
  - (a) actuators dynamics,
  - (b) cable flexibilities (for pendulum-like models),
  - (c) multiple cables *vs* single cable (kinematics at the payload attachment),
  - (d) 3D effects (for 2D designs), like payload rotations involving nonlinear inertial torques.

### 1.4 Lagrange Dynamics in Finite Dimensions

Let a mechanical system with  $n$  degrees of freedom be given, let  $q \in \mathbb{R}^n$  be the vector of its generalised coordinates. The Lagrange equations are given as:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q \quad (1)$$

where  $L(q, \dot{q})$  is the Lagrangian function,  $L(q, \dot{q}) = T(q, \dot{q}) - U(q)$ ,  $T(q, \dot{q})$  is the system's kinetic energy,  $U(q)$  is its potential energy,  $Q$  is the vector of generalized forces (that perform work on the coordinate  $q$ ). It is assumed that  $q$  is minimal, which means that there are no redundant coordinates, and (1) describes the system's dynamics without the need to add any equality (bilateral) holonomic constraints. Usually it is much easier to write the dynamics with redundant coordinates, and adding bilateral constraints that represent the joints. As is well-known [2], the Lagrange dynamics can be rewritten as:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = Q \quad (2)$$

where:  $M(q) = M^\top(q) \succ 0$  is the mass matrix, or inertia matrix (it may be just  $\succcurlyeq 0$ ),  $C(q, \dot{q})\dot{q}$  contains centrifugal and Coriolis nonlinear generalised forces,  $g(q) = \frac{\partial U}{\partial q}$  is the vector of forces that derive from a potential (gravity, elasticity, *etc*). The next property is also well-known (since at least the 1980s in the Automatic Control and the Robotics scientific communities):

**Property 1** *The matrix  $C(q, \dot{q})$ , when written with the so-called Christoffel's symbols<sup>1</sup>, verifies*

$$\dot{M}(q, \dot{q}) = C(q, \dot{q}) + C^\top(q, \dot{q}) \Leftrightarrow \dot{M}(q, \dot{q}) - 2C(q, \dot{q}) \text{ is skew-symmetric,} \quad (3)$$

where  $\dot{M}(q, \dot{q})$  is the matrix obtained from  $M(q)$  by differentiating each one of its entries with respect to time.

Property (3) is a key property for the design of passivity-based controllers [2], it implies passivity but it is not implied by passivity. Another fundamental property is that the system (1), seen as an input/output operator  $Q \mapsto \dot{q}$ , and under a boundedness condition on the potential energy, is passive, *i.e.*,  $V(q(t), \dot{q}(t)) - V(q(0), \dot{q}(0)) \leq \int_0^t Q^\top(s)\dot{q}(s)ds$  along the trajectories of the system, and where  $V(q, \dot{q}) = T(q, \dot{q}) + U(q)$  is the total mechanical energy.

**Remark 1** *In this manuscript we focus mainly on the calculation of the mass matrix  $M(q)$ . The Coriolis/centrifugal forces can be deduced from it using the classical Christoffel's symbols as  $C(q, \dot{q}) = (\sum_{k=1}^n \Gamma_{ijk}\dot{q}_k)_{ij}$ , where  $\Gamma_{ijk} = \frac{1}{2} \left( \frac{\partial m_{ij}}{\partial q_k} + \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{kj}}{\partial q_i} \right)$  [2, Lemma 6.16]. In spite of the fact that such calculations may not be straightforward in general, they are doable.*

## 1.5 Main Specific Features for Control Design and Analysis

Some essential and specific properties of the overhead crane dynamics are the following ones:

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<sup>1</sup>Indeed there is a general not a unique way to write the matrix  $C(q, \dot{q})$  in  $C(q, \dot{q})\dot{q}$ .

1. **Underactuation:** the degree of underactuation  $d_{unac}$  is equal to the number of dof, minus the number of independent torque inputs. In cranes like in Figure 1, there are  $1 + 2N$  (2D case with varying lengths), or  $1 + N$  (2D case with fixed lengths), or  $2 + 3N$  (3D case with varying lengths) or  $2 + 2N$  (3D with fixed lengths) degrees of freedom. The number of independent inputs varies from 1 (trolley controller) to 3 (trolley and length controllers). Thus  $d_{unac}$  is usually very large.
2. **Passivity:** the passive outputs usually do not correspond to the output to be controller, mainly due to the noncollocation. A passive mapping can be recovered by defining another suitable output function [18, 5].
3. **Input collocation or noncollocation:** the control input usually is the force applied to the trolley, hence only the trolley's state yields a collocated input. Usually it is desired to control the cable's tip (or the payload mounted at the cable's tip), which yields a noncollocated input.
4. **High flexibility:** cables are highly flexible systems, different from beams since their bending flexibility is very large.
5. **Links with variable length:** though this is not very often considered in the overhead crane control literature, using the cable's length in the control design may be of great help. This is used in robotic systems involving cables [18].

*All these features render the control problem hard, and make it necessary to use dynamical couplings for control design.*

## 2 Rigid Links with Varying Length (2D Operating Space)

Let us consider a system as in Figures 1 or 2. This is an overhead crane with a finite-dimensional approximation of a cable, made of  $N$  rigid links (a series kinematic chain with pivot joints,  $N = 3$  in the figure). The basic assumptions are:

1. Each rigid link is assumed here to possess a varying length  $l_i(t)$ . Indeed links may possess some longitudinal stiffness (see section 7), or be assumed perfectly rigid but with some winch mechanism (it could be assumed that at least  $l_1$  is acted upon by a control input, realised with some winding devices at  $A_0$ ).
2. Links are massless, the masses are lumped at the joints, that one may name nodes (points  $A_i$ ,  $1 \leq i \leq N$ ).
3. The trolley has mass  $m > 0$  and is acted upon by a force  $F(t)$ , that may be a control input.

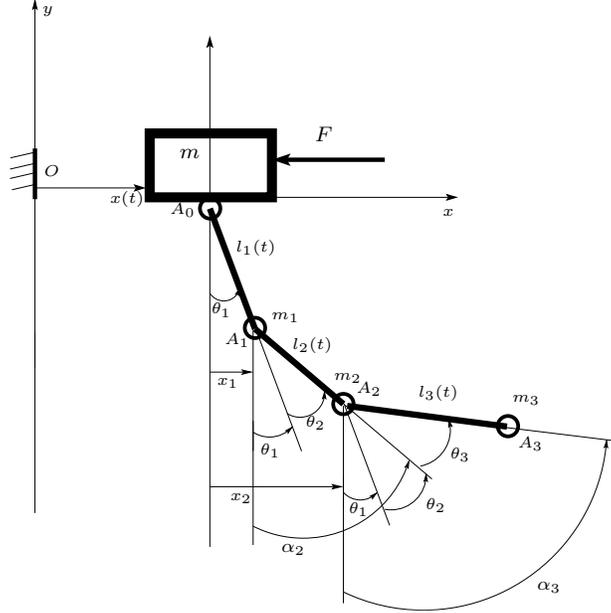


Figure 1: Overhead crane with pendulum-like cable (controlled links' lengths).

4. The payload has mass  $m_N > 0$ , each node has a mass  $m_i \geq 0$ .
5. Since cables, contrarily to beams, usually have negligible bending stiffness, there is no angular springs at the joints, or these stiffnesses are very small.
6. Sagging (static) effects are neglected, contrarily to cable models for cable-cars in which they are a prominent effect.
7. Sometimes it may be justified to assume that the payload mass  $m_N$  is much bigger than the nodes  $A_i$  masses (said otherwise, the cable's mass is much smaller than the payload mass), so that the latter may be taken equal to zero in the dynamics [5, 18]. This is particularly true for cranes carrying very large payloads of several tenths of tons.

**Remark 2** *A slightly more general model of rigid links modelled as rigid slender rods may be considered. Rotational kinetic energies  $\frac{1}{2}J_i(t)\dot{\theta}_i^2$ ,  $1 \leq i \leq N$ , add terms  $J_i(t)\ddot{\theta}_i + \dot{J}_i(t)\dot{\theta}_i$  in the Lagrange dynamics. They modify the matrix  $M_{\theta\theta}(q)$  and the nonlinear terms. If  $J_i(t) = J_i(l(t))$ , then  $J_i(l)\ddot{\theta}_i + \frac{\partial J_i}{\partial l_i}\dot{l}_i\dot{\theta}_i$  have to be considered. However multibody approaches may not be suitable for modelling variable length cables with winding mechanism at  $A_0$ .*

Let us derive kinetic energies and potential energy. The Lagrange equations cannot be written explicitly in the general case (*i.e.*,  $N \geq 2$ ) since the many couplings make it quite

cumbersome to write expressions for the Coriolis and centrifugal nonlinearities. This may be the reason why such pendulum-like models are limited to  $N = 1$  and  $N = 2$  in the Automatic Control literature. However some properties can be deduced from energies. In the general case with varying lengths, the system has  $2N + 1$  degrees of freedom, and the chosen generalised coordinates vector is  $q = (x, \theta_1, \dots, \theta_N, l_1, \dots, l_N)^\top \in \mathbb{R}^{2N+1}$ .

### 3 Kinetic energies

- Kinetic energy of the trolley:

$$T_{trol} = \frac{1}{2}m\dot{x}^2. \quad (4)$$

so that

$$\frac{d}{dt} \frac{\partial T_{trol}}{\partial \dot{x}} = m\ddot{x}. \quad (5)$$

- Translational kinetic energy of each point-mass  $m_i$ : let the translational velocity of point  $A_i$  be denoted as  $V_{A_i} \in \mathbb{R}^3$  in the  $(O, x, y, z)$  Galilean frame. For instance,  $OA_1 = \begin{pmatrix} x + l_1 \sin(\theta_1) \\ -l_1 \cos(\theta_1) \\ 0 \end{pmatrix}$  where  $OA_0 = (x(t), 0, 0)^\top$ . Thus

$$\begin{aligned} V_{A_1} &= \begin{pmatrix} \dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1) \\ l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix} + l_1 \dot{\theta}_1 \begin{pmatrix} \cos(\theta_1) \\ \sin(\theta_1) \\ 0 \end{pmatrix} + \dot{l}_1 \begin{pmatrix} \sin(\theta_1) \\ -\cos(\theta_1) \\ 0 \end{pmatrix}, \end{aligned} \quad (6)$$

and  $OA_2 = \begin{pmatrix} x + l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \\ -l_1 \cos(\theta_1) - l_2 \cos(\theta_1 + \theta_2) \\ 0 \end{pmatrix}$ , *i.e.*,

$$\begin{aligned} V_{A_2} &= \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix} + l_1 \dot{\theta}_1 \begin{pmatrix} \cos(\theta_1) \\ \sin(\theta_1) \\ 0 \end{pmatrix} + \dot{l}_1 \begin{pmatrix} \sin(\theta_1) \\ -\cos(\theta_1) \\ 0 \end{pmatrix} \\ &\quad + l_2 (\dot{\theta}_1 + \dot{\theta}_2) \begin{pmatrix} \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \\ 0 \end{pmatrix} + \dot{l}_2 \begin{pmatrix} \sin(\theta_1 + \theta_2) \\ -\cos(\theta_1 + \theta_2) \\ 0 \end{pmatrix} \end{aligned} \quad (7)$$

Similarly:

$$\begin{aligned}
V_{A_3} = & \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix} + l_1 \dot{\theta}_1 \begin{pmatrix} \cos(\theta_1) \\ \sin(\theta_1) \\ 0 \end{pmatrix} + \dot{l}_1 \begin{pmatrix} \sin(\theta_1) \\ -\cos(\theta_1) \\ 0 \end{pmatrix} \\
& + l_2 (\dot{\theta}_1 + \dot{\theta}_2) \begin{pmatrix} \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \\ 0 \end{pmatrix} + \dot{l}_2 \begin{pmatrix} \sin(\theta_1 + \theta_2) \\ -\cos(\theta_1 + \theta_2) \\ 0 \end{pmatrix} \\
& + l_3 (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) \begin{pmatrix} \cos(\theta_1 + \theta_2 + \theta_3) \\ \sin(\theta_1 + \theta_2 + \theta_3) \\ 0 \end{pmatrix} + \dot{l}_3 \begin{pmatrix} \sin(\theta_1 + \theta_2 + \theta_3) \\ -\cos(\theta_1 + \theta_2 + \theta_3) \\ 0 \end{pmatrix}
\end{aligned} \tag{8}$$

Clearly there is a recursive structure for linear velocities of joints nodes  $A_i$ :

$$V_{A_i} = \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \\ \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \\ 0 \end{pmatrix} + \begin{pmatrix} \sum_{k=1}^i \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \\ -\sum_{k=1}^i \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \\ 0 \end{pmatrix} \tag{9}$$

The translational kinetic energy of each point mass  $A_i$  is  $T_{V_{A_i}} = \frac{1}{2} m_i V_{A_i}^\top V_{A_i}$ :

$$\begin{aligned}
T_{V_{A_i}} = & \frac{1}{2} m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right)^2 \\
& + \frac{1}{2} m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right)^2
\end{aligned} \tag{10}$$

and the total translational kinetic energy of all mass-points is:

$$\begin{aligned}
T_{trans} = & \frac{1}{2} \sum_{i=1}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right)^2 \\
& + \frac{1}{2} \sum_{i=1}^N m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right)^2.
\end{aligned} \tag{11}$$

Therefore the total kinetic energy of the system is

$$T(q, \dot{q}) = T_{trol} + T_{trans}. \tag{12}$$

## 4 Lagrange Dynamics of Overhead Cranes in 2D Operating Spaces

In this manuscript the mass matrix is denoted as:

$$M(q) = \begin{pmatrix} M_{x\theta}(q) & M_{x\theta l}(q) \\ M_{lx\theta}(q) & M_{ll}(q) \end{pmatrix} \in \mathbb{R}^{(1+2N) \times (1+2N)}, \tag{13}$$

with  $M_{x\theta}(q) \in \mathbb{R}^{(1+N) \times (1+N)}$ ,  $M_{ll}(q) \in \mathbb{R}^{N \times N}$ ,  $M_{x\theta l}(q) = M_{lx\theta}^\top(q) \in \mathbb{R}^{(1+N) \times N}$ . It is also possible to go a step further with:

$$M_{x\theta}(q) = \begin{pmatrix} M_{xx}(q) & \bar{M}_{x\theta}(q) \\ \bar{M}_{x\theta}^\top(q) & M_{\theta\theta}(q) \end{pmatrix} \quad (14)$$

with  $M_{xx}(q) \in \mathbb{R}$ ,  $\bar{M}_{x\theta}(q) \in \mathbb{R}^{1 \times N}$ ,  $M_{\theta\theta}(q) \in \mathbb{R}^{N \times N}$ .

This expression of the mass matrix corresponds to the choice of the generalized coordinates as  $q = (x, \theta_1, \dots, \theta_N, l_1, \dots, l_N)^\top \in \mathbb{R}^{2N+1}$ . Other choices can be made, like those splitting the coordinates into actuated  $q_a$  and nonactuated ones  $q_{na}$ , which is a classical way of doing in the Control literature [22].

## 4.1 Calculation of $\frac{d}{dt} \frac{\partial T}{\partial \dot{q}}$

This is the term that allows us to derive the mass matrix.

### 4.1.1 Calculation of $\frac{d}{dt} \frac{\partial T}{\partial \dot{x}}$

Let us examine  $\frac{\partial T}{\partial \dot{x}} = \frac{\partial T_{trol}}{\partial \dot{x}} + \frac{\partial T_{trans}}{\partial \dot{x}} = m\dot{x} + \frac{\partial T_{trans}}{\partial \dot{x}}$ , and:

$$\frac{\partial T_{trans}}{\partial \dot{x}} = \sum_{i=1}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \quad (15)$$

The derivative  $\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{x}}$  gives the expression for the first row (and the first column by symmetry) of the mass matrix  $M(q)$ , with couplings between  $\ddot{x}$ ,  $\ddot{l}_k$  and angles acceleration  $\ddot{\theta}_i$ .

**Remark 3** Let  $M = 1$ , then from (15):

$$\frac{\partial T_{trans}}{\partial \dot{x}} = m_1 \left( \dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1) \right) \quad (16)$$

from which

$$\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{x}} = m_1 \left( \ddot{x} + l_1 \ddot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \dot{\theta}_1 \cos(\theta_1) - l_1 \dot{\theta}_1^2 \sin(\theta_1) + \ddot{l}_1 \sin(\theta_1) + \dot{l}_1 \dot{\theta}_1 \cos(\theta_1) \right) \quad (17)$$

In the general case from (15):

$$\begin{aligned} \frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{x}} &= \sum_{i=1}^N m_i \ddot{x} \\ &+ \sum_{i=1}^N m_i \sum_{k=1}^i \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \\ &- l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \sin \left( \sum_{j=1}^k \theta_j \right) \\ &+ \sum_{i=1}^N m_i \sum_{k=1}^i \ddot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \end{aligned} \quad (18)$$

and  $\frac{d}{dt} \frac{\partial T}{\partial \dot{x}} = m\ddot{x} + \frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{x}}$ . Both (17) and (18) show the couplings between  $\ddot{x}$ ,  $\ddot{\theta}_i$  and  $\ddot{l}_k$ , *i.e.*, off-diagonal entries of the first row (and first column by symmetry) of the mass matrix. Let us derive the form of the first row of the mass matrix using (18). Clearly:

$$m_{11}(q) = m + \sum_{i=1}^N m_i \quad (19)$$

- For  $2 \leq p \leq N + 1$  (the factor of  $\ddot{\theta}_{p-1}$  in the first row of the mass matrix):

$$m_{1p}(q) = \sum_{i=p-1}^N m_i \sum_{k=p-1}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right) \quad (20)$$

In particular the factor of  $\ddot{\theta}_N$  is equal to  $m_{1(N+1)}(q) = m_N l_N \cos(\sum_{j=1}^N \theta_j)$ , and the factor of  $\ddot{\theta}_1$  is equal to  $m_{12}(q) = \sum_{i=1}^N m_i \sum_{k=1}^i l_k \cos(\sum_{j=1}^k \theta_j)$ .

- For  $N + 2 \leq p \leq 2N + 1$  (the factor of  $\ddot{l}_{p-N-1}$  in the first row of the mass matrix):

$$m_{1p}(q) = \sum_{i=p-N-1}^N m_i \sin\left(\sum_{j=1}^{p-N-1} \theta_j\right), \quad (21)$$

In particular the factor of  $\ddot{l}_N$  (*i.e.*,  $p = 2N + 1$ ) is equal to  $m_{1(2N+1)}(q) = m_N \sin(\sum_{j=1}^N \theta_j)$ , and the factor of  $\ddot{l}_1$  (*i.e.*,  $p = N + 2$ ) is equal to  $m_{1(N+2)}(q) = \sum_{i=1}^N m_i \sin(\theta_1)$ . It appears clearly from (21) that the inertial couplings between  $\ddot{x}$  and  $\ddot{l}$  are weak in a neighborhood of the cable's vertical posture where angles are small and  $\sin(\theta_i) \approx \theta_i$ .

**Small angles:** If the cable stays in a neighborhood of the vertical position (the  $y$  axis), thus very small angles, we may consider  $\sin(\theta_i) = \theta_i$ , and  $\cos(\theta_i) = 1$ , and same for the sums. Then (18) simplifies to:

$$\begin{aligned} \frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{x}} &= \sum_{i=1}^N m_i \ddot{x} + \sum_{i=1}^N m_i \sum_{k=1}^i \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \\ &- l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \left( \sum_{j=1}^k \theta_j \right) + \sum_{i=1}^N m_i \sum_{k=1}^i \ddot{l}_k \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \end{aligned} \quad (22)$$

*Comment:* even if angles approach zero, couplings in the Coriolis/centrifugal nonlinearities are kept through the velocity terms  $\dot{l}_k \dot{\theta}_i$ . This is not the case if lengths are constant. If we further take  $\theta_1 = 0$  (a zero-order tangent linearisation):

$$\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{x}} = \sum_{i=1}^N m_i \ddot{x} + \sum_{i=1}^N m_i \sum_{k=1}^i \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \quad (23)$$

**Small angles and large payload mass:** If furthermore  $m_N \gg m_i$ , then:

$$\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{x}} = m_N \ddot{x} + m_N \sum_{k=1}^N \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \quad (24)$$

It happens from both (21) and (24) that the coupling with  $\ddot{l}_k$  disappears in this case (heavy payload, small deviation from the vertical axis with zero order approximation). In case of first order approximation these couplings are proportional to  $\theta_i$ . But the  $\dot{l}_k \dot{\theta}_i$  terms are still present.

#### 4.1.2 Calculation of $\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_n}$ , $1 \leq n \leq N$

Let us calculate now  $\frac{\partial T}{\partial \dot{\theta}_n} = \frac{\partial T_{trans}}{\partial \dot{\theta}_n}$ ,  $1 \leq n \leq N$ .

$$\begin{aligned} \frac{\partial T_{trans}}{\partial \dot{\theta}_n} &= \sum_{i=1}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \frac{\partial}{\partial \dot{\theta}_n} \left\{ \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right\} \\ &\quad + \sum_{i=1}^N m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \frac{\partial}{\partial \dot{\theta}_n} \left\{ \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right\} \end{aligned} \quad (25)$$

**Remark 4** If  $N = 1$  then (25) becomes:

$$\begin{aligned} \frac{\partial T_{trans}}{\partial \dot{\theta}_1} &= m_1 \left( \dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1) \right) \frac{\partial}{\partial \dot{\theta}_1} \left\{ l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \theta_1 \right\} \\ &\quad + m_1 \left( l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1) \right) \frac{\partial}{\partial \dot{\theta}_1} \left\{ l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1) \right\} \\ &= m_1 \left( \dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1) \right) l_1 \cos(\theta_1) \\ &\quad + m_1 \left( l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1) \right) l_1 \sin(\theta_1) \\ &= m_1 l_1 \cos(\theta_1) \dot{x} + m_1 l_1^2 \dot{\theta}_1 \end{aligned} \quad (26)$$

From (26):

$$\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{\theta}_1} = m_1 l_1 \cos(\theta_1) \ddot{x} + m_1 \dot{l}_1 \cos(\theta_1) \dot{x} - m_1 l_1 \dot{\theta}_1 \sin(\theta_1) \dot{x} + 2m_1 l_1 \dot{l}_1 \dot{\theta}_1 + m_1 l_1^2 \ddot{\theta}_1. \quad (27)$$

Let us start with

$$\begin{aligned} &\frac{\partial}{\partial \dot{\theta}_n} \left\{ \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right\} \\ &= \frac{\partial}{\partial \dot{\theta}_n} \left\{ \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right\} \\ &= \sum_{k=1}^i l_k \frac{\partial}{\partial \dot{\theta}_n} \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \\ &= \begin{cases} \sum_{k=n}^i l_k \cos \left( \sum_{j=1}^k \theta_j \right) & \text{if } i \geq n \\ 0 & \text{if } i < n \end{cases} \end{aligned} \quad (28)$$

and

$$\begin{aligned}
& \frac{\partial}{\partial \dot{\theta}_n} \left\{ \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right\} \\
&= \frac{\partial}{\partial \dot{\theta}_n} \left\{ \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right\} \\
&= \sum_{k=1}^i l_k \frac{\partial}{\partial \dot{\theta}_n} \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \\
&= \begin{cases} \sum_{k=n}^i l_k \sin \left( \sum_{j=1}^k \theta_j \right) & \text{if } i \geq n \\ 0 & \text{if } i < n \end{cases}
\end{aligned} \tag{29}$$

Thus:

$$\begin{aligned}
\frac{\partial T_{trans}}{\partial \theta_n} &= \sum_{i=n}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \sum_{k=n}^i l_k \cos \left( \sum_{j=1}^k \theta_j \right) \\
&+ \sum_{i=n}^N m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \sum_{k=n}^i l_k \sin \left( \sum_{j=1}^k \theta_j \right)
\end{aligned} \tag{30}$$

Then from (30) it is inferred:

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{\theta}_n} &= \sum_{i=n}^N m_i \left( \ddot{x} + \sum_{k=1}^i \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right. \\
&\quad \left. - l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \sin \left( \sum_{j=1}^k \theta_j \right) + \ddot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \sum_{k=n}^i l_k \cos \left( \sum_{j=1}^k \theta_j \right) \\
&+ \sum_{i=n}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \sum_{k=n}^i \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) - l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \\
&+ \sum_{i=n}^N m_i \left( \sum_{k=1}^i \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \cos \left( \sum_{j=1}^k \theta_j \right) \right. \\
&\quad \left. + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \ddot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \sum_{k=n}^i l_k \sin \left( \sum_{j=1}^k \theta_j \right) \\
&+ \sum_{i=n}^N m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \sum_{k=n}^i \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right)
\end{aligned} \tag{31}$$

• The entry  $m_{(1+n)1}(q)$  in the mass matrix (which multiplies  $\ddot{x}$  in the row corresponding to  $\ddot{\theta}_n$ , and is thus an entry of  $\bar{M}_{x\theta}(q)$  in (14)) is:

$$m_{(1+n)1}(q) = \sum_{i=n}^N m_i \sum_{k=n}^i l_k \cos \left( \sum_{j=1}^k \theta_j \right) \tag{32}$$

For  $n = 1$  we get  $m_{21}(q) = \sum_{i=1}^N m_i \sum_{k=1}^i l_k \cos \left( \sum_{j=1}^k \theta_j \right)$ , which is equal to  $m_{12}(q)$  in (20). And so on by symmetry of the mass matrix for  $n \geq 2$ .

- The entry  $m_{(1+n)p}(q)$ ,  $2 \leq p \leq N + 1$  which corresponds to  $\ddot{\theta}_{p-1}$  in the mass matrix' row  $M_{(1+n)\bullet}(q)$  of  $\ddot{\theta}_n$  (in the submatrix  $M_{\theta\theta}(q)$  in (14)) is given by:

$$m_{(1+n)p}(q) = \sum_{i=\max(n,p-1)}^N m_i \sum_{k=p-1}^i l_k \left\{ \cos\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right)\right) + \sin\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \sin\left(\sum_{j=1}^k \theta_j\right)\right) \right\}. \quad (33)$$

This is equal to  $m_{p(1+n)}(q)$  by symmetry of  $M(q)$ . In the case  $N = 2$ , it can be checked that (33) gives the expressions for  $m_{22}(q)$ ,  $m_{23}(q)$  and  $m_{33}(q)$  in (139) and (140), respectively. The proof proceeds from examination of the terms in (31) that contain angle accelerations in the row  $n + 1$ ,  $M_{(n+1)\bullet}(q)$ , and are equal to:

$$\sum_{i=n}^N m_i \left\{ \sum_{k=1}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{j=1}^k \ddot{\theta}_j\right) \left(\sum_{k=n}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right)\right) + \sum_{k=1}^i l_k \sin\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{j=1}^k \ddot{\theta}_j\right) \left(\sum_{k=n}^i l_k \sin\left(\sum_{j=1}^k \theta_j\right)\right) \right\} \quad (34)$$

Factorizing inside the term between brackets inside the summation in (34):

$$\sum_{i=n}^N m_i \sum_{k=1}^i l_k \left\{ \cos\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right)\right) + \sin\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \sin\left(\sum_{j=1}^k \theta_j\right)\right) \right\} \left(\sum_{j=1}^k \ddot{\theta}_j\right) \quad (35)$$

Due to the last summation  $\sum_{j=1}^k \ddot{\theta}_j$ ,  $\ddot{\theta}_1$  appears in all the terms of the sum  $\sum_{k=1}^i$ , hence in all the terms of the sum  $\sum_{i=n}^N m_i$  for any  $1 \leq n \leq N$ . Hence for all  $1 \leq n \leq N$ , the term multiplying  $\ddot{\theta}_1$  is equal to

$$m_{(1+n)2}(q) = \sum_{i=n}^N m_i \sum_{k=1}^i l_k \left\{ \cos\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right)\right) + \sin\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \sin\left(\sum_{j=1}^k \theta_j\right)\right) \right\}, \quad (36)$$

where it is noted that  $\max(n, 1) = n$ . Consider now  $\ddot{\theta}_2$ . It does not appear for  $k = 1$  in the summation  $\sum_{j=1}^k \ddot{\theta}_j$  in (35), and it appears in all the terms of the summation  $\sum_{k=2}^i$ , provided that  $i \geq 2$ . Hence for all  $2 \leq n \leq N$  the term multiplying  $\ddot{\theta}_2$  is equal to

$$m_{(1+n)3}(q) = \sum_{i=n}^N m_i \sum_{k=2}^i l_k \left\{ \cos\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right)\right) + \sin\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \sin\left(\sum_{j=1}^k \theta_j\right)\right) \right\}, \quad (37)$$

or we may write that for all  $1 \leq n \leq N$ :

$$m_{(1+n)3}(q) = \sum_{i=\max(n,2)}^N m_i \sum_{k=2}^i l_k \left\{ \cos\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right)\right) + \sin\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \sin\left(\sum_{j=1}^k \theta_j\right)\right) \right\},$$

Consider now the entry of the  $(1+n)$ th row  $M_{(1+n)\bullet}(q)$ , which multiplies  $\ddot{\theta}_p$ , *i.e.*,  $m_{(1+n)(1+p)}(q)$ :  $\ddot{\theta}_p$  appears in  $\sum_{j=1}^k \ddot{\theta}_j$  only if  $k \geq p \Rightarrow i \geq p$ . Thus for  $p \leq n \leq N$ :

$$m_{(1+n)(p+1)}(q) = \sum_{i=n}^N m_i \sum_{k=p}^i l_k \left\{ \cos\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right)\right) + \sin\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \sin\left(\sum_{j=1}^k \theta_j\right)\right) \right\}.$$

or equivalently for all  $1 \leq n \leq N$ :

$$m_{(1+n)(p+1)}(q) = \sum_{i=\max(n,p)}^N m_i \sum_{k=p}^i l_k \left\{ \cos\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right)\right) + \sin\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \sin\left(\sum_{j=1}^k \theta_j\right)\right) \right\}.$$

The acceleration  $\ddot{\theta}_N$  in (34) appears in the summation (34) only once in  $\sum_{j=1}^k \ddot{\theta}_j$  for  $k = N$ , hence for  $i = N$ , therefore the entries of the last column of  $M_{\theta\theta}(q)$ , which is part of  $M_{\bullet(1+N)}(q)$ , are equal to (in this case  $\max(n, p) = \max(n, N) = N$ ):

$$m_{(1+n)(1+N)}(q) = m_N l_N \left\{ \cos\left(\sum_{j=1}^N \theta_j\right) \left(\sum_{k=n}^N l_k \cos\left(\sum_{j=1}^k \theta_j\right)\right) + \sin\left(\sum_{j=1}^N \theta_j\right) \left(\sum_{k=n}^N l_k \sin\left(\sum_{j=1}^k \theta_j\right)\right) \right\}$$

for  $1 \leq n \leq N$ . For  $n = N$ :

$$m_{(1+N)(1+N)}(q) = m_N l_N^2 \left\{ \cos\left(\sum_{j=1}^N \theta_j\right) \cos\left(\sum_{j=1}^N \theta_j\right) + \sin\left(\sum_{j=1}^N \theta_j\right) \sin\left(\sum_{j=1}^N \theta_j\right) \right\} = m_N l_N^2. \quad (38)$$

This is in agreement with  $m_{33}(q) = m_{33}$  in (140). The other diagonal terms of  $M_{\theta\theta}(q)$  are equal to:

$$m_{(1+n)(n+1)}(q) = \sum_{i=n}^N m_i \sum_{k=n}^i l_k \left\{ \cos\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right)\right) + \sin\left(\sum_{j=1}^k \theta_j\right) \left(\sum_{k=n}^i l_k \sin\left(\sum_{j=1}^k \theta_j\right)\right) \right\},$$

since  $\max(n, n) = n$ .

• The entry  $m_{(1+n)p}(q)$ ,  $N+2 \leq p \leq 2N+1$  which corresponds to  $\ddot{l}_{p-N-1}$  in the mass matrix' row of  $\ddot{\theta}_n$  is given by:

$$m_{(1+n)p}(q) = \sum_{i=\max(n,p-N-1)}^N m_i \left\{ \sin\left(\sum_{j=1}^{p-N-1} \theta_j\right) \sum_{k=n}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right) - \cos\left(\sum_{j=1}^{p-N-1} \theta_j\right) \sum_{k=n}^i l_k \sin\left(\sum_{j=1}^k \theta_j\right) \right\} \quad (39)$$

For the case  $N = 2$ , (39) gives  $m_{24}(q)$ ,  $m_{25}(q)$  and  $m_{34}(q)$ ,  $m_{35}(q)$  in (139) and (140). For instance  $m_{35}(q)$  is obtained from (39) setting  $n = 2$  and  $p = 5$ , hence  $\max(n, p - N -$

1) = 2 and  $m_{35}(q) = m_2 (\sin(\theta_1 + \theta_2)l_2 \cos(\theta_1 + \theta_2) - \cos(\theta_1 + \theta_2)l_2 \sin(\theta_1 + \theta_2)) = 0$ . Let us examine the terms in (31) which contain lengths accelerations  $\ddot{l}_k$ :

$$\sum_{i=n}^N m_i \sum_{k=1}^i \left\{ \sin\left(\sum_{j=1}^k \theta_j\right) \sum_{k=n}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right) - \cos\left(\sum_{j=1}^k \theta_j\right) \sum_{k=n}^i l_k \sin\left(\sum_{j=1}^k \theta_j\right) \right\} \ddot{l}_k \quad (40)$$

The acceleration  $\ddot{l}_1$  appears in all the summations  $\sum_{k=1}^i$  so that

$$m_{(1+n)(N+2)}(q) = \sum_{i=n}^N m_i \left\{ \sin(\theta_1) \sum_{k=n}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right) - \cos(\theta_1) \sum_{k=n}^i l_k \sin\left(\sum_{j=1}^k \theta_j\right) \right\}.$$

Next,  $\ddot{l}_2$  appears in all summations with  $i \geq 2$ , hence:

$$m_{(1+n)(N+3)}(q) = \sum_{\max(n,2)}^N m_i \left\{ \sin\left(\sum_{j=1}^2 \theta_j\right) \sum_{k=n}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right) - \cos\left(\sum_{j=1}^2 \theta_j\right) \sum_{k=n}^i l_k \sin\left(\sum_{j=1}^k \theta_j\right) \right\}$$

and so on for the terms  $\ddot{l}_k$ ,  $3 \leq k \leq N$ , from which (39) follows.

Therefore from sections 4.1.1 and 4.1.2 a complete characterization of the submatrices  $M_{xx}$ ,  $\bar{M}_{x\theta}$ ,  $M_{\theta\theta}$  is obtained in the general case.

**Large payload mass:** Let us assume that  $m_N \gg m_n$ ,  $1 \leq n \leq N$  (the cable's mass is negligible compared to the payload's mass). Then:

$$\begin{aligned} \frac{\partial T_{trans}}{\partial \dot{\theta}_n} &= m_N \left( \dot{x} + \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) + \dot{l}_k \sin\left(\sum_{j=1}^k \theta_j\right) \right) \\ &\times \frac{\partial}{\partial \dot{\theta}_n} \left\{ \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) + \dot{l}_k \sin\left(\sum_{j=1}^k \theta_j\right) \right\} \\ &+ m_N \left( \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin\left(\sum_{j=1}^k \theta_j\right) - \dot{l}_k \cos\left(\sum_{j=1}^k \theta_j\right) \right) \\ &\times \frac{\partial}{\partial \dot{\theta}_n} \left\{ \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin\left(\sum_{j=1}^k \theta_j\right) - \dot{l}_k \cos\left(\sum_{j=1}^k \theta_j\right) \right\} \end{aligned} \quad (41)$$

which is quite different from (25), showing the influence of the deformations introduced by the joints at  $A_n$ . Let us proceed with some calculations:

$$\begin{aligned} \frac{\partial}{\partial \dot{\theta}_n} \left\{ \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) + \dot{l}_k \sin\left(\sum_{j=1}^k \theta_j\right) \right\} &= \frac{\partial}{\partial \dot{\theta}_n} \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) \\ &= \sum_{k=n}^N l_k \frac{\partial}{\partial \dot{\theta}_n} \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) = \sum_{k=n}^N l_k \cos\left(\sum_{j=1}^k \theta_j\right). \end{aligned}$$

And similarly

$$\frac{\partial}{\partial \dot{\theta}_n} \left\{ \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right\} = \sum_{k=i}^N l_k \sin \left( \sum_{j=1}^k \theta_j \right).$$

Therefore:

$$\begin{aligned} \frac{\partial T_{trans}}{\partial \dot{\theta}_n} &= m_N \left( \dot{x} + \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \sum_{k=n}^N l_k \cos \left( \sum_{j=1}^k \theta_j \right) \\ &\quad + m_N \left( \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \sum_{k=n}^N l_k \sin \left( \sum_{j=1}^k \theta_j \right). \end{aligned} \tag{42}$$

Consequently:

$$\begin{aligned} \frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{\theta}_n} &= m_N \left( \ddot{x} + \sum_{k=1}^N \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right. \\ &\quad \left. - l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \sin \left( \sum_{j=1}^k \theta_j \right) + \ddot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \sum_{k=n}^N l_k \cos \left( \sum_{j=1}^k \theta_j \right) \\ &\quad + m_N \left( \dot{x} + \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \left( \sum_{k=n}^N -l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) + \sum_{k=n}^N \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad + m_N \left( \sum_{k=1}^N \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right. \\ &\quad \left. + l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \cos \left( \sum_{j=1}^k \theta_j \right) - \ddot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \sum_{k=n}^N l_k \sin \left( \sum_{j=1}^k \theta_j \right) \\ &\quad + m_N \left( \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \left( \sum_{k=n}^M l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \sum_{k=n}^M \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \end{aligned} \tag{43}$$

Let us calculate the components of the row  $M_{(1+n)\bullet}(q)$ .

- First entry (inertial term of  $\ddot{x}$ , compare with (32)):

$$m_{(1+n)1}(q) = m_N \sum_{k=n}^N l_k \cos \left( \sum_{j=1}^k \theta_j \right) \tag{44}$$

- Entry  $p$  (inertial term of  $\ddot{\theta}_{p-1}$ ),  $2 \leq p \leq N + 1$ :

$$\begin{aligned} m_{(1+n)p}(q) &= m_N \left( \sum_{k=n}^N l_k \cos(\sum_{j=1}^k \theta_j) \right) \sum_{k=p-1}^N l_k \cos(\sum_{j=1}^k \theta_j) \\ &+ m_N \left( \sum_{k=n}^N l_k \sin(\sum_{j=1}^k \theta_j) \right) \sum_{k=p-1}^N l_k \sin(\sum_{j=1}^k \theta_j) \end{aligned} \quad (45)$$

Indeed in (43) angular accelerations appear in the terms:

$$\begin{aligned} &m_N \left( \sum_{k=n}^N l_k \cos(\sum_{j=1}^k \theta_j) \right) \sum_{k=1}^N l_k \cos(\sum_{j=1}^k \theta_j) (\sum_{j=1}^k \ddot{\theta}_j) \\ &+ m_N \left( \sum_{k=n}^N l_k \sin(\sum_{j=1}^k \theta_j) \right) \sum_{k=1}^N l_k \sin(\sum_{j=1}^k \theta_j) (\sum_{j=1}^k \ddot{\theta}_j). \end{aligned}$$

The acceleration  $\ddot{\theta}_1$  appears in all the summations, so that

$$\begin{aligned} m_{(1+n)2}(q) &= m_N \left( \sum_{k=n}^N l_k \cos(\sum_{j=1}^k \theta_j) \right) \sum_{k=1}^N l_k \cos(\sum_{j=1}^k \theta_j) \\ &+ m_N \left( \sum_{k=n}^N l_k \sin(\sum_{j=1}^k \theta_j) \right) \sum_{k=1}^N l_k \sin(\sum_{j=1}^k \theta_j), \end{aligned}$$

which can be obtained from (36) setting  $m_i = 0$  for all  $1 \leq i \leq N - 1$ . The acceleration  $\ddot{\theta}_1$  appears in all the summations but the first one for  $k = 1$  in the terms  $\sum_{k=1}^N l_k \cos(\sum_{j=1}^k \theta_j) (\sum_{j=1}^k \ddot{\theta}_j)$  and  $\sum_{k=1}^N l_k \sin(\sum_{j=1}^k \theta_j) (\sum_{j=1}^k \ddot{\theta}_j)$ , so that:

$$\begin{aligned} m_{(1+n)3}(q) &= m_N \left( \sum_{k=n}^N l_k \cos(\sum_{j=1}^k \theta_j) \right) \sum_{k=2}^N l_k \cos(\sum_{j=1}^k \theta_j) \\ &+ m_N \left( \sum_{k=n}^N l_k \sin(\sum_{j=1}^k \theta_j) \right) \sum_{k=2}^N l_k \sin(\sum_{j=1}^k \theta_j), \end{aligned}$$

which can also be obtained from (37) setting  $m_i = 0$  for all  $1 \leq i \leq N - 1$ . It is deduced that

$$\begin{aligned} m_{(1+n)p}(q) &= m_N \left( \sum_{k=n}^N l_k \cos(\sum_{j=1}^k \theta_j) \right) \sum_{k=p-1}^N l_k \cos(\sum_{j=1}^k \theta_j) \\ &+ m_N \left( \sum_{k=n}^N l_k \sin(\sum_{j=1}^k \theta_j) \right) \sum_{k=p-1}^N l_k \sin(\sum_{j=1}^k \theta_j). \end{aligned}$$

The inertial component  $m_{(1+n)(1+n)}(q)$  for  $\ddot{\theta}_n$  in the row  $M_{(1+n)\bullet}(q)$  is equal to:

$$\begin{aligned} m_{(1+n)(1+n)}(q) &= m_N \left( \sum_{k=n}^N l_k \cos(\sum_{j=1}^k \theta_j) \right) \sum_{k=n}^N l_k \cos(\sum_{j=1}^k \theta_j) \\ &+ m_N \left( \sum_{k=n}^N l_k \sin(\sum_{j=1}^k \theta_j) \right) \sum_{k=n}^N l_k \sin(\sum_{j=1}^k \theta_j) \\ &= m_N \left( \sum_{k=n}^N l_k \cos(\sum_{j=1}^k \theta_j) \right)^2 + m_N \left( \sum_{k=n}^N l_k \sin(\sum_{j=1}^k \theta_j) \right)^2 \end{aligned}$$

For  $n = N$  (row  $M_{(1+N)\bullet}(q)$ ), the diagonal entry is

$$\begin{aligned} m_{(1+N)(1+N)}(q) &= m_N \left( \sum_{k=N}^N l_k \cos(\sum_{j=1}^k \theta_j) \right)^2 + m_N \left( \sum_{k=N}^N l_k \sin(\sum_{j=1}^k \theta_j) \right)^2 \\ &= m_N l_N^2 \cos^2(\sum_{j=1}^N \theta_j) + m_N l_N^2 \sin^2(\sum_{j=1}^N \theta_j) = m_N l_N^2, \end{aligned}$$

which is (38).

- Entry  $p$  (inertial term of  $\ddot{l}_{p-N-1}$ ),  $N + 2 \leq p \leq 2N + 1$ :

$$m_{(1+n)p}(q) = m_N \sin \left( \sum_{j=1}^{p-N-1} \theta_j \right) \sum_{k=n}^N l_k \cos \left( \sum_{j=1}^k \theta_j \right) - m_N \cos \left( \sum_{j=1}^{p-N-1} \theta_j \right) \sum_{k=n}^N l_k \sin \left( \sum_{j=1}^k \theta_j \right). \quad (46)$$

The expression in (39) boils down to (46) when  $m_i = 0$  for all  $1 \leq i \leq N - 1$ . Indeed in (43) lengths accelerations appear in the terms:

$$m_N \sum_{k=1}^N \ddot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \sum_{k=n}^N l_k \cos \left( \sum_{j=1}^k \theta_j \right) - m_N \sum_{k=1}^N \ddot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \sum_{k=n}^N l_k \sin \left( \sum_{j=1}^k \theta_j \right). \quad (47)$$

*A first comment is that the inertial terms for accelerations  $\ddot{l}_k$  are all proportional to angles sines, hence the inertial couplings between lengths and angles accelerations are weak in the neighborhood of the vertical posture. This is already the case in (21) for inertial couplings between  $\ddot{x}$  and  $\ddot{l}_k$ .*

The factor of  $\ddot{l}_1$  is equal to  $m_N \sin(\theta_1) \sum_{k=n}^N l_k \cos(\sum_{j=1}^k \theta_j) - m_N \cos(\theta_1) \sum_{k=n}^N l_k \sin(\sum_{j=1}^k \theta_j)$ , the factor of  $\ddot{l}_2$  is equal to  $m_N \sin(\theta_1 + \theta_2) \sum_{k=n}^N l_k \cos(\sum_{j=1}^k \theta_j) - m_N \cos(\theta_1 + \theta_2) \sum_{k=n}^N l_k \sin(\sum_{j=1}^k \theta_j)$ , and so on.

**Small angles and large payload mass:** Let us compute  $\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{\theta}_n}$  for small angles ( $\sin(\theta_i) = \theta_i$  and  $\cos(\theta_i) = 1$ ) and  $m_N \gg m_i$ , using (43).

$$\begin{aligned} \frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{\theta}_n} &= m_N \left( \sum_{k=n}^N l_k \right) \left( \ddot{x} + \sum_{k=1}^N \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \right. \\ &\quad \left. - l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \left( \sum_{j=1}^k \theta_j \right) + \ddot{l}_k \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \right) \\ &\quad + m_N \left( \dot{x} + \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) + \dot{l}_k \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \left( \sum_{k=n}^N -l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \left( \sum_{j=1}^k \theta_j \right) + \sum_{k=n}^N \dot{l}_k \right) \\ &\quad + m_N \left( \sum_{k=n}^N l_k \left( \sum_{j=1}^k \theta_j \right) \right) \left( \sum_{k=1}^N \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \left( \sum_{j=1}^k \theta_j \right) \right. \\ &\quad \left. + l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 - \ddot{l}_k + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad + m_M \left( \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \right) \left( \sum_{k=n}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) + \sum_{k=n}^N \dot{l}_k \left( \sum_{j=1}^k \theta_j \right) \right) \end{aligned} \quad (48)$$

The corresponding mass matrix entries can be deduced from (48) or from the above expressions. In order to further simplify and to better understand the couplings around the vertical

posture, let us take  $\sin(\theta_i) = 0$ :

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{\theta}_n} &= m_N \left( \sum_{k=n}^N l_k \right) \left( \ddot{x} + \sum_{k=1}^N 2\dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \right) \\
&+ m_N \left( \sum_{k=n}^M \dot{l}_k \right) \left( \dot{x} + \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \right) \\
&- m_N \left( \sum_{k=1}^M \dot{l}_k \right) \left( \sum_{k=i}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \right)
\end{aligned} \tag{49}$$

**Remark 5** (*inertial couplings*) It is visible in both (48) and (49) that the couplings between the  $l$  dynamics and the  $\theta$  dynamics, are very small in the mass matrix  $M(q)$  since the terms  $\dot{l}_k$  are multiplied by the angles sines. But, couplings between these two dynamics exist in the Coriolis/centrifugal nonlinearities with products  $\dot{l}_k \dot{\theta}_i$ . This may be an important property for control.

**Remark 6** In the case  $N = 1$  and small angles, from (27) or from (49):

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_1} = m_1 l_1 \ddot{x} + m_1 l_1^2 \ddot{\theta}_1 + m_1 \dot{l}_1 \dot{x} - m_1 l_1 \dot{\theta}_1 \theta_1 \dot{x} + 2m_1 l_1 \dot{l}_1 \dot{\theta}_1. \tag{50}$$

#### 4.1.3 Calculation of $\frac{d}{dt} \frac{\partial T}{\partial \dot{l}_n}$ , $1 \leq n \leq N$

Let us now compute  $\frac{\partial T}{\partial \dot{l}_n} = \frac{\partial T_{trans}}{\partial \dot{l}_n}$  for  $1 \leq n \leq N$ . This gives the submatrices  $M_u(q)$  and  $M_{lx\theta}(q)$  in (13). However  $M_{lx\theta}(q) = M_{x\theta l}^\top(q)$  which is already characterized in sections 4.1.1 and 4.1.2, so the calculations are not done once again in this section. Let us start from (11).

$$\begin{aligned}
\frac{\partial T_{trans}}{\partial \dot{l}_n} &= \sum_{i=1}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\times \left( \frac{\partial}{\partial \dot{l}_n} \sum_{k=1}^i \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&- \sum_{i=1}^N m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\times \left( \frac{\partial}{\partial \dot{l}_n} \sum_{k=1}^i \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right).
\end{aligned} \tag{51}$$

Now  $\frac{\partial}{\partial \dot{l}_n} \sum_{k=1}^i \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) = \sin \left( \sum_{j=1}^n \theta_j \right)$  if  $i \geq n$ , = 0 otherwise, and similarly for  $\frac{\partial}{\partial \dot{l}_n} \sum_{k=1}^i \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right)$ . Thus:

$$\begin{aligned}
\frac{\partial T_{trans}}{\partial \dot{l}_n} &= \sum_{i=n}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \sin \left( \sum_{j=1}^n \theta_j \right) \\
&- \sum_{i=n}^N m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \cos \left( \sum_{j=1}^n \theta_j \right).
\end{aligned} \tag{52}$$

Consequently:

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{l}_n} &= \sum_{i=n}^N m_i \frac{d}{dt} \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \sin \left( \sum_{j=1}^n \theta_j \right) \\
&+ \sum_{i=n}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \left( \sum_{j=1}^n \dot{\theta}_j \right) \cos \left( \sum_{j=1}^n \theta_j \right) \\
&- \sum_{i=n}^N m_i \frac{d}{dt} \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \cos \left( \sum_{j=1}^n \theta_j \right) \\
&+ \sum_{i=n}^N m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \left( \sum_{j=1}^n \dot{\theta}_j \right) \sin \left( \sum_{j=1}^n \theta_j \right)
\end{aligned} \tag{53}$$

**Remark 7** If  $N = 1$  we get the following:

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{l}_1} &= m_1 \frac{d}{dt} \left( \dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1) \right) \sin(\theta_1) + m_1 \left( \dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1) \right) \dot{\theta}_1 \cos(\theta_1) \\
&\quad - m_1 \frac{d}{dt} \left( l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1) \right) \cos(\theta_1) + m_1 \left( l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1) \right) \dot{\theta}_1 \sin(\theta_1) \\
&= m_1 \left( \ddot{x} + \dot{l}_1 \dot{\theta}_1 \cos(\theta_1) + l_1 \ddot{\theta}_1 \cos(\theta_1) - l_1 \dot{\theta}_1^2 \sin(\theta_1) + \ddot{l}_1 \sin(\theta_1) + \dot{l}_1 \dot{\theta}_1 \cos(\theta_1) \right) \sin(\theta_1) \\
&\quad + m_1 \left( \dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1) \right) \dot{\theta}_1 \cos(\theta_1) \\
&\quad - m_1 \left( \dot{l}_1 \dot{\theta}_1 \sin(\theta_1) + l_1 \ddot{\theta}_1 \sin(\theta_1) + l_1 \dot{\theta}_1^2 \cos(\theta_1) - \ddot{l}_1 \cos(\theta_1) + \dot{l}_1 \dot{\theta}_1 \sin(\theta_1) \right) \cos(\theta_1) \\
&\quad + m_1 \left( l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1) \right) \dot{\theta}_1 \sin(\theta_1) \\
&= m_1 \sin(\theta_1) \ddot{x} + m_1 \ddot{l}_1 + m_1 \dot{x} \dot{\theta}_1 \cos(\theta_1).
\end{aligned} \tag{54}$$

Let us go on with (53):

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{l}_n} &= \sum_{i=n}^N m_i \left( \ddot{x} + \sum_{k=1}^i \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right. \\
&\quad \left. - l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \sin \left( \sum_{j=1}^k \theta_j \right) + \ddot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \sin \left( \sum_{j=1}^n \theta_j \right) \\
&+ \sum_{i=n}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \left( \sum_{j=1}^n \dot{\theta}_j \right) \cos \left( \sum_{j=1}^n \theta_j \right) \\
&- \sum_{i=n}^N m_i \left( \sum_{k=1}^i \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right. \\
&\quad \left. + l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \cos \left( \sum_{j=1}^k \theta_j \right) - \ddot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \cos \left( \sum_{j=1}^n \theta_j \right) \\
&+ \sum_{i=n}^N m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \left( \sum_{j=1}^n \dot{\theta}_j \right) \sin \left( \sum_{j=1}^n \theta_j \right)
\end{aligned} \tag{55}$$

**Comments on the mass matrix:** It is clear from (55) that the terms multiplying  $\ddot{x}$  and  $\ddot{\theta}_i$  are all proportional to sines. Hence they demonstrate weak couplings between the  $x - \theta$  dynamics and the  $l$  dynamics at the acceleration level (*i.e.*, the submatrices  $M_{lx\theta}(q)$  and  $M_{l\theta}(q)$  in (58)). In the mass matrix row corresponding to  $\ddot{l}_n$ , couplings with  $\ddot{x}$  and with  $\ddot{\theta}_i$  are proportional to  $\sin(\sum_{j=1}^n \theta_j)$ , thus they are weak in the neighborhood of the vertical position. Using (55) let us calculate the entry  $m_{nn}(q)$  of the mass matrix, corresponding to  $\ddot{l}_n$ :

$$m_{nn}(q) = \sum_{i=n}^N m_i \left( \sin^2 \left( \sum_{j=1}^n \theta_j \right) + \cos^2 \left( \sum_{j=1}^n \theta_j \right) \right) = \sum_{i=l}^N m_i. \tag{56}$$

In order to focus on the submatrix  $M_{ll}(q)$  only, let us keep only the acceleration terms  $\ddot{l}_k$  in (55). This gives:

$$\begin{aligned}
&\sum_{i=n}^N m_i \left( \sum_{k=1}^i \ddot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \sin \left( \sum_{j=1}^n \theta_j \right) + \sum_{i=l}^N m_i \left( \sum_{k=1}^i \ddot{n}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \cos \left( \sum_{j=1}^n \theta_j \right) \\
&= \sum_{i=n}^N m_i \left( \sum_{k=1}^i \ddot{l}_k \left( \sin \left( \sum_{j=1}^k \theta_j \right) \sin \left( \sum_{j=1}^n \theta_j \right) + \cos \left( \sum_{j=1}^k \theta_j \right) \cos \left( \sum_{j=1}^n \theta_j \right) \right) \right)
\end{aligned} \tag{57}$$

Let us recall that the mass matrix is written as

$$M(q) = \begin{pmatrix} M_{x\theta}(q) & M_{x\theta l}(q) \\ M_{lx\theta}(q) & M_{ll}(q) \end{pmatrix}, \quad (58)$$

$M_{x\theta} \in \mathbb{R}^{(1+N) \times (1+N)}$ ,  $M_{x\theta l} \in \mathbb{R}^{(1+N) \times N}$ ,  $M_{ll} \in \mathbb{R}^{N \times N}$ ,  $M_{lx\theta} = M_{x\theta l}^\top$ . Thus equation (57) is rewritten:  $(m_{n1}, m_{n2}, \dots, m_{nN}) \ddot{l}$ ,  $l \triangleq (l_1, l_2, \dots, l_N)^\top$ , where  $(m_{n1}, m_{n2}, \dots, m_{nN})$  is the  $n$ th row of  $M_{ll}(q)$ , *i.e.*, the  $(1+N+n)$ th row of  $M(q)$ ,  $1 \leq n \leq N$ .

Let us compute the term  $m_{np}(q)$ , with  $1 \leq p \leq N$ ,  $p \neq n$ , which collects all terms multiplying  $\ddot{l}_p$  in (57). Let us start with  $p = 1$ . Since  $\ddot{l}_1$  appears in all the sums  $\sum_{i=1}^k$ , it follows that:

$$\begin{aligned} m_{n1}(q) &= \sum_{i=n}^N m_i (\sin(\theta_1) \sin(\sum_{j=1}^n \theta_j) + \cos(\theta_1) \cos(\sum_{j=1}^n \theta_j)) \\ &= \sum_{i=n}^N m_i \cos(\sum_{j=1}^n \theta_j - \theta_1) \end{aligned} \quad (59)$$

Now if  $p \leq n$ ,  $\ddot{l}_p$  appears also in all the sums  $\sum_{k=1}^i$ , and we get:

$$\begin{aligned} m_{np}(q) &= \sum_{i=n}^N m_i (\sin(\sum_{j=1}^p \theta_j) \sin(\sum_{j=1}^n \theta_j) + \cos(\sum_{j=1}^p \theta_j) \cos(\sum_{j=1}^n \theta_j)) \\ &= \sum_{i=n}^N m_i \cos(\sum_{j=1}^n \theta_j - \sum_{j=1}^p \theta_j) \end{aligned} \quad (60)$$

If  $p \geq n$ ,  $\ddot{l}_p$  appears only in the sums  $\sum_{i=p}^N$ , *i.e.*, in the terms

$$\sum_{i=p}^N m_i \left( \sum_{k=1}^i \ddot{l}_k \sin(\sum_{j=1}^k \theta_j) \right) \sin(\sum_{j=1}^n \theta_j) + \sum_{i=n}^N m_i \left( \sum_{k=1}^i \ddot{l}_k \cos(\sum_{j=1}^k \theta_j) \right) \cos(\sum_{j=1}^n \theta_j).$$

Thus:

$$\begin{aligned} m_{np}(q) &= \sum_{i=p}^N m_i (\sin(\sum_{j=1}^p \theta_j) \sin(\sum_{j=1}^n \theta_j) + \cos(\sum_{j=1}^p \theta_j) \cos(\sum_{j=1}^n \theta_j)) \\ &= \sum_{i=p}^N m_i \cos(\sum_{j=1}^n \theta_j - \sum_{j=1}^p \theta_j) \end{aligned} \quad (61)$$

To summarize:

$$m_{np}(q) = \sum_{i=\max(n,p)}^N m_i \cos(\sum_{j=1}^n \theta_j - \sum_{j=1}^p \theta_j) \quad (62)$$

Using the notation as in sections 4.1.1 and 4.1.2 where  $p$  denotes the column of the mass matrix  $M(q)$ , thus  $N+2 \leq p \leq 2N+1$ ,  $1 \leq n \leq N$ , and the mass matrix' entry corresponding to  $m_{np}(q)$  in (62) is:

$$m_{(1+N+n)p}(q) = \sum_{i=\max(n,p-N-1)}^N m_i \cos(\sum_{j=1}^n \theta_j - \sum_{j=1}^{p-N-1} \theta_j). \quad (63)$$

For instance for  $N = 2$ ,  $n = 2$  and  $p = 5$  correspond to  $m_{55}(q) = \sum_{i=\max(2,2)=2}^2 m_i \cos(\sum_{j=1}^2 \theta_j - \sum_{j=1}^2 \theta_j) = m_2$ ,  $n = 2$  and  $p = 4$  correspond to  $m_{54}(q) = \sum_{i=\max(2,1)=2}^2 m_i \cos(\sum_{j=1}^2 \theta_j -$

$\sum_{j=1}^1 \theta_j = m_2 \cos(\theta_2)$ , which are equal to the ones in (142). And  $n = 1$ ,  $p = 5$  gives  $m_{45}(q) = \sum_{i=\max(1,2)=2}^2 m_i \cos(\sum_{j=1}^1 \theta_j - \sum_{j=1}^2 \theta_j) = m_2 \cos(\theta_2)$ , while  $n = 1$  and  $p = 4$  gives  $m_{44}(q) = \sum_{i=\max(1,1)=1}^2 m_i \cos(\sum_{j=1}^1 \theta_j - \sum_{j=1}^1 \theta_j) = m_1 + m_2$ , which are equal to the ones in (141).

**Remark 8** *These results indicate that if small angles are considered, then the matrix  $M_U(q)$  is constant.*

**Large payload mass:** Let us assume as above that  $m_N \gg m_i$  in (55). This yields:

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{\text{trans}}}{\partial \dot{l}_n} &= m_N \sin(\sum_{j=1}^n \theta_j) \left( \ddot{x} + \sum_{k=1}^N \dot{l}_k (\sum_{j=1}^k \dot{\theta}_j) \cos(\sum_{j=1}^k \theta_j) + l_k (\sum_{j=1}^k \ddot{\theta}_j) \cos(\sum_{j=1}^k \theta_j) \right. \\
&\quad \left. - l_k (\sum_{j=1}^k \dot{\theta}_j)^2 \sin(\sum_{j=1}^k \theta_j) + \ddot{l}_k \sin(\sum_{j=1}^k \theta_j) + \dot{l}_k (\sum_{j=1}^k \dot{\theta}_j) \cos(\sum_{j=1}^k \theta_j) \right) \\
&\quad + m_N \left( \dot{x} + \sum_{k=1}^N l_k (\sum_{j=1}^k \dot{\theta}_j) \cos(\sum_{j=1}^k \theta_j) + \dot{l}_k \sin(\sum_{j=1}^k \theta_j) \right) (\sum_{j=1}^n \dot{\theta}_j) \cos(\sum_{j=1}^n \theta_j) \\
&\quad - m_N \cos(\sum_{j=1}^n \theta_j) \left( \sum_{k=1}^N \dot{l}_k (\sum_{j=1}^k \dot{\theta}_j) \sin(\sum_{j=1}^k \theta_j) + l_k (\sum_{j=1}^k \ddot{\theta}_j) \sin(\sum_{j=1}^k \theta_j) \right. \\
&\quad \left. + l_k (\sum_{j=1}^k \dot{\theta}_j)^2 \cos(\sum_{j=1}^k \theta_j) - \ddot{l}_k \cos(\sum_{j=1}^k \theta_j) + \dot{l}_k (\sum_{j=1}^k \dot{\theta}_j) \sin(\sum_{j=1}^k \theta_j) \right) \\
&\quad + m_N \left( \sum_{k=1}^N l_k (\sum_{j=1}^k \dot{\theta}_j) \sin(\sum_{j=1}^k \theta_j) - \dot{l}_k \cos(\sum_{j=1}^k \theta_j) \right) (\sum_{j=1}^n \dot{\theta}_j) \sin(\sum_{j=1}^n \theta_j)
\end{aligned} \tag{64}$$

In this case we obtain:

$$m_{nn}(q) = \sum_{i=n}^N m_i \left( \sin^2(\sum_{j=1}^n \theta_j) + \cos^2(\sum_{j=1}^n \theta_j) \right) = \sum_{i=n}^N m_i. \tag{65}$$

The other entries are obtained from (64) or directly from the general case above.

**Large payload mass and small angles:** Let us start with the first order linearization  $\sin(\theta_i) \approx \theta_i$ :

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{l}_n} &= m_N \left( \sum_{j=1}^n \theta_j \right) \left( \ddot{x} + \sum_{k=1}^N \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \right. \\
&\quad \left. - l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \left( \sum_{j=1}^k \theta_j \right) + \ddot{l}_k \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \right) \\
&\quad + m_N \left( \dot{x} + \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) + \dot{l}_k \left( \sum_{j=1}^k \theta_j \right) \right) \left( \sum_{j=1}^n \dot{\theta}_j \right) \\
&\quad - m_N \left( \sum_{k=1}^N \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \left( \sum_{j=1}^k \theta_j \right) \right. \\
&\quad \left. + l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 - \ddot{l}_k + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad + m_N \left( \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \right) \left( \sum_{j=1}^n \dot{\theta}_j \right) \left( \sum_{j=1}^n \theta_j \right)
\end{aligned} \tag{66}$$

and when  $\sin(\theta_i) \approx 0$ :

$$\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{l}_n} = m_N \left( \dot{x} + \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \right) \left( \sum_{j=1}^n \dot{\theta}_j \right) - m_N \left( \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 - \ddot{l}_k \right) \tag{67}$$

from which (54) is recovered.

**Remark 9** *It follows from (67) that the assumption made in this paragraph yields a sub-matrix  $M_{ll}(q) = M_{ll}$  with all entries equal to  $m_N$ , hence it has rank 1 and the approximated mass matrix is low rank. See also remarks 13 and 14.*

## 4.2 Calculation of $\frac{\partial T}{\partial q}$

Nonlinear terms stemming from  $\frac{\partial T}{\partial \theta_i}$  and  $\frac{\partial T}{\partial l_k}$  are present in the Lagrange dynamics, see (11) (notice that  $\frac{\partial T}{\partial x} = 0$ ).

### 4.2.1 Calculation of $\frac{\partial T}{\partial \theta_n}$ , $1 \leq n \leq N$

Clearly  $\frac{\partial T}{\partial \theta_n} = \frac{\partial T_{trans}}{\partial \theta_n}$ . Let us recall that:

$$\begin{aligned}
T_{trans} &= \frac{1}{2} \sum_{i=1}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right)^2 \\
&\quad + \frac{1}{2} \sum_{i=1}^N m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right)^2.
\end{aligned} \tag{68}$$

Consequently:

$$\begin{aligned}
\frac{\partial T_{trans}}{\partial \theta_n} &= \sum_{i=1}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \frac{\partial}{\partial \theta_n} \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \frac{\partial}{\partial \theta_i} \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad + \sum_{i=1}^N m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \frac{\partial}{\partial \theta_n} \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \frac{\partial}{\partial \theta_n} \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&= \sum_{i=n}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \left( \sum_{k=n}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \frac{\partial}{\partial \theta_i} \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \frac{\partial}{\partial \theta_i} \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad + \sum_{i=n}^N m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \left( \sum_{k=n}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \frac{\partial}{\partial \theta_n} \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \frac{\partial}{\partial \theta_n} \cos \left( \sum_{j=1}^k \theta_j \right) \right)
\end{aligned} \tag{69}$$

Now for  $k \geq n$ , one has  $\frac{\partial}{\partial \theta_n} \sin \left( \sum_{j=1}^k \theta_j \right) = \frac{\partial}{\partial \theta_n} \left( \sum_{j=1}^k \theta_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) = \cos \left( \sum_{j=1}^k \theta_j \right)$ , and  $\frac{\partial}{\partial \theta_n} \cos \left( \sum_{j=1}^k \theta_j \right) = -\frac{\partial}{\partial \theta_n} \left( \sum_{j=1}^k \theta_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) = -\sin \left( \sum_{j=1}^k \theta_j \right)$ .

Thus:

$$\begin{aligned}
\frac{\partial T_{trans}}{\partial \theta_n} &= \sum_{i=n}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \left( \sum_{k=n}^i -l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad + \sum_{i=n}^N m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \left( \sum_{k=n}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right)
\end{aligned} \tag{70}$$

**Remark 10** In the case  $N = 1$ , (70) becomes:

$$\begin{aligned}
\frac{\partial T_{trans}}{\partial \theta_1} &= m_1 \left( \dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1) \right) \left( -l_1 \dot{\theta}_1 \sin(\theta_1) + \dot{l}_1 \cos(\theta_1) \right) \\
&\quad + m_1 \left( l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1) \right) \left( l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1) \right) \\
&= m_1 \dot{x} \left( -l_1 \dot{\theta}_1 \sin(\theta_1) + \dot{l}_1 \cos(\theta_1) \right) \\
&\quad + m_1 \left( -l_1^2 \dot{\theta}_1^2 \cos(\theta_1) \sin(\theta_1) + l_1 \dot{l}_1 \dot{\theta}_1 \cos^2(\theta_1) - l_1 \dot{l}_1 \dot{\theta}_1 \sin^2(\theta_1) + \dot{l}_1^2 \sin(\theta_1) \cos(\theta_1) \right) \\
&\quad + m_1 \left( l_1^2 \dot{\theta}_1^2 \sin(\theta_1) \cos(\theta_1) + l_1 \dot{l}_1 \dot{\theta}_1 \sin^2(\theta_1) - l_1 \dot{l}_1 \dot{\theta}_1 \cos^2(\theta_1) - \dot{l}_1^2 \sin(\theta_1) \cos(\theta_1) \right) \\
&= -m_1 \dot{x} l_1 \dot{\theta}_1 \sin(\theta_1) + m_1 \dot{x} \dot{l}_1 \cos(\theta_1).
\end{aligned} \tag{71}$$

**Large payload mass  $m_N \gg m_i$  and small angles:** From (70) with  $\sin(\theta_i) \approx \theta_i$ :

$$\begin{aligned} \frac{\partial T_{trans}}{\partial \theta_n} &= m_N \left( \dot{x} + \sum_{k=1}^N l_k (\sum_{j=1}^k \dot{\theta}_j) + \dot{l}_k (\sum_{j=1}^k \theta_j) \right) \left( \sum_{k=n}^N -l_k (\sum_{j=1}^k \dot{\theta}_j) (\sum_{j=1}^k \theta_j) + \dot{l}_k \right) \\ &\quad + m_N \left( \sum_{k=1}^N l_k (\sum_{j=1}^k \dot{\theta}_j) (\sum_{j=1}^k \theta_j) - \dot{l}_k \right) \left( \sum_{k=n}^N l_k (\sum_{j=1}^k \dot{\theta}_j) + \dot{l}_k (\sum_{j=1}^k \theta_j) \right) \end{aligned} \quad (72)$$

From (70) with  $\sin(\theta_i) \approx 0$ :

$$\begin{aligned} \frac{\partial T_{trans}}{\partial \theta_n} &= m_N (\dot{x} + \sum_{k=1}^N l_k (\sum_{j=1}^k \dot{\theta}_j) (\sum_{k=n}^N \dot{l}_k) - m_M (\sum_{k=1}^N \dot{l}_k) (\sum_{k=1}^N l_k \sum_{j=1}^k \dot{\theta}_j)) \\ &= m_N (\sum_{k=n}^N \dot{l}_k) (\dot{x} + \sum_{k=1}^N l_k \sum_{j=1}^k \dot{\theta}_j - \sum_{k=n}^N l_k \sum_{j=1}^k \dot{\theta}_j) = m_N (\sum_{k=n}^N \dot{l}_k) \dot{x} \end{aligned} \quad (73)$$

Both (72) and (73) reduce to (71) when  $M = 1$ .

#### 4.2.2 Calculation of $\frac{\partial T}{\partial l_n}$ , $1 \leq n \leq N$

Here  $\frac{\partial T}{\partial l_n} = \frac{\partial T_{trans}}{\partial l_n}$ . First:

$$\begin{aligned} \frac{\partial T_{trans}}{\partial l_n} &= \sum_{i=1}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \frac{\partial}{\partial l_n} \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \\ &\quad + \sum_{i=1}^N m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \frac{\partial}{\partial l_n} \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right). \end{aligned} \quad (74)$$

Proceeding as above:

$$\begin{aligned} \frac{\partial T_{trans}}{\partial l_n} &= \sum_{i=1}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \frac{\partial}{\partial l_n} \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \\ &\quad + \sum_{i=1}^N m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \frac{\partial}{\partial l_n} \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \\ &= \sum_{i=n}^N m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \left( \sum_{j=1}^n \dot{\theta}_j \right) \cos \left( \sum_{j=1}^n \theta_j \right) \\ &\quad + \sum_{i=n}^N m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \left( \sum_{j=1}^n \dot{\theta}_j \right) \sin \left( \sum_{j=1}^n \theta_j \right) \end{aligned} \quad (75)$$

**Large payload mass**  $m_N \ggg m_i$ ,  $1 \leq i \leq N - 1$ :

$$\begin{aligned} \frac{\partial T_{trans}}{\partial l_n} &= m_N \left( \dot{x} + \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \left( \sum_{j=1}^n \dot{\theta}_j \right) \cos \left( \sum_{j=1}^n \theta_j \right) \\ &+ m_N \left( \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \left( \sum_{j=1}^n \dot{\theta}_j \right) \sin \left( \sum_{j=1}^n \theta_j \right) \end{aligned} \quad (76)$$

**Large payload mass**  $m_N \ggg m_i$ ,  $1 \leq i \leq N - 1$ , **and small angles:** With the first order approximation:

$$\begin{aligned} \frac{\partial T_{trans}}{\partial l_n} &= m_N \left( \dot{x} + \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) + \dot{l}_k \left( \sum_{j=1}^k \theta_j \right) \right) \left( \sum_{j=1}^n \dot{\theta}_j \right) \\ &+ m_N \left( \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \right) \left( \sum_{j=1}^n \dot{\theta}_j \right) \left( \sum_{j=1}^n \theta_j \right) \end{aligned} \quad (77)$$

and with the zero order approximation:

$$\frac{\partial T_{trans}}{\partial l_n} = m_N \left( \dot{x} + \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \right) \left( \sum_{j=1}^n \dot{\theta}_j \right) \quad (78)$$

**Remark 11** If  $N = 1$  from (75):

$$\begin{aligned} \frac{\partial T_{trans}}{\partial l_1} &= m_1 \left( l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1) \right) \dot{\theta}_1 \sin(\theta_1) + m_1 \left( \dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1) \right) \dot{\theta}_1 \cos(\theta_1) \\ &= m_1 \dot{x} \dot{\theta}_1 \cos(\theta_1) + m_1 l_1 \dot{\theta}_1^2 \end{aligned} \quad (79)$$

and from (77):

$$\frac{\partial T_{trans}}{\partial l_1} = m_1 \dot{x} \dot{\theta}_1 + m_1 l_1 \dot{\theta}_1^2 + m_1 l_1 \dot{\theta}_1^2 \theta_1^2, \quad (80)$$

(an error of order 2 in  $\theta_1$  appears), and from (78):

$$\frac{\partial T_{trans}}{\partial l_1} = m_1 \dot{x} \dot{\theta}_1 + m_1 l_1 \dot{\theta}_1^2. \quad (81)$$

### 4.3 Potential Energy

If no elastic torsional joint stiffness is modeled, this reduces to the gravity potential energy of each link, that of the trolley being constant chosen equal to zero.

$$U_g = - \sum_{i=1}^N m_i g \sum_{k=1}^i l_k \cos \left( \sum_{j=1}^k \theta_j \right) \quad (82)$$

Thus:

$$\begin{aligned}\frac{\partial U_g}{\partial \theta_n} &= -\frac{\partial}{\partial \theta_n} \sum_{i=1}^N m_i g \sum_{k=1}^i l_k \cos(\sum_{j=1}^k \theta_j) \\ &= -\sum_{i=n}^N m_i g \sum_{k=n}^i l_k \frac{\partial}{\partial \theta_n} \cos(\sum_{j=1}^k \theta_j) = \sum_{i=n}^N m_i g \sum_{k=n}^i l_k \sin(\sum_{j=1}^k \theta_j)\end{aligned}\quad (83)$$

so that for small angles  $\frac{\partial U_g}{\partial \theta_n} = 0$  (the potential energy is almost constant), and for  $m_N \gg m_i$ :  $\frac{\partial U_g}{\partial \theta_n} = m_N g \sum_{k=1}^N l_k \sin(\sum_{k=1}^M \theta_k)$ . And

$$\begin{aligned}\frac{\partial U_g}{\partial l_n} &= -\frac{\partial}{\partial l_n} \sum_{i=1}^N m_i g \sum_{k=1}^i l_k \cos(\sum_{j=1}^k \theta_j) \\ &= -\sum_{i=1}^N m_i g \frac{\partial}{\partial l_n} \sum_{k=1}^i l_k \cos(\sum_{j=1}^k \theta_j) = -\sum_{i=n}^N m_i g \frac{\partial}{\partial l_n} \sum_{k=n}^i l_k \cos(\sum_{j=1}^k \theta_j) \\ &= -\sum_{i=n}^N m_i g \cos(\sum_{j=1}^n \theta_j)\end{aligned}\quad (84)$$

For small angles  $\frac{\partial U_g}{\partial l_n} = -\sum_{i=n}^N m_i g$ . For  $m_N \gg m_i$  this gives  $\frac{\partial U_g}{\partial l_n} = -m_N g$ .

**Remark 12** In case  $N = 1$  the general expression (83) gives:

$$\frac{\partial U_g}{\partial \theta_1} = m_1 g l_1 \sin(\theta_1) \quad (85)$$

and from (84) for  $N = 1$ :

$$\frac{\partial U_g}{\partial l_1} = -m_1 g \cos(\theta_1) \quad (86)$$

**Joint flexibility:** If joint flexibility is added at joints  $A_i$ ,  $1 \leq i \leq N$ , then the additional potential energy is  $U_{flex}(\theta) = \frac{1}{2} \sum_{j=0}^{N-1} \kappa_j \theta_{j+1}^2 = \frac{1}{2} \theta^\top \mathcal{K} \theta$ , where  $\kappa_j \geq 0$  is the angular stiffness at joint  $A_j$  and  $\mathcal{K} = \text{diag}(\kappa_i)$ ,  $0 \leq i \leq N-1$ . Thus:

$$\frac{\partial U_{flex}(\theta)}{\partial \theta_i} = \kappa_{i-1} \theta_i, \quad 1 \leq i \leq N. \quad (87)$$

This introduces no coupling between the coordinates, contrarily to what occurs in flexible joint manipulators [19, 3, 29]. In the coordinate angles  $\alpha_i = \sum_{j=1}^i \theta_j$ ,  $1 \leq i \leq N$ ,  $\alpha = J\theta$  with  $J$  in (155),  $\theta = J^{-1}\alpha$  with  $J^{-1}$  in (156), hence  $U_{flex}(\alpha) = \frac{1}{2} \theta^\top \mathcal{K} \theta = \frac{1}{2} \alpha^\top J^{-\top} \mathcal{K} J^{-1} \alpha$ . In

view of (156):  $J^{-1}\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 - \alpha_1 \\ \alpha_3 - \alpha_2 \\ \vdots \\ \alpha_N - \alpha_{N-1} \end{pmatrix}$ , hence  $U_{flex}(\alpha) = \frac{1}{2} \kappa_0 \alpha_1^2 + \sum_{i=1}^{N-1} \kappa_i (\alpha_{i+1} - \alpha_i)^2$ . It is

inferred that

$$\frac{\partial U_{flex}(\alpha)}{\partial \alpha_1} = \kappa_0 \alpha_1 - \kappa_1 (\alpha_2 - \alpha_1) = (\kappa_0 + \kappa_1) \alpha_1 - \kappa_1 \alpha_2, \quad (88)$$

for  $2 \leq j \leq N - 1$ :

$$\frac{\partial U_{flex}(\alpha)}{\partial \alpha_j} = \kappa_{j-1}(\alpha_j - \alpha_{j-1}) - \kappa_j(\alpha_{j+1} - \alpha_j) = (\kappa_{j-1} + \kappa_j)\alpha_j - \kappa_j\alpha_{j+1} - \kappa_{j-1}\alpha_{j-1}, \quad (89)$$

and

$$\frac{\partial U_{flex}(\alpha)}{\partial \alpha_N} = \kappa_{N-1}(\alpha_N - \alpha_{N-1}), \quad (90)$$

so that

$$\frac{\partial U_{flex}(\alpha)}{\partial \alpha} = \begin{pmatrix} \kappa_0 + \kappa_1 & -\kappa_1 & 0 & \dots & & & 0 \\ -\kappa_1 & \kappa_1 + \kappa_2 & -\kappa_2 & 0 & \dots & & 0 \\ 0 & -\kappa_2 & \kappa_2 + \kappa_3 & -\kappa_3 & 0 & \dots & 0 \\ \vdots & & & & & & \vdots \\ \vdots & & & & & & \vdots \\ 0 & \dots & & & 0 & -\kappa_{N-2} & \kappa_{N-2} + \kappa_{N-1} & -\kappa_{N-1} \\ 0 & \dots & & & 0 & & -\kappa_{N-1} & \kappa_{N-1} \end{pmatrix} \quad (91)$$

This time the joint flexibility introduces a triangular structure in the torques that derive from the elasticity potential, since the row  $1 + n$  of the Lagrange dynamics,  $1 \leq n \leq N$ , corresponding to  $M_{(1+n)\bullet}(q_\alpha)$  and  $\ddot{\alpha}_n$ , contains the flexibility torque  $(-\kappa_{n-1}\alpha_{n-1} + (\kappa_{n-1} + \kappa_n)\alpha_n) - \kappa_n\alpha_{n+1}$ . One can think of using a backstepping-like control design using the fictitious input  $\alpha_{n+1}$ . However, as seen in Lemma 1, the *vis-à-vis* terms in the submatrix  $M_{\alpha\alpha}(q_\alpha)$  always contain strong inertial couplings between  $\ddot{\alpha}_n$  and the other angular accelerations. Therefore the global triangular structure of Spong's model for flexible-joint manipulators [19, 3] does not exist in such overhead crane systems.

## 4.4 Recapitulation

The Lagrange dynamics of the overhead crane is given by:

$$\frac{d}{dt} \frac{\partial(T_{trol} + T_{trans} - U_g)}{\partial \dot{q}} - \frac{\partial(T_{trol} + T_{trans} - U_g - U_{flex})}{\partial q} = Q, \quad (92)$$

where  $Q_i$  works on  $q_i$ . More precisely:

$$\begin{cases} \frac{d}{dt} \frac{\partial(T_{trol} + T_{trans})}{\partial \dot{x}} = Q_x = F \\ \frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{\theta}} - \frac{\partial(T_{trans} - U_g - U_{flex})}{\partial \theta} = Q_\theta = 0 \\ \frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{l}} - \frac{\partial(T_{trans} - U_g - U_{flex})}{\partial l} = Q_l = (F_1, 0, \dots, 0)^\top \end{cases} \quad (93)$$

It may be useful to introduce viscous friction (Rayleigh dissipation [2, Definition 6.12]) at the joints, and some joint stiffness as well. This may allow to avoid chaotic behaviours of conservative pendula. Such dynamical behaviour is unlikely to occur in real cables where a small amount of dissipation and of bending stiffness is always present.

## 5 The Case $N = 2$

This may be of interest in practice for cranes with two cables related by a hook. It could be of interest in such a case to investigate energy-based controllers, which have been derived in case  $N = 1$  [26]. For clarity let us rewrite the above energies for  $N = 2$ :

$$T_{trol} = \frac{1}{2}m\dot{x}^2. \quad (94)$$

$$\begin{aligned} T_{trans} &= \frac{1}{2} \sum_{i=1}^2 m_i \left( \dot{x} + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right)^2 \\ &+ \frac{1}{2} \sum_{i=1}^2 m_i \left( \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right)^2 \end{aligned} \quad (95)$$

Let us split  $T_{trans}$  it into three parts:

$$T_{trans,1} = \frac{1}{2}m_1(\dot{x} + l_1\dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1))^2 + \frac{1}{2}m_1(l_1\dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1))^2 \quad (96)$$

and

$$T_{trans,2} = \frac{1}{2}m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right)^2 \quad (97)$$

and

$$T_{trans,3} = \frac{1}{2}m_2 \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right)^2. \quad (98)$$

The potential energy due to gravity is:

$$U_g = - \sum_{i=1}^2 m_i g \sum_{k=1}^i l_k \cos \left( \sum_{j=1}^k \theta_j \right) = -m_1 g l_1 \cos(\theta_1) - m_2 g (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \quad (99)$$

Let us calculate once again the Lagrange equations. All the terms in (94)–(99) are easily handled, excepted for (97) and (98) with which we start now.

## 5.1 Calculation of $\frac{d}{dt} \frac{\partial T_{trans,2}}{\partial \dot{q}}$

Let us consider:

$$\frac{\partial T_{trans,2}}{\partial \dot{x}} = m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \sum_{j=1}^k \dot{\theta}_j \cos\left(\sum_{j=1}^k \theta_j\right) + \dot{l}_k \sin\left(\sum_{j=1}^k \theta_j\right) \right) \quad (100)$$

Hence:

$$\begin{aligned} \frac{d}{dt} \frac{\partial T_{trans,2}}{\partial \dot{x}} &= m_2 \left( \ddot{x} + \sum_{k=1}^2 \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) \right. \\ &\quad \left. + \sum_{k=1}^2 -l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \sin\left(\sum_{j=1}^k \theta_j\right) + \ddot{l}_k \sin\left(\sum_{j=1}^k \theta_j\right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) \right) \end{aligned} \quad (101)$$

We infer that the first row of the mass matrix contains the terms:

$$\begin{aligned} m_{11}(q) &: m_2 \\ m_{12}(q) &: m_2(l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\ m_{13}(q) &: m_2 l_2 \cos(\theta_1 + \theta_2) \\ m_{14}(q) &: m_2 \sin(\theta_1) \\ m_{15}(q) &: m_2 \sin(\theta_1 + \theta_2) \end{aligned} \quad (102)$$

Now let us pass to the next coordinate:

$$\begin{aligned} \frac{\partial T_{trans,2}}{\partial \dot{\theta}_1} &= m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) + \dot{l}_k \sin\left(\sum_{j=1}^k \theta_j\right) \right) \\ &\quad \times \frac{\partial}{\partial \dot{\theta}_1} \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) \right) \\ &= m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) + \dot{l}_k \sin\left(\sum_{j=1}^k \theta_j\right) \right) \\ &\quad \times \left( \sum_{k=1}^2 l_k \frac{\partial}{\partial \dot{\theta}_1} \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) \right) \\ &= m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) + \dot{l}_k \sin\left(\sum_{j=1}^k \theta_j\right) \right) \\ &\quad \times (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \end{aligned} \quad (103)$$

Consequently:

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans,2}}{\partial \dot{\theta}_1} &= m_2 \left( \ddot{x} + \frac{d}{dt} \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
&+ m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \left( \dot{l}_1 \cos(\theta_1) - l_1 \dot{\theta}_1 \sin(\theta_1) + \dot{l}_2 \cos(\theta_1 + \theta_2) - l_2 (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2) \right) \\
&= m_2 \left( \ddot{x} + \sum_{k=1}^2 \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \ddot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \\
&\quad - l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \sin \left( \sum_{j=1}^k \theta_j \right) \left( l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \right) \\
&+ m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \left( \dot{l}_1 \cos(\theta_1) - l_1 \dot{\theta}_1 \sin(\theta_1) + \dot{l}_2 \cos(\theta_1 + \theta_2) - l_2 (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2) \right)
\end{aligned} \tag{104}$$

From (104) it can be deduced terms that belong to the second row of the mass matrix (recall  $q = (x, \theta_1, \theta_2, l_1, l_2)^\top$ ) and which come from  $\frac{d}{dt} \frac{\partial T_{trans,2}}{\partial \dot{\theta}_1}$  :

$$\begin{aligned}
m_{21}(q) &: m_2 (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
m_{22}(q) &: m_2 (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2))^2 \\
m_{23}(q) &: m_2 l_2 \cos(\theta_1 + \theta_2) (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
m_{24}(q) &: m_2 \sin(\theta_1) (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
m_{25}(q) &: m_2 \sin(\theta_1 + \theta_2) (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2))
\end{aligned} \tag{105}$$

Similarly:

$$\begin{aligned}
\frac{\partial T_{trans,2}}{\partial \dot{\theta}_2} &= m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \frac{\partial}{\partial \dot{\theta}_2} \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&= m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \left( \sum_{k=1}^2 l_k \frac{\partial}{\partial \dot{\theta}_2} \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&= m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times l_2 \cos(\theta_1 + \theta_2)
\end{aligned} \tag{106}$$

Consequently:

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans,2}}{\partial \dot{\theta}_2} &= m_2 \left( \ddot{x} + \sum_{k=1}^2 \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right. \\
&\quad \left. - l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \sin \left( \sum_{j=1}^k \theta_j \right) + \ddot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times l_2 \cos(\theta_1 + \theta_2) \\
&\quad + m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \left( \dot{l}_2 \cos(\theta_1 + \theta_2) - l_2 (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2) \right)
\end{aligned} \tag{107}$$

From (107) it can be deduced terms that belong to the third row of the mass matrix and which come from  $\frac{d}{dt} \frac{\partial T_{trans,2}}{\partial \dot{\theta}_2}$  :

$$\begin{aligned}
m_{31}(q) &: m_2 l_2 \cos(\theta_1 + \theta_2) \\
m_{32}(q) &: m_2 (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) l_2 \cos(\theta_1 + \theta_2) \\
m_{33}(q) &: m_2 l_2^2 \cos^2(\theta_1 + \theta_2) \\
m_{34}(q) &: m_2 \sin(\theta_1) l_2 \cos(\theta_1 + \theta_2) \\
m_{35}(q) &: m_2 \sin(\theta_1 + \theta_2) l_2 \cos(\theta_1 + \theta_2)
\end{aligned} \tag{108}$$

Now terms of the fourth row of the mass matrix:

$$\begin{aligned}
\frac{\partial T_{trans,2}}{\partial \dot{l}_1} &= m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\quad \times \frac{\partial}{\partial \dot{l}_1} \left( \sum_{k=1}^2 \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&= m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \sin(\theta_1)
\end{aligned} \tag{109}$$

Hence

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans,2}}{\partial \dot{l}_1} &= m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \dot{\theta}_1 \cos(\theta_1) \\
&\quad + m_2 \sin(\theta_1) \left( \ddot{x} + \sum_{k=1}^2 \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right. \\
&\quad \left. - l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \sin \left( \sum_{j=1}^k \theta_j \right) + \ddot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right)
\end{aligned} \tag{110}$$

From (110) it can be deduced terms that belong to the fourth row of the mass matrix and which come from  $\frac{d}{dt} \frac{\partial T_{trans,2}}{\partial \dot{l}_1}$  :

$$\begin{aligned}
m_{41}(q) &: m_2 \sin(\theta_1) \\
m_{42}(q) &: m_2 \sin(\theta_1)(l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
m_{43}(q) &: m_2 \sin(\theta_1)l_2 \cos(\theta_1 + \theta_2) \\
m_{44}(q) &: m_2 \sin^2(\theta_1) \\
m_{45}(q) &: m_2 \sin(\theta_1) \sin(\theta_1 + \theta_2)
\end{aligned} \tag{111}$$

Let us end with terms of the fifth row of the mass matrix:

$$\begin{aligned}
\frac{\partial T_{trans,2}}{\partial \dot{l}_2} &= m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\times \frac{\partial}{\partial \dot{l}_2} \left( \sum_{k=1}^2 \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&= m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \sin(\theta_1 + \theta_2).
\end{aligned} \tag{112}$$

Thus:

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans,2}}{\partial \dot{l}_2} &= m_2 \left( \dot{x} + \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2) \\
&+ m_2 \left( \ddot{x} + \sum_{k=1}^2 \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right. \\
&\left. - l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \sin \left( \sum_{j=1}^k \theta_j \right) + \ddot{l}_k \sin \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right) \sin(\theta_1 + \theta_2).
\end{aligned} \tag{113}$$

From (113) it can be deduced terms that belong to the fifth row of the mass matrix and which come from  $\frac{d}{dt} \frac{\partial T_{trans,2}}{\partial \dot{l}_2}$  :

$$\begin{aligned}
m_{51}(q) &: m_2 \sin(\theta_1 + \theta_2) \\
m_{52}(q) &: m_2 \sin(\theta_1 + \theta_2)(l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
m_{53}(q) &: m_2 \sin(\theta_1 + \theta_2)l_2 \cos(\theta_1 + \theta_2) \\
m_{54}(q) &: m_2 \sin(\theta_1 + \theta_2) \sin(\theta_1) \\
m_{55}(q) &: m_2 \sin^2(\theta_1 + \theta_2)
\end{aligned} \tag{114}$$

## 5.2 Calculation of $\frac{d}{dt} \frac{\partial T_{trans,3}}{\partial \dot{q}}$

Calculations are similar to those of  $\frac{d}{dt} \frac{\partial T_{trans,2}}{\partial \dot{q}}$ .

$$\frac{\partial T_{trans,3}}{\partial \dot{x}} = 0 \tag{115}$$

hence

$$\frac{d}{dt} \frac{\partial T_{trans,3}}{\partial \dot{x}} = 0, \quad (116)$$

This does not contribute to the first row of the mass matrix.

$$\begin{aligned} \frac{\partial T_{trans,3}}{\partial \theta_1} &= m_2 \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \frac{\partial}{\partial \theta_1} \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\ &= m_2 \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \left( \sum_{k=1}^2 l_k \frac{\partial}{\partial \theta_1} \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\ &= m_2 \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \end{aligned} \quad (117)$$

Consequently:

$$\begin{aligned} \frac{d}{dt} \frac{\partial T_{trans,3}}{\partial \dot{\theta}_1} &= m_2 \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times \left( \dot{l}_1 \sin(\theta_1) + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_2 \sin(\theta_1 + \theta_2) + l_2 \dot{\theta}_1 + \dot{\theta}_2 \right) \cos(\theta_1 + \theta_2) \\ &\quad + m_2 \left( \sum_{k=1}^2 \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right. \\ &\quad \left. + l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \cos \left( \sum_{j=1}^k \theta_j \right) - \ddot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \sum_{j=1}^k \dot{\theta}_j \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\ &\quad \times (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \end{aligned} \quad (118)$$

From (118) it can be deduced terms that belong to the second row of the mass matrix (recall  $q = (x, \theta_1, \theta_2, l_1, l_2)^\top$ ) and which come from  $\frac{d}{dt} \frac{\partial T_{trans,3}}{\partial \dot{\theta}_1}$  :

$$\begin{aligned} m_{21}(q) &: 0 \\ m_{22}(q) &: m_2 (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2))^2 \\ m_{23}(q) &: m_2 l_2 \sin(\theta_1 + \theta_2) (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \\ m_{24}(q) &: -m_2 \cos(\theta_1) (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \\ m_{25}(q) &: -m_2 \cos(\theta_1 + \theta_2) (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \end{aligned} \quad (119)$$

Similarly:

$$\begin{aligned}
\frac{\partial T_{trans,3}}{\partial \dot{\theta}_2} &= m_2 \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\times \frac{\partial}{\partial \dot{\theta}_2} \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&= m_2 \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&\times \left( \sum_{k=1}^2 l_k \frac{\partial}{\partial \dot{\theta}_2} \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&= m_2 l_2 \sin(\theta_1 + \theta_2) \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right). \tag{120}
\end{aligned}$$

Consequently:

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans,3}}{\partial \dot{\theta}_2} &= m_2 l_2 (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2) \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&+ m_2 l_2 \sin(\theta_1 + \theta_2) \frac{d}{dt} \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&= m_2 l_2 (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2) \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \\
&+ m_2 l_2 \sin(\theta_1 + \theta_2) \left( \sum_{k=1}^2 \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right. \\
&\left. + l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \cos \left( \sum_{j=1}^k \theta_j \right) - \ddot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right) \tag{121}
\end{aligned}$$

From (121) it can be deduced terms that belong to the third row of the mass matrix and which come from  $\frac{d}{dt} \frac{\partial T_{trans,3}}{\partial \dot{\theta}_2}$  :

$$\begin{aligned}
m_{31}(q) &: 0 \\
m_{32}(q) &: m_2 l_2 \sin(\theta_1 + \theta_2) (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \\
m_{33}(q) &: m_2 l_2^2 \sin^2(\theta_1 + \theta_2) \\
m_{34}(q) &: -m_2 l_2 \sin(\theta_1 + \theta_2) \cos(\theta_1) \\
m_{35}(q) &: -m_2 l_2 \sin(\theta_1 + \theta_2) \cos(\theta_1 + \theta_2) \tag{122}
\end{aligned}$$

Let us pass now to the fourth row of the mass matrix (the  $l_1$ -dynamics):

$$\frac{\partial T_{trans,3}}{\partial \dot{l}_1} = -m_2 \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \cos(\theta_1). \tag{123}$$

Hence:

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans,3}}{\partial \dot{l}_1} &= m_2 \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \dot{\theta}_1 \sin(\theta_1) \\
&- m_2 \left( \sum_{k=1}^2 \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right. \\
&\left. + l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \cos \left( \sum_{j=1}^k \theta_j \right) - \ddot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right) \cos(\theta_1)
\end{aligned} \tag{124}$$

From (124) it can be deduced terms that belong to the fourth row of the mass matrix and which come from  $\frac{d}{dt} \frac{\partial T_{trans,3}}{\partial \dot{l}_1}$  :

$$\begin{aligned}
m_{41}(q) &: 0 \\
m_{42}(q) &: -m_2 \cos(\theta_1)(l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \\
m_{43}(q) &: -m_2 \cos(\theta_1)l_2 \sin(\theta_1 + \theta_2) \\
m_{44}(q) &: m_2 \cos^2(\theta_1) \\
m_{45}(q) &: m_2 \cos(\theta_1) \cos(\theta_1 + \theta_2)
\end{aligned} \tag{125}$$

Let us end with terms of the fifth row of the mass matrix:

$$\frac{\partial T_{trans,3}}{\partial \dot{l}_2} = -m_2 \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) \cos(\theta_1 + \theta_2). \tag{126}$$

Thus:

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans,3}}{\partial \dot{l}_2} &= m_2 \left( \sum_{k=1}^2 l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) - \dot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2) \\
&- m_2 \left( \sum_{k=1}^2 \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right. \\
&\left. + l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \cos \left( \sum_{j=1}^k \theta_j \right) - \ddot{l}_k \cos \left( \sum_{j=1}^k \theta_j \right) + \dot{l}_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right) \cos(\theta_1 + \theta_2)
\end{aligned} \tag{127}$$

From (127) it can be deduced terms that belong to the fifth row of the mass matrix and which come from  $\frac{d}{dt} \frac{\partial T_{trans,3}}{\partial \dot{l}_2}$  :

$$\begin{aligned}
m_{51}(q) &: 0 \\
m_{52}(q) &: -m_2 \cos(\theta_1 + \theta_2)(l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \\
m_{53}(q) &: -m_2 \cos(\theta_1 + \theta_2)l_2 \sin(\theta_1 + \theta_2) \\
m_{54}(q) &: m_2 \cos(\theta_1 + \theta_2) \cos(\theta_1) \\
m_{55}(q) &: m_2 \cos^2(\theta_1 + \theta_2)
\end{aligned} \tag{128}$$

### 5.3 Calculation of $\frac{d}{dt} \frac{\partial T_{trans,1}}{\partial \dot{q}}$

From (96):

$$\frac{\partial T_{trans,1}}{\partial \dot{x}} = m_1(\dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1)) \tag{129}$$

Therefore:

$$\frac{d}{dt} \frac{\partial T_{trans,1}}{\partial \dot{x}} = m_1(\ddot{x} + \dot{l}_1 \dot{\theta}_1 \cos(\theta_1) + l_1 \ddot{\theta}_1 \cos(\theta_1) - l_1 \dot{\theta}_1^2 \sin(\theta_1) + \ddot{l}_1 \sin(\theta_1) + \dot{l}_1 \dot{\theta}_1 \cos(\theta_1)) \quad (130)$$

We infer that the first row of the mass matrix contains the terms:

$$\begin{aligned} m_{11}(q) &: m_1 \\ m_{12}(q) &: m_1 l_1 \cos(\theta_1) \\ m_{13}(q) &: 0 \\ m_{14}(q) &: m_1 \sin(\theta_1) \\ m_{15}(q) &: 0 \end{aligned} \quad (131)$$

Let us now pass to the second row of the mass matrix:

$$\frac{\partial T_{trans,1}}{\partial \dot{\theta}_1} = m_1 l_1 \cos(\theta_1)(\dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1)) + m_1 l_1 \sin(\theta_1)(l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1)) \quad (132)$$

from which:

$$\begin{aligned} \frac{d}{dt} \frac{\partial T_{trans,1}}{\partial \dot{\theta}_1} &= m_1(\dot{l}_1 \cos(\theta_1) - l_1 \dot{\theta}_1 \sin(\theta_1))(\dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1)) \\ &+ m_1 l_1 \cos(\theta_1)(\ddot{x} + \dot{l}_1 \dot{\theta}_1 \cos(\theta_1) + l_1 \ddot{\theta}_1 \cos(\theta_1) - l_1 \dot{\theta}_1^2 \sin(\theta_1) + \ddot{l}_1 \sin(\theta_1) + \dot{l}_1 \dot{\theta}_1 \cos(\theta_1)) \\ &+ m_1(\dot{l}_1 \sin(\theta_1) + l_1 \dot{\theta}_1 \cos(\theta_1))(l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1)) \\ &+ m_1 l_1 \sin(\theta_1)(\dot{l}_1 \dot{\theta}_1 \sin(\theta_1) + l_1 \ddot{\theta}_1 \sin(\theta_1) + l_1 \dot{\theta}_1^2 \cos(\theta_1) - \ddot{l}_1 \cos(\theta_1) + \dot{l}_1 \dot{\theta}_1 \sin(\theta_1)) \end{aligned} \quad (133)$$

The terms corresponding to the second row of the mass matrix are:

$$\begin{aligned} m_{21}(q) &: m_1 l_1 \cos(\theta_1) \\ m_{22}(q) &: m_1 l_1^2 \\ m_{23}(q) &: 0 \\ m_{24}(q) &: 0 \\ m_{25}(q) &: 0 \end{aligned} \quad (134)$$

Further,  $\frac{\partial T_{trans,1}}{\partial \dot{\theta}_2} = 0$ , and  $\frac{\partial T_{trans,1}}{\partial \dot{l}_2} = 0$ : these terms do not contribute to the third and fifth rows of the mass matrix.

$$\frac{\partial T_{trans,1}}{\partial \dot{l}_1} = m_1 \sin(\theta_1)(\dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1)) - m_1 \cos(\theta_1)(l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1)) \quad (135)$$

Hence:

$$\begin{aligned} \frac{d}{dt} \frac{\partial T_{trans,1}}{\partial \dot{l}_1} &= m_1 \dot{\theta}_1 \cos(\theta_1)(\dot{x} + l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1)) \\ &+ m_1 \sin(\theta_1)(\ddot{x} + \dot{l}_1 \dot{\theta}_1 \cos(\theta_1) + l_1 \ddot{\theta}_1 \cos(\theta_1) - l_1 \dot{\theta}_1^2 \sin(\theta_1) + \ddot{l}_1 \sin(\theta_1) + \dot{l}_1 \dot{\theta}_1 \cos(\theta_1)) \\ &+ m_1 \dot{\theta}_1 \sin(\theta_1)(l_1 \dot{\theta}_1 \sin(\theta_1) - \dot{l}_1 \cos(\theta_1)) \\ &- m_1 \cos(\theta_1)(\dot{l}_1 \dot{\theta}_1 \sin(\theta_1) + l_1 \ddot{\theta}_1 \sin(\theta_1) + l_1 \dot{\theta}_1^2 \cos(\theta_1) - \ddot{l}_1 \cos(\theta_1) + \dot{l}_1 \dot{\theta}_1 \sin(\theta_1)) \end{aligned} \quad (136)$$

The terms corresponding to the fourth row of the mass matrix are:

$$\begin{aligned}
m_{41}(q) &: m_1 \sin(\theta_1) \\
m_{42}(q) &: 0 \\
m_{43}(q) &: 0 \\
m_{44}(q) &: m_1 \\
m_{45}(q) &: 0
\end{aligned} \tag{137}$$

Since  $\frac{\partial T_{trans,1}}{\partial \dot{l}_2} = 0$  and  $\frac{\partial T_{trans,1}}{\partial \dot{\theta}_2} = 0$  the contributions to the third and fifth rows of the mass matrix are zero.

#### 5.4 Recapitulation: Calculation of $M(q)$

Let us gather the above results. This gives:

$$\begin{aligned}
m_{11}(q) &= m + m_1 + m_2 \\
m_{12}(q) &= m_1 l_1 \cos(\theta_1) + m_2 (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
m_{13}(q) &= m_2 l_2 \cos(\theta_1 + \theta_2) \\
m_{14}(q) &= (m_1 + m_2) \sin(\theta_1) \\
m_{15}(q) &= m_2 \sin(\theta_1 + \theta_2)
\end{aligned} \tag{138}$$

$$\begin{aligned}
m_{21}(q) &= m_1 l_1 \cos(\theta_1) + m_2 (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) = m_{12}(q) \\
m_{22}(q) &= m_1 l_1^2 + m_2 (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2))^2 + m_2 (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2))^2 \\
&= m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 (\sin(\theta_1) \sin(\theta_1 + \theta_2) + \cos(\theta_1) \cos(\theta_1 + \theta_2)) \\
&= m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 \cos(\theta_2) \\
m_{23}(q) &= m_2 l_2 \sin(\theta_1 + \theta_2) (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) + m_2 l_2 \cos(\theta_1 + \theta_2) (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
&= m_2 l_1 l_2 \cos(\theta_2) + m_2 l_2^2 \\
m_{24}(q) &= m_2 \sin(\theta_1) (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) - m_2 \cos(\theta_1) (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) \\
&= m_2 l_2 (\sin(\theta_1) \cos(\theta_1 + \theta_2) - \cos(\theta_1) \sin(\theta_1 + \theta_2)) \\
&= -m_2 l_2 \sin(\theta_2) \\
m_{25}(q) &= -m_2 \cos(\theta_1 + \theta_2) (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) + m_2 \sin(\theta_1 + \theta_2) (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
&= m_2 l_1 (-\cos(\theta_1 + \theta_2) \sin(\theta_1) + \sin(\theta_1 + \theta_2) \cos(\theta_1)) \\
&= m_2 l_1 \sin(\theta_2)
\end{aligned} \tag{139}$$

$$\begin{aligned}
m_{31}(q) &= m_2 l_2 \cos(\theta_1 + \theta_2) = m_{13}(q) \\
m_{32}(q) &= m_2 l_2 \sin(\theta_1 + \theta_2) (l_1 \sin(\theta_1 + l_2 \sin(\theta_1 + \theta_2)) + m_2 l_2 \cos(\theta_1 + \theta_2) (l_1 \cos(\theta_1 + l_2 \cos(\theta_1 + \theta_2))) \\
&= m_2 l_1 l_2 \cos(\theta_2) + m_2 l_2^2 = m_{23}(q) \\
m_{33}(q) &= m_2 l_2^2 \sin^2(\theta_1 + \theta_2) + m_2 l_2^2 \cos^2(\theta_1 + \theta_2) \\
&= m_2 l_2^2 \\
m_{34}(q) &= -m_2 l_2 \sin(\theta_1 + \theta_2) \cos(\theta_1) + m_2 l_2 \cos(\theta_1 + \theta_2) \sin(\theta_1) \\
&= -m_2 l_2 \sin(\theta_2) \\
m_{35}(q) &= 0
\end{aligned} \tag{140}$$

$$\begin{aligned}
m_{41}(q) &= (m_1 + m_2) \sin(\theta_1) = m_{14}(q) \\
m_{42}(q) &= -m_2 \cos(\theta_1) (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) + m_2 \sin(\theta_1) (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
&= m_2 l_2 (\sin(\theta_1) \cos(\theta_1 + \theta_2) - \cos(\theta_1) \sin(\theta_1 + \theta_2)) \\
&= -m_2 l_2 \sin(\theta_2) = m_{24}(q) \\
m_{43}(q) &= -m_2 l_2 \cos(\theta_1) \sin(\theta_1 + \theta_2) + m_2 l_2 \sin(\theta_1) \cos(\theta_1 + \theta_2) \\
&= -m_2 l_2 \sin(\theta_2) = m_{34}(q) \\
m_{44}(q) &= m_1 + m_2 \cos^2(\theta_1) + m_2 \sin^2(\theta_1) = m_1 + m_2 \\
m_{45}(q) &= m_2 \cos(\theta_1) \cos(\theta_1 + \theta_2) + m_2 \sin(\theta_1) \sin(\theta_1 + \theta_2) \\
&= m_2 \cos(\theta_2)
\end{aligned} \tag{141}$$

$$\begin{aligned}
m_{51}(q) &= m_2 \sin(\theta_1 + \theta_2) = m_{15}(q) \\
m_{52}(q) &= -m_2 \cos(\theta_1 + \theta_2) (l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)) + m_2 \sin(\theta_1 + \theta_2) (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
&= m_2 l_1 (\sin(\theta_1 + \theta_2) \cos(\theta_1) - \cos(\theta_1 + \theta_2) \sin(\theta_1)) \\
&= m_2 l_1 \sin(\theta_2) = m_{25}(q) \\
m_{53}(q) &= 0 = m_{35}(q) \\
m_{54}(q) &= m_2 \sin(\theta_1) \sin(\theta_1 + \theta_2) + m_2 \cos(\theta_1) \cos(\theta_1 + \theta_2) \\
&= m_2 \cos(\theta_2) = m_{45}(q) \\
m_{55}(q) &= m_2 \cos^2(\theta_1 + \theta_2) + m_2 \sin^2(\theta_1 + \theta_2) = m_2
\end{aligned} \tag{142}$$

It is verified that indeed  $M(q) = M^\top(q)$ . Also when  $l_1$  and  $l_2$  are constant the mass matrix given in [27, 28] is retrieved, noting that  $\theta_1 + \theta_2$  is the angle  $\theta_2$  in [27, 28], which modifies some terms of the mass matrix.

### 5.4.1 $m_2 \gg m_1$

This can be simplified if  $m_2 \gg m_1$ , if for instance  $m_1$  is the hook's mass while  $m_2$  is a much bigger tool plus object mass. Then  $m_1 = 0$  can be taken in  $M(q)$ :

$$\begin{aligned}
m_{11}(q) &= m + m_2 \\
m_{12}(q) &= m_2(l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) \\
m_{13}(q) &= m_2 l_2 \cos(\theta_1 + \theta_2) \\
m_{14}(q) &= m_2 \sin(\theta_1) \\
m_{15}(q) &= m_2 \sin(\theta_1 + \theta_2)
\end{aligned} \tag{143}$$

$$\begin{aligned}
m_{21}(q) &= m_2(l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) = m_{12}(q) \\
m_{22}(q) &= m_2(l_1^2 + l_2^2 + 2l_1 l_2 \cos(\theta_2)) \\
m_{23}(q) &= m_2 l_1 l_2 \cos(\theta_2) + m_2 l_2^2 \\
m_{24}(q) &= -m_2 l_2 \sin(\theta_2) \\
m_{25}(q) &= m_2 l_1 \sin(\theta_2)
\end{aligned} \tag{144}$$

$$\begin{aligned}
m_{31}(q) &= m_2 l_2 \cos(\theta_1 + \theta_2) = m_{13}(q) \\
m_{32}(q) &= m_2 l_1 l_2 \cos(\theta_2) + m l_2^2 = m_{23}(q) \\
m_{33}(q) &= m_2 l_2^2 \\
m_{34}(q) &= -m_2 l_2 \sin(\theta_2) \\
m_{35}(q) &= 0
\end{aligned} \tag{145}$$

$$\begin{aligned}
m_{41}(q) &= m_2 \sin(\theta_1) = m_{14}(q) \\
m_{42}(q) &= -m_2 l_2 \sin(\theta_2) = m_{24}(q) \\
m_{43}(q) &= -m_2 l_2 \sin(\theta_2) = m_{34}(q) \\
m_{44}(q) &= m_2 \\
m_{45}(q) &= m_2 \cos(\theta_2)
\end{aligned} \tag{146}$$

$$\begin{aligned}
m_{51}(q) &= m_2 \sin(\theta_1 + \theta_2) = m_{15}(q) \\
m_{52}(q) &= m_2 l_1 \sin(\theta_2) = m_{25}(q) \\
m_{53}(q) &= 0 = m_{35}(q) \\
m_{54}(q) &= m_2 \cos(\theta_2) = m_{45}(q) \\
m_{55}(q) &= m_2 \cos^2(\theta_1 + \theta_2) + m_2 \sin^2(\theta_1 + \theta_2) = m_2
\end{aligned} \tag{147}$$

**Remark 13** In this case the matrix  $M_{II}(q)$  has its determinant equal to  $m_2^2(1 - \cos^2(\theta_2))$  which vanishes at  $\theta_2 = 0$ . On the contrary if  $m_1 > 0$  then  $M_{II}(q)$  has rank 2 from (141) and (142).

### 5.4.2 Small Angles

If the small angles assumption is made with the zero order approximation ( $\theta_1 \approx 0$  and  $\theta_2 \approx 0$ ), then the mass matrix simplifies to:

$$\begin{aligned}
m_{11}(q) &= m + m_1 + m_2 \\
m_{12}(q) &= m_1 l_1 + m_2(l_1 + l_2) \\
m_{13}(q) &= m_2 l_2 \\
m_{14}(q) &= 0 \\
m_{15}(q) &= 0
\end{aligned} \tag{148}$$

$$\begin{aligned}
m_{21}(q) &= m_1 l_1 + m_2(l_1 + l_2) = m_{12}(q) \\
m_{22}(q) &= m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 \\
m_{23}(q) &= m_2 l_1 l_2 + m_2 l_2^2 \\
m_{24}(q) &= 0 \\
m_{25}(q) &= 0
\end{aligned} \tag{149}$$

$$\begin{aligned}
m_{31}(q) &= m_2 l_2 = m_{13}(q) \\
m_{32}(q) &= m_2 l_1 l_2 + m_2 l_2^2 = m_{23}(q) \\
m_{33}(q) &= m_2 l_2^2 \\
m_{34}(q) &= 0 \\
m_{35}(q) &= 0
\end{aligned} \tag{150}$$

$$\begin{aligned}
m_{41}(q) &= 0 = m_{14}(q) \\
m_{42}(q) &= 0 = m_{24}(q) \\
m_{43}(q) &= 0 = m_{34}(q) \\
m_{44}(q) &= m_1 + m_2 \\
m_{45}(q) &= m_2
\end{aligned} \tag{151}$$

$$\begin{aligned}
m_{51}(q) &= 0 = m_{15}(q) \\
m_{52}(q) &= 0 = m_{25}(q) \\
m_{53}(q) &= 0 = m_{35}(q) \\
m_{54}(q) &= m_2 = m_{45}(q) \\
m_{55}(q) &= m_2
\end{aligned} \tag{152}$$

Hence:

$$M(q) = \begin{pmatrix} m + m_1 + m_2 & m_1 l_1 + m_2(l_1 + l_2) & m_2 l_2 & 0 & 0 \\ m_1 l_1 + m_2(l_1 + l_2) & m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 & m_2 l_1 l_2 + m_2 l_2^2 & 0 & 0 \\ m_2 l_2 & m_2 l_1 l_2 + m_2 l_2^2 & m_2 l_2^2 & 0 & 0 \\ 0 & 0 & 0 & m_1 + m_2 & m_2 \\ 0 & 0 & 0 & m_2 & m_2 \end{pmatrix} \tag{153}$$

There is a decoupling between the  $(x, \theta_1, \theta_2)$  dynamics and the  $(l_1, l_2)$  dynamics in the mass matrix which has a block diagonal structure.

**Remark 14** (*large payload mass*) As alluded to above in Remarks 13 and 9, it follows from (153) that if  $m_2 \gg m_1$  then the determinant of the mass matrix gets close to zero. Indeed in the limit as  $m_1 = 0$  the submatrix  $M_{ll}$  has rank 1.

## 6 Mass Matrix Structure

### 6.1 Coordinate $q = (x, \theta_1, \dots, \theta_N, l_1, \dots, l_N)^\top$

Using (19), (20), (21), (24), (33), (39), Remarks 5, 9, 13, 14, (55), (56)–(61), (63), (194), the following holds in the general case:

**Lemma 1** *Let  $N \geq 1$  and  $q = (x, \theta_1, \theta_2, \dots, \theta_N, l_1, l_2, \dots, l_N)^\top$ . Consider the mass matrix in (13) and (14).*

1. *If small angles are considered (the cable stays in a neighborhood of the vertical position) with a zero-order approximation, i.e.,  $\sin(\theta_i) \approx \theta_i \approx 0$ , then the mass matrix in (58) has the block-diagonal form:*

$$M(q) = \begin{pmatrix} M_{x\theta}(q) & 0 \\ 0 & M_{ll} \end{pmatrix} \quad (154)$$

where  $0 \prec M_{x\theta}(q) \in \mathbb{R}^{(1+N) \times (1+N)}$ ,  $0 \prec M_{ll} \in \mathbb{R}^{N \times N}$ .

2. *If the first order approximation is chosen, i.e.,  $\sin(\theta_i) \approx \theta_i$ , then the off-block-diagonal terms  $M_{x\theta l}$  and  $M_{lx\theta}$  are proportional to the angles  $\theta_i$  and sums of them, and  $M_{ll}$  is still constant.*
3. *The matrix  $M_{ll}(q)$  always has constant diagonal entries  $m_{nn}$  equal to  $\sum_{j=1}^n m_j$ . The matrix  $M_{xx}(q)$  is the entry  $m_{11}$  and is constant.*
4. *The matrix  $M_{x\theta}(q)$  has entries proportional to angles cosines, or to the sum of cosines and sines, and thus remains bounded away from zero in the neighborhood of the cable's vertical posture (roughly speaking, the inertial couplings between the trolley's and the angle's accelerations, are not weak around the vertical posture).*
5. *The submatrix  $M_{\theta\theta}(q)$  has entries made of sums of terms functions of products of cosines only, and of products of sines only. Hence they remain nonlinear in the angles (including terms like  $\theta_i \theta_j$ ) if the first order approximation is chosen. Also the inertial couplings inside  $M_{\theta\theta}(q)$  are strong, in the sense that they remain bounded away from zero even in the neighborhood of the cable's vertical posture.*
6. *The submatrix  $\bar{M}_{x\theta}(q)$  has entries linear in  $l_k$  if the first and second order approximations are chosen.*

7. *The small angles assumption (even the zero order approximation) does not imply that the Coriolis/centrifugal nonlinearities vanish in the case of varying lengths: products  $\dot{l}_j \dot{\theta}_k$  and  $l_k \theta_j$  are still present.*
8. *In case of large payload mass, the mass matrix becomes bad-conditionned, since  $M_U(q)$  loses its rank around the vertical position when  $m_i = 0$ ,  $1 \leq i \leq N - 1$ , and has low rank when in addition the zero order approximation for small angles is made.*

This lemma is important because it shows that when the cable is close to the vertical posture, the  $(x, \theta)$ - $l$  couplings become very small in the mass matrix. They clearly could not be inverted to design a controller. Item 7 can be verified for instance in (194) below.

## 6.2 Other Generalized Coordinates

### 6.2.1 Angles $\alpha_i$ , $1 \leq i \leq N$

Most of the authors work with the angles  $\alpha_i = \sum_{j=1}^i \theta_j$ , with generalized coordinate  $q_\alpha = (x, \alpha_1, \dots, \alpha_N, l_1, \dots, l_N)^\top$ , see Figure 1. The transformation matrix  $q_\alpha = Tq$  is given by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & I_N \end{pmatrix}, J \in \mathbb{R}^{N \times N}, \text{ with :}$$

$$J = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & & & & \vdots \\ 1 & 1 & 1 & \dots & 1 & 1 & 0 \\ 1 & 1 & 1 & \dots & & 1 & 1 \end{pmatrix} \quad (155)$$

The matrix  $J$  is full-rank Toeplitz [1, Definition 3.1.3], and its inverse is [1, Fact 3.18.11]:

$$J^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 0 & 0 & \dots & & 0 \\ 0 & -1 & 1 & 0 & \dots & & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & \dots & 0 & -1 & 1 & 0 \\ 0 & 0 & \dots & & 0 & -1 & 1 \end{pmatrix} \quad (156)$$

Then

$$\begin{aligned}
M_\alpha(q_\alpha) &= T^{-\top} M(q) T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & J^{-\top} & 0 \\ 0 & 0 & I_N \end{pmatrix} M(q) \begin{pmatrix} 1 & 0 & 0 \\ 0 & J^{-1} & 0 \\ 0 & 0 & I_N \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & J^{-\top} & 0 \\ 0 & 0 & I_N \end{pmatrix} \begin{pmatrix} M_{x\theta}(q) & 0 \\ 0 & M_u \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & J^{-1} & 0 \\ 0 & 0 & I_N \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & J^{-\top} \end{pmatrix} M_{x\theta}(q) \begin{pmatrix} 1 & 0 \\ 0 & J^{-1} \end{pmatrix} & 0 \\ 0 & M_u \end{pmatrix} = \begin{pmatrix} M_{x\alpha}(q_\alpha) & 0 \\ 0 & M_u \end{pmatrix}.
\end{aligned} \tag{157}$$

Therefore the conclusions of Lemma 1 still hold in the new coordinates.

**Remark 15 (Differences with flexible-joint manipulators)** *The mass matrix in coordinates  $q$  or  $q_\alpha$ , has a block-diagonal structure, with small off-diagonal blocks in the neighborhood of the cable's vertical posture. But, this does not seem to put the system's dynamics in an amenable form for control design, as it is for instance the case for Spong's model of flexible joint manipulators. One reason is the number and location of the inputs in (93): the underactuated structure of overhead cranes and of flexible-joint manipulators, totally differ one from each other. It is noteworthy that Spong's model for flexible-joint manipulators, is obtained by neglecting the kinetic energy of actuators' shafts, which is a justified assumption in many applications, and allows one to get rid of some inertial couplings. Doing a similar "triangularization" assumption in  $M_{\theta\theta}(q)$  or  $M_{\alpha\alpha}(q_\alpha)$  is not possible. In this sense the overhead crane dynamics is closer to the complete flexible-joint manipulators dynamics where off-diagonal blocks are present in the mass matrix [29]. It is known that the backstepping method no longer applies in this case.*

### 6.2.2 Fixed Lengths $l_k$ , $2 \leq k \leq N$ , Controlled $l_1(t)$

In most cases the winding device is mounted at node  $A_0$  and we may lump the cable's length variation in the first link, *i.e.*,  $l_1(t)$  is varying while the other links are constant (this is not the case if we consider links' longitudinal elasticity as in section 7). The kinetic energy in (11) becomes equal to:

$$\begin{aligned}
T_{trans} &= \frac{1}{2} \sum_{i=1}^N m_i \left( \dot{x} + \dot{l}_1 \sin(\theta_1) + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) \right)^2 \\
&\quad + \frac{1}{2} \sum_{i=1}^N m_i \left( -\dot{l}_1 \cos(\theta_1) + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin \left( \sum_{j=1}^k \theta_j \right) \right)^2.
\end{aligned} \tag{158}$$

The generalized coordinate reduces to  $q = (x, \theta_1, \dots, \theta_N, l_1)^\top \in \mathbb{R}^{2+N}$ . From (31) it is inferred:

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{\theta}_n} &= \sum_{i=n}^N m_i \left( \ddot{x} + 2\dot{l}_1 \dot{\theta}_1 \cos(\theta_1) + \ddot{l}_1 \sin(\theta_1) + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) \right. \\
&\quad \left. - \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \sin\left(\sum_{j=1}^k \theta_j\right) \right) \left( \sum_{k=n}^i l_k \cos\left(\sum_{j=1}^k \theta_j\right) \right) \\
&\quad + \sum_{i=n}^N m_i \left( \dot{x} + \dot{l}_1 \sin(\theta_1) + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) \right) \\
&\quad \times \left( \sum_{k=n}^i -l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin\left(\sum_{j=1}^k \theta_j\right) + \begin{cases} \dot{l}_1 \cos(\theta_1) & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases} \right) \\
&\quad + \sum_{i=n}^N m_i \left( 2\dot{l}_1 \dot{\theta}_1 \sin(\theta_1) - \ddot{l}_1 \cos(\theta_1) + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \cos\left(\sum_{j=1}^k \theta_j\right) \right. \\
&\quad \left. + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \sin\left(\sum_{j=1}^k \theta_j\right) \right) \left( \sum_{k=n}^i l_k \sin\left(\sum_{j=1}^k \theta_j\right) \right) \\
&\quad + \sum_{i=n}^N m_i \left( -\dot{l}_1 \cos(\theta_1) + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin\left(\sum_{j=1}^k \theta_j\right) \right) \\
&\quad \times \left( \sum_{k=n}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) + \begin{cases} \dot{l}_1 \sin(\theta_1) & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases} \right)
\end{aligned} \tag{159}$$

From (55) it is inferred:

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{l}_1} &= \sum_{i=1}^N m_i \left( \ddot{x} + 2\dot{l}_1 \dot{\theta}_1 \cos(\theta_1) + \ddot{l}_1 \sin(\theta_1) + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) \right. \\
&\quad \left. - l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \sin\left(\sum_{j=1}^k \theta_j\right) \right) \sin(\theta_1) \\
&\quad + \sum_{i=1}^N m_i \left( \dot{x} + \dot{l}_1 \sin(\theta_1) + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos\left(\sum_{j=1}^k \theta_j\right) \right) \dot{\theta}_1 \cos(\theta_1) \\
&\quad - \sum_{i=1}^N m_i \left( 2\dot{l}_1 \dot{\theta}_1 \sin(\theta_1) - \ddot{l}_1 \cos(\theta_1) + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \sin\left(\sum_{j=1}^k \theta_j\right) \right. \\
&\quad \left. + l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \cos\left(\sum_{j=1}^k \theta_j\right) \right) \cos(\theta_1) \\
&\quad + \sum_{i=1}^N m_i \left( -\dot{l}_1 \cos(\theta_1) + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin\left(\sum_{j=1}^k \theta_j\right) \right) \dot{\theta}_1 \sin(\theta_1)
\end{aligned} \tag{160}$$

Finally from (18) it is inferred:

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{x}} &= \sum_{i=1}^N m_i \ddot{x} \\
&+ \sum_{i=1}^N m_i \left( \dot{l}_1 \dot{\theta}_1 \cos(\theta_1) + \sum_{k=1}^i l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \cos \left( \sum_{j=1}^k \theta_j \right) - l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \sin \left( \sum_{j=1}^k \theta_j \right) \right) \\
&+ \sum_{i=1}^N m_i (\ddot{l}_1 \sin(\theta_1) + \dot{l}_1 \dot{\theta}_1 \cos(\theta_1))
\end{aligned} \tag{161}$$

The three equations (159), (160) and (161) are sufficient to deduce  $M(q)$ .

• Row  $M_{1\bullet}(q) = M_{\bullet 1}^\top(q)$ : from (161) it follows that:

$$m_{11}(q) = m + \sum_{i=1}^N m_i \tag{162}$$

$$m_{1(N+2)}(q) = \sin(\theta_1) \sum_{i=1}^N m_i \tag{163}$$

which is (21) for  $p = N + 2$ , and for  $2 \leq n \leq N + 1$  ( $m_{1n}(q)$  is the factor of  $\ddot{\theta}_{n-1}$  in  $M_{1\bullet}(q)\ddot{q}$ ):

$$m_{1n}(q) = \sum_{i=n-1}^N m_i \sum_{k=n-1}^i l_k \cos \left( \sum_{j=1}^k \theta_j \right), \tag{164}$$

which is (20).

• Row  $M_{n\bullet}(q)$ ,  $2 \leq n \leq N + 1$ : this time it follows from (159) that the entry multiplying  $\ddot{\theta}_{p-1}$ ,  $2 \leq p \leq N + 1$ , is:

$$m_{np}(q) = \sum_{i=\max(n-1, p-1)}^N m_i \sum_{k=p-1}^i l_k \left\{ \cos \left( \sum_{j=1}^k \theta_j \right) \left( \sum_{k=n-1}^i l_k \cos \left( \sum_{j=1}^k \theta_j \right) \right) + \sin \left( \sum_{j=1}^k \theta_j \right) \left( \sum_{k=n-1}^i l_k \sin \left( \sum_{j=1}^k \theta_j \right) \right) \right\}. \tag{165}$$

Indeed the terms involving angles accelerations  $\ddot{\theta}_i$  in (159) are exactly equal to (34), hence the mass matrix entries are those in (33) (there is a shift from  $1 + n$  to  $n$  between (33) and (165) because we considered  $1 \leq n \leq N$  in section 4.1.2 while we consider  $2 \leq n \leq N + 1$  now). And the entry that multiplies  $\ddot{x}$  in the row  $M_{n\bullet}(q)$ ,  $2 \leq n \leq N + 1$ , which corresponds to  $\ddot{\theta}_{n-1}$ , is:

$$m_{n1}(q) = \sum_{i=n-1}^N m_i \sum_{k=n-1}^i l_k \cos \left( \sum_{j=1}^k \theta_j \right) \tag{166}$$

One sees that this is equal to (164) which confirms the symmetry of  $M(q)$  with  $M_{1\bullet}(q) = M_{\bullet 1}^\top(q)$ . Finally the factor of  $\ddot{l}_1$  is:

$$\begin{aligned} m_{n(N+2)}(q) &= \sum_{i=n-1}^N m_i \left( \sin(\theta_1) \sum_{k=n-1}^i l_k \cos(\sum_{j=1}^k \theta_j) - \cos(\theta_1) \sum_{k=n-1}^i l_k \sin(\sum_{j=1}^k \theta_j) \right) \\ &= \sum_{i=n-1}^N m_i \sum_{k=n-1}^i l_k \sin(\theta_1 - \sum_{j=1}^k \theta_j) \end{aligned} \quad (167)$$

• Row  $M_{(N+2)\bullet}(q)$ : from (160) it follows that the entry multiplying  $\ddot{x}$  is:

$$m_{(N+2)1}(q) = \sin(\theta_1) \sum_{i=1}^N m_i, \quad (168)$$

which is (163), and for  $2 \leq p \leq N+1$ :

$$m_{(N+2)p}(q) = \sum_{i=p-1}^N m_i \sum_{k=p-1}^i l_k \sin(\theta_1 - \sum_{j=1}^k \theta_j) \quad (169)$$

Indeed the terms involving angular accelerations in (160) are

$$\sum_{i=1}^N m_i \sum_{k=1}^i l_k \left( \cos(\sum_{j=1}^k \theta_j) \sin(\theta_1) - \sin(\sum_{j=1}^k \theta_j) \cos(\theta_1) \right) \sum_{j=1}^k \ddot{\theta}_j \quad (170)$$

$\ddot{\theta}_1$  is in all the summation terms so that:

$$\begin{aligned} m_{(N+2)2}(q) &= \sum_{i=1}^N m_i \sum_{k=1}^i l_k \left( \cos(\sum_{j=1}^k \theta_j) \sin(\theta_1) - \sin(\sum_{j=1}^k \theta_j) \cos(\theta_1) \right) \\ &= \sum_{i=1}^N m_i \sum_{k=1}^i l_k \sin(\theta_1 - \sum_{j=1}^k \theta_j) \end{aligned} \quad (171)$$

which is (167) with  $n = 2$ . Now for  $\ddot{\theta}_2$ :

$$\begin{aligned} m_{(N+2)3}(q) &= \sum_{i=2}^N m_i \sum_{k=2}^i l_k \left( \cos(\sum_{j=1}^k \theta_j) \sin(\theta_1) - \sin(\sum_{j=1}^k \theta_j) \cos(\theta_1) \right) \\ &= \sum_{i=2}^N m_i \sum_{k=2}^i l_k \sin(\theta_1 - \sum_{j=1}^k \theta_j) \end{aligned} \quad (172)$$

which is (167) with  $n = 3$ . And so on. Finally the diagonal term for  $\ddot{l}_1$  is:

$$m_{(N+2)(N+2)}(q) = \sum_{i=1}^N m_i. \quad (173)$$

Therefore a complete characterization of the mass matrix is obtained.

**Remark 16** In case  $N = 2$  the  $4 \times 4$  mass matrix is equal to:

$$M(q) = \begin{pmatrix} m + m_1 + m_2 & m_1 l_1 \cos(\theta_1) + m_2(l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) & m_2 l_2 \cos(\theta_1 + \theta_2) & (m_1 + m_2) \sin(\theta_1) \\ m_1 l_1 \cos(\theta_1) + m_2(l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)) & m_1 l_1^2 + m_2(l_1^2 + l_2^2) + 2m_1 l_1 l_2 \cos(\theta_2) & m_2 l_2^2 + m_2 l_1 l_2 \cos(\theta_1) & -m_2 l_2 \sin(\theta_2) \\ m_2 l_2 \cos(\theta_1 + \theta_2) & m_2 l_2^2 + m_2 l_1 l_2 \cos(\theta_2) & m_2 l_2^2 & -m_2 l_2 \sin(\theta_2) \\ (m_1 + m_2) \sin(\theta_1) & -m_2 l_2 \sin(\theta_2) & -m_2 l_2 \sin(\theta_2) & m_1 + m_2 \end{pmatrix} \quad (174)$$

which is equal to the principal  $4 \times 4$  submatrix of  $M(q)$  in section 5.4. This shows that the mass matrix with some of the lengths considered as being fixed, is obtained from the one with all varying lengths, by dropping the corresponding rows and columns.

Referring to (13) and (14):  $M_{xx}(q) = m_{11}$ ,  $M_{ll}(q) = m_{(N+2)(N+2)}(q)$ ,  $M_{\theta\theta}(q)$  has entries in (165), the coupling column  $M_{x\theta l}(q) = (m_{1(N+2)}(q), m_{2(N+2)}(q), \dots, m_{(N+1)(N+2)}(q))^\top$ , and the coupling row  $\bar{M}_{x\theta}(q) = (m_{12}(q), \dots, m_{1(N+1)}(q)) \in \mathbb{R}^{1 \times N}$  in (164). It is inferred that the conclusions of Lemma 1 still hold.

**Lemma 2** Let  $q = (x, \theta_1, \dots, \theta_N, l_1)^\top$ , then the mass matrix in (13) and (14) has the following properties:

1.  $M_{xx}$  and  $M_{ll}$  are scalar constant positive entries.
2. The coupling column  $M_{x\theta l}(q) \in \mathbb{R}^{(N+1) \times 1}$  is weak in the neighborhood of the cable's vertical posture, since it is proportional to angles sines.

Let us further split the matrix  $M_{x\theta l}(q) = \begin{pmatrix} M_{xl}(q) \\ M_{\theta l}(q) \end{pmatrix}$ , so that:

$$M(q) = \begin{pmatrix} M_{xx} & \bar{M}_{x\theta} & M_{xl} \\ \bar{M}_{x\theta}^\top & M_{\theta\theta} & M_{\theta l} \\ M_{xl}^\top & M_{\theta l}^\top & M_{ll} \end{pmatrix} \quad (175)$$

Item 1 in Lemma 2 allows us to perform Spong's transformation for actuated and unactuated coordinates [22]. Indeed  $\ddot{\theta} = M_{\theta\theta}^{-1}(-\bar{M}_{x\theta}^\top \ddot{x} - M_{\theta l} \ddot{l}_1 - NL_\theta(q, \dot{q}))$ , hence:

$$\underbrace{(M_{xx} - \bar{M}_{x\theta} M_{\theta\theta}^{-1} \bar{M}_{x\theta}^\top)}_{\triangleq \mathcal{M}_x} \ddot{x} + \underbrace{(M_{xl} - \bar{M}_{x\theta} M_{\theta\theta}^{-1} M_{\theta l})}_{\triangleq \mathcal{M}_{xl_1}} \ddot{l}_1 - \bar{M}_{x\theta} M_{\theta\theta}^{-1} NL_\theta(q, \dot{q}) + NL_x(q, \dot{q}) = F \quad (176)$$

and

$$(M_{xl}^\top - M_{\theta l}^\top M_{\theta\theta}^{-1} \bar{M}_{x\theta}^\top) \ddot{x} + \underbrace{(M_{ll} - M_{\theta l}^\top M_{\theta\theta}^{-1} M_{\theta l})}_{\triangleq \mathcal{M}_{l_1}} \ddot{l}_1 - M_{\theta l}^\top M_{\theta\theta}^{-1} NL_\theta(q, \dot{q}) + NL_{l_1}(q, \dot{q}) = F_1 \quad (177)$$

where  $\mathcal{M}_x \succ 0$  and  $\mathcal{M}_{l_1} \succ 0$  from the Schur complement Theorem. Rearranging the matrix in (175) and applying again the Schur complement Theorem, it follows that:

**Lemma 3** *Let us consider (176) (177), then:*

$$\mathcal{M}(q) \triangleq \begin{pmatrix} \mathcal{M}_x(q) & \mathcal{M}_{x l_1}(q) \\ \mathcal{M}_{x l_1}^\top(q) & \mathcal{M}_{l_1}(q) \end{pmatrix} = \mathcal{M}^\top(q) \succ 0 \quad (178)$$

Consequently the system in (176) (177) with state  $(x, \dot{x}, l_1, \dot{l}_1)^\top$  is controllable, linearizable by state feedback. Then the crucial and classical question is: what about the system of unactuated coordinates:

$$\ddot{\theta} = M_{\theta\theta}^{-1}(-\bar{M}_{x\theta}^\top \ddot{x} - M_{\theta l} \ddot{l}_1 - NL_\theta(q, \dot{q})) \quad (179)$$

From Lemma 2 the term  $M_{\theta l}(q)$  vanishes at the vertical cable's posture and is proportional to  $\theta_i$  and sums of angles in a neighborhood of it. Thus around the vertical posture only  $\bar{M}_{x\theta}^\top \ddot{x}$  remains available as a control input to this subdynamics, and some control action may also exist thanks to  $NL_\theta(q, \dot{q})$ . Said otherwise:

**Proposition 1** *Let  $q = (x, \theta_1, \dots, \theta_N, l_1)^\top$ , with all links' lengths from  $l_2$  to  $l_n$  being constant. The only way to control the  $\theta$  dynamics (179) in the neighborhood of the cable's vertical posture, is through  $\ddot{x}$  and/or some velocity couplings in the nonlinearities  $NL_\theta(q, \dot{q})$ , as visible in (159).*

Let us take a close look at (159), for  $n = N$ :

$$\begin{aligned} \frac{d}{dt} \frac{\partial T_{\text{trans}}}{\partial \dot{\theta}_N} &= m_N \left( \ddot{x} + 2\dot{l}_1 \dot{\theta}_1 \cos(\theta_1) + \ddot{l}_1 \sin(\theta_1) + \sum_{k=1}^N l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \cos(\sum_{j=1}^k \theta_j) \right. \\ &\quad \left. - \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \sin(\sum_{j=1}^k \theta_j) \right) l_N \cos(\sum_{j=1}^N \theta_j) \\ &\quad + m_N \left( \dot{x} + \dot{l}_1 \sin(\theta_1) + \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos(\sum_{j=1}^k \theta_j) \right) \\ &\quad \times \left( -l_N (\sum_{j=1}^N \dot{\theta}_j) \sin(\sum_{j=1}^N \theta_j) + \begin{cases} \dot{l}_1 \cos(\theta_1) & \text{if } N = 1 \\ 0 & \text{if } N \geq 2 \end{cases} \right) \\ &\quad + m_N \left( 2\dot{l}_1 \dot{\theta}_1 \sin(\theta_1) - \ddot{l}_1 \cos(\theta_1) + \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right)^2 \cos(\sum_{j=1}^k \theta_j) \right. \\ &\quad \left. + l_k \left( \sum_{j=1}^k \ddot{\theta}_j \right) \sin(\sum_{j=1}^k \theta_j) \right) l_N \sin(\sum_{j=1}^N \theta_j) \\ &\quad + m_N \left( -\dot{l}_1 \cos(\theta_1) + \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin(\sum_{j=1}^k \theta_j) \right) \\ &\quad \times \left( l_N (\sum_{j=1}^N \dot{\theta}_j) \cos(\sum_{j=1}^N \theta_j) + \begin{cases} \dot{l}_1 \sin(\theta_1) & \text{if } N = 1 \\ 0 & \text{if } N \geq 2 \end{cases} \right) \end{aligned} \quad (180)$$

and at (70), for  $n = N$  also:

$$\begin{aligned}
\frac{\partial T_{trans}}{\partial \theta_n} &= m_N \left( \dot{x} + \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \cos(\sum_{j=1}^k \theta_j) + \dot{l}_1 \sin(\theta_1) \right) \\
&\times \left( -l_N \left( \sum_{j=1}^N \dot{\theta}_j \right) \sin(\sum_{j=1}^N \theta_j) + \begin{cases} \dot{l}_1 \cos(\theta_1) & \text{if } N = 1 \\ 0 & \text{if } N \geq 2 \end{cases} \right) \\
&+ m_N \left( \sum_{k=1}^N l_k \left( \sum_{j=1}^k \dot{\theta}_j \right) \sin(\sum_{j=1}^k \theta_j) - \dot{l}_1 \cos(\theta_1) \right) \\
&\times \left( l_N \left( \sum_{j=1}^N \dot{\theta}_j \right) \cos(\sum_{j=1}^N \theta_j) + \begin{cases} \dot{l}_1 \sin(\theta_1) & \text{if } N = 1 \\ 0 & \text{if } N \geq 2 \end{cases} \right)
\end{aligned} \tag{181}$$

These two expressions allows us to calculate the Coriolis/centrifugal velocity quadratic couplings in the Lagrange dynamics row  $\frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{\theta}_n} - \frac{\partial T_{trans}}{\partial \theta_n}$ . The terms involving  $\dot{l}_1$  are:

$$\begin{aligned}
&2m_N \dot{l}_1 l_N \dot{\theta}_1 \cos(\theta_1) \cos(\sum_{j=1}^N \theta_j) + m_N \dot{l}_1 \sin(\theta_1) \left( -l_N (\sum_{j=1}^N \dot{\theta}_j) \sin(\sum_{j=1}^N \theta_j) + \begin{cases} \dot{l}_1 \cos(\theta_1) & \text{if } N = 1 \\ 0 & \text{if } N \geq 2 \end{cases} \right) \\
&+ 2m_N \dot{l}_1 l_N \dot{\theta}_1 \sin(\theta_1) \sin(\sum_{j=1}^N \theta_j) - m_N \dot{l}_1 \cos(\theta_1) \left( l_N (\sum_{j=1}^N \dot{\theta}_j) \cos(\sum_{j=1}^N \theta_j) + \begin{cases} \dot{l}_1 \sin(\theta_1) & \text{if } N = 1 \\ 0 & \text{if } N \geq 2 \end{cases} \right) \\
&- m_N \dot{l}_1 l_N \sin(\theta_1) \left( \sum_{j=1}^N \dot{\theta}_j \right) \sin(\sum_{j=1}^N \theta_j) - m_N \dot{l}_1 l_N \cos(\theta_1) \left( \sum_{j=1}^N \dot{\theta}_j \right) \cos(\sum_{j=1}^N \theta_j)
\end{aligned} \tag{182}$$

The terms involving  $\dot{x}$  vanish. For  $N = 1$ , equation (182) boils down to:

$$\begin{aligned}
&2m_1 \dot{l}_1 l_1 \dot{\theta}_1 \cos^2(\theta_1) + m_1 \dot{l}_1 \sin(\theta_1) \left( -l_1 \dot{\theta}_1 \sin(\theta_1) + \dot{l}_1 \cos(\theta_1) \right) \\
&+ 2m_1 \dot{l}_1 l_1 \dot{\theta}_1 \sin^2(\theta_1) - m_1 \dot{l}_1 \cos(\theta_1) \left( l_1 \dot{\theta}_1 \cos(\theta_1) + \dot{l}_1 \sin(\theta_1) \right) \\
&- m_1 \dot{l}_1 l_1 \sin^2(\theta_1) \dot{\theta}_1 - m_1 \dot{l}_1 l_1 \cos^2(\theta_1) \dot{\theta}_1 \\
&= 2m_1 \dot{l}_1 l_1 \dot{\theta}_1.
\end{aligned} \tag{183}$$

For  $N \geq 2$ :

$$\begin{aligned}
&2m_N \dot{l}_1 l_N \dot{\theta}_1 (\cos(\theta_1) \cos(\sum_{j=1}^N \theta_j) + \sin(\theta_1) \sin(\sum_{j=1}^N \theta_j)) \\
&- 2m_N \dot{l}_1 l_N \left( \sum_{j=1}^N \dot{\theta}_j \right) (\cos(\theta_1) \cos(\sum_{j=1}^N \theta_j) + \sin(\theta_1) \sin(\sum_{j=1}^N \theta_j)) \\
&= 2m_N \dot{l}_1 l_N \left( \dot{\theta}_1 \cos(\sum_{j=2}^N \theta_j) - \left( \sum_{j=1}^N \dot{\theta}_j \right) \cos(\sum_{j=2}^N \theta_j) \right) \\
&= -2m_N \dot{l}_1 l_N \left( \sum_{j=2}^N \dot{\theta}_j \right) \cos(\sum_{j=2}^N \theta_j)
\end{aligned} \tag{184}$$

**Lemma 4** Let  $q = (x, \theta_1, \dots, \theta_N, l_1)^\top$ , with all links' lengths from  $l_2$  to  $l_n$  being constant. The Coriolis/centrifugal velocity couplings in the Lagrange dynamics, corresponding to the  $\theta_N$  dynamics (the row  $M_{(N+1)\bullet}(q)$ ), and involving  $\dot{l}_1$  in the vector  $(C(q, \dot{q})\dot{q})_{1+N}$ , are given by:

$$2m_1\dot{l}_1l_1\dot{\theta}_1, \quad (185)$$

for  $N = 1$ , and

$$-2m_N\dot{l}_1l_N\left(\sum_{j=2}^N\dot{\theta}_j\right)\cos\left(\sum_{j=2}^N\theta_j\right) \quad (186)$$

for  $N \geq 2$ .

Clearly quadratic terms in velocities  $\dot{\theta}_i$  are present in the nonlinearities also.

### 6.2.3 Horizontal Positions of the Mass-Points

Let us analyse the system when the coordinates are the horizontal coordinates of each point-mass  $A_i$ :  $x_i = x + \sum_{j=1}^i l_j \sin(\sum_{k=1}^j \theta_k)$ , and the angles  $\theta_i$ , see Figure 1. The small angle and large payload mass assumptions are made to ease the calculations of the transformation (which otherwise involves trigonometric functions):  $x_i = x + \sum_{j=1}^i l_j (\sum_{k=1}^j \theta_k)$ . It is noteworthy that this type of coordinates together with the small angles assumptions, are used in some PDEs approaches (wave, or string equations) [7, 6]. The new generalized coordinate is  $q_x = (x, \theta_1, \dots, \theta_N, x_1, \dots, x_N)^\top$ , and the transformation from  $q$  to  $q_x$  is nonlinear:  $q_x = Q(q)$ , hence  $\dot{q}_x = \frac{\partial Q}{\partial q}(q)\dot{q}$  and  $\ddot{q}_x = \frac{\partial^2 Q}{\partial q^2}(q)\ddot{q} + \frac{d}{dt}\left(\frac{\partial Q}{\partial q}\right)(q, \dot{q})\dot{q}$ . In the new coordinates  $M_x(q_x) = \frac{\partial Q}{\partial q}^{-\top} M(q) \frac{\partial Q}{\partial q}^{-1}$ . We have:

$$\frac{\partial Q}{\partial q}(q) = \begin{pmatrix} I_{N+1} & 0 \\ 1 & & & & \\ \vdots & Q_L & Q_\theta & & \\ 1 & & & & \end{pmatrix}, \quad (187)$$

with:

$$Q_L = \begin{pmatrix} l_1 & 0 & \dots & & & 0 \\ l_1 + l_2 & l_2 & 0 & \dots & & 0 \\ l_1 + l_2 + l_3 & l_2 + l_3 & l_3 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ \vdots & & & & l_{N-1} & 0 \\ \sum_{k=1}^N l_k & \sum_{k=2}^N l_k & \dots & l_{N-1} + l_N & l_N \end{pmatrix} \quad (188)$$

and

$$Q_\theta = \begin{pmatrix} \theta_1 & 0 & \dots & 0 \\ \theta_1 & \theta_1 + \theta_2 & 0 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & 0 \\ \theta_1 & \theta_1 + \theta_2 & \dots & & \sum_{k=1}^N \theta_k \end{pmatrix} \quad (189)$$

Clearly  $Q_\theta$  is not invertible at  $\theta_1 = 0$ , *etc*, its determinant being equal to  $\theta_1(\theta_1 + \theta_2) \dots (\sum_{k=1}^N \theta_k)$ , and neither is  $\frac{\partial Q}{\partial q}(q)$  at these angles, because the lengths  $l_k$  cannot be obtained from  $q_x$  at those coordinates values. From the matrix inversion Lemma:

$$\frac{\partial Q^{-1}}{\partial q}(q) = \begin{pmatrix} I_{N+1} & 0 \\ -Q_\theta^{-1} \bar{Q}_L & Q_\theta^{-1} \end{pmatrix} \quad (190)$$

with  $\bar{Q}_L = \begin{pmatrix} 1 \\ \vdots & Q_L \\ 1 \end{pmatrix}$ , and  $(\dot{x}_1, \dots, \dot{x}_N)^\top = \bar{Q}_L \dot{\theta} + Q_\theta \dot{L}$ ,  $L = (l_1, \dots, l_N)^\top$ . Therefore:

$$M_x(q_x) = \begin{pmatrix} I_{N+1} & -(Q_\theta^{-1} \bar{Q}_L)^\top \\ 0 & Q_\theta^{-\top} \end{pmatrix} \begin{pmatrix} M_{x\theta}(q) & M_{x\theta l}(q) \\ M_{lx\theta}(q) & M_{ll}(q) \end{pmatrix} \begin{pmatrix} I_{N+1} & 0 \\ -Q_\theta^{-1} \bar{Q}_L & Q_\theta^{-1} \end{pmatrix} \quad (191)$$

In order to analyze the dynamics using the positions  $x_i$  let us try to start directly with the kinetic energy, doing some basic assumptions as done in the PDEs approaches [7, 6], like constant total length, that we may translate here by constant length, and pure horizontal motion of the nodes. In this case  $q = (x, x_1, x_2, \dots, x_N)^\top$ ,  $V_{A_i} = \dot{x} + \dot{x}_i$ , and the total kinetic energy is  $T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \sum_{i=1}^N m_i \dot{x}_i^2$ . In this case the mass matrix is linear invariant with  $M(q) = \text{diag}(m, m_1, \dots, m_N)$ , and the Lagrange dynamics is easily obtained.

## 7 Links with Linear Longitudinal Elasticity (2D Operating Space)

Flexibility at the joints have already been studied in section 4.3. Now the same system is considered as in Figure 1, with elastic links to approximate extensible cables. This is depicted in Figure 2. Each link has a longitudinal linear elasticity with stiffness  $k_i > 0$ ,  $1 \leq i \leq N$ . The same framework as in the foregoing sections is adopted, but the potential energy is augmented with terms  $\frac{1}{2} k_i (l_i - l_{i,r})^2$  (assuming that springs are at rest for  $l_i = l_{i,r}$ ). This model is close in spirit to the lumped-mass models developed in [4, 18], but nonlinearities are considered here. It is also easy to add some viscous friction (linear spring-dashpot model)

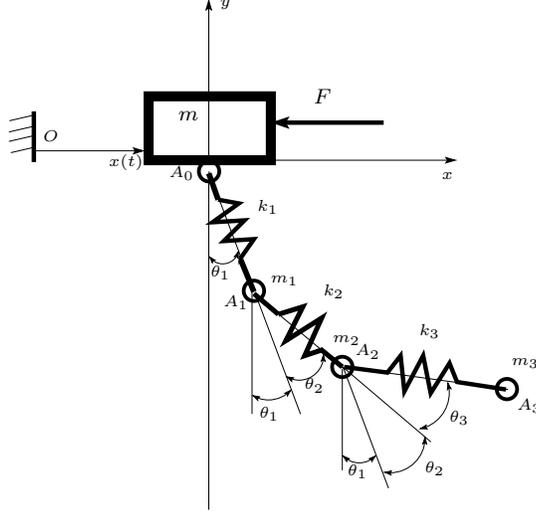


Figure 2: Overhead crane with pendulum-like cable (elastic links).

$c_i \dot{l}_i$ , which is some kind of Rayleigh dissipation [2]. Remind that  $l = (l_1, l_2, \dots, l_N)^\top$ , one has:

$$\frac{\partial U_{elas}}{\partial l} = (k_1(l_1 - l_{1,r}), \dots, k_N(l_N - l_{N,r}))^\top \quad (192)$$

Thus (93) is changed to:

$$\left\{ \begin{array}{l} (a) \quad \frac{d}{dt} \frac{\partial(T_{trol} + T_{trans})}{\partial \dot{x}} = Q_x = F \\ (b) \quad \frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{\theta}} - \frac{\partial(T_{trans} - U_g - U_{flex})}{\partial \theta} + C_\theta \dot{\theta} = Q_\theta = 0 \\ (c) \quad \frac{d}{dt} \frac{\partial T_{trans}}{\partial \dot{l}} - \frac{\partial(T_{trans} - U_g - U_{flex} - U_{elas})}{\partial l} + C_l \dot{l} = Q_l = (F_1, 0, \dots, 0)^\top \end{array} \right. \quad (193)$$

with  $C_l = \text{diag}(c_i)$ , the form of  $C_\theta \succcurlyeq 0$  depends on the choice of the coordinates.

**Remark 17 (Linear chains)** *The overhead crane dynamics and control may be seen as an extension of the linear chains dynamics and control [20, 21]: it is a nonlinear chain that is much more complex than its linear counterpart, where (193) (a) and (b) are absent.*

**Remark 18 (Influence of flexibilities)** *The flexibility potential energy due to links' elasticity, does not fundamentally modify the system's structure for control, as it does for instance for flexible joint kinematic chains with triangular structure, rendering the system suitable for tracking control via backstepping and passivity-based inputs, as proposed for the first time in [19, 3]. It is noteworthy also that the triangular structure for flexible-joint chains is obtained at the price of neglecting the kinetic energy of some parts, since otherwise the mass matrix*

is “dense” [29]. As said above, joint flexibilities in the joints  $A_i$  is not a reasonable modelling assumption for most cables, since cables usually possess a very low bending stiffness. Moreover this does not seem to bring anything useful to the control problem (at least with the chosen  $q$ ).

From the above the Lagrange equations are (which are also given in [26, Equations (6) (7) (8)] and [25, Equations (1) (2) (3)]):

$$\left\{ \begin{array}{l} (a) (m + m_1)\ddot{x} + m_1 \left( l_1 \cos(\theta_1)\ddot{\theta}_1 + \sin(\theta_1)\ddot{l}_1 + 2\dot{l}_1\dot{\theta}_1 \cos(\theta_1) - l_1\dot{\theta}_1^2 \sin(\theta_1) \right) = F \\ (b) m_1 l_1 \cos(\theta_1)\ddot{x} + m_1 l_1^2 \ddot{\theta}_1 + 2m_1 \dot{l}_1 l_1 \dot{\theta}_1 + m_1 g l_1 \sin(\theta_1) = 0 \\ (c) m_1 \sin(\theta_1)\ddot{x} + m_1 \ddot{l}_1 - m_1 g \cos(\theta_1) - m_1 l_1 \dot{\theta}_1^2 = F_1, \end{array} \right. \quad (194)$$

where  $F_1$  is the force exerted on the “cable” by some winding device mounted at joint  $A_0$ . Property (3) can be checked. Here  $Q = (F, 0, F_1)^\top$  and passivity holds with supply rate  $w(Q, \dot{q}) = F\dot{x} + F_1\dot{l}_1$ . The mass matrix for  $q = (x, \theta_1, l_1)^\top$  is:

$$M(q) = \begin{pmatrix} m + m_1 & m_1 l_1 \cos(\theta_1) & m_1 \sin(\theta_1) \\ m_1 l_1 \cos(\theta_1) & m_1 l_1^2 & 0 \\ m_1 \sin(\theta_1) & 0 & m_1 \end{pmatrix} = \begin{pmatrix} M_{x\theta_1} & M_{x\theta_1 l_1} \\ M_{x\theta_1 l_1}^\top & M_{ll} \end{pmatrix}. \quad (195)$$

with  $M_{ll} = m_1$ . Following [22], if the coordinate is rearranged as  $q_a = (x, l_1)^\top$  and  $q_{na} = \theta_1$ , then:

$$M(q) = \begin{pmatrix} m + m_1 & m_1 \sin(\theta_1) & m_1 l_1 \cos(\theta_1) \\ m_1 \sin(\theta_1) & m_1 & 0 \\ m_1 l_1 \cos(\theta_1) & 0 & m_1 l_1^2 \end{pmatrix} = \begin{pmatrix} M_{x l_1} & M_{x l_1 \theta_1} \\ M_{x l_1 \theta_1}^\top & M_{\theta_1} \end{pmatrix}, \quad (196)$$

with  $M_{x l_1 \theta_1} = m_1 l_1 \cos(\theta_1)$ ,  $M_{\theta_1} = m_1 l_1^2$ .

## 8 Some Remarks

### 8.1 Rigid Links

If instead of lumped-mass model we consider that the links are rigid bodies with center of mass  $G_i$  situated inside the segments  $A_i A_{i+1}$ , and with moments of inertia  $J_i$ , then the above developments have to be modified accordingly. Their kinetic energies are the sum of the translational gravity centers energy  $T_{G_i} = \frac{1}{2} m_i V_{G_i}^\top V_{G_i}$ , plus the rotational kinetic energies  $\frac{1}{2} J_i \dot{\theta}_i^2$ .

## 8.2 3D Operational Space

The derivation of the dynamics in 3D is quite cumbersome as soon as  $N \geq 3$ . It is conjectured that most of the above properties of the mass matrix, still hold true in 3D.

## 8.3 Finite-Element Models

Let  $N \rightarrow +\infty$ . If the total mass  $m_{tot}$  is constant, clearly  $m_i \approx \frac{m_{tot}}{N} \rightarrow 0$  for all  $1 \leq i \leq N - 1$  (the payload mass  $m_N$  stays the same). This justifies the study of the cases  $m_N \gg m_i$ . Lumped-mass models and Rayleigh-Ritz discretization are compared in [16], where it is concluded that: *It was shown through linear frequency responses that the Rayleigh-Ritz model more accurately represents the natural frequencies of the system compared to the lumped-mass method, when using a small number of basis functions or lumped masses. This highlights a major advantage of using the Rayleigh-Ritz method to model flexible CDPMs, rather than lumped-mass methods.* As alluded to in the introduction of this report, other FEM approaches should be tried for overhead cranes control, like the one in [23]. Indeed it seems to be able to combine the multibody approach with global nonlinearities, and the FEM discretizations which take into account more vibrational modes than multibody models. It takes the form of a multibody model (see [23, Equations (83)–(85)]), though this is no Lagrange dynamics. The main issues concerning its applicability to feedback control, are about the measurement of the state vector  $v_{AB}$  and of the variable  $\mathbf{d}$  in [23, Equation (85)], and the properties of this dynamics (passivity, *etc*).

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