HAL open science

Galois theory for analogical classifiers

Miguel Couceiro, Erkko Lehtonen

▶ To cite this version:

Miguel Couceiro, Erkko Lehtonen. Galois theory for analogical classifiers. Annals of Mathematics and Artificial Intelligence, In press, 10.1007/s10472-023-09833-6 . hal-03665625v2

HAL Id: hal-03665625

https://inria.hal.science/hal-03665625v2

Submitted on 14 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

Galois theory for analogical classifiers

Miguel Couceiro[1*] and Erkko Lehtonen[2,3]

[1*]Université de Lorraine, CNRS, LORIA, F-54000, France.
[2]Centro de Matemática e Aplicações, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, Caparica, 2829-516, Portugal.
[3]Mathematics Department, Khalifa University, P.O. Box 127788, Abu Dhabi, United Arab Emirates.

*Corresponding author(s). E-mail(s): miguel.couceiro@loria.fr;
Contributing authors: e.lehtonen@fct.unl.pt;

Abstract

Analogical proportions are 4-ary relations that read "A is to B as C is to D". Recent works have highlighted the fact that such relations can support a specific form of inference, called analogical inference. This inference mechanism was empirically proved to be efficient in several reasoning and classification tasks. In the latter case, it relies on the notion of analogy preservation. In this paper, we explore this relation between formal models of analogy and the corresponding classes of analogy preserving functions, and we establish a Galois theory of analogical classifiers. We illustrate the usefulness of this Galois framework over Boolean domains, and we explicitly determine the closed sets of analogical classifiers, i.e., classifiers that are compatible with the analogical inference, for each pair of Boolean analogies.

Keywords: Analogical proportion, analogical reasoning, analogical classifier, Galois theory

1 Introduction and motivation

Analogical reasoning (AR) is a remarkable capability of human thought that exploits parallels between situations of different nature to infer plausible conclusions, by relying simultaneously on similarities and dissimilarities. Machine learning (ML) and artificial intelligence (AI) have tried to develop AR, mostly based on cognitive considerations, and to integrate it in a variety of ML tasks, such as natural language processing (NLP), preference learning and recommendation [1–4]. Also, analogical extrapolation (inference) can solve difficult reasoning tasks such as *scholastic aptitude tests* and *visual question answering* [5, 6]. Inference based on AR can also support dataset augmentation (analogical extension and extrapolation) for model learning, especially in environments with few labeled examples [7]. Furthermore, AR can also be performed at a meta level for transfer learning [8, 9] where the idea is to take advantage of what has been learned on a source domain in order to improve the learning process in a target domain related to the source domain. Moreover, analogy making can provide useful explanations that rely on the parallel example-counterexample [10] and guide counterfactual generation [11].

However, early works lacked theoretical and formalizational support. The situation started to change about a decade ago when researchers adopted the view of analogical proportions as statements of the form "a relates to b as c relates to d", usually denoted $a : b :: c : d$. Such proportions are at the root of the analogical inference mechanism, and several formalisms to study this mechanism have been proposed, which follow different axiomatic and logical approaches [12, 13]. For instance, Lepage [14] introduces the following 4 postulates in the linguistic context as a guideline for formal models of analogical proportions: *symmetry* (if $a : b :: c : d$, then $c : d :: a : b$), *central permutation* (if $a : b :: c : d$, then $a : c :: b : d$), *strong inner reflexivity* (if $a : a :: c : d$, then $d = c$), and *strong reflexivity* (if $a : b :: a : d$, then $d = b$). Such postulates appear reasonable in the word domain, but they can be criticized in other application domains. For instance, in a setting where two distinct conceptual spaces are involved, as in *wine* : *French* :: *beer* : *Belgian* where two different spaces "drinks" and "nationality" are considered, the central permutation is not tolerable.

A key task associated with AR is *analogy solving*, i.e. finding or extrapolating, for a given triple a, b, c a value x such that $a : b :: c : x$ is a valid analogy. In fact, this task can be seen as central in case-based reasoning (CBR) [15]. Given a set $\mathcal{P}$ of problems, a set $\mathcal{S}$ of solutions and a set CB of problem-solution tuples $(x, y) \in \mathcal{P} \times \mathcal{S}$, called *cases*, the CBR task consists in finding a solution $y_{\mathrm{t}} \in \mathcal{S}$ to a given target problem $x_{\mathrm{t}} \in \mathcal{P}$. The CBR methodology splits this problem into several steps, the two most important being (1) *retrieval:* select k "relevant" source cases in the case-base CB according to some criteria related to the target problem; and (2) *adaptation*: reuse the k retrieved cases for proposing a solution to the target problem. The adaptation step obviously depends on the number of retrieved source cases. For $k = 1$, the desired solution y_{t} corresponds to the solution of the analogical equation $x : y :: x_{\mathrm{t}} : z$.

For higher values of k, different models of analogy on $\mathcal{P}$ and $\mathcal{S}$ could be taken into account. For instance, when $k = 3$, the retrieval task consists in finding a triple of cases (x_1, y_1), (x_2, y_2) and (x_3, y_3), such that $x_1 : x_2 :: x_3 : x_\mathrm{t}$ is valid and such that $y_1 : y_2 :: y_3 : z$ is solvable in y_t [16]. In this setting the desired y_t would then be one of such solutions.

The latter idea was extended to analogy based classification [17] where objects are viewed as attribute tuples (instances) $\mathbf{x} = (x_1, \ldots, x_n)$. Similarly, if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are in analogical proportion for most of their attributes, and class labels are known for $\mathbf{a}, \mathbf{b}, \mathbf{c}$ but unknown for $\mathbf{d}$, then one may infer the label for $\mathbf{d}$ as a solution of an analogical proportion equation. All these applications rely on the same idea: if four instances $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are in analogical proportion for most of the attributes describing them, then it may still be the case for the other attributes $f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}), f(\mathbf{d})$ (for some function f). This principle is called *analogical inference principle* (AIP).

Theoretically, it is quite challenging to find and characterize situations where AIP can be soundly applied. A first step toward explaining the analogical mechanism consists in characterizing the set of functions f for which AIP is sound (*i.e.*, no error occurs) no matter which triplets of examples are used. In the case of Boolean attributes and for the model of proportional analogy, it was shown in [7] that these so-called "analogy-preserving" (AP) functions coincide exactly with the set of affine Boolean functions. Moreover, it was also shown that, when the function is not affine, the prediction accuracy remains high if the function is close to being affine [18]. In fact, it was shown that if f is ε-approximately affine (i.e., f is at Hamming distance at most ε from the class of affine functions), then the average error rate of f is at most 4ε. These results were recently extended in [19] to nominal (finite) underlying sets when taking the *minimal model* of analogy, i.e., only patterns of the form $x : x :: y : y$ and $x : y :: x : y$, in both the domain and codomain of classifiers.

Intuitively, this class will change when adopting different models of analogy. In this paper, we investigate the relation between formal models of analogy and the corresponding class of AP functions. We establish a formal correspondence between them and describe a *Galois theory of analogical classifiers*. Moreover, we revisit 8 formal models, including those of analogy, reverse analogy, paralogy and inverse paralogy of [20] in the Boolean case, and for each pair of formal models, we explicitly determine the corresponding closed sets of analogical classifiers.

2 Galois theories for functions

Throughout the paper, we use the notation $[n]$ for the set $\{1, \ldots, n\}$ of the first n positive integers.

Let A and B be nonempty sets. A *function of several arguments* from A to B is a mapping $f\colon A^n \to B$ for some natural number n called the *arity* of f. Denote by $\mathcal{F}_{AB}^{(n)}$ the set of all n-ary functions of several arguments from A to B, and let $\mathcal{F}_{AB} := \bigcup_{n\in\mathbb{N}} \mathcal{F}_{AB}^{(n)}$. In the case when $A = B$ we speak of *operations*

on A, and we use the notation $\mathcal{O}_A^{(n)} := \mathcal{F}_{AA}^{(n)}$ and $\mathcal{O}_A := \mathcal{F}_{AA}$. For any set $\mathcal{C} \subseteq \mathcal{F}_{AB}$, the *n-ary part* of $\mathcal{C}$ is $\mathcal{C}^{(n)} := \mathcal{C} \cap \mathcal{F}_{AB}^{(n)}$.

If $f \in \mathcal{F}_{BC}^{(n)}$ and $g_1, \ldots, g_n \in \mathcal{F}_{AB}^{(m)}$, then the *composition* $f(g_1, \ldots, g_n)$ belongs to $\mathcal{F}_{AC}^{(m)}$ and is defined by the rule

$$f(g_1, \ldots, g_n)(\mathbf{a}) := f(g_1(\mathbf{a}), \ldots, g_n(\mathbf{a})) \quad \text{for all } \mathbf{a} \in A^m.$$

The i-th n-ary *projection* $\mathrm{pr}_i^{(n)} \in \mathcal{O}_A^{(n)}$ is defined by $\mathrm{pr}_i^{(n)}(a_1, \ldots, a_n) := a_i$ for all $a_1, \ldots, a_n \in A$. We denote by $\mathcal{J}_A$ the set of all projections on A.

The notion of functional composition can be extended to sets of functions as follows. Let $\mathcal{C} \subseteq \mathcal{F}_{BC}$ and $\mathcal{K} \subseteq \mathcal{F}_{AB}$. The *composition* of $\mathcal{C}$ with $\mathcal{K}$ is the set

$$\mathcal{C}\mathcal{K} := \{h \in \mathcal{F}_{AC} \mid \exists m, n \in \mathbb{N},\ f \in \mathcal{C}^{(n)},\ g_1, \ldots, g_n \in \mathcal{K}^{(m)},\ h = f(g_1, \ldots, g_n)\}.$$

A *clone* on A is a set $\mathcal{C} \subseteq \mathcal{O}_A$ that is closed under composition and contains all projections, in symbols, $\mathcal{C}\mathcal{C} \subseteq \mathcal{C}$ and $\mathcal{J}_A \subseteq \mathcal{C}$. For $\mathcal{F} \subseteq \mathcal{O}_A$, we denote by $\langle \mathcal{F} \rangle$ the clone generated by $\mathcal{F}$, i.e., the smallest clone on A containing $\mathcal{F}$.

Let $f \in \mathcal{F}_{AB}^{(n)}$ and $g \in \mathcal{F}_{AB}^{(m)}$. We say that f is a *minor* of g, and we write $f \leq g$, if $f \in \{g\}\mathcal{J}_A$, or, equivalently, there exists a $\sigma\colon [m] \to [n]$ such that

$$f(a_1, \ldots, a_n) = g(a_{\sigma(1)}, \ldots, a_{\sigma(m)}) \quad \text{for all } a_1, \ldots, a_n \in A.$$

The minor relation $\leq$ is a quasi-order (a reflexive and transitive relation) on $\mathcal{F}_{AB}$. Downsets of $(\mathcal{F}_{AB}, \leq)$ are called *minor-closed* classes or *minions*. Equivalently, a set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ is a minion if $\mathcal{C}\mathcal{J}_A \subseteq \mathcal{C}$.

A set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ is *m-locally closed* if for all $f \in \mathcal{F}_{AB}$ (say f is n-ary), it holds that $f \in \mathcal{C}$ whenever for every finite subset $S \subseteq A^n$ of size at most m, there exists a $g \in \mathcal{C}$ such that $f|_S = g|_S$. A set $\mathcal{C}$ is said to be *locally closed* if it is m-locally closed for every positive integer m.

Subsets of A^m are called m-ary *relations* on A. Denote by $\mathcal{R}_A^{(m)}$ the set of all m-ary relations on A, and let $\mathcal{R}_A := \bigcup_{m \in \mathbb{N}} \mathcal{R}_A^{(m)}$. Let $f \in \mathcal{O}_A^{(n)}$ and $R \in \mathcal{R}_A^{(m)}$. We say that the function f *preserves* the relation R (or f is a *polymorphism* of R, or R is an *invariant* of f), and we write $f \triangleright R$, if for all $\mathbf{a}_1, \ldots, \mathbf{a}_n \in R$, we have $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \in R$. Here $f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ means the componentwise application of f to the tuples, i.e., if $\mathbf{a}_i = (a_{i1}, \ldots, a_{im})$ for $i \in [n]$, then

$$f(\mathbf{a}_1, \ldots, \mathbf{a}_n) := (f(a_{11}, \ldots, a_{n1}), \ldots, f(a_{1m}, \ldots, a_{nm})).$$

The preservation relation $\triangleright$ induces a Galois connection between the sets $\mathcal{O}_A$ and $\mathcal{R}_A$ of operations and relations on A. Its polarities are the maps $\mathrm{Pol}\colon \mathcal{P}(\mathcal{R}_A) \to \mathcal{P}(\mathcal{O}_A)$ and $\mathrm{Inv}\colon \mathcal{P}(\mathcal{O}_A) \to \mathcal{P}(\mathcal{R}_A)$ given by the following rules:

for all $\mathcal{R} \subseteq \mathcal{R}_A$ and $\mathcal{F} \subseteq \mathcal{O}_A$,

$$\operatorname{Pol} \mathcal{R} := \{f \in \mathcal{O}_A \mid \forall R \in \mathcal{R} \colon f \triangleright R\},$$
$$\operatorname{Inv} \mathcal{F} := \{R \in \mathcal{R}_A \mid \forall f \in \mathcal{F} \colon f \triangleright R\}.$$

Under this Galois connection, the closed sets of operations are precisely the locally closed clones. The closed sets of relations, known as *relational clones,* are precisely the locally closed sets of relations that contain the empty relation and the binary equality relation and are closed under formation of "primitively positively definable relations" (*i.e.*, definable by existentially quantified positive conjunctive formulas). This was first shown for finite base sets by Bodnarchuk, Kaluzhnin, Kotov, Romov [21, 22] and Geiger [23] and later extended for arbitrary sets by Szabó [24] and Pöschel [25].

The preservation relation can be adapted for functions of several arguments from A to B; we now need to consider pairs of relations. Let

$$\mathcal{R}_{AB}^{(m)} := \mathcal{R}_A^{(m)} \times \mathcal{R}_B^{(m)} \quad \text{and} \quad \mathcal{R}_{AB} := \bigcup_{m \in \mathbb{N}} \mathcal{R}_{AB}^{(m)}$$

be the set of all (m-ary) *relational constraints* from A to B.

Let $f \in \mathcal{F}_{AB}^{(n)}$ and $(R, S) \in \mathcal{R}_{AB}^{(m)}$. We say that f *preserves* (R, S) (or f is a *polymorphism* of (R, S), or (R, S) is an *invariant* of f), and we write $f \triangleright (R, S)$, if for all $\mathbf{a}_1, \ldots, \mathbf{a}_n \in R$, we have $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \in S$. As before, the preservation relation $\triangleright$ induces a Galois connection between the sets $\mathcal{F}_{AB}$ and $\mathcal{R}_{AB}$ of functions and relational constraints from A to B. Its polarities are the maps $\operatorname{Pol} \colon \mathcal{P}(\mathcal{R}_{AB}) \to \mathcal{P}(\mathcal{F}_{AB})$ and $\operatorname{Inv} \colon \mathcal{P}(\mathcal{F}_{AB}) \to \mathcal{P}(\mathcal{R}_{AB})$ given by the following rules: for all $\mathcal{Q} \subseteq \mathcal{R}_{AB}$ and $\mathcal{F} \subseteq \mathcal{F}_{AB}$,

$$\operatorname{Pol} \mathcal{Q} := \{f \in \mathcal{F}_{AB} \mid \forall (R, S) \in \mathcal{Q} \colon f \triangleright (R, S)\},$$
$$\operatorname{Inv} \mathcal{F} := \{(R, S) \in \mathcal{R}_{AB} \mid \forall f \in \mathcal{F} \colon f \triangleright (R, S)\}.$$

The sets $\operatorname{Pol} \mathcal{Q}$ and $\operatorname{Inv} \mathcal{F}$ are said to be *defined* by $\mathcal{Q}$ and $\mathcal{F}$, respectively. Sets of functions of the form $\operatorname{Pol} \mathcal{Q}$ for some $\mathcal{Q} \subseteq \mathcal{R}_{AB}$ and sets of relational constraints of the form $\operatorname{Inv} \mathcal{F}$ for some $\mathcal{F} \subseteq \mathcal{F}_{AB}$ are said to be *definable* by relational constraints and functions, respectively.

The closed sets of functions under this Galois connection were described for finite base sets by Pippenger [26] and later for arbitrary sets by Couceiro and Foldes [27]. This result was refined by Couceiro [28] for sets of functions definable by relations of restricted arity.

Theorem 1 ([27, 28]) *Let A and B be arbitrary nonempty sets, and let $C \subseteq \mathcal{F}_{AB}$.*

(i) *$\mathcal{C}$ is definable by constraints if and only if $\mathcal{C}$ is a locally closed minion.*

(ii) *$\mathcal{C}$ is definable by constraints of arity m if and only if $\mathcal{C}$ is an m-locally closed minion.*

The closed sets of relational constraints were described in terms of closure conditions that parallel those for relational clones.

The description of the dual objects of constraints on possibly infinite sets A and B was also provided in [27] and inspired by those given by Geiger [23], Szabó [24], Pöschel [25] and Pippenger [26], and given in terms of positive primitive first-order relational definitions applied simultaneously on antecedents and consequents. Sets of constraints that are closed under such formation schemes are said to be *closed under conjunctive minors*. Moreover, every function satisfies the empty $(\varnothing, \varnothing)$ and the equality $(=_A, =_B)$ constraints, and if a function f satisfies a constraint (R, S), then f also satisfies its *relaxations*, *i.e.*, constraints (R', S') such that $R' \subseteq R$ and $S' \supseteq S$.

As for functions, in the infinite case, we also need to consider a "local closure" condition to describe the dual closed sets of relational constraints on A and B. A set $\mathcal{Q}$ of constraints on A and B is *n-locally closed* if it contains every relaxation of its members whose antecedent has size at most n, and it is *locally closed* if it is n-locally closed for every positive integer n.

Theorem 2 ([27, 28]) *For arbitrary nonempty sets A and B, and let $\mathcal{Q} \subseteq \mathcal{R}_{AB}$ be a set of relational constraints on A and B.*

(i) *$\mathcal{Q}$ is definable by some set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ if and only if it is locally closed, contains the binary equality and the empty constraints, and it is closed under relaxations and conjunctive minors.*

(ii) *$\mathcal{Q}$ is definable by some set $\mathcal{C} \subseteq \mathcal{F}_{AB}^{(n)}$ of n-ary functions if and only it is n-locally closed, contains the binary equality and the empty constraints, and it is closed under relaxations and conjunctive minors.*

Let $\mathcal{K} \subseteq \mathcal{F}_{AB}$ and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be clones on A and B. We say that $\mathcal{K}$ is *stable under right composition with $\mathcal{C}_1$ if $\mathcal{K}\mathcal{C}_1 \subseteq \mathcal{K}$*, and we say that $\mathcal{K}$ is *stable under left composition with $\mathcal{C}_2$* if $\mathcal{C}_2\mathcal{K} \subseteq \mathcal{K}$. We say that $\mathcal{K}$ is *$(\mathcal{C}_1, \mathcal{C}_2)$-stable* or a *$(\mathcal{C}_1, \mathcal{C}_2)$-clonoid*, if $\mathcal{K}\mathcal{C}_1 \subseteq \mathcal{K}$ and $\mathcal{C}_2\mathcal{K} \subseteq \mathcal{K}$.

Motivated by earlier results on linear definability of equational classes of Boolean functions [29], later refined by Couceiro and Lehtonen [30], which were described in terms of stability under compositions with the clone of constant preserving affine functions, Couceiro and Foldes [31] introduced a Galois framework for describing sets of functions $\mathcal{F} \subseteq \mathcal{F}_{AB}$ stable under right and left compositions with clones $\mathcal{C}_1$ on A and $\mathcal{C}_2$ on B, respectively.

For that they restricted the defining dual objects to relational constraints (R, S) where R and S are invariant under $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, i.e., $R \in \operatorname{Inv} \mathcal{C}_1$ and $S \in \operatorname{Inv} \mathcal{C}_2$. These were referred to as *$(\mathcal{C}_1, \mathcal{C}_2)$-constraints*. We denote by $\mathcal{R}_{AB}^{(\mathcal{C}_1, \mathcal{C}_2)}$ the set of all $(\mathcal{C}_1, \mathcal{C}_2)$-constraints.

Theorem 3 ([31]) *Let A and B be arbitrary nonempty sets, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ clones on A and B, respectively. A set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ is definable by some set of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints if and only if $\mathcal{C}$ is locally closed and stable under right and left composition with $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, i.e., it is a locally closed $(\mathcal{C}_1, \mathcal{C}_2)$-clonoid.*

Dually, a set $\mathcal{Q}$ of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints is definable by a set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ if $\mathcal{Q} = \operatorname{Inv} \mathcal{C} \cap \mathcal{R}_{AB}^{(\mathcal{C}_1,\mathcal{C}_2)}$.

To describe the dual closed sets of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints, Couceiro and Foldes [31] observed that conjunctive minors of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints are themselves $(\mathcal{C}_1, \mathcal{C}_2)$-constraints. However, this is not the case for relaxations. They thus proposed the following variants of local closure and of constraint relaxations.

A set $\mathcal{Q}_0$ of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints is said to be $(\mathcal{C}_1, \mathcal{C}_2)$-*locally closed* if the set $\mathcal{Q}$ of all relaxations of the various constraints in $\mathcal{Q}_0$ is locally closed. A relaxation (R_0, S_0) of a relational constraint (R, S) is said to be a $(\mathcal{C}_1, \mathcal{C}_2)$-*relaxation* if (R_0, S_0) is a $(\mathcal{C}_1, \mathcal{C}_2)$-constraint.

Theorem 4 ([31]) *Let A and B be arbitrary nonempty sets, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ clones on A and B, respectively. A set $\mathcal{Q}$ of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints is definable by some set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ if and only if it is $(\mathcal{C}_1, \mathcal{C}_2)$-locally closed and contains the binary equality constraint, the empty constraint, and it is closed under $(\mathcal{C}_1, \mathcal{C}_2)$-relaxations and conjunctive minors.*

Further examples and variants of these Galois theories for function classes are present, *e.g.*, in [32–34].

In this paper, we will focus on relational constraints whose antecedent and consequent are derived from analogies, and that we will refer to as *analogical constraints*. We will denote the set of all analogical constraints from A to B by $\mathcal{A}_{AB}$.

3 Formal models of analogy

In this section we briefly survey different axiomatic settings to formally define analogies and different approaches to address the two main problems dealing with AR, namely, analogy making and solving.

Multiple attempts have been made to formalize and manipulate analogies, starting as early as De Saussure's work in 1916 [35], but there is not a consensual view on the topic. Many works rely on the common view of analogy as a geometric proportion ($a \times d = b \times c$) or as an arithmetic proportion ($b - a = d - c$), which can be thought of as a *parallelogram rule* in a vector space. This view led to an axiomatic approach [12] in which analogies are quaternary relations satisfying 3 postulates: reflexivity, symmetry, and central permutation. These postulates imply many other properties: *identity* ($a : a :: b : b$), *internal reversal* (if $a : b :: c : d$, then $b : a :: d : c$), *extreme permutation* (if $a : b :: c : d$, then $d : b :: c : a$), etc.

Another frequently accepted postulate is *uniqueness* stating that if there exists d such that $a : b :: c : d$, then d is unique. As argued in [36], the latter postulate is debatable and the authors illustrate it through linguistic examples, *e.g.*, *wine* : *French* :: *beer* : x as *Belgian*, *Czech* and *German* seem reasonable. This is further illustrated in [37] through arithmetic examples. Take, for instance, $20 : 4 :: 30 : x$ that has a clear solution $x = 6$. However,

$x = 9$ is another solution since $(10 \cdot 2) : 2^2 :: (10 \cdot 3) : 3^2$. Independently, it has also been observed that uniqueness and central permutation are not compatible in simple analogy models in non-Euclidean domains [38].

The framework [37] assumes that pairs (a, b) and (c, d) are interpreted over two, possibly different, algebras $\mathbb{A} = (A, F)$ and $\mathbb{B} = (B, F)$ with the same functional signature L, and called respectively the *source* and *target domains*. Analogies are then modeled by common rewriting transformations called *justifications* of the form $s \to t$, where both s and t are L-terms in such a way that $a = s^{\mathbb{A}}(\mathbf{e}_1)$ and $b = t^{\mathbb{A}}(\mathbf{e}_1)$, for some $\mathbf{e}_1 \in \mathbb{A}^{|\mathbf{x}|}$, and $c = s^{\mathbb{B}}(\mathbf{e}_2)$ and $d = t^{\mathbb{B}}(\mathbf{e}_2)$, for some $\mathbf{e}_2 \in \mathbb{B}^{|\mathbf{x}|}$. A quadruple $(a, b, c, d) \in A^2 \times B^2$ is then said to be an *analogical proportion*, denoted $a : b :: c : d$, if there are no $a', b' \in A$ and $c', d' \in B$ such that

1. the set of justifications of (a, b, c, d') strictly contains that of (a, b, c, d),
2. the set of justifications of (b, a, d, c') strictly contains that of (b, a, d, c),
3. the set of justifications of (c, d, a, b') strictly contains that of (c, d, a, b),
4. the set of justifications of (d, c, b, a') strictly contains that of (d, c, b, a).

Given $\mathbb{A}$ and $\mathbb{B}$, the relation comprising all analogical proportions will be referred to as a *formal model of analogy* or, simply, as an *analogy*. Note that every analogy fulfills internal reversal and extreme permutation.

This framework accommodates many formal models of analogies, including the *factorial* view of [39] and the *functional* view of [40, 41], except that it is not bound by the central permutation postulate. Such a framework is close to Gentner's symbolic model of analogical reasoning [42] based on *structure mapping theory* and first implemented in [43]. Both share the view that analogies are compatible with structure preserving maps. However, the latter prefers knowledge connected facts to isolated ones, and the former fails to satisfactorily account for analogies over different conceptual spaces as in *wine* : *French* :: *beer* : *Belgian*. For an in-depth discussion on different axiomatic settings and conceptual spaces, see, *e.g.*, [40, 44].

In this paper, we will focus on the case where the source and target domains coincide (i.e., $\mathbb{B} = \mathbb{A}$), and we will denote the set of all analogies (on A) by $\mathcal{A}_A$, and we may omit the subscript A when it is clear from the context. For an analogy $R \in \mathcal{A}_A$ we will adopt the more specific notation $R(a, b, c, d)$ instead of the usual notation $a : b :: c : d$, since we will be dealing simultaneously with multiple models of analogy.

Example 5 We will consider several formal models of analogy on the two-element set $\{0, 1\}$. In particular, we will focus on the following 5 relations as running examples throughout the paper. Miclet and Prade's [45] definition of Boolean proportion corresponds to R_1, Klein's [46] definition corresponds to R_2, while relations R_3, R_4 and R_5 are respectively referred to in [20] as reverse analogy, paralogy and inverse paralogy. We present a relation as a matrix whose columns are the tuples belonging to the relation.

$$R_1 := \begin{pmatrix} 0&1&1&0&0&1 \\ 0&1&0&1&0&1 \\ 0&0&1&0&1&1 \\ 0&0&0&1&1&1 \end{pmatrix} \quad R_2 := \begin{pmatrix} 0&1&1&0&1&0&0&1 \\ 0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1 \end{pmatrix}$$

$$R_3 := \begin{pmatrix} 0&1&0&1&0&1 \\ 0&1&0&1&1&0 \\ 0&1&1&0&1&0 \\ 0&1&1&0&0&1 \end{pmatrix} \quad R_4 := \begin{pmatrix} 0&1&1&0&0&1 \\ 0&1&0&1&1&0 \\ 0&1&0&1&0&1 \\ 0&1&1&0&1&0 \end{pmatrix} \quad R_5 := \begin{pmatrix} 1&0&1&0&0&1 \\ 1&0&0&1&1&0 \\ 0&1&0&1&0&1 \\ 0&1&1&0&1&0 \end{pmatrix}$$

As can be seen with the help of Fact 13, projections and negated projections are polymorphisms of all these five relations. Moreover, constant functions are polymorphisms of the first four but not of R_5.

4 Galois theory for analogical classifiers

As mentioned in the Introduction, analogical inference yields competitive results in classification and recommendation tasks. However, the justification of why and when a classifier is compatible with the analogical inference principle (AIP) remained rather obscure until the work of Couceiro *et al.* [7]. In this paper the authors considered the minimal Boolean analogy model (see R_4 in Example 5) and addressed the problem of determining those *Boolean classifiers for which the AIP always holds*, that is, for which there are no classification errors. Surprisingly, they showed that they correspond to "analogy preserving" (see Definition 6 below) and that they constitute the clone of affine functions. This result was later generalized to binary classification tasks on nominal (finite) domains in [19] where the authors considered the more stringent notion of "hard analogy preservation". By taking the same minimal analogy model made only of analogical proportions of the form $a : a :: b : b$ and $a : b :: a : b$ on both the domain and the label set, the authors showed that in this case the sets of hard analogy preserving functions constitute Burle's clones [47].

These preliminary results ask for a better understanding of analogical classifiers, and in this paper we seek a general theory of classifiers compatible with AIP that is not dependent on the underlying sets nor on particular models of analogy. More precisely, we generalize the existing literature by establishing a Galois theory of analogical classifiers which we then use to explicitly describe the sets of Boolean analogical classifiers with respect to the pairs (R, S) of the known Boolean models R and S of analogy. We first recall the notion of analogy preservation and establish some useful results that allow us to use the universal algebraic toolbox.

Definition 6 Let A and B be sets, and let R and S be analogical proportions defined on the two sets, respectively. A function $f\colon A^n \to B$ is *analogy-preserving* (AP for short) relative to (R, S) if for all $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in A^n$, the following implication holds:

$$\big(R(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \quad \text{and} \quad S\text{-solv}(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}))\big) \implies S(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}), f(\mathbf{d})),$$

where $R(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is a shorthand for $(a_i, b_i, c_i, d_i) \in R$ for all $i \in [n]$ and S-solv$(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}))$ means that there exists an $x \in B$ such that $S(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}), x)$. Denote by $\mathsf{AP}(R, S)$ the set of all analogy-preserving functions relative to (R, S).

This relation between functions and formal models of analogy gives rise to a Galois connection whose closed sets of functions correspond exactly to the classes of analogical classifiers that we now describe.

We start by stating and proving some useful results.

Proposition 7 *Let R and S be analogical proportions defined on sets A and B, respectively. Then $\mathsf{AP}(R, S) = \mathrm{Pol}(R, S')$, where*

$$S' := S \cup \{(a, b, c, d) \in B^4 \mid \nexists x \in B \colon (a, b, c, x) \in S\}. \tag{1}$$

Proof The condition of Definition 6 can be written equivalently as follows: for all $\mathbf{a}_1, \ldots, \mathbf{a}_n \in R$, $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \in S'$. This is exactly what it means that f preserves (R, S'). Therefore, $\mathsf{AP}(R, S) = \mathrm{Pol}(R, S')$. □

Example 8 In continuation to Example 5, the extended relations as in Proposition 7 are the following:

$$R_1' = R_1 \cup \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad R_2' = R_2, \quad R_3' = R_3 \cup \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$R_4' = R_4 \cup \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad R_5' = R_5 \cup \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Note that

- $R_1' = R_2' \cup \{1000, 0111\}$;
- $R_3' = R_2' \cup \{0100, 1011\}$;
- $R_4' = R_2' \cup \{0010, 1101\}$;
- $R_5' = R_2' \cup \{0001, 1110\}$.

Moreover, all are closed under negation.

To fully describe the sets of the form $\mathrm{Pol}(R, S')$ we need to introduce some variants of the closure conditions discussed in Section 2

Let $\mathcal{R}$ be set of m-ary relations on A. An $m \times n$ matrix D whose columns belong to a relation $R \in \mathcal{R}$, is called an $\mathcal{R}$-*locality*. Let $\mathcal{Q} \subseteq \mathcal{R}_{AB}$, and let $\mathcal{Q}_1 := \{R \in \mathcal{R}_A \mid \exists S \in \mathcal{R}_B$ such that $(R, S) \in \mathcal{Q}\}$. A set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ is $\mathcal{Q}$-*locally closed* if for all $f \in \mathcal{F}_{AB}$ (say f is n-ary), it holds that $f \in \mathcal{C}$ whenever for every $\mathcal{Q}_1$-locality D, either

1. there exists a $g \in \mathcal{C}$ such that $fD = gD$, or
2. for any relation R in $\mathcal{Q}_1$ such that $D \preceq R$ and for any

$$T \in \{S \in \mathcal{R}_B \mid (R, S) \in \mathcal{Q}, \mathcal{C}R \subseteq S\}$$

we have that $fR \subseteq T$.

Let $\mathcal{A}'_B := \{S' \mid S \in \mathcal{A}_B\}$, and let $\mathcal{A}_{AB} := \mathcal{A}_A \times \mathcal{A}'_B$. We refer to the elements of $\mathcal{A}_{AB}$ as *analogical constraints* from A to B. The set of analogical constraints that are $(\mathcal{C}_1, \mathcal{C}_2)$-constraints will be denoted by

$$\mathcal{A}_{AB}^{(\mathcal{C}_1,\mathcal{C}_2)} := \mathcal{A}_{AB} \cap \mathcal{R}_{AB}^{(\mathcal{C}_1,\mathcal{C}_2)}.$$

A set $\mathcal{C}$ is said to be $(\mathcal{C}_1, \mathcal{C}_2)$-*analogically locally closed* if it is $\mathcal{A}_{AB}^{(\mathcal{C}_1,\mathcal{C}_2)}$-locally closed. Note that $\mathcal{A}_{AB} = \mathcal{A}_{AB}^{(\mathcal{J}_A,\mathcal{J}_B)}$, and in this case we simply say that $\mathcal{C}$ is *analogically locally closed.*

Theorem 9 *Let A and B be arbitrary nonempty sets, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be clones on A and B, respectively.*

1. *A set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ is definable by analogical $(\mathcal{C}_1, \mathcal{C}_2)$-constraints if and only if it is a $(\mathcal{C}_1, \mathcal{C}_2)$-analogically locally closed $(\mathcal{C}_1, \mathcal{C}_2)$-clonoid.*
2. *A set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ is definable by analogical constraints if and only if it is an analogically locally closed minion.*

Proof Note that the second statement is a particular case of the first: just take $\mathcal{C}_1$ and $\mathcal{C}_2$ to be the clones of projections on A and B, respectively. We will thus prove the first statement.

To prove that the conditions are necessary, observe that $\mathcal{C}$ is a $(\mathcal{C}_1, \mathcal{C}_2)$-clonoid by Theorem 3. It thus remains to show that it is $(\mathcal{C}_1, \mathcal{C}_2)$-analogically locally closed. Let $f \notin \mathcal{C}$, say of arity n. Hence, there is an analogical $(\mathcal{C}_1, \mathcal{C}_2)$-constraint $(R, S) \in \mathcal{A}_{AB}^{(\mathcal{C}_1,\mathcal{C}_2)}$ that is satisfied by every function in $g \in \mathcal{C}$, but not by f. Let $\mathbf{a}_1, \ldots, \mathbf{a}_n \in R$ such that $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \notin S$. Consider the $(\mathcal{A}_{AB}^{(\mathcal{C}_1,\mathcal{C}_2)})_1$-locality $D = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$. As every $g \in \mathcal{C}^{(n)}$ satisfies the constraint (R, S), we have that $fD \neq gD$. Also, $D \preceq R \in (\mathcal{A}_{AB}^{(\mathcal{C}_1,\mathcal{C}_2)})_1$ and it is clear that $S \in \{S_0 \in \mathcal{R}_B \mid (R, S_0) \in \mathcal{A}_{AB}^{(\mathcal{C}_1,\mathcal{C}_2)}, \mathcal{C}R \subseteq S_0\}$, and we have $fR \not\subseteq S$ since $fD \notin S$.

To prove that the conditions are sufficient, we follow a similar strategy to that in [31] and show that for every n-ary $f \notin \mathcal{C}$, there is an analogical $(\mathcal{C}_1, \mathcal{C}_2)$-constraint $(R, S) \in \mathcal{A}_{AB}^{(\mathcal{C}_1,\mathcal{C}_2)}$ that is satisfied by every function $g \in \mathcal{C}$, but not by f. The set of such analogical $(\mathcal{C}_1, \mathcal{C}_2)$-constraints will then define $\mathcal{C}$.

So suppose that $f \notin \mathcal{C}$. Since $\mathcal{C}$ is $(\mathcal{C}_1, \mathcal{C}_2)$-analogically locally closed, there is an $(\mathcal{A}_{AB}^{(\mathcal{C}_1,\mathcal{C}_2)})_1$-locality D such that $fD \neq gD$, for every n-ary $g \in \mathcal{C}$, and there exist $R \in (\mathcal{A}_{AB}^{(\mathcal{C}_1,\mathcal{C}_2)})_1$ with $D \preceq R$, and $S \in \{S_0 \in \mathcal{R}_B \mid (R, S_0) \in \mathcal{A}_{AB}^{(\mathcal{C}_1,\mathcal{C}_2)}, \mathcal{C}R \subseteq S_0\}$ such that $fR \notin S$.

Since (R, S) is an analogical $(\mathcal{C}_1, \mathcal{C}_2)$-constraint that is satisfied by every function $g \in \mathcal{C}$ (as $\mathcal{C}R \subseteq T$) but not by f, this constitutes the desired constraint separating f and $\mathcal{C}$, and the proof is thus complete. □

Dually, a set $\mathcal{Q}$ of analogical $(\mathcal{C}_1, \mathcal{C}_2)$-constraints is definable by a set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ if $\mathcal{Q} = \operatorname{Inv} \mathcal{C} \cap \mathcal{A}_{AB}^{(\mathcal{C}_1, \mathcal{C}_2)}$. The description of the dual closed sets of analogical constraints is then an immediate consequence of Theorem 4.

Theorem 10 *Let A and B be arbitrary nonempty sets, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be clones on A and B, respectively.*

1. *A set $\mathcal{Q}$ of analogical $(\mathcal{C}_1, \mathcal{C}_2)$-constraints is definable by some set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ if and only if there exists a set $\mathcal{Q}_0$ of constraints from A to B that is $(\mathcal{C}_1, \mathcal{C}_2)$-locally closed and contains the binary equality and the empty constraints, and it is closed under $(\mathcal{C}_1, \mathcal{C}_2)$-relaxations and conjunctive minors, such that $\mathcal{Q} = \mathcal{A}_{AB} \cap \mathcal{Q}_0$.*
2. *A set $\mathcal{Q}$ of analogical constraints from A to B is definable by some set $\mathcal{C} \subseteq \mathcal{F}_{AB}$ if and only if there exists a set $\mathcal{Q}_0$ of constraints from A to B that is locally closed and contains the binary equality and the empty constraints, and it is closed under relaxations and conjunctive minors, such that $\mathcal{Q} = \mathcal{A}_{AB} \cap \mathcal{Q}_0$.*

Note that the Galois framework that we propose is general and it is not restricted to analogies. To illustrate, we will also consider relations that were identified by Antić in an earlier version of [48].

Example 11 Antić [48] considered, in addition to Klein's and the minimal model, the following 3 relations as formal models of analogies on the two-element set $\{0, 1\}$:

$$R_6 := \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$R_7 := \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad R_8 := \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

which give rise to the following extensions:

$$R_6' = R_6, , \quad R_7' = R_7 \cup \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R_8' = R_8 \cup \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that projections and constant functions are polymorphisms of all three of R_6, R_7, and R_8. Moreover, negated projections are polymorphisms of R_6 but not of R_7 nor of R_8.

$\mathsf{AP}(R,S)$		R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
						S			
R	R_1	L	L	L	L	L	L	$\Omega(1)$	$\Omega(1)$
	R_2	L	L	L	L	L	L	C	C
	R_3	L	L	L	L	L	L	C	C
	R_4	L	L	L	L	L	L	C	C
	R_5	L	L	L	L	L	L	C	C
	R_6	C	C	C	C	C	$\Omega(1)$	C	C
	R_7	C	C	C	C	C	$\Omega(1)$	I	N
	R_8	C	C	C	C	C	$\Omega(1)$	N	I

Table 1 Summary of results. Notation: L denotes the class of linear (affine) functions, $\Omega(1)$ the class of projections, negations, and constants, N the class of negations and constants, I the class of projections and constants, and C the class of constants.

For the sake of illustration, we also take these 3 relations into account, and compute the corresponding AP classes for all possible combinations (R_i, R_j), for $i, j \in [8]$.

5 Explicit description of Boolean analogical classifiers

Recall the formal models of analogy R_i, $i \in [8]$, given in Examples 5 and 11, on the two-element set $\{0,1\}$. In this section we make use of the Galois theory described in Section 4 to determine the classes of analogical classifiers $\mathsf{AP}(R_i, R_j) = \mathrm{Pol}(R_i, R'_j)$ for all $i, j \in [8]$ (see Examples 8 and 11 for the relations R'_j and Equation (1) for the definition of the extension S' of a relation S). The results are summarised in Table 1, together with our notation for various classes of Boolean functions.

5.1 Preliminary results

For $a \in \{0,1\}$, let $\overline{a} := 1 - a$, and for $\mathbf{a} = (a_1, \ldots, a_n) \in \{0,1\}^n$, let $\overline{\mathbf{a}} := (\overline{a_1}, \ldots, \overline{a_n})$. We will also use $\mathbf{a}^{\mathrm{n}_I}$, $I \subseteq [n]$, for the tuple obtained from $\mathbf{a}$ by negating the components a_i, $i \in I$. Let $f\colon \{0,1\}^n \to \{0,1\}$. The (*outer*) *negation* $\overline{f}$, the *inner negation* f^{n}, the *partial inner negation* f^{n_I}, $I \subseteq [n]$, and the *dual* f^{d} of f are the n-ary Boolean functions given by the rules

$$\begin{aligned} \overline{f}(\mathbf{a}) &:= \overline{f(\mathbf{a})}, \quad f^{\mathrm{n}}(\mathbf{a}) := f(\overline{\mathbf{a}}), \quad f^{\mathrm{n}_I}(\mathbf{a}) := f(\mathbf{a}^{\mathrm{n}_I}), \text{ and} \\ f^{\mathrm{d}}(\mathbf{a}) &:= \overline{f(\overline{\mathbf{a}})}, \quad \text{for all } \mathbf{a} \in \{0,1\}^n. \end{aligned}$$

For $C \subseteq \Omega$, we let $\overline{C} := \{\overline{f} \mid f \in C\}$, $C^{\mathrm{n}} := \{f^{\mathrm{n}} \mid f \in C\}$, $C^{\mathrm{d}} := \{f^{\mathrm{d}} \mid f \in C\}$.

Up to permutation of arguments, the binary Boolean functions are the following:

- the constant 0 and 1 functions, denoted respectively by 0 and 1,
- the first projection $\mathrm{pr}_1\colon (x_1, x_2) \mapsto x_1$ and its negation $\neg_1 = \overline{\mathrm{pr}_1}$,
- the conjunction $\wedge$ and its negation $\uparrow$,

- the disjunction $\vee$ and its negation $\downarrow$,
- the implication $\rightarrow$ and its negation $\nrightarrow$, and
- the addition $+$ modulo 2 and its negation $\leftrightarrow$.

Note that $\uparrow$ and $\downarrow$ are often referred to as *Sheffer functions* as each one of them can generate the class of all Boolean functions by taking compositions and variable substitutions.

The ternary Boolean functions include the triple sum

$$\oplus_3 \colon (x_1, x_2, x_3) \mapsto (x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$$

and its negation $\overline{\oplus_3} = \oplus_3^{\mathrm{n}} = \oplus_3^{\mathrm{n}\{i\}}$, $i \in [3]$, and the median function

$$\begin{aligned}\mu \colon (x_1, x_2, x_3) &\mapsto (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3)\\ &= (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_2 \vee x_3)\\ &= \oplus_3(x_1 \wedge x_2, x_1 \wedge x_3, x_2 \wedge x_3)\end{aligned}$$

and its negation $\overline{\mu} = \mu^{\mathrm{n}}$. Note that $\mu^{\mathrm{n}\{i\}} = \overline{\mu^{\mathrm{n}[3]\setminus\{i\}}}$.

We also make use of the following terminology. For a relation $R \subseteq \{0,1\}^m$, we define its *negation* $\overline{R}$ by $\overline{R} := \{\overline{\mathbf{a}} \mid \mathbf{a} \in R\}$.

Lemma 12 $\mathrm{Pol}(R, S) = \mathrm{Pol}(\overline{R}, \overline{S})^{\mathrm{d}}$.

Proof We need to show that $f \triangleright (R, S)$ if and only if $f^{\mathrm{d}} \triangleright (\overline{R}, \overline{S})$. Suppose that $f \triangleright (R, S)$, and let $M \prec \overline{R}$. Hence, $\overline{M} \in R$ and we have $f^{\mathrm{d}} M = f^{\mathrm{d}} \overline{\overline{M}} = \overline{f\overline{M}} \in \overline{S}$, and thus $f^{\mathrm{d}} \triangleright (\overline{R}, \overline{S})$. The converse implication follows by the same argument. $\square$

Fact 13 Let R and S be m-ary relations on $\{0,1\}$.

(i) $\mathrm{id} \in \mathrm{Pol}(R, S)$ if and only if $R \subseteq S$.

(ii) $\neg \in \mathrm{Pol}(R, S)$ if and only if $\overline{R} \subseteq S$.

(iii) For $a \in \{0,1\}$, if $(a, \ldots, a) \in S$, then $a \in \mathrm{Pol}(R, S)$.

Observe that the constant tuples $\mathbf{0}$ and $\mathbf{1}$ belong to R_i, for every $i \in [8]\setminus\{5\}$, and thus every such R_i is invariant under I, i.e., $\mathsf{I}R_i \subseteq R_i$. Hence, for every $i \in [8] \setminus \{5\}$ and $j \in [8]$, $\mathrm{Pol}(R_i, R'_j)$ is stable under right composition with I. This leads us to considering the following notion.

A function f is said to be a $\mathcal{C}$-*minor* of a function g if $f \in g\mathcal{C}$. Recall that in the particular case when $\mathcal{C} = \mathcal{J}_{\{0,1\}}$, f is called a minor of g. The functions f and g are said to be *equivalent*, denoted by $f \equiv g$, if f is a minor of g and g is a minor of f. For further background on these notions and variants see, *e.g.*, [26, 49–54].

Note that the stability of $\mathrm{Pol}(R_i, R'_j)$ under right composition with $\mathcal{C}$ means that if a $\mathcal{C}$-minor f of a function g does not belong to $\mathrm{Pol}(R_i, R'_j)$, then neither

does g. This observation will be used repeatedly in the proofs of the results that will follow.

Lemma 14 *Let R and S be m-ary relations on $\{0,1\}$ such that $\mathsf{I} \subseteq \operatorname{Pol} R$. Then the following statements hold.*

(i) *If* $\mathrm{id} \notin \operatorname{Pol}(R,S')$ *or* $\neg \notin \operatorname{Pol}(R,S')$, *then* $+, \leftrightarrow, \rightarrow, \nrightarrow \notin \operatorname{Pol}(R,S')$.

(ii) *If* $+ \notin \operatorname{Pol}(R,S')$, *then* $\oplus_3 \notin \operatorname{Pol}(R,S')$.

(iii) *If* $\wedge \notin \operatorname{Pol}(R_i,R_j')$ *or* $\vee \notin \operatorname{Pol}(R,S')$, *then* $\mu \notin \operatorname{Pol}(R,S')$.

Proof The proof is obtained by verifying that the functions in the antecedent of the implications are I-minors of those in the consequent of the implication. We prove this claim explicitly for (i), and leave the remaining for the reader.

(i) For every $x \in \{0,1\}$, $\mathrm{id}(x) = +(x,0) = \leftrightarrow(x,1) = \rightarrow(1,x) = \nrightarrow(x,0)$. Similarly, we also have $\neg(x) = +(x,1) = \leftrightarrow(x,0) = \neg(x) = \rightarrow(x,0) = \nrightarrow(1,x)$.

(ii) and (iii) are shown similarly. □

Lemma 15 *Let R and S be m-ary relations on $\{0,1\}$ such that $\mathsf{I} \subseteq \operatorname{Pol} R$. Then it holds that if* $\mathrm{id}, \neg \notin \operatorname{Pol}(R,S)$, *then* $\operatorname{Pol}(R,S) = \mathsf{C}$.

Proof Just note that the only functions that do not have I-minors in $\{\mathrm{id}, \neg\}$ are the constant functions. The result thus follows from the observations above. □

5.2 Description of Boolean analogical classifiers

In this section we consider all pairs of relations R_i, $i \in [8]$, and explicitly compute the corresponding sets of analogical classifiers.

Proposition 16 $\mathsf{AP}(R_2,R_2) = \operatorname{Pol}(R_2,R_2') = \mathsf{L}$.

Proof Since $R_2 = R_2'$, it follows immediately that $\operatorname{Pol}(R_2,R_2') = \operatorname{Pol} R_2$ is a clone, and it is well known that $\operatorname{Pol} R_2 = \mathsf{L}$. □

Lemma 17 *For every Boolean function $f \notin \mathsf{L}$, we have that*

(i) *f has an I_{c}-minor in $\{\wedge, \vee, \uparrow, \downarrow, \nrightarrow, \rightarrow, \mu, \mu^{\mathrm{n}\{1\}}, \mu^{\mathrm{n}\{1,2\}}, \mu^{\mathrm{n}}\}$,*

(ii) *f has an I-minor in $\{\wedge, \vee, \uparrow, \downarrow, \nrightarrow, \rightarrow\}$,*

(iii) *f has an I^*-minor in $\{\wedge, \vee, \mu\}$, and*

(iv) *f has an $\Omega(1)$-minor in $\{\wedge, \vee\}$.*

Proof This result follows quite easily from Zverovich's description of the class L in terms of "forbidden subfunctions" [55, Theorem 2(H9)]; in fact, the claims about I_c- and I-minors (statements (i) and (ii)) appear as such in Zverovich's paper, and the claims about I^*- and $\Omega(1)$-minors (statements (iii) and (iv)) are easy consequences. For the sake of illustration, we provide a detailed proof of the claim about I-minors. We make use of the well-known fact that every Boolean function can be represented by a unique multilinear polynomial over the 2-element field $GF(2)$.

So suppose that $f \notin \mathsf{L}$, i.e., there is a monomial in the multilinear representation of f with degree[1] at least 2. Take a minimal monomial with at least 2 variables; without loss of generality, suppose that it is $x_1x_2x_3\cdots x_j$.

Consider the I-minor f' of f obtained by:

- substituting 1 for each $x_3, \ldots, x_j$, and
- substituting 0 for each $x_{j+1}, \ldots, x_n$ in the polynomial representation of f.

It is not difficult to verify that $f' = x_1x_2 + ax_1 + bx_2 + c$, for some $a, b, c \in \{0, 1\}$, and

- $f' \equiv \wedge$, if $a = b = c = 0$;
- $f' \equiv \uparrow$, if $a = b = 0$ and $c = 1$;
- $f' \equiv \vee$, if $a = b = 1$ and $c = 0$;
- $f' \equiv \downarrow$, if $a = b = c = 1$;
- $f' \equiv \nrightarrow$, if $a = c = 1$ and $b = 0$, or if $b = c = 1$ and $a = 0$;
- $f' \equiv \rightarrow$, if $a = 1$ and $b = c = 0$, or if $b = 1$ and $a = c = 0$.

This completes the proof of the claim about I-minors.

The claim about I^*-minors follows from statement (i): any function not in L has an I_c-minor in $\{\wedge, \vee, \uparrow, \downarrow, \nrightarrow, \rightarrow, \mu, \mu^{\mathrm{n}\{1\}}, \mu^{\mathrm{n}\{1,2\}}, \mu^{\mathrm{n}}\}$. From each one of these functions, we can obtain by negating variables one of the functions in the set $\{\wedge, \vee, \mu\}$. This proves the claim about I^*-minors.

As for the claim about $\Omega(1)$-minors, we make use of the fact that $\Omega(1) = \mathsf{I} \cup \mathsf{I}^*$. By statement (ii), any function in L has an I-minor in $\{\wedge, \vee, \uparrow, \downarrow, \nrightarrow, \rightarrow\}$, and by negating arguments, we obtain from each one of these a function in the set $\{\wedge, \vee\}$. □

Lemma 18 *For all* $i \in [5]$, $j \in [6]$, $\mathsf{L} \subseteq \mathsf{AP}(R_i, R_j) = \mathrm{Pol}(R_i, R'_j)$.

Proof Observe that $R_i \subseteq R_2$ and $R'_j \supseteq R'_2 = R_2$. Therefore (R_i, R'_j) is a relaxation of (R_2, R'_2), so it follows from Theorem 2 and Proposition 16 that $\mathrm{Pol}(R_i, R'_j) \subseteq \mathrm{Pol}(R_2, R'_2) = \mathsf{L}$. □

Lemma 19 *For all* $i, j \in [8]$, $\mathsf{AP}(R_i, R_j) = \mathrm{Pol}(R_i, R'_j) \subseteq \mathsf{L}$.

Proof Observe first the following.

Obs. 1: For each $i \in [8]$ and for each $k \in [4]$, there exist distinct tuples $\mathbf{a} = (a_1, a_2, a_3, a_4)$ and $\mathbf{b} = (b_1, b_2, b_3, b_4)$ in R_i such that $w(\mathbf{a}) = w(\mathbf{b}) = 2$ and $a_k =$

[1] The degree of a monomial is the sum of the exponents of the variables that appear in it, and the degree of a polynomial is the largest degree among its monomials.

$b_k = 1$. (Here $w(\mathbf{a})$ denotes the Hamming weight of $\mathbf{a}$, i.e., the number of nonzero entries in $\mathbf{a}$.)

Obs. 2: For each $i \in [8]$, if $\mathbf{a} \in R_i$ and $w(\mathbf{a}) = 2$, then $\overline{\mathbf{a}} \in R_i$.

Obs. 3: For each $j \in [8]$, R'_j contains at most one of the tuples $(0,0,1,0)$ and $(0,0,0,1)$, and at most one of the tuples $(1,1,0,1)$ and $(1,1,1,0)$.

Now, for any $i, j \in [8]$, we can conclude the following. By Obs. 1, there exist tuples $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in R_i$, each of weight 2, such that $\mathbf{a} \wedge \mathbf{b} = (0,0,1,0)$ and $\mathbf{c} \wedge \mathbf{d} = (0,0,0,1)$ By Obs. 2, we have also $\overline{\mathbf{a}}, \overline{\mathbf{b}}, \overline{\mathbf{c}}, \overline{\mathbf{d}} \in R_i$, and by De Morgan's laws we have $\overline{\mathbf{a}} \vee \overline{\mathbf{b}} = (1,1,0,1)$ and $\overline{\mathbf{c}} \vee \overline{\mathbf{d}} = (1,1,1,0)$. This, together with Obs. 3 shows that $\wedge, \vee \notin \mathrm{Pol}(R_i, R'_j)$. □

Proposition 20 *For all $i \in [5]$, $j \in [6]$, $\mathsf{AP}(R_i, R_j) = \mathrm{Pol}(R_i, R'_j) = \mathsf{L}$.*

Proof This follows immediately from Lemmata 18 and 19. □

Proposition 21 *For $j \in \{7, 8\}$, we have $\mathsf{AP}(R_1, R_j) = \Omega(1)$.*

Proof We prove $\mathsf{AP}(R_1, R_7) = \Omega(1)$; the remaining case $\mathsf{AP}(R_1, R_8) = \Omega(1)$ can be proved similarly using the fact that $R'_8 = \overline{R'_7}$. We have $\mathsf{AP}(R_1, R_7) \subseteq \mathsf{L}$ by Lemma 19, so it suffices to determine which members of L belong to $\mathsf{AP}(R_1, R_7)$. Since $\overline{R_1} = R_1 \subseteq R_7$, Fact 13 asserts that $\mathsf{AP}(R_1, R_7)$ contains the projections and the negated projections; moreover, since $\mathbf{0}, \mathbf{1} \in R'_7$, it also contains the constant functions.

To complete the proof, it suffices to show that $+, \leftrightarrow, \oplus_3, \overline{\oplus_3} \notin \mathsf{AP}(R_1, R_7)$. For that, let $\mathbf{a} := 1010$, $\mathbf{b} = 0011$ and observe that $\mathbf{a}, \mathbf{b}, \mathbf{0} \in R_1$ and $\mathbf{a} + \mathbf{b} = 1001 \notin R'_7$ and $\oplus_3(\mathbf{a}, \mathbf{b}, \mathbf{0}) = 1001 \notin R'_7$. Using the fact that R_1 is invariant under negation, we also have $\leftrightarrow = +^{\mathrm{n}_{\{1\}}} \notin \mathsf{AP}(R_1, R_7)$ and similarly for $\overline{\oplus_3} = \oplus_3^{\mathrm{n}_{\{1\}}} \notin \mathsf{AP}(R_1, R_7)$. □

Proposition 22 *For $i \in \{2, 3, 4, 5\}$, $j \in \{7, 8\}$, we have $\mathsf{AP}(R_i, R_j) = \mathsf{C}$.*

Proof We have $\mathsf{AP}(R_i, R_j) \subseteq \mathsf{L}$ by Lemma 19, so it suffices to determine which members of L belong to $\mathsf{AP}(R_1, R_7)$. Let $\mathbf{a} := 0110$, $\mathbf{b} := 1001$, and notice that $\{\mathbf{a}, \mathbf{b}\} \subseteq R_i$ and $\{\mathbf{a}, \mathbf{b}\} \nsubseteq R'_j$; hence $\mathsf{AP}(R_i, R_j)$ does not contain projections nor negated projections by Fact 13. However, $\mathsf{AP}(R_i, R_j)$ contains the constant functions because $\mathbf{0}, \mathbf{1} \in R'_j$.

It remains to show that $+, \leftrightarrow \notin \mathsf{AP}(R_i, R_j)$. For $i \in \{2, 3, 4\}$, we have $\mathbf{0} \in R_i$ and $\{\mathbf{a}+\mathbf{0}, \mathbf{b}+\mathbf{0}\} = \{\mathbf{a} \leftrightarrow \mathbf{0}, \mathbf{b} \leftrightarrow \mathbf{0}\} = \{\mathbf{a}, \mathbf{b}\} \nsubseteq R'_j$. For $i = 5$, we have $0101.1100, 0011 \in R_i$ and $\{0101 + 1100, 0101 + 0011\} = \{0101 \leftrightarrow 1100, 0101 \leftrightarrow 0011\} = \{\mathbf{a}, \mathbf{b}\} \nsubseteq R'_j$. □

Lemma 23 *For all $i \in \{6, 7, 8\}$, $j \in [8]$, $\mathsf{AP}(R_i, R_j) \subseteq \mathsf{AP}(R_1, R_j)$.*

Proof Observe that $R_1 \subseteq R_i$. Therefore (R_1, R'_j) is a relaxation of (R_i, R'_j), so it follows from Theorem 2 that $\mathsf{AP}(R_i, R_j) = \mathrm{Pol}(R_i, R'_j) \subseteq \mathrm{Pol}(R_1, R'_j) = \mathsf{AP}(R_1, R_j)$. □

Proposition 24 *For all* $i \in \{6, 7, 8\}$, $j \in [5]$, $\mathsf{AP}(R_i, R_j) = \mathrm{Pol}(R_i, R'_j) = \mathsf{C}$.

Proof By Lemma 19 (or by Lemma 23 and Proposition 20), we have $\mathsf{AP}(R_i, R_j) \subseteq \mathsf{L}$, so it suffices to determine which members of L belong to $\mathsf{AP}(R_i, R_j) = \mathrm{Pol}(R_i, R'_j)$. We have $\{1000, 0100\} \subseteq R_i$ or $\{0111, 1011\} \subseteq R_i$ but $\{1000, 0100\} \nsubseteq R'_j$ and $\{0111, 1011\} \nsubseteq R'_j$, which implies by Fact 13 that the projections and the negated projections are not in $\mathrm{Pol}(R_i, R'_j)$. Since $\mathbf{0}, \mathbf{1} \in R'_j$, the constant functions belong to $\mathrm{Pol}(R_i, R'_j)$.

It remains to show that $+, \leftrightarrow \notin \mathrm{Pol}(R_i, R'_j)$. But this follows immediately from Lemma 14(i), because $\mathbf{0}, \mathbf{1} \in R_i$ and hence $\mathsf{I} \subseteq \mathrm{Pol}\, R_i$. □

Lemma 25 *For every* $f \in \mathsf{L} \setminus \Omega(1)$, *we have that*

(i) f *has an* I_c*-minor in* $\{+, \leftrightarrow, \oplus_3, \overline{\oplus_3}\}$,

(ii) f *has an* I*-minor in* $\{+, \leftrightarrow\}$,

(iii) f *has an* I^**-minor in* $\{+, \oplus_3\}$, *and*

(iv) f *has an* $\Omega(1)$*-minor in* $\{+\}$.

Proof Straightforward verification. □

Lemma 26 *For* $i \in \{7, 8\}$, $j \in [8]$, $\mathsf{AP}(R_6, R_j) \subseteq \mathsf{AP}(R_i, R_j)$.

Proof For $i \in \{7, 8\}$, it holds that $R_i \subseteq R_6$. Therefore (R_i, R'_j) is a relaxation of (R_6, R'_j), so it follows from Theorem 2 that $\mathsf{AP}(R_6, R_j) = \mathrm{Pol}(R_6, R'_j) \subseteq \mathrm{Pol}(R_i, R'_j) = \mathsf{AP}(R_i, R_j)$. □

Proposition 27 *For all* $i \in \{6, 7, 8\}$, $\mathsf{AP}(R_i, R_6) = \mathrm{Pol}(R_i, R'_6) = \Omega(1)$.

Proof By Lemma 23 and Proposition 20, we have $\mathrm{Pol}(R_i, R'_6) \subseteq \mathrm{Pol}(R_1, R'_6) = \mathsf{L}$ for all $i \in \{6, 7, 8\}$. By Fact 13, we have $0, 1, \mathrm{id}, \neg \in \mathrm{Pol}(R_i, R'_6)$, i.e., $\Omega(1) \subseteq \mathrm{Pol}(R_i, R'_6)$ for all $i \in \{6, 7, 8\}$. It remains to prove the converse inclusion $\mathrm{Pol}(R_i, R'_6) \subseteq \Omega(1)$.

We have $+, \leftrightarrow \notin \mathrm{Pol}(R_7, R'_6)$, because $1000, 1010 \in R_7$ but $1000 + 1010 = 0010 \notin R'_6$ and $1000 \leftrightarrow 1010 = 1101 \notin R'_6$. Similarly, $+, \leftrightarrow \notin \mathrm{Pol}(R_8, R'_6)$, because $0111, 0101 \in R_7$ but $0111 + 0101 = 0010 \notin R'_6$ and $0111 \leftrightarrow 0101 = 1101 \notin R'_6$. Now, for $i \in \{7, 8\}$, Lemma 25 implies that $\mathrm{Pol}(R_i, R'_6) \cap (\mathsf{L} \setminus \Omega(1)) = \emptyset$ because $\mathrm{Pol}(R_i, R'_6)$ is closed under I-minors; hence $\mathrm{Pol}(R_i, R'_6) \subseteq \Omega(1)$. Furthermore, by Lemma 26, we have $\mathrm{Pol}(R_6.R'_6) \subseteq \mathrm{Pol}(R_7, R'_6) \subseteq \Omega(1)$. □

Proposition 28 $\mathsf{AP}(R_6, R'_7) = \mathsf{C}$, $\mathsf{AP}(R_7, R'_7) = \mathsf{I}$, $\mathsf{AP}(R_8, R'_7) = \mathsf{N}$, $\mathsf{AP}(R_6, R'_8) = \mathsf{C}$, $\mathsf{AP}(R_7, R'_8) = \mathsf{N}$, $\mathsf{AP}(R_8, R'_8) = \mathsf{I}$.

Proof By Lemma 23 and Proposition 21, $\mathrm{Pol}(R_i, R'_j) \subseteq \mathrm{Pol}(R_1, R'_j) = \Omega(1)$ for all $i \in \{6, 7, 8\}$, $j \in \{7, 8\}$. Therefore it suffices to determine which members of $\Omega(1)$ belong to $\mathrm{Pol}(R_i, R'_j)$ in each case. From Fact 13 we obtain immediately the following: $0, 1 \in \mathrm{Pol}(R_i, R'_j)$ for all $i \in \{6, 7, 8\}$ and $j \in \{7, 8\}$; $\mathrm{id}, \neg \notin \mathrm{Pol}(R_6, R_j)$ for $j \in \{7, 8\}$; and for all $i, j \in \{7, 8\}$, $\mathrm{id} \in \mathrm{Pol}(R_i, R'_j)$ if and only if $i = j$, and $\neg \in \mathrm{Pol}(R_i, R'_j)$ if and only if $i \neq j$. This proves the claimed equalities. □

6 Conclusion and perspectives

In this paper we addressed and tackled the question of determining classes of analogical classifiers. Our approach makes use of model theoretic and universal algebraic tools that we used to establish a general Galois theory of such analogical classifiers that does not depend on the underlying domains nor the formal models of analogy considered. In the particular case of Boolean analogies, in which several formal models of analogy have been identified, we made use of this Galois framework to explicitly describe, for each pair of such relations, the respective sets of analogical classifiers.

As future work, we intend to further explore different formal models of analogy that may be obtained by considering different algebraic signatures or different logical frameworks. For instance, here we only considered the so-called "homogeneous logical proportions" [56] and it could be of interest to explore also the "heterogeneous logical proportions". Also, in [48], Antić only considered reducts of the 2-element Boolean algebra for which he provided all Boolean analogies. However, other algebraic signatures and their reducts could be considered, such as that of median algebras and fields. These may give rise to further models of analogy considered over different underlying domains, for which we may obtain other sets of analogical classifiers. Further related questions of model theoretic and universal algebraic flavour will also be addressed.

Also, here we focused on classifiers that are compatible with AIP and for which no classification errors occur. We intend to extend the study of the behavior of analogical classifiers in the spirit of [18] by analyzing the relative classification errors with respect to the distance to the classes $\mathsf{AP}(R, S)$, and by empirically studying the performance of analogical classifiers on real world datasets. This empirical study goes beyond the scope of this contribution and will be addressed in a future collaboration.

Acknowledgments. The authors wish to thank Esteban Marquer, Pierre-Alexandre Murena and Henri Prade for the insightful discussions and useful suggestions for improving this manuscript. The authors are also grateful to the anonymous referees for their constructive remarks which helped improve the presentation and expand the results of this paper.

Declarations

Funding

This research was partially supported by TAILOR, a project funded by EU Horizon 2020 research and innovation program under GA No 952215, the Inria Project Lab "Hybrid Approaches for Interpretable AI" (HyAIAI), and the ANR project "Analogies: from theory to tools and applications" (AT2TA), ANR-22-CE23-0023.

This work was also partially funded by National Funds through the FCT – Fundação para a Ciência e a Tecnologia, I.P., under the scope of the project UIDB/00297/2020 (Center for Mathematics and Applications) and the project PTDC/MAT-PUR/31174/2017.

Conflict of interest/Competing interests

The authors have no competing interests to declare that are relevant to the content of this article.

Ethics approval

Not applicable.

Consent to participate

Not applicable.

Consent for publication

Not applicable.

Availability of data and materials

Data sharing not applicable to this article as not datasets were generated or analysed during the current study.

Code availability

Not applicable.

Authors' contributions

Both authors contributed equally to all aspects of this work.

References

[1] Alsaidi, S., Decker, A., Lay, P., Marquer, E., Murena, P.-A., Couceiro, M.: A neural approach for detecting morphological analogies. In: IEEE 8th DSAA, pp. 1–10 (2021)

[2] Fahandar, M.A., Hüllermeier, E.: Learning to rank based on analogical reasoning. In: AAAI-18, pp. 2951–2958 (2018)

[3] Fahandar, M.A., Hüllermeier, E.: Analogical embedding for analogy-based learning to rank. In: IDA 2021. LNCS, vol. 12695, pp. 76–88. Springer (2021)

[4] Mitchell, M.: Abstraction and Analogy-Making in Artificial Intelligence (2021). https://arxiv.org/abs/2102.10717

[5] Sadeghi, F., Zitnick, C.L., Farhadi, A.: Visalogy: Answering visual analogy questions. In: NIPS 2015, pp. 1882–1890 (2015)

[6] Peyre, J., Laptev, I., Schmid, C., Sivic, J.: Detecting unseen visual relations using analogies. In: IEEE ICCV 2019, pp. 1981–1990 (2019)

[7] Couceiro, M., Hug, N., Prade, H., Richard, G.: Analogy-preserving functions: A way to extend Boolean samples. In: IJCAI 2017, pp. 1575–1581. ijcai.org

[8] Cornuéjols, A., Murena, P.-A., Olivier, R.: Transfer learning by learning projections from target to source. In: IDA 2020. LNCS, vol. 12080, pp. 119–131. Springer (2020)

[9] Alsaidi, S., Decker, A., Lay, P., Marquer, E., Murena, P.-A., Couceiro, M.: On the transferability of neural models of morphological analogies. In: AIMLAI@ECML/PKDD 2021, pp. 76–89

[10] Hüllermeier, E.: Towards analogy-based explanations in machine learning. In: MDAI 2020. LNCS, vol. 12256, pp. 205–217. Springer (2020)

[11] Keane, M.T., Smyth, B.: Good counterfactuals and where to find them: A case-based technique for generating counterfactuals for explainable AI (XAI). In: ICCBR 2020. LNCS, vol. 12311, pp. 163–178. Springer (2020)

[12] Lepage, Y.: Analogy and formal languages. Electron. Notes Theor. Comput. Sci. **53**, 180–191 (2001)

[13] Miclet, L., Bayoudh, S., Delhay, A.: Analogical dissimilarity: Definition, algorithms and two experiments in machine learning. J. Artif. Intell. Res. **32**, 793–824 (2008)

[14] Lepage, Y.: De L'analogie Rendant Compte de la Commutation en Linguistique, (2003). https://tel.archives-ouvertes.fr/tel-00004372

[15] Richter, M.M., Weber, R.O.: Case-based Reasoning, Springer (2016)

[16] Lieber, J., Nauer, E., Prade, H.: When revision-based case adaptation

meets analogical extrapolation. In: ICCBR 2021. LNCS, vol. 12877, pp. 156–170. Springer (2021)

[17] Bounhas, M., Prade, H., Richard, G.: Analogy-based classifiers for nominal or numerical data. IJAR **91**, 36–55 (2017)

[18] Couceiro, M., Hug, N., Prade, H., Richard, G.: Behavior of analogical inference w.r.t. Boolean functions. In: IJCAI 2018, pp. 2057–2063. ijcai.org

[19] Couceiro, M., Lehtonen, E., Miclet, L., Prade, H., Richard, G.: When nominal analogical proportions do not fail. In: SUM 2020,. LNCS, vol. 12322, pp. 68–83. Springer (2020)

[20] Prade, H., Richard, G.: From analogical proportion to logical proportions. Logica Universalis **7**(4), 441–505 (2013)

[21] Bodnarchuk, V.G., Kaluzhnin, L.A., Kotov, V.N., Romov, B.A.: Galois theory for Post algebras I. Kibernetika **5**(3), 1–10 (1969)

[22] Bodnarchuk, V.G., Kaluzhnin, L.A., Kotov, V.N., Romov, B.A.: Galois theory for Post algebras II. Kibernetika **5**(5), 1–9 (1969)

[23] Geiger, D.: Closed systems of functions and predicates. Pacific J. Math. **27**, 95–100 (1968)

[24] Szabó, L.: Concrete representation of related structures of universal algebras. i. Acta Sci. Math. (Szeged) **40**, 175–184 (1978)

[25] Pöschel, R.: Concrete representation of algebraic structures and a general Galois theory. In: Kautschitsch, H., Müller, W.B., Nöbauer, W. (eds.) Contributions to General Algebra (Proc. Klagenfurt Conf., Klagenfurt, 1978), pp. 249–272. Johannes Heyn, Klagenfurt (1979)

[26] Pippenger, N.: Galois theory for minors of finite functions. Discrete Math. **254**(1-3), 405–419 (2002)

[27] Couceiro, M., Foldes, S.: On closed sets of relational constraints and classes of functions closed under variable substitutions. Algebra Universalis **54**, 149–165 (2005)

[28] Couceiro, M.: On Galois connections between external operations and relational constraints: arity restrictions and operator decompositions. Acta Sci. Math. **72**(1-2), 15–35 (2006)

[29] Couceiro, M., Foldes, S.: Definability of Boolean function classes by linear equations over **GF**(2). Discrete Appl. Math. **142**(1-3), 29–34 (2004)

[30] Couceiro, M., Lehtonen, E.: Stability of Boolean function classes with respect to clones of linear functions. Order. To appear.

[31] Couceiro, M., Foldes, S.: Function classes and relational constraints stable under compositions with clones. Discuss. Math., Gen. Algebra Appl. **29**(2), 109–121 (2009)

[32] Lehtonen, E.: Characterization of preclones by matrix collections. Asian-Eur. J. Math. **3**(3), 457–473 (2010)

[33] Lehtonen, E.: Closed classes of functions, generalized constraints, and clusters. Algebra Universalis **63**(2-3), 203–234 (2010)

[34] Couceiro, M., Lehtonen, E.: Galois theory for sets of operations closed under permutation, cylindrification, and composition. Algebra Universalis **67**(3), 273–297 (2012)

[35] De Saussure, F.: Cours de Linguistique Générale, Lausanne – Paris, Payot (1st edition) (1916)

[36] Lim, S., Prade, H., Richard, G.: Classifying and completing word analogies by machine learning. Internat. J. Approx. Reason. **132**, 1–25 (2021)

[37] Antić, C.: Analogical proportions. Ann. Math. Artif. Intell. **90**(6), 595–644 (2022)

[38] Murena, P.-A., Cornuéjols, A., Dessalles, J.-L.: Opening the parallelogram: Considerations on non-Euclidean analogies. In: International Conference on Case-Based Reasoning, pp. 597–611 (2018). Springer

[39] Stroppa, N., Yvon, F.: An analogical learner for morphological analysis. In: CoNLL 2005, ACL, pp. 120–127 (2005)

[40] Barbot, N., Miclet, L., Prade, H.: Analogy between concepts. Artif. Intell. **275**, 487–539 (2019)

[41] Murena, P.-A., Al-Ghossein, M., Dessalles, J.-L., Cornuéjols, A.: Solving analogies on words based on minimal complexity transformation. In: IJCAI, pp. 1848–1854 (2020)

[42] Gentner, D., Hoyos, C.: Analogy and abstraction. Top. Cogn. Sci. **9**(3), 672–693 (2017)

[43] Falkenhainer, B., Forbus, K.D., Gentner, D.: The structure-mapping engine: Algorithm and examples. Artif. Intell. **41**(1), 1–63 (1989)

[44] Afantenos, S.D., Kunze, T., Lim, S., Prade, H., Richard, G.: Analogies between sentences: Theoretical aspects - preliminary experiments. In:

ECSQARU 2021, pp. 3–18 (2021)

[45] Miclet, L., Prade, H.: Handling analogical proportions in classical logic and fuzzy logics settings. In: C. Sossai, G.C. (ed.) Proc. 10th Europ. Conf. on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2009), pp. 638–650. Springer, Berlin/Heidelberg (2009)

[46] Klein, S.: Culture, mysticism & social structure and the calculation of behavior. In: Proc. 5th Europ. Conf. in Artificial Intelligence (ECAI'82), pp. 141–146 (1982)

[47] Burle, G.A.: Classes of k-valued logic which contain all functions of a single variable. Diskret. Analiz, Novosibirsk **10**, 3–7 (1967)

[48] Antić, C.: Boolean proportions. arXiv (2021). https://arxiv.org/abs/2109.00388v1

[49] Lehtonen, E.: Descending chains and antichains of the unary, linear, and monotone subfunction relations. Order **23**, 129–142 (2006)

[50] Lehtonen, E., Nešetřil, J.: Minors of Boolean functions with respect to clique functions and hypergraph homomorphisms. European J. Combin. **31**(8), 1981–1995 (2010)

[51] Lehtonen, E., Szendrei, Á.: Partial orders induced by quasilinear clones. In: Contributions to General Algebra 20, pp. 51–84. Verlag Johannes Heyn, Klagenfurt (2012)

[52] Lehtonen, E., Szendrei, Á.: Equivalence of operations with respect to discriminator clones. Discrete Math. **309**(4), 673–685 (2009)

[53] Lehtonen, E., Szendrei, Á.: Clones with finitely many relative $\mathcal{R}$-classes. Algebra Universalis **65**(2), 109–159 (2011)

[54] Lehtonen, E., Szendrei, Á.: The submaximal clones on the three-element set with finitely many relative $\mathcal{R}$-classes. Discuss. Math. Gen. Algebra Appl. **30**(1), 7–33 (2010)

[55] Zverovich, I.: Characterizations of closed classes of Boolean functions in terms of forbidden subfunctions and Post classes. Discrete Appl. Math. **149**, 200–218 (2005)

[56] Prade, H., Richard, G.: Homogenous and heterogeneous logical proportions. IfCoLog J. Logics Appl. **1**(1), 1–52 (2014)