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► To cite this version:

| Marc Lambert, Silvère Bonnabel, Francis Bach. The continuous-discrete variational Kalman filter
| (CD-VKF). 2022. hal-03665666v1

HAL Id: hal-03665666

<https://inria.hal.science/hal-03665666v1>

Preprint submitted on 12 May 2022 (v1), last revised 3 Sep 2022 (v2)

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The continuous-discrete variational Kalman filter (CD-VKF)

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Abstract

In this paper we seek to estimate the state of a continuous-time dynamical process governed by a nonlinear stochastic differential equation, observed through discrete-time measurements. As the Bayesian posterior density is difficult to compute, we use variational inference (VI) – a method from machine learning – to approximate it. This is achieved by seeking the closest Gaussian density to the posterior, in the sense of the Kullback-Leibler divergence. The obtained algorithm, called the continuous-discrete variational Kalman filter (CD-VKF), provides implicit formulas that solve the considered problem. Our hope is that such a Kalman filter variant may prove more stable as it optimizes a closeness to the target distribution. We first clarify the connections between many Kalman filter variants and VI, then develop closed form approximate formulas for the CD-VKF. Our algorithm gives state of the art performances on the problem of reentry tracking of a space capsule.

1 INTRODUCTION

Continuous-discrete estimation problems naturally arise in numerous applications, such as radar tracking, guidance and navigation, see, e.g., [25, 6]. This is because nonlinear dynamical models often stem from physics, whose laws are continuous-time, whereas the observations come digitally in discrete time through sensors' measurements. Moreover, working with discrete-time observations allows for seamlessly accommodating timestamped observations emanating from heterogeneous sensors. Besides, to account for discrepancies between actual motion and the model, as well as sensors' errors, it is customary to assume noisy dynamics and measurements, leading to nonlinear stochastic differential equations (SDE)s. Throughout the paper, we will consider a system with a state x_t defined in continuous time and the observation y_{t+T} defined in discrete time $t + T$, where T is the observation period. The state and the observation are related by a system of equation of the form:

$$dx_t = f(x_t, t)dt + L(x_t, t)d\beta \quad \text{state-propagation} \quad (1)$$

$$y_{t+T} = h(x_{t+T}) + \varepsilon, \quad \text{state-observation} \quad (2)$$

where $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ represents the dynamics (drift), $L(x_t, t)$ is the magnitude of the noise that potentially dependent on the state, β is a Brownian motion such that the small increments satisfy $\delta\beta \sim \mathcal{N}(0, Q\delta t)$, $h : \mathbb{R}^d \rightarrow \mathbb{R}^p$ is the observation function, and $\varepsilon \sim \mathcal{N}(0, R)$ is the observation noise. The covariance matrix R may also depend on the state. This is the case, for example, for a radar where the angular and range precisions depend on the distance to the system through the signal-to-noise-ratio [25].

Bayesian filtering aim to estimate the distribution of the state variable x_t conditionally to the past observations. Propagate the distribution of x_t from t to $t + T$ require to solve the Fokker-Planck partial differential equation. This equation may not be solved in closed form and one needs to resort to approximations. Moreover, we need to update the state at time $t + T$ accounting for the observation y_{t+T} , that is, conditioning upon

it. The conditional, or posterior, distribution may not be computed in a closed form and need also to be approximated. Popular approaches consist in the well-known extended Kalman filter (EKF), the unscented Kalman filter (UKF) [10], or cubature Kalman filters (CKF) [1] that seek to sequentially approximate the maximum a posteriori (MAP) estimate of the state variable x_t , along with the associated covariance.

In the present paper, we follow a slightly different approach that seeks to approximate a distribution p by the closest Gaussian distribution q in the sense of the left Kullback-Leibler divergence [11] defined by:

$$\begin{aligned} KL(q||p) &= \int q(x) \log \frac{q(x)}{p(x)} dx \\ &= H(q) - \mathbb{E}_q[\log p(x)], \end{aligned} \tag{3}$$

where H is the negative entropy defined by $H(q) = \int q(x) \log q(x) dx$. This setting is known as variational Gaussian approximation, see [5, 19], or variational inference [8]. We will apply it for both: the propagation in continuous time of the distribution of the state x_t through the Fokker-Planck equation and the computation of the posterior distribution during the state update in discrete time:

- Our first task is to approximate the SDE (1) through variational Gaussian approximation. This is a variational formulation of Gaussian assumed density approximation (see [21], chapter 9). To our knowledge, a few main advances were done on the subject. In [4] the problem of propagation and conditioning is tackled jointly through variational optimization of a continuous-discrete process, and non-closed form formulas are found owing to the presence of continuous Lagrange multipliers. In [20], an approach based on Fokker-Planck is advocated, and then the intractable formulas are simplified through an heuristic. In this paper, we essentially reformulate the variational approach of [4] by focusing only on propagation between updates to obtain closed-form formula.
- Our second task is to use variational inference for updates, that is, to account for observations. To this aim we use a recursive version of variational Gaussian approximation [19], see also our recent paper [13].

Combining propagation and update, we obtain a novel real-time algorithm we call the continuous-discrete variational Kalman filter (CD-VKF). On the tracking problem we have considered, this filter is shown to outperform the extended Kalman filter (EKF) and to compete with the unscented Kalman filter (UKF). Besides, our hope is that it be less prone to divergence than other Kalman filter variants, since it optimizes a closeness to a target distribution at all times, yielding strong indications of potential stability. Beyond this contribution, we also draw connections between variational inference and various popular filters, and provide clarifications.

The organization of the paper is as follows: in Section 2 we formally introduce our new filter as the solution of the two mentioned variational approximation problems. In Section 3 we relate existing Kalman filters to left and right KL variational problems, and discuss how they are related to the problems we solve. In Section 4 we derive the optimal solution for the variational Gaussian propagation problem in continuous time and the variational Gaussian update in discrete time. A practical implementation is proposed in Section 5 where a new square-root form is derived. The proposed algorithm is assessed on a reentry tracking of a space capsule and compared with CD-EKF and CD-UKF.

2 Situating the problem

We now define formally the two variational problem we solve and state our main theorem which introduce our algorithm as the solution of these two problems.

2.1 Variational state propagation

To define our first problem, let's consider the Euler-Marayama discretization with time step δt of the non linear SDE (1), yielding the transition:

$$p(x_{t+\delta t}|x_t) = \mathcal{N}(x_t + f(x_t)\delta t, L(t)QL(t)^T \delta t), \quad (4)$$

where the drift L is supposed independent of x .

Following [4], we seek to approximate (1) by a linear SDE of the form

$$dx_t = (A(t)x_t + b(t))dt + \Lambda(t)d\beta_t. \quad (5)$$

A similar discretization yields a transition for this equation of the form

$$q(x_{t+\delta t}|x_t) = \mathcal{N}(x_t + A(t)x_t\delta t + b(t)\delta t, \Lambda(t)Q\Lambda(t)^T \delta t). \quad (6)$$

This sequence defines a recursive variational Gaussian process by letting the variational parameters $A(t)$, $b(t)$ and $\Lambda(t)$ be optimal at each time in the following sense.

Problem 1 (variational Gaussian propagation): To best approximate (1) by an equation of the form (5), find at all time t the parameters $A(t), b(t), \Lambda(t)$ which solve

$$\min_{A(t), b(t), \Lambda(t)} KL(q(x_{t+\delta t}, x_t), p(x_{t+\delta t}, x_t)), \quad (7)$$

in the limit $\delta t \rightarrow 0$, with p corresponding to the transition (4) associated with the original nonlinear equation, and q corresponding to the transition (6) associated with its linear approximation governed by $A(t), b(t), \Lambda(t)$.

2.2 Variational state update

The second problem is to incorporate an observation when it arrives. Let us consider the non linear observation model (2). Given an approximative Gaussian prior $x_{t+T|t} \sim \mathcal{N}(\mu_{t+T|t}, P_{t+T|t})$ as a result of the previous variational propagation scheme, at time $t + T$, and a single observation y_{t+T} , we seek a variational Gaussian approximation to the assumed posterior

$$p(x_{t+T}|y_{t+T}) \propto p(y_{t+T}|x_{t+T|t})q(x_{t+T|t}). \quad (8)$$

Problem 2 (variational Gaussian update): Find the Gaussian distribution conditionally to the new single observation $q(x_{t+T}|y_{t+T}) = \mathcal{N}(\mu_{t+T}, P_{t+T})$ which approximate the posterior (8):

$$\arg \min_{\mu_{t+T}, P_{t+T}} KL(q(x_{t+T}|y_{t+T}), p(x_{t+T}|y_{t+T})). \quad (9)$$

This scheme has already been explored in our previous work and we called our solution the recursive variational Gaussian approximation [13].

2.3 Main result

As will be proved in the sequel, it turns that formulas may be derived for both problems. This results in the following algorithm called the continuous-discrete variational Kalman filter (CD-VKF).

CD-VKF Algorithm:

while $t \geq 0$ **do**

(1) Variational Gaussian propagation from t to $t + T$

The solution to the linear equation (5) with optimal parameters solving Problem 1 has a Gaussian distribution $q = \mathcal{N}(\mu_{t+T|t}, P_{t+T|t})$ satisfying between two successive observations

$$\begin{pmatrix} \mu_{t+T|t} \\ P_{t+T|t} \end{pmatrix} = \int_t^{t+T} \begin{pmatrix} \dot{\mu}(s) \\ \dot{P}(s) \end{pmatrix} ds \text{ where:}$$

$$\dot{\mu}(t) = \mathbb{E}_{q(\mu, P)}[f(x, t)];$$

$$\dot{P}(t) = \mathbb{E}_{q(\mu, P)}[J_f(x)]P + P\mathbb{E}_{q(\mu, P)}[J_f(x)]^T + L(t)QL(t)^T;$$

where $J_f(x)$ is the (square) Jacobian matrix of the dynamics drift.

(2) Variational Gaussian update at $t + T$

The solution to the Problem 2 is $q = \mathcal{N}(\mu, P)$ st:

$$v(x) = J_h(x)^T R^{-1} (y_{t+T} - h(x));$$

$$\mu = \mu_{t+T|t} + P_{t+T|t} \mathbb{E}_{q(\mu, P)}[v(x)];$$

$$P = P_{t+T|t} + \frac{1}{2} \mathbb{E}_{q(\mu, P)}[(x - \mu)v(x)^T]P_{t+T|t} + \frac{1}{2} P_{t+T|t} \mathbb{E}_{q(\mu, P)}[(x - \mu)v(x)^T]^T;$$

where $J_h(x)$ is the (rectangular) Jacobian matrix of observation map.

$$t \leftarrow t + T;$$

end

Theorem 1 (CD-VKF). *We consider the non linear setting (1) and (2). Given an initial prior $x_0 \sim \mathcal{N}(\mu_0, P_0)$ at $t = 0$, and observations arriving at a fixed period y_{kT} , we approximate the true conditional distribution process by exactly solving the approximation problems described in Problems 1 and 2 above. We then obtain a real-time algorithm we call the continuous-discrete variational Kalman filter (CD-VKF), as displayed below.*

The proof, along with more details and comments, will be given in Sections 4. The hope that underlies our approach is that as this filter solves optimization Problems 1 and 2, the estimates are better controlled than, say, those of an EKF, so that more stability can be expected. But before getting further into CD-VKF, which will be the topic of Section 4, we would like to review and make some interesting connections between variational inference and Kalman filter variants.

3 A review of Kalman filters in the light of variational inference

3.1 Left vs right KL divergence

Our variational problems are stated for the left Kullback-Leibler divergence (3). In Kalman filtering the problem are generally formulated for the other side. Indeed, from the definition (3) we see that the KL divergence is not symmetric, if q is our variational distribution the divergence (3) will be called the “left KL”. As opposed, we may define the right KL as:

$$\begin{aligned} KL(p(x)||q(x)) &= \int p(x) \log \frac{p(x)}{q(x)} dx \\ &= C - \mathbb{E}_p[\log q(x)], \end{aligned} \tag{10}$$

where $C = H(p)$ is now a constant independent of our variable q . Both KLs involve expectations: in the left KL the expectation is under a Gaussian q , whereas in the right one it is under an unknown distribution we seek to approximate. Properties of integrals under Gaussian distributions make the left KL attractive and may lead often to closed form updates in particular for inference. Moreover the left KL gives a lower bound on the log-marginal likelihood and is used in variational batch optimization in which each iteration is guaranteed not to decrease the bound [8]. Finally, when we consider the Gaussian approximation of a sharp distribution, the left KL is known to better catch the mode than the right one.

However, a large class of non linear Kalman filters are based on moment matching and, as such, are implicitly defined with the right KL not the left one. To see how moment matching is related to KL divergence, let us compute a right KL Gaussian approximation for $q(x) = \mathcal{N}(x|\mu, P)$:

$$KL(p(x)||q(x)) = C - \mathbb{E}_p\left[-\frac{1}{2}(x - \mu)^T P^{-1}(x - \mu)\right] + \frac{1}{2} \log \det P.$$

To search the optimal distribution q , we search the critical points by cancelling the derivatives of the KL with respect to the variational parameters, here μ and P . Using the relation $\nabla_P \log \det P = P^{-1}$ and $\nabla_P x^T P^{-1} x = -P^{-1} x x^T P^{-1}$, we find:

$$\mu = \mathbb{E}_p[x] \text{ and } P = \mathbb{E}_p[(x - \mu)(x - \mu)^T], \quad (11)$$

which is exactly moment matching between p and q . However, computing the expectations under p is not easy.

In (3), the KL divergence is defined w.r.t. a generic distribution $p(x)$, but the KL divergence may be computed also on joint distributions $p(x, y)$, conditional distributions $p(x|y)$, etc. It turns out that, interestingly, all these variants lead to different Kalman filters. Joint and conditional divergences can be related through the following formula:

$$KL(q(z, x)||p(z, x)) = KL(q(x)||p(x)) + \mathbb{E}_{q(x)}[KL(q(z|x)||p(z|x))]. \quad (12)$$

Finally, to better understand the nature of the left and right KL, we may also consider the formalism of Bayesian networks or graphical models [8]. A variational theory has been developed for graphical models [8] and is the subject of active research. The left KL divergence is associated to variational message passing algorithms [24], and the right one to the expectation-propagation algorithm [15] for factor graphs.

We now reconsider several non linear Kalman filters from the point view of variational approximation, and define more precisely how our approach is related to existing methods.

3.2 Variational Gaussian propagation

If the state x follows the SDE (1), the evolution of its distribution p satisfies the Fokker-Planck equation (FPE):

$$\begin{aligned} \frac{\partial p(x, t)}{\partial t} = & - \sum_i \frac{\partial}{\partial x_i} f_i(x, t) p(x, t) \\ & + \sum_{i, j} \frac{\partial^2}{\partial x_i \partial x_j} [L(x, t) Q L(x, t)^T]_{i, j} p(x, t). \end{aligned} \quad (13)$$

The Fokker-Planck equation 13 is a partial differential equation which may not be solved in the general case. However, since we are interested in Gaussian approximations, we need to compute only the two first moments

of the distribution. We can then derive a set of two ordinary differential equations (ODE) from the FPE using the definition of the mean and covariance matrix as expectation under p :

$$\begin{aligned}\frac{d}{dt}\mu_t &= \mathbb{E}[f(x,t)] \\ \frac{d}{dt}P_t &= \mathbb{E}[f(x,t)(x - \mu_t)^T] + \mathbb{E}[(x - \mu_t)f(x,t)^T] \\ &\quad + \mathbb{E}[L(x,t)QL(x,t)^T].\end{aligned}\tag{14}$$

Unfortunately, these ODEs are implicit and involve expectations under the unknown target distribution p . The CD-UKF filter [20] is based on the following heuristic: the expectations are computed under the current Gaussian q instead of unknown p , using quadrature rules provided in the UKF filter [10]. Computing expectations under the Gaussian distribution q rather than p appears as an heuristic, however the resulting form resembles the one obtained by [4] using a wholly different method based on the left KL approximation of the non linear process. This relation suggests that there is a deeper connection between approximated Fokker-Planck equations and variational approximations, as mentioned in [20], but without further explanation. In Section 4, we will clarify this connection and lay the theoretical foundations to justify it, as this heuristic approximation done in CD-UKF corresponds exactly to the solution of the recursive variational Gaussian process as defined in (7).

3.3 Variational Gaussian update

The problem of inference is to approximate the posterior distribution $p(x|y)$ with a Gaussian $q(x) = \mathcal{N}(x|\mu, P)$. Bayes rule gives the exact formula for the posterior distribution $p(x|y) \propto p(y|x)q_0(x)$. If the observation noise is Gaussian $\varepsilon \sim \mathcal{N}(0, R)$, the conditional distribution of observation is always Gaussian $p(y|x) \sim \mathcal{N}(h(x), R)$ but $p(y|x)p(x) = p(x, y)$ is not a Gaussian due to the potential non-linearity of h . To approximate the posterior with a Gaussian, different classes of methods may be considered:

1. Taylor-linearization of the observation

$$h(x) \approx h(\mu_0) + \nabla h|_{\mu_0}x = b + Hx.\tag{15}$$

2. Statistical linearization of the observation

$$h(x) \approx Hx + b,\tag{16}$$

where H and b minimize the least square loss:

$$\mathbb{E}_{q_0}[(h(x) - Hx - b)^T (h(x) - Hx - b)].$$

3. Approximation of $p(x|y, x_0)$ with moment-matching

$$\mu = \int xp(x|y)dx = \frac{1}{Z} \int xp(y|x)q_0(x)\tag{17}$$

$$P = \frac{1}{Z} \int xxp(y|x)q_0(x) - \mu\mu^T$$

$$\text{where } Z = \int p(y|x)q_0(x)dx.$$

4. Approximation of $p(x, y|x_0)$ with moment-matching

$$p(x, y|x_0) \approx \quad (18)$$

$$\mathcal{N} \left(\begin{bmatrix} \mathbb{E}_{x_0}[x] \\ \mathbb{E}_{x_0}[y] \end{bmatrix}, \begin{bmatrix} \text{Cov}_{x_0}(x, x) & \text{Cov}_{x_0}(x, y) \\ \text{Cov}_{x_0}(y, x) & \text{Cov}_{x_0}(y, y) \end{bmatrix} \right)$$

where:

$$\mathbb{E}_{x_0}[a] = \mathbb{E}_{x_0 \sim q_0} \mathbb{E}[a|x_0] \quad (19)$$

$$\text{Cov}_{x_0}(a, b) = \mathbb{E}_{x_0 \sim q_0} \mathbb{E}[ab^T|x_0] - \mathbb{E}_{x_0}[a] \mathbb{E}_{x_0}[b]^T.$$

The first approach is used in the celebrated extended Kalman filter (EKF), and is the most used in practice because it does not need to compute any expectation. The second one is used in the quadrature Kalman filter (QKF) [3], and was associated with the Gauss-Hermite quadrature rules. It is related to the left KL. The third approach is used in the assumed density filter [18], and was extended for batch machine learning and factor graphs through expectation-propagation algorithms [15]. This approach is derived with the right KL (10). Finally, the last approach corresponds to filters based on the innovation process like the unscented Kalman filter (UKF) [10], or the cubature Kalman filters (CKF) [1]. They are related also to the right KL (10) but applied to the joint distribution.

When the posterior distribution $p(x|y)$ is not directly computed with moment matching (17), it is derived from the conditional formulas:

$$\begin{aligned} \mathbb{E}_{x_0}[x|y] &= \mathbb{E}_{x_0}[x] + \text{Cov}_{x_0}(x, y) \text{Cov}_{x_0}(y, y)^{-1} (y - \mathbb{E}_{x_0}[y]) \\ \text{Cov}_{x_0}(x|y) &= \text{Cov}_{x_0}(x, x) - \text{Cov}_{x_0}(x, y) \text{Cov}_{x_0}(y, y)^{-1} \text{Cov}_{x_0}(y, x). \end{aligned}$$

In the non linear case (UKF, CKF), the expectations may be approximated with quadrature rules [10] for some well-chosen sigma points x_i : $\mathbb{E}[f(x)] \approx \sum_i w_i f(x_i)$, see [14] for a survey on quadrature rules. In the linearized case (EKF or QKF) we have $y = Hx + b + \mathcal{N}(0, R)$, the expectations are available in closed form, and the conditional formulas yield the classical Kalman update equations:

$$\begin{aligned} \mu &= \mu_0 + P_0 H^T (R + H P_0 H^T)^{-1} (y - H \mu_0 - b) \\ &= \mu_0 + K E \\ P &= P_0 - P_0 H^T (R + H P_0 H^T)^{-1} H P_0 \\ &= P_0 - K H P_0, \end{aligned} \quad (20)$$

where $\mu = \mathbb{E}[x|y]$, $P = \text{Cov}(x|y)$, K is the Kalman gain, and $E = y - H \mu_0 - b$ is the estimation error. In practice the error E is replaced by the non linearized quantity $y - h(\mu_0)$. To sum up, the moment matching (17) minimizes directly the right KL divergence for our quantity of interest $p(x|y)$ while the innovation based method (18) finds indirectly the posterior by first minimizing the right KL on the joint distribution $p(x, y)$, and then applying the conditional formulas. Finally, statistical linearization (16) also first approximates the conditional distribution with a linear one using the left KL and then using conditional formulas. This discussion is synthesized in Table 1.

3.4 Summary

We summarize our review in Table 1, where M-M stand for moment matching methods, J-M-M for moment matching of the joint distribution, SLR for statistical linearization, VI for variational inference, MM-GP for moment-matching based Gaussian process and V-GP for variational (left KL) based Gaussian process. The associated equations are given in parentheses; $KL(\cdot||p)$ stands for the left KL and $KL(p||\cdot)$ for the right one.

Table 1: Non linear Kalman filters as variational minimizers

Method	KL	Filters	References
MM (17)	$KL(p(x y, x_0) \cdot)$	ADF, E-P	[18], [15]
VI (29)	$KL(\cdot p(x y, x_0))$	VGA, VB R-VGA CD-VKF	[5], [19], [22] [13] this paper
J-MM (18)	$KL(p(x, y x_0) \cdot)$	UKF, CKF	[10], [1]
SLR (16)	$KL(\cdot p(y x_0)p(x_0))$	QKF	[3]
MM-GP	$KL(p(x_{t+1}, x_t) \cdot)$	G-ADF	[7]
V-GP (25)	$KL(\cdot p(x_{t+1}, x_t))$	SLR-GP CD-UKF CD-CKF CD-VKF	[23], [4] [20] [2] this paper

4 Derivation of Theorem 1

We now prove our main theorem: first we solve the variational Gaussian propagation problem (7) in continuous time, then the variational Gaussian update problem (9) in discrete time.

4.1 Solution to the variational propagation problem (7)

Let us consider a small step δt such that the Euler-Marayama approximation of the SDE (1) is :

$$x_{t+1} = x_t + f(x_t)\delta t + L(x_t, t)\delta\beta_t; \quad \delta\beta_t \sim \mathcal{N}(0, \delta t Q).$$

Starting from q_t the propagated distribution is:

$$\begin{aligned} p(x_{t+1}) &= \mathbb{E}_{q_t}[p(x_{t+1}|x_t)] \\ &= \mathbb{E}_{q_t}[\mathcal{N}(x_{t+1}|x_t + f(x_t)\delta t, \delta t V(x_t))], \end{aligned}$$

where $V(x_t)$ is a compact notation for $L(x_t, t)QL(x_t, t)^T$. The variational approximation (3) of this marginal with a Gaussian q_{t+1} for the left KL is:

$$KL(q(x_{t+1})||p(x_{t+1})) = H(q_{t+1}) - \mathbb{E}_{q_{t+1}}[\log p(x_{t+1})], \quad (21)$$

where H is the negative entropy. Unfortunately, the optimal parameters are not available in a closed form in the non linear case. To circumvent this problem, we consider the auxiliary problem which consists in approximating the non linear process with a linear process as proposed in [3].

Remark 1 (On the right KL version for propagation). *Contrary to the left KL problem (21), the right KL problem can be tackled directly using recursive moment matching to yield the Gaussian assumed density filter (G-ADF) [7]. However, the resulting moments do not satisfy the ODE given by Ito formulas (14), and gives inconsistent results between Ito and Stratonovich calculus [7].*

4.1.1 Variational linearization of the process

Our linear process for a small step δt is parametrized by $b(t)$, $A(t)$ and $\Lambda(t)$, to simplify the notation we will omit the time variable:

$$q(x_{t+1}|x_t) \sim \mathcal{N}(x_{t+1}|x_t + Ax_t\delta t + b\delta t, \delta t\Lambda),$$

and our auxiliary problem is now given by a divergence between two joint distributions:

$$\min_{b, A, \Lambda} KL(q(x_{t+1}, x_t) || p(x_{t+1}, x_t)).$$

The KL may be rewritten using the formula (12):

$$\begin{aligned} & KL(q(x_{t+1}, x_t) || p(x_{t+1}, x_t)) \\ &= \mathbb{E}_{q_t} [KL(q(x_{t+1} | x_t) || p(x_{t+1} | x_t))], \end{aligned}$$

where the first term in (12) vanished since the prior is supposed to follow the same Gaussian q_t . We obtain an expected divergence between two Gaussians which takes the form:

$$\begin{aligned} & \mathbb{E}_{q_t} \left[\frac{1}{2} Tr \Lambda V(x_t)^{-1} + \frac{1}{2} \log \det V(x_t) - \frac{1}{2} \log \det \Lambda - \frac{1}{2} d \right. \\ & \left. + \frac{\delta t}{2} (f(x_t) - Ax_t - b)^T V(x_t)^{-1} (f(x_t) - Ax_t - b) \right]. \end{aligned}$$

The advantage of using the left KL appears clearly here where the mean variational parameters A and b are decoupled from the covariance variational parameter Λ . Taking the derivatives with respect to Λ , A and b , and using the relation $\nabla_{\Lambda} Tr \Lambda V^{-1} = V^{-1}$ and $\nabla_{\Lambda} \log \det \Lambda = \Lambda^{-1}$, we obtain the following set of equations:

$$\Lambda = \mathbb{E}_{q_t} [(L(x_t, t) Q L(x_t, t)^T)^{-1}]^{-1} \quad (22)$$

$$\mathbb{E}_{q_t} [V(x_t)^{-1} A x_t x_t^T] = \mathbb{E}_{q_t} [V(x_t)^{-1} (f(x_t) - b) x_t^T] \quad (23)$$

$$\mathbb{E}_{q_t} [V(x_t)^{-1} A x_t] = \mathbb{E}_{q_t} [V(x_t)^{-1} (f(x_t) - b)]. \quad (24)$$

To obtain a closed form expression we suppose that the drift term is independent of the state x_t , i.e., $L(x_t, t) = L(t)$. The set of equations becomes:

$$\begin{aligned} \Lambda &= L(t) Q L(t)^T \\ A &= \mathbb{E}_{q_t} [f(x_t) (x_t - \mu_t)^T] P^{-1} \\ b &= \mathbb{E}_{q_t} [f(x_t)] - A \mu_t. \end{aligned} \quad (25)$$

4.1.2 Variational Gaussian propagation

We now have a linear process which is given by a one-step update as:

$$x_{t+1} = x_t + A x_t \delta t + b \delta t + \delta \beta_t; \quad \delta \beta_t \sim \mathcal{N}(0, \delta t \Lambda).$$

It is straightforward to compute its mean and covariance:

$$\begin{aligned} \mu_{t+1} &= \mathbb{E}[x_{t+1}] \\ &= \mu_t + A \mu_t \delta t + b \delta t \\ P_{t+1} &= \mathbb{E}[(x_{t+1} - \mu_{t+1})(x_{t+1} - \mu_{t+1})^T] \\ &= P_t + \delta t A P_t + \delta t P_t A^T + \delta^2 t A P_t A^T + \delta t \Lambda. \end{aligned}$$

Taking the limit $\delta t \rightarrow 0$ and replacing A , b and Λ by their expressions (25), we find a set of two coupled ODEs:

$$\begin{aligned} \dot{\mu} &= \mathbb{E}_{q(\mu, P)} [f(x, t)] \\ \dot{P} &= \mathbb{E}_{q(\mu, P)} [f(x_t) (x_t - \mu)^T] \\ &\quad + \mathbb{E}_{q(\mu, P)} [(x_t - \mu) f(x_t)^T] + L(t) Q L(t)^T. \end{aligned} \quad (26)$$

It is quite remarkable how the resulting form is closed to the one derived from the Fokker-Planck equations (14), but the expectation are now taken on q , as proposed initially by Sarkka & al. [20] as a heuristics. These ODEs are implicit and when integrated with an explicit scheme defines a recursive variational Gaussian process.

Finally, if we suppose f sufficiently smooth, we may use the properties of Gaussians $\nabla_x \mathcal{N}(x|\mu, P) = -P^{-1}(x - \mu) \mathcal{N}(x|\mu, P)$, and integration by part to let appear the Jacobian matrix $\nabla_x f(x)$ in (26):

$$\mathbb{E}_q[f(x)(x - \mu)^T] = \mathbb{E}_q[\nabla_x f(x)]P, \quad (27)$$

which is a generalization of the Stein Lemma to multivariate functions (see [26]). Hence the form given in Theorem 1.

4.2 Solution to the variational update problem (9)

The variational Gaussian inference problem is to update a prior, say, q_0 , with the new observation y such that the approximated posterior satisfies:

$$\mu, P = \arg \min \quad KL(q(x|y)||p(x|y)), \quad (28)$$

where $p(x|y) \propto p(y|x)q_0(x)$ from Bayes rule, and $p(y|x)$ corresponds to the general non linear observation model (2) with state-dependent covariance $p(y|x) = \mathcal{N}(h(x), R(x))$.

We may consider also the same auxiliary problem than in the previous section minimizing $KL(q(x, y)||p(x, y))$, where $q(y|x) = \mathcal{N}(y|Hx + b, M)$ corresponds to a linear model. Using exactly the same derivation and supposing R is independent of x , we can find exactly the solution given by inference with statistical linearization (16) based on least square loss minimization [3]. However, using variational inference and Bayes rule we tackle directly the initial problem:

$$\begin{aligned} KL(q(x|y)||p(x|y)) &= \int q(x|y) \log \frac{q(x|y)}{p(x|y)} dx \\ &= \int q(x|y) \log \frac{q(x|y)}{p(y|x)q_0(x)} dx + c \\ &= H(q) - \mathbb{E}_q[\log p(y|x)] - \mathbb{E}_q[\log q_0(x)] + c, \end{aligned} \quad (29)$$

where H is the negative entropy and c is a constant independent of q . This approach was developed in our previous work in the context of online machine learning [13]. Zeroing the derivatives with respect to μ and P gives the following updates (see [13], Theorem 1 for details):

$$\begin{aligned} \mu &= \mu_0 - P_0 \nabla_\mu \mathbb{E}_q[-\log p(y|x)] \\ P^{-1} &= P_0^{-1} + 2 \nabla_P \mathbb{E}_q[-\log p(y|x)]. \end{aligned}$$

These updates can be simplified using the properties of Gaussians:

$$\begin{aligned} \nabla_\mu \mathcal{N}(x|\mu, P) &= -\nabla_x \mathcal{N}(x|\mu, P) \\ \nabla_P \mathcal{N}(x|\mu, P) &= \frac{1}{2} \nabla_x^2 \mathcal{N}(x|\mu, P). \end{aligned}$$

If we suppose p sufficiently smooth, we may use integration by part as in Section 4.1 to let appear the derivatives of the negative likelihood. The updates are now:

$$\mu = \mu_0 - P_0 \mathbb{E}_q[\nabla_x(-\log p(y|x))] \quad (30)$$

$$P^{-1} = P_0^{-1} + \mathbb{E}_q[\nabla_x^2(-\log p(y|x))]. \quad (31)$$

Remark 2 (Online natural gradient). *These updates can be interpreted as an averaged version of the online natural gradient which is equivalent to the extended Kalman filter (EKF) as shown in [17]. In particular the update (31) resembles the information update in EKF but the (stochastic) Fisher information matrix $-\nabla_x^2 \log p(y|x)$ is averaged w.r.t. x and can be of higher rank than the EKF information matrix. The connexion between these updates and optimization algorithms is discussed in detail in [13] (Section 4) and in [12] (Appendix F).*

The update (31) can be reformulated in an Hessian free version using a second integration by part:

$$\begin{aligned}
P^{-1} &= P_0^{-1} + P^{-1} \mathbb{E}_q[(x - \mu) \nabla_x - \log p(y|x)^T] \\
&= P_0^{-1} (\mathbb{I} - \mathbb{E}_q[(x - \mu) \nabla_x - \log p(y|x)^T])^{-1} \\
P &= P_0 + \mathbb{E}_q[(x - \mu) \nabla_x \log p(y|x)^T] P_0 \\
&= P_0 + \frac{1}{2} \mathbb{E}_q[(x - \mu) \nabla_x \log p(y|x)^T] P_0 \\
&\quad + \frac{1}{2} P_0 \mathbb{E}_q[(x - \mu) \nabla_x \log p(y|x)^T]^T,
\end{aligned} \tag{32}$$

where the last equation use that $\mathbb{E}_q[(x - \mu) \nabla_x \log p(y|x)^T] P_0$ is symmetric since P is symmetric. It allows us to ensure that this update always provides a symmetric matrix. We now apply this scheme to the particular case of Gaussian observations. The probability $p(y|x)$ depends on x through the observation function $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$. The derivatives of $-\log p$ can be obtained using the chain rule:

$$\begin{aligned}
\nabla_x \log p(y|x) &= \nabla_x \left[-\frac{1}{2} (y - h(x))^T R^{-1} (y - h(x)) \right] \\
&= J_h(x)^T R^{-1} (y - h(x)),
\end{aligned}$$

where $J_h(x) \in \mathcal{M}(m, n)$ is the jacobian matrix of h and the products are standard matrix product. Putting the gradient in (32), we obtain the update for inference given in Theorem 1.

5 Practical implementation

We now propose an implementation of the updates described in Theorem 1.

5.1 Main difficulties

We need to tackle several difficulties before obtaining a tractable filter:

- **Solving the implicit scheme:** The theoretical scheme is implicit, since the expectation \mathbb{E}_q depends on the unknown parameters μ and P . The two ODE equations can be vectorized to take into account the coupling between the variables, and solved by an explicit Runge-Kutta scheme to obtained the propagated moments μ_{t+T} , P_{t+T} up to the next observation date. To compute the inference updates we “open the loop” and substitute distribution q with its previous estimate. An alternative is to use “extragradients” to approximate the implicit scheme as it is done in proximal optimization and in the iterated Kalman filter (see [12] Appendix F).
- **Computing the expectations:** The expectations can be approximated with quadrature rules using sigma points as proposed in [20] and [1]. $2d + 1$ sigma points are spread around the covariance as follows $x_i = \mu + c_i R e_i$, $i = 0, \dots, 2d$, where $e_i|_{i=1, \dots, n}$ are basis vectors and $e_i|_{i=n+1, \dots, 2n}$ their opposite, R is the covariance square root (given by Cholesky decomposition) and c_i are spreading factors. The quadrature rules are given by $\mathbb{E}[f(x)] \approx \sum_{i=0}^{2n} w_i f(\mu + c_i R e_i)$.

- **Ensuring symmetry and positive definiteness:** The covariance matrix may no longer be symmetric or positive after propagation and update. We detail below a square root form to remedy the problem.

5.2 Proposed square-root form

The filter might be used without resorting a square-root form. However, it improves the numerical behavior.

5.2.1 Propagation

For the propagation, we use the continuous-time square-root version developed in [16] and applied in [20]. We consider a lower triangular matrix R such that $P = RR^T$. We note $\dot{P} = F(R)$ the ODE on P expressed as a function of R . For example using the sigma point approach $x_i = \mu + c_i R e_i$ in (26), we can let appear the variable R explicitly. The ODE can be rewritten:

$$\frac{dP}{dt} = \frac{d(RR)}{dt} = \frac{dR}{dt}R^T + R\frac{dR^T}{dt} = F(R), \quad (33)$$

multiplying by R^{-1} on the left and R^{-T} on the right gives:

$$R^{-1}\frac{dR}{dt} + \frac{dR^T}{dt}R^{-T} = R^{-1}F(R)R^{-T}. \quad (34)$$

The solution is given by:

$$R^{-1}\frac{dR}{dt} = \text{Tria}(R^{-1}X(R)R^{-T}) \quad (35)$$

$$\frac{dR}{dt} = R\text{Tria}(R^{-1}X(R)R^{-T}), \quad (36)$$

where $\text{Tria}(A)$ gives the lower triangular matrix L corresponding to A such that $A = L + L^T$ where $L_{i,i} = \frac{1}{2}A_{i,i}$, $L_{i,j} = A_{i,j}$ if $i > j$ and $L_{i,j} = 0$ elsewhere.

5.2.2 Update

Considering the prior is available in a square-root form $P_0 = R_0R_0^T$, we search $P = RR^T$. The update (32) can be rewritten in a square root form as follows:

$$\begin{aligned} RR^T &= R_0R_0^T + \frac{1}{2}\mathbb{E}_q[(x - \mu)v(x)^T]R_0R_0^T \\ &\quad + \frac{1}{2}R_0R_0^T\mathbb{E}_q[v(x)(x - \mu)^T] \\ \text{where } v(x) &= \nabla_x \log p(y|x) = J_h(x)^T R^{-1}(y - h(x)). \end{aligned}$$

Let us “open the loop” and consider integral under q_0 instead of q such that we can insert the sigma points $x_i = \mu_0 + c_i R_0 e_i$:

$$\begin{aligned} RR^T &= R_0R_0^T + \frac{1}{2}\sum_{i=1}^N w_i (c_i R_0 e_i) v(\mu_0 + c_i R_0 e_i)^T R_0R_0^T \\ &\quad + \frac{1}{2}\sum_{i=1}^N w_i R_0R_0^T v(\mu_0 + c_i R_0 e_i)^T (c_i R_0 e_i)^T. \end{aligned}$$

We can now factorize terms as follows:

$$RR^T = R_0(\mathbb{I} + \frac{1}{2}A + \frac{1}{2}A^T)R_0^T \quad (37)$$

$$\text{where } A = \sum_{i=1}^N w_i c_i e_i v(\mu + c_i R_0 e_i)^T R_0.$$

We have now a form $RR^T = R_0QR_0^T$, where Q is a symmetric matrix. We can compute the Cholesky decomposition of $Q = LL^T$ to obtain the square-root updates: $R = R_0L$.

5.3 Experimentations

We consider the reentry tracking problem described in [9]. This problem describes a space capsule which reenters the atmosphere with high velocity and drag making the tracking challenging for a radar. The state is in dimension 5 and composed of 2D-cartesian positions and velocities plus a drag coefficient a : $X = (x(t) \ y(t) \ \dot{x}(t) \ \dot{y}(t) \ a(t))$. a is defined such that the aerodynamic coefficient of the vehicle is $\beta = \beta_0 \exp(a)$, where β_0 is our prior value. This coefficient has an unknown dynamic and is modelled as a Brownian noise $\dot{a}(t) = w(t)$. We use the same initial conditions for the state and covariance as in [9] but to make the problem more challenging we consider sparser observations taken every 0.5s and higher noise: a standard deviation of 100m in range and 100mrad in angle is considered. We have made the schemes explicit without using extragredients. The ODEs are integrated with a Runge-Kutta scheme of order 4 with a step 0.25s (two steps between observations). All the expectations under Gaussians are computed with quadrature rules based on the UKF sigma points [10].

In Figure 1 we show the covariance estimated by the CD-VKF. The standard deviation is consistent with the true trajectories and the filter estimates well the drag coefficient a of the vehicle.

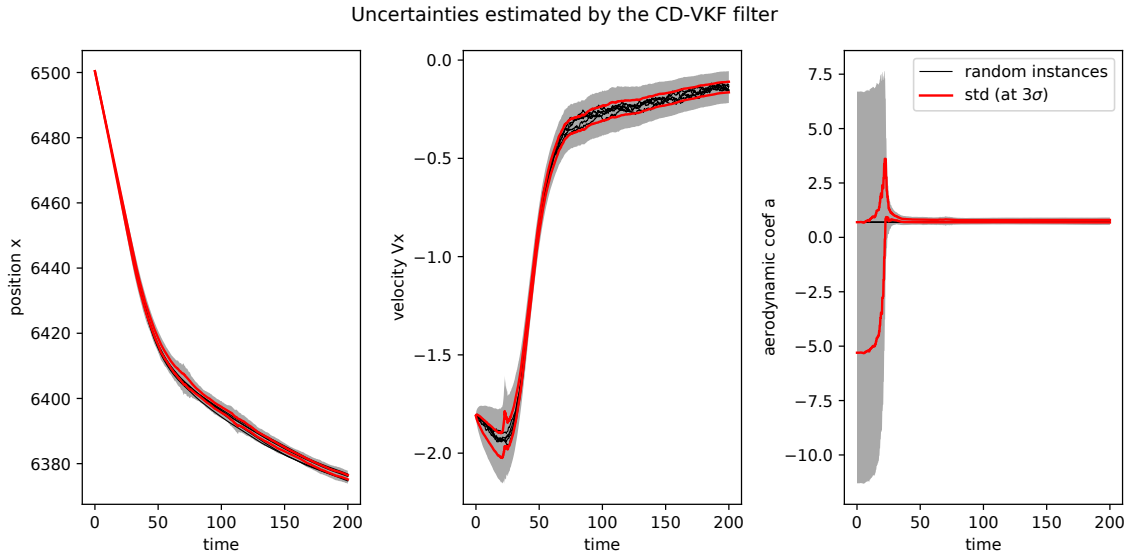


Figure 1: Uncertainties estimated by CD-VKF on the components $x(t)$, $\dot{x}(t)$ and a . The standard deviation around the mean is shown in red at 3σ and in grey at 9σ . The envelopes contain well the sampled real trajectories. The drag coefficient converges to the true value at around 0.6.

In Figure 2 we show the root means squared error (RMSE) for the CD-EKF, the CD-UKF and the CD-VKF. The CD-VKF gives identical results to the CD-UKF on this example, and both outperform the CD-EKF.

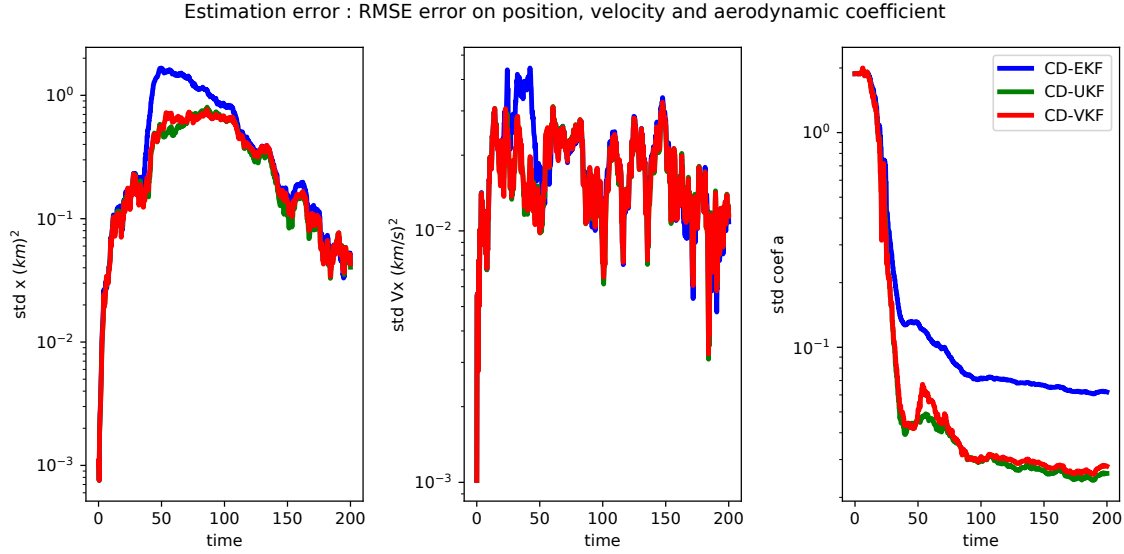


Figure 2: RMSE error achieved by the CD-EKF, CD-UKF and CD-VKF for the components $x(t)$, $\dot{x}(t)$ and a . CD-VKF and CD-UKF yield similar results and outperform the CD-EKF.

The sources of the code are available on Github on the following repository:
<https://github.com/marc-h-lambert/CD-VKF>.

6 CONCLUSION

After a classification of Kalman filters as recursive minimizers for the left or right KL, we have introduced a new variational Kalman filter based on the left KL for both propagation and update. The left KL divergence comes with attractive properties of monotony. Our new algorithm is optimal for the variational optimization problems (7) and (9) and offers a new alternative to moment-matching based filters like UKF. As a left KL minimizer, this new Kalman filter is implicit in essence and comes with many options for approximate implementation (implicit Runge-Kutta, Verlet method, ...) and update (iterate filtering, mirror proximal methods, ...). Understanding which method is best adapted to these recursive schemes is an open problem.

7 ACKNOWLEDGMENTS

This work was funded by the French Defence procurement agency (DGA) and by the French government under management of Agence Nationale de la Recherche as part of the “Investissements d’avenir” program, reference ANR-19-P3IA-0001(PRAIRIE 3IA Institute). We also acknowledge support from the European Research Council (grant SEQUOIA 724063). The authors would like to thank Théophile Cantelobre for reviewing and proof reading the draft version of this manuscript.

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