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# On stability of mechanical systems with homogeneous and delayed forces

A. Aleksandrov, D. Efimov

## ARTICLE HISTORY

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## ABSTRACT

The problem of delay-independent stability is studied for a class of mechanical systems under dissipative, non-conservative and potential forces, which are described by homogeneous terms. The obtained conditions are extended to the case of switched force model under arbitrary and restricted commutation laws. The proof is based on analysis of a complete Lyapunov–Krasovskii functional (common and multiple ones are considered for switched scenario). The proposed stability conditions are illustrated by simulation examples.

## KEYWORDS

Mechanical systems; Homogeneous systems; Time-delay systems

## 1. Introduction

The analysis and control design problems for time-delay mechanical systems are challenging and practically important (Arajo & Santos, 2020; Hu & Wang, 2002). Appearance of delay may be related with distributed nature of the process, or coming from distant location of actuators and sensors, as it is usually the case in haptic devices controlled through a communication network (Gil, Shez, Hulin, Preusche, & Hirzinger, 2009). Frequently, linear time-delay models of mechanical systems are used (Keller, 2018) due to a relative simplicity of their analysis, but in many situations the nonlinearity of forces has to be taken into account, e.g., in delayed Duffing equation (Rusinek, Weremczuk, Kecik, & Warminski, 2014), where delay appears to represent a regenerative force effect, or in magnetic bearing systems (Post, 2000).

In general, stability investigation of time-delay systems is a rather sophisticated issue (Fridman, 2014; Hale, 1977; Kolmanovskii & Myshkis, 1999), and for nonlinear systems there is no structured way to select a Lyapunov–Krasovskii functional for analysis. Exceptions exist for particular classes of systems, as it is demonstrated in a recent work for homogeneous time-delay dynamics in (Efimov & Aleksandrov, 2021).

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Alexander Aleksandrov is with Saint Petersburg State University, 7-9 Universitetskaya nab., 199034 Saint Petersburg, Russia and Institute for Problems of Mechanical Engineering, Russian Academy of Sciences, 199178 Saint Petersburg, Russia.

Denis Efimov is with Inria, Univ. Lille, CNRS, UMR 9189 - CRISTAL, F-59000 Lille, France and ITMO University, 49 av. Kronverkskiy, 197101 Saint Petersburg, Russia.

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For this class systems the independent on delay kind of stability is recurrent (Efimov, Polyakov, Perruquetti, & Richard, 2016).

Considering switching systems is another popular extension for modeling of complex mechanical processes, where commutation may come from dependence of the force shape on the coordinates, from time variation of parameters, from impacts or reconfiguration of multi-component plants. Utility of this modeling frameworks is originated by a well-developed theory of stability analysis for switched systems (Liberzon, 2003). Usually, a common Lyapunov–Krasovskii functional allows the stability conditions to be obtained under arbitrary switching, while the multiple Lyapunov–Krasovskii functional approach is coupled with (dwell-time) restrictions on the commutation sequence (Aleksandrov, Chen, Kosov, & Zhang, 2011; Aleksandrov, Kosov, & Chen, 2011; Aleksandrov, Kosov, & Platonov, 2012; Liberzon, 2003).

The topic of this work lies on the intersection of all above mentioned issues: delay-independent stability analysis for a class of (switched) mechanical systems with nonlinear homogeneous forces. The results are based on investigation of a complete Lyapunov–Krasovskii functional. The derived applicability conditions are mild and easily verifiable, they are only related with the form of the forces. Contrarily the conventional results obtained for linear models, where delay may destroy the system stability (Fridman, 2014; Keller, 2018), it is shown in the present work that for a strongly nonlinear systems the delay impact is only quantitative (i.e., appearance of a lag cannot lead to instability, but it may reduce the domain of convergence and attenuate the decay rates).

The outline of the paper is as follows. Preliminaries are surveyed in Section 2. The detailed problem statement and basic hypotheses are presented in Section 3. Sections 4 and 5 are devoted to the construction of Lyapunov–Krasovskii functional for the cases without and with commutation, respectively. Illustrative examples with the simulation results are shown in Section 6. Final conclusions are summarized in Section 7.

## 2. Preliminaries

The set of real numbers is denoted by  $\mathbb{R}$ ,  $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$ ,  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times n}$  are used to denote the space of vectors and matrices of dimensions  $n$  and  $n \times n$ , respectively. The Euclidean norm of a vector  $x \in \mathbb{R}^n$  is denoted by  $\|x\|$ , and  $|s|$  is the absolute value for a scalar  $s \in \mathbb{R}$ . Denote by  $C^1([-\tau, 0], \mathbb{R}^n)$  for  $\tau > 0$  the space of continuously differentiable functions  $\varphi(\xi) : [-\tau, 0] \rightarrow \mathbb{R}^n$  with the uniform norm

$$\|\varphi\|_\tau = \max_{\xi \in [-\tau, 0]} (\|\varphi(\xi)\| + \|\dot{\varphi}(\xi)\|).$$

The Young’s inequality claims that for any  $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+$  and  $\gamma > 0$ ,  $\delta > 0$  (Fridman, 2014):

$$\mathbf{a}^\gamma \mathbf{b}^\delta \leq \frac{1}{p} \mathbf{a}^{\gamma p} + \frac{p-1}{p} \mathbf{b}^{\frac{\delta p}{p-1}}$$

for any  $p > 1$ , while Hölder’s inequality for any  $f, g : I \rightarrow \mathbb{R}$  with  $I \subset \mathbb{R}$  ensures that

(Fridman, 2014):

$$\int_I |f(s)g(s)|ds \leq \left( \int_I |f(s)|^p ds \right)^{\frac{1}{p}} \left( \int_I |g(s)|^{\frac{p}{p-1}} ds \right)^{\frac{p-1}{p}}$$

for any  $p > 1$ .

Using the properties of homogeneous functions the following result can be obtained (Efimov & Aleksandrov, 2021):

**Lemma 1.** *Let  $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+$  and  $\ell > 0$ ,  $\alpha > 0$ ,  $\beta > 0$ ,  $\gamma > 0$ ,  $\delta > 0$  be given, then*

$$\mathbf{a}^\alpha + \mathbf{b}^\beta - \ell \mathbf{a}^\gamma \mathbf{b}^\delta \geq 0$$

*provided that*

- 1)  $\max\{\mathbf{a}^\alpha, \mathbf{b}^\beta\} \geq \ell^{\frac{1}{1-\frac{\gamma}{\alpha}-\frac{\delta}{\beta}}}$  and  $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} < 1$ ,
- 2)  $\max\{\mathbf{a}^\alpha, \mathbf{b}^\beta\} \leq \ell^{\frac{1}{1-\frac{\gamma}{\alpha}-\frac{\delta}{\beta}}}$  and  $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} > 1$ .

For definitions of various conventional stability concepts used in the paper the reader is referenced to (Fridman, 2014; Hale, 1977; Kolmanovskii & Myshkis, 1999).

### 3. Statement of the Problem

Consider a mechanical system whose dynamics are defined for  $t \geq t_0 \geq 0$  by the equation

$$A\ddot{x}(t) + B\dot{x}(t) + \frac{\partial \Pi_1(x(t))}{\partial x} + \frac{\partial \Pi_2(x(t-\tau))}{\partial x} + F(t, x) = 0, \quad (1)$$

where  $x(t), \dot{x}(t) \in \mathbb{R}^n$  are vectors of generalized coordinates and velocities, respectively;  $A, B \in \mathbb{R}^{n \times n}$  are constant positive definite matrices;  $\Pi_1, \Pi_2 : \mathbb{R}^n \rightarrow \mathbb{R}$  are continuously differentiable and homogeneous (with respect to the standard dilation) functions of the order  $\mu + 1$ ,  $\mu > 0$ , i.e.,

$$\Pi_i(\lambda x) = \lambda^{\mu+1} \Pi_i(x), \quad \forall x \in \mathbb{R}^n, \forall \lambda > 0, i = 1, 2;$$

the vector function  $F(t, x)$  is continuous for  $t \geq 0$  and  $x \in \mathbb{R}^n$ ;  $\tau$  is a constant positive delay. Assume that initial functions for solutions of (1) belong to the space  $C^1([-\tau, 0], \mathbb{R}^n)$ . Let  $x_t$  be the restriction of a solution  $x(t)$  of (1) to the segment  $[t-\tau, t]$ , i.e.,  $x_t : \xi \rightarrow x(t+\xi)$ ,  $\xi \in [-\tau, 0]$ .

We will impose the following constraints on the force fields in (1):

**Assumption 1.** *The function  $\Pi(x) = \Pi_1(x) + \Pi_2(x)$  is positive definite.*

Note that  $\Pi$  can be interpreted as a potential energy of (1).

**Assumption 2.** *The force  $-F(t, x)$  is non-conservative, i.e.,  $x^\top F(t, x) = 0$  for all  $t \geq 0$ ,  $x \in \mathbb{R}^n$ .*

Hence, the considered mechanical system is under the action of dissipative, potential and non-conservative forces, and the term  $\partial \Pi_2(x(t-\tau))/\partial x$  can be interpreted as a

control with a constant delay in the feedback loop. The positional forces in (1) have potential and non-conservative components, which corresponds to the general scenario (Zubov, 1970). Note also that Assumption 1 includes the cases when in the absence of the delayed part (without control) the system (1) is unstable.

**Remark 1.** Assumption 2 implies that  $F(t, 0) = 0$ , see (Rouche, Habets, & Laloy, 1977). Hence, the system (1) admits the steady-state solution

$$x(t) = \dot{x}(t) = 0, \quad t \geq 0. \quad (2)$$

We will look for conditions ensuring the delay-independent asymptotic stability of the equilibrium position (2) or delay-independent ultimate boundedness for solutions of (1).

It is worth noting that asymptotic stability conditions for the corresponding delay-free system ( $\tau = 0$ ) were obtained in (Aleksandrov, Chen, et al., 2011; Aleksandrov, Kosov, & Chen, 2011) with the aid of a special construction of a strong Lyapunov function. On the other hand, in (Aleksandrov, Zhabko, & Pecherskiy, 2015; Efimov & Aleksandrov, 2021), an approach to the construction of a complete-type Lyapunov–Krasovskii functional for some classes of nonlinear time-delay systems was developed. In the present contribution, we will apply the results of (Aleksandrov, Chen, et al., 2011; Aleksandrov, Kosov, & Chen, 2011; Aleksandrov et al., 2015; Efimov & Aleksandrov, 2021) to the analysis of dynamics of the system (1).

#### 4. Constructing a Complete-Type Lyapunov–Krasovskii Functional

We will consider the cases where the vector of non-conservative forces satisfies one of the following conditions:

**Assumption 3.** *There exist numbers  $\Delta > 0$ ,  $c > 0$  and  $\lambda > (\mu + 1)/2$  such that  $\|F(t, x)\| \leq c\|x\|^\lambda$  for all  $t \geq 0$ ,  $\|x\| < \Delta$ .*

**Assumption 4.** *There exist numbers  $\Delta > 0$ ,  $c > 0$ ,  $d > 0$  and  $\lambda < (\mu + 1)/2$  such that  $\|F(t, x)\| \leq c\|x\|^\lambda$  for all  $t \geq 0$ ,  $\|x\| > \Delta$  and  $\|F(t, x)\| \leq d$  for all  $t \geq 0$ ,  $\|x\| \leq \Delta$ .*

The difference between assumptions 3 and 4 consists in the domain, where the potential forces are related with the non-conservative one: close to the origin or far enough, respectively. Examples of such nonlinear forces satisfying assumptions 1–4 can be found in (Blekhman, 2007; Burton & Rahman, 1986; Gendelman & Lamarque, 2005; Rakaric & Kovacic, 2011).

**Theorem 1.** *Let assumptions 1 and 2 be fulfilled and consider (1) with any  $\tau > 0$ :*

- *if  $\mu > 1$  and Assumption 3 is valid, then the equilibrium position (2) is asymptotically stable;*
- *if  $\mu < 1$  and Assumption 4 is valid, then solutions of (1) are uniformly ultimately bounded.*

**Proof.** First, using results of (Aleksandrov, Chen, et al., 2011; Aleksandrov, Kosov, & Chen, 2011), choose a Lyapunov function candidate as follows:

$$V(x, \dot{x}) = \frac{\alpha}{2} \dot{x}^\top A \dot{x} + \frac{1}{2} x^\top B x + x^\top A \dot{x},$$

where  $\alpha$  is a positive parameter. There exist positive numbers  $\alpha_0, a_1, a_2, a_3, a_4$  such that

$$\alpha a_1 \|\dot{x}\|^2 + a_2 \|x\|^2 \leq V(x, \dot{x}) \leq \alpha a_3 \|\dot{x}\|^2 + a_4 \|x\|^2$$

for  $\alpha \geq \alpha_0$  and all  $x, \dot{x} \in \mathbb{R}^n$ .

Considering the derivative of  $V$  along the solutions of the system (1), we obtain:

$$\begin{aligned} \dot{V} &= -\alpha \dot{x}^\top(t) B \dot{x}(t) + \dot{x}^\top(t) A \dot{x}(t) - \alpha \dot{x}^\top(t) \left( \frac{\partial \Pi_1(x(t))}{\partial x} + \frac{\partial \Pi_2(x(t-\tau))}{\partial x} \right) \\ &\quad - \alpha \dot{x}^\top(t) F(t, x(t)) - x^\top(t) \frac{\partial \Pi(x(t))}{\partial x} + x^\top(t) \left( \frac{\partial \Pi_2(x(t))}{\partial x} - \frac{\partial \Pi_2(x(t-\tau))}{\partial x} \right) \\ &\leq -(\alpha a_5 - a_6) \|\dot{x}(t)\|^2 + \alpha a_7 \|\dot{x}(t)\| (\|x(t)\|^\mu + \|x(t-\tau)\|^\mu) \\ &\quad + \alpha \|\dot{x}(t)\| \|F(t, x(t))\| - a_8 \|x(t)\|^{\mu+1} + x^\top(t) \left( \frac{\partial \Pi_2(x(t))}{\partial x} - \frac{\partial \Pi_2(x(t-\tau))}{\partial x} \right), \end{aligned}$$

where  $a_5, a_6, a_7, a_8$  are positive constants, and the Euler theorem on homogeneous functions was used:  $x^\top \frac{\partial \Pi(x)}{\partial x} = (\mu + 1) \Pi(x)$ .

Next, according to the approach developed in (Efimov & Aleksandrov, 2021), construct a Lyapunov–Krasovskii functional by the formula

$$\begin{aligned} \tilde{V}(x_t) &= V(x(t), \dot{x}(t)) - x^\top(t) \int_{t-\tau}^t \frac{\partial \Pi_2(x(\xi))}{\partial x} d\xi \\ &\quad + \int_{t-\tau}^t (\beta + \gamma(\xi - t + \tau)) \|x(\xi)\|^{\mu+1} d\xi, \end{aligned} \quad (3)$$

where  $\beta$  and  $\gamma$  are positive parameters. If  $\alpha \geq \alpha_0$ , then

$$\begin{aligned} &\alpha a_1 \|\dot{x}(t)\|^2 + a_2 \|x(t)\|^2 + \beta \int_{t-\tau}^t \|x(\xi)\|^{\mu+1} d\xi - a_9 \|x(t)\| \int_{t-\tau}^t \|x(\xi)\|^\mu d\xi \leq \tilde{V}(x_t) \\ &\leq \alpha a_3 \|\dot{x}(t)\|^2 + a_4 \|x(t)\|^2 + (\beta + \gamma\tau) \int_{t-\tau}^t \|x(\xi)\|^{\mu+1} d\xi + a_9 \|x(t)\| \int_{t-\tau}^t \|x(\xi)\|^\mu d\xi, \end{aligned}$$

where  $a_9 > 0$ . Differentiating the functional (3) along the solutions of (1), we obtain

$$\begin{aligned} \dot{\tilde{V}} &\leq -(\alpha a_5 - a_6) \|\dot{x}(t)\|^2 + \alpha a_7 \|\dot{x}(t)\| (\|x(t)\|^\mu + \|x(t-\tau)\|^\mu) \\ &\quad + \alpha \|\dot{x}(t)\| \|F(t, x(t))\| - (a_8 - \beta - \gamma\tau) \|x(t)\|^{\mu+1} - \beta \|x(t-\tau)\|^{\mu+1} \end{aligned}$$

$$+a_9\|\dot{x}(t)\| \int_{t-\tau}^t \|x(\xi)\|^\mu d\xi - \gamma \int_{t-\tau}^t \|x(\xi)\|^{\mu+1} d\xi.$$

Let  $\alpha_0 \geq 2a_6/a_5$  and  $2(\beta + \gamma\tau) < a_8$ , with the aid of Hlder's inequality, we arrive at the estimates:

$$\begin{aligned} & \alpha a_1 \|\dot{x}\|^2 + a_2 \|x\|^2 + \beta \int_{t-\tau}^t \|x(\xi)\|^{\mu+1} d\xi \\ -a_{10} \|x(t)\| \left( \int_{t-\tau}^t \|x(\xi)\|^{\mu+1} d\xi \right)^{\frac{\mu}{\mu+1}} & \leq \tilde{V}(x_t) \leq \alpha a_3 \|\dot{x}\|^2 + a_4 \|x\|^2 \\ +(\beta + \gamma\tau) \int_{t-\tau}^t \|x(\xi)\|^{\mu+1} d\xi + a_{10} \|x(t)\| & \left( \int_{t-\tau}^t \|x(\xi)\|^{\mu+1} d\xi \right)^{\frac{\mu}{\mu+1}} \end{aligned}$$

and

$$\begin{aligned} \dot{\tilde{V}} & \leq -\frac{1}{2}\alpha a_5 \|\dot{x}(t)\|^2 - \frac{1}{2}a_8 \|x(t)\|^{\mu+1} + \alpha a_7 \|\dot{x}(t)\| (\|x(t)\|^\mu + \|x(t-\tau)\|^\mu) \\ & + \alpha \|\dot{x}(t)\| \|F(t, x(t))\| - \beta \|x(t-\tau)\|^{\mu+1} \\ & + a_{10} \|\dot{x}(t)\| \left( \int_{t-\tau}^t \|x(\xi)\|^{\mu+1} d\xi \right)^{\frac{\mu}{\mu+1}} - \gamma \int_{t-\tau}^t \|x(\xi)\|^{\mu+1} d\xi \end{aligned}$$

for  $\alpha \geq \alpha_0$ , where  $a_{10}$  is another positive constant.

Using Lemma 1 and Young's inequality and taking into account assumptions 3 and 4, it is easy to show the existence of positive numbers  $\delta_1, \delta_2$  such that

- if  $\mu > 1$ , then

$$\begin{aligned} \alpha a_1 \|\dot{x}(t)\|^2 + \frac{1}{2}a_2 \|x(t)\|^2 + \frac{1}{2}\beta \int_{t-\tau}^t \|x(\xi)\|^{\mu+1} d\xi & \leq \tilde{V}(x_t) \quad (4) \\ \leq \alpha a_3 \|\dot{x}(t)\|^2 + 2a_4 \|x(t)\|^2 + 2(\beta + \gamma\tau) \int_{t-\tau}^t \|x(\xi)\|^{\mu+1} d\xi, \end{aligned}$$

$$\dot{\tilde{V}} \leq -\frac{1}{4}\alpha a_5 \|\dot{x}(t)\|^2 - \frac{1}{4}a_8 \|x(t)\|^{\mu+1} - \frac{1}{2}\gamma \int_{t-\tau}^t \|x(\xi)\|^{\mu+1} d\xi \quad (5)$$

for  $\|\dot{x}(t)\|^2 + \|x(t)\|^2 + \int_{t-\tau}^t \|x(\xi)\|^{\mu+1} d\xi < \delta_1$ ;

- if  $\mu < 1$ , then the inequalities (4), (5) hold for  $\|\dot{x}(t)\|^2 + \|x(t)\|^2 + \int_{t-\tau}^t \|x(\xi)\|^{\mu+1} d\xi > \delta_2$ .

Hence (see (Efimov & Aleksandrov, 2021)), in such a case

$$\dot{\tilde{V}} \leq -h_1 \tilde{V}^{\frac{\mu+1}{2}} \quad (6)$$

for  $\mu > 1$  in the domain where  $\tilde{V} < \delta_3$ , and

$$\dot{\tilde{V}} \leq -h_2 \tilde{V}^{\frac{\mu+1}{2}} \quad (7)$$

for  $\mu < 1$  in the domain where  $\tilde{V} > \delta_4$  for some positive numbers  $h_1, h_2, \delta_3, \delta_4$ . This completes the proof.  $\square$

**Remark 2.** There are dissipative and potential stabilizing forces in (1). Theorem 1 guarantees the asymptotic stability of the equilibrium position (2) or ultimate boundedness of solutions for any positive delay and any non-conservative force satisfying Assumption 3 or Assumption 4, respectively. It is important to highlight that, contrarily the existing results on the stability and ultimate boundedness by homogeneous approximations (see (Zubov, 1962)), if  $\mu > 1$  ( $\mu < 1$ ), then the order of non-conservative forces may be less (greater) than that of the potential forces. Thus, Theorem 1 complements the analysis based on the first-order approximation.

With the aid of the differential inequalities (6) and (7), estimates on the convergence rates for solutions of the system (1) can be derived.

**Corollary 1.** *Let assumptions 1 and 2 be fulfilled,*

• *if  $\mu > 1$ , Assumption 3 is valid and for a solution  $x(t)$  of (1) the inequalities  $t_0 \geq 0$ ,  $\tilde{V}(x_{t_0}) < \delta_3$  hold, then*

$$\begin{aligned} & \alpha a_1 \|\dot{x}(t)\|^2 + \frac{1}{2} a_2 \|x(t)\|^2 \\ & \leq \tilde{V}(x_{t_0}) \left(1 + \frac{1}{2} h_1 (\mu - 1) \tilde{V}^{\frac{\mu-1}{2}}(x_{t_0}) (t - t_0)\right)^{-\frac{2}{\mu-1}} \end{aligned}$$

for  $t \geq t_0$ ;

• *if  $\mu < 1$ , Assumption 4 is valid and for a solution  $x(t)$  of (1) the inequalities  $t_0 \geq 0$ ,  $\tilde{V}(x_{t_0}) > \delta_4$  hold, then*

$$\alpha a_1 \|\dot{x}(t)\|^2 + \frac{1}{2} a_2 \|x(t)\|^2 \leq \delta_4$$

for  $t \geq t_0 + T$ , where

$$T = \frac{2}{h_2(1-\mu)} \left( \tilde{V}^{\frac{1-\mu}{2}}(x_{t_0}) - \delta_4^{\frac{1-\mu}{2}} \right).$$

**Proof.** The claim can be obtained as a direct consequence of (6) and (7) by calculating their solutions analytically.  $\square$

## 5. Mechanical Systems with Switched Force Fields

In the present section, we will apply the proposed approach to the analysis of dynamics of mechanical systems with switched forces, where commutation may represent instantaneous modification of the force shape or parameter values.

Consider the system

$$A\ddot{x}(t) + B\dot{x}(t) + \frac{\partial \Pi_1^{(\sigma)}(x(t))}{\partial x} + \frac{\partial \Pi_2(x(t-\tau))}{\partial x} + F^{(\sigma)}(t, x) = 0, \quad (8)$$

where  $\sigma = \sigma(t) : [0, +\infty) \mapsto \{1, \dots, N\}$  is a piecewise constant right-continuous function defining switching signal, integer  $N > 0$  determines the number of subsystems in (8),  $\Pi_1^{(s)} : \mathbb{R}^n \rightarrow \mathbb{R}$  are continuously differentiable and homogeneous (with respect to the standard dilation) functions of the order  $\mu + 1$ , vector functions  $F^{(s)}(t, x)$  are continuous for  $t \geq 0$ ,  $x \in \mathbb{R}^n$ ,  $s = 1, \dots, N$ ; and the remaining notation is the same as for (1).

Let  $\sigma(t)$  be a non-Zeno switching law. This means (see (Liberzon, 2003)) that on every bounded time interval the set of discontinuity instants for  $\sigma(t)$  is finite. Such a type of switching signals will be called admissible ones. Denote the respective switching instants by  $\theta_j$ ,  $j = 1, 2, \dots$ , which satisfy the property  $0 < \theta_1 < \theta_2 < \dots$  under the absence of Zeno behavior. For brevity, the characteristics of the orchestrating signal  $\sigma(t)$  are formulated as for a function of time, but a similar consideration can be done if the switching is governed by a function of state.

**Assumption 5.** *Functions  $\Pi^{(s)}(x) = \Pi_1^{(s)}(x) + \Pi_2(x)$  are positive definite,  $s = 1, \dots, N$ .*

**Assumption 6.** *Forces  $-F^{(s)}(t, x)$  are non-conservative, i.e.,  $x^\top F^{(s)}(t, x) = 0$  for all  $t \geq 0$ ,  $x \in \mathbb{R}^n$ ,  $s = 1, \dots, N$ .*

Thus, in (8), non-conservative forces and the delay-free part of potential forces are switched, whereas the delayed and dissipative components of forces are non-switched. In addition, consider the following assumptions:

**Assumption 7.** *There exist numbers  $\Delta > 0$ ,  $c > 0$  and  $\lambda_s > (\mu + 1)/2$  such that  $\|F^{(s)}(t, x)\| \leq c\|x\|^{\lambda_s}$  for all  $t \geq 0$ ,  $\|x\| < \Delta$  and  $s = 1, \dots, N$ .*

**Assumption 8.** *There exist numbers  $\Delta > 0$ ,  $c > 0$ ,  $d > 0$  and  $\lambda_s < (\mu + 1)/2$  such that  $\|F^{(s)}(t, x)\| \leq c\|x\|^{\lambda_s}$  for all  $t \geq 0$ ,  $\|x\| > \Delta$  and  $\|F^{(s)}(t, x)\| \leq d$  for all  $t \geq 0$ ,  $\|x\| \leq \Delta$ ,  $s = 1, \dots, N$ .*

Assumptions 5–8 are direct projections to the case of (8) of assumptions 1–4, respectively.

**Theorem 2.** *Let assumptions 5 and 6 be fulfilled and consider (8) with any  $\tau > 0$  and any admissible switching law  $\sigma$ :*

- *if  $\mu > 1$  and Assumption 7 is valid, then the equilibrium position (2) is asymptotically stable;*
- *if  $\mu < 1$  and Assumption 8 is valid, then solutions of (8) are uniformly ultimately bounded.*

**Proof.** Similarly to the proof of Theorem 1, it is easy to verify that the functional (3) is a common Lyapunov–Krasovskii functional for the family of subsystems associated with the switched system (8). By standard argumentation (Liberzon, 2003), existence of such a functional guarantees the asymptotic stability of the equilibrium position (2) for  $\mu > 1$  and uniform ultimate boundedness of solutions for  $\mu < 1$  for any admissible switching signal  $\sigma$ .  $\square$

Next, consider the case where all the forces in the considered mechanical system are switched:

$$A\ddot{x}(t) + B^{(\sigma)}\dot{x}(t) + \frac{\partial \Pi_1^{(\sigma)}(x(t))}{\partial x} + \frac{\partial \Pi_2^{(\sigma)}(x(t-\tau))}{\partial x} + F^{(\sigma)}(t, x) = 0, \quad (9)$$

where  $B^{(s)}$  is a constant positive definite matrix of dissipative forces;  $\Pi_2^{(s)} : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuously differentiable and homogeneous (with respect to the standard dilation) function of the order  $\mu + 1$ ,  $s = 1, \dots, N$ ; and the remaining notation is the same as for (1) and (8).

**Assumption 9.** *Functions  $\Pi^{(s)}(x) = \Pi_1^{(s)}(x) + \Pi_2^{(s)}(x)$  are positive definite,  $s = 1, \dots, N$ .*

**Assumption 10.** *The switching instants  $\theta_1, \theta_2, \dots$  are given, but we have no information on the order of commutation of the subsystems in (9).*

In this case, to derive delay-independent conditions of asymptotic stability and ultimate boundedness, we will use multiple Lyapunov–Krasovskii functionals, whose application needs additional constraints to be imposed on switching signal (frequently, dwell-time restrictions are used (Liberzon, 2003), which however cannot be applied in highly nonlinear cases as here (Efimov, Panteley, & Loria, 2008), and have to be revisited).

Construct a family of Lyapunov–Krasovskii functionals

$$\begin{aligned} \tilde{V}_s(x_t) &= \frac{1}{2}\alpha_s \dot{x}^\top(t) A \dot{x}(t) + \frac{1}{2}x^\top(t) B^{(s)} x(t) \\ &\quad + x^\top(t) A \dot{x}(t) - x^\top(t) \int_{t-\tau}^t \frac{\partial \Pi_2^{(s)}(x(\xi))}{\partial x} d\xi \\ &\quad + \int_{t-\tau}^t (\beta_s + \gamma_s(\xi - t + \tau)) \|x(\xi)\|^{\mu+1} d\xi, \quad s = 1, \dots, N, \end{aligned}$$

where  $\alpha_s, \beta_s, \gamma_s$  are positive parameters. Denote by  $\dot{\tilde{V}}_s(x_t)$  the derivative of  $\tilde{V}_s(x_t)$  along the solutions of the  $s$ th subsystem from the family associated with (9),  $s = 1, \dots, N$ , and introduce a norm-like variable

$$\omega(x_t) = \|\dot{x}(t)\|^2 + \|x(t)\|^2 + \int_{t-\tau}^t \|x(\xi)\|^{\mu+1} d\xi.$$

With the aid of the same arguments as in the proof of Theorem 1, it can be shown that, for an appropriate choice of  $\alpha_s, \beta_s, \gamma_s$  (see the proof of Theorem 1 for the method of their selection), there exist positive numbers  $\tilde{\delta}_1, \tilde{\delta}_2, b_l, d_l, l = 1, \dots, 4$ , such that

- if  $\mu > 1$ , then

$$\begin{aligned} b_1 \omega(x_t) &\leq \tilde{V}_s(x_t) \leq b_2 \omega(x_t), \\ \dot{\tilde{V}}_s(x_t) &\leq -b_3 \tilde{V}_s^{\frac{\mu+1}{2}}(x_t), \quad \tilde{V}_s(x_t) \leq b_4 \tilde{V}_r(x_t) \end{aligned} \quad (10)$$

for  $\omega(x_t) < \tilde{\delta}_1$ ,  $s, r = 1, \dots, N$ ;

- if  $\mu < 1$ , then

$$\begin{aligned} d_1\omega(x_t) &\leq \tilde{V}_s(x_t) \leq d_2\omega(x_t), \\ \dot{\tilde{V}}_s(x_t) &\leq -d_3\tilde{V}_s^{\frac{\mu+1}{2}}(x_t), \quad \tilde{V}_s(x_t) \leq d_4\tilde{V}_r(x_t) \end{aligned} \quad (11)$$

for  $\omega(x_t) > \tilde{\delta}_2$ ,  $s, r = 1, \dots, N$ .

**Theorem 3.** *Let assumptions 6, 9, 10 be fulfilled and consider (9) with any  $\tau > 0$  and an admissible  $\sigma$ :*

- if  $\mu > 1$ , Assumption 7 is valid and

$$\sum_{j=1}^{k-1} b_4^{\frac{(\mu-1)(j-k)}{2}} (\theta_{j+1} - \theta_j) \rightarrow \infty \quad \text{as } k \rightarrow \infty, \quad (12)$$

then the equilibrium position (2) of (9) is asymptotically stable;

- if  $\mu < 1$ , Assumption 8 is valid and

$$\sum_{j=1}^{k-1} d_4^{\frac{(\mu-1)j}{2}} (\theta_{m+j+1} - \theta_{m+j}) \rightarrow \infty \quad \text{as } k \rightarrow \infty \quad (13)$$

uniformly with respect to  $m \in \{0, 1, \dots\}$ , then solutions of (9) are uniformly ultimately bounded.

**Proof.** Let  $\mu > 1$ . Consider the system (9) with a switching law satisfying the condition (12). Define a piecewise continuously differentiable Lyapunov–Krasovskii functional by the formula  $\tilde{V}(t, x_t) = \tilde{V}_{\sigma(t)}(x_t)$ . Let  $x(t, \varphi, t_0)$  be a solution of (9) such that  $x(t_0 + \xi, \varphi, t_0) = \varphi(\xi)$  for  $\xi \in [-\tau, 0]$ ,  $\omega(x_t(\varphi, t_0)) < \tilde{\delta}_1$  for  $t \in [t_0, \bar{t}]$ , where  $t_0 \geq 0$  with some  $\bar{t} > t_0$ . Find  $m, k \in \{0, 1, 2, \dots\}$  satisfying the conditions  $t_0 \in [\theta_m, \theta_{m+1})$ ,  $\bar{t} \in [\theta_{k+m}, \theta_{k+m+1})$ . Using (10) we arrive at the following estimates:

$$b_1^{-\frac{\mu-1}{2}} \omega^{-\frac{\mu-1}{2}}(x_{\bar{t}}(\varphi, t_0)) \geq \tilde{v}^{-\frac{\mu-1}{2}}(\bar{t}) \geq \tilde{v}^{-\frac{\mu-1}{2}}(t_0) + \frac{\mu-1}{2} b_3(\bar{t} - t_0)$$

$$\geq b_2^{-\frac{\mu-1}{2}} \omega^{-\frac{\mu-1}{2}}(\varphi) + \frac{\mu-1}{2} b_3(\bar{t} - t_0) \quad \text{for } k = 0;$$

$$b_1^{-\frac{\mu-1}{2}} \omega^{-\frac{\mu-1}{2}}(x_{\bar{t}}(\varphi, t_0)) \geq \tilde{v}^{-\frac{\mu-1}{2}}(\bar{t}) \geq b_4^{-\frac{(\mu-1)k}{2}} \tilde{v}^{-\frac{\mu-1}{2}}(t_0)$$

$$+ \frac{\mu-1}{2} b_3 \left( b_4^{-\frac{(\mu-1)k}{2}} (\theta_{m+1} - t_0) + b_4^{-\frac{(\mu-1)(k-1)}{2}} (\theta_{m+2} - \theta_{m+1}) + \dots \right.$$

$$\left. + b_4^{-\frac{(\mu-1)}{2}} (\theta_{m+k} - \theta_{m+k-1}) + \bar{t} - \theta_{m+k} \right)$$

$$\begin{aligned} &\geq b_4^{-\frac{(\mu-1)k}{2}} b_2^{-\frac{\mu-1}{2}} \omega^{-\frac{\mu-1}{2}}(\varphi) + \frac{\mu-1}{2} b_3 \left( b_4^{-\frac{(\mu-1)(k-1)}{2}} (\theta_{m+2} - \theta_{m+1}) + \dots \right. \\ &\quad \left. + b_4^{-\frac{(\mu-1)}{2}} (\theta_{m+k} - \theta_{m+k-1}) \right) \text{ for } k > 0, \end{aligned}$$

where  $\tilde{v}(t) = \tilde{V}(t, x_t(\varphi, t_0))$ . Similarly to the proof of Theorem 2 in (Aleksandrov et al., 2012), it can be shown that for (12) these inequalities imply delay-independent asymptotic stability of the steady-state solution (2).

Next, consider the system (9) with  $\mu < 1$  and a switching signal satisfying the condition (13). Let  $x(t, \varphi, t_0)$  be a solution of (9) such that  $x(t_0 + \xi, \varphi, t_0) = \varphi(\xi)$  for  $\xi \in [-\tau, 0]$ ,  $\omega(x_t(\varphi, t_0)) > \delta_2$  for  $t \in [t_0, \bar{t}]$ , where  $0 \leq t_0 < \bar{t}$  and  $t_0 \in [\theta_m, \theta_{m+1})$ ,  $\bar{t} \in [\theta_{k+m}, \theta_{k+m+1})$  for some  $m, k \in \{0, 1, 2, \dots\}$ . Denoting  $\tilde{v}(t) = \tilde{V}(t, x_t(\varphi, t_0))$  and applying the inequalities (11), we obtain the following estimates:

$$\begin{aligned} d_1^{\frac{1-\mu}{2}} \omega^{\frac{1-\mu}{2}}(x_{\bar{t}}(\varphi, t_0)) &\leq \tilde{v}^{\frac{1-\mu}{2}}(\bar{t}) \leq \tilde{v}^{\frac{1-\mu}{2}}(t_0) - \frac{1-\mu}{2} d_3(\bar{t} - t_0) \\ &\leq d_2^{\frac{1-\mu}{2}} \omega^{\frac{1-\mu}{2}}(\varphi) - \frac{1-\mu}{2} d_3(\bar{t} - t_0) \text{ for } k = 0; \\ d_1^{\frac{1-\mu}{2}} \omega^{\frac{1-\mu}{2}}(x_{\bar{t}}(\varphi, t_0)) &\leq \tilde{v}^{\frac{1-\mu}{2}}(\bar{t}) \leq d_4^{\frac{(1-\mu)k}{2}} \tilde{v}^{\frac{1-\mu}{2}}(t_0) \\ &- \frac{1-\mu}{2} d_3 \left( d_4^{\frac{(1-\mu)k}{2}} (\theta_{m+1} - t_0) + d_4^{\frac{(1-\mu)(k-1)}{2}} (\theta_{m+2} - \theta_{m+1}) + \dots \right. \\ &\quad \left. + d_4^{\frac{(1-\mu)}{2}} (\theta_{m+k} - \theta_{m+k-1}) + \bar{t} - \theta_{m+k} \right) \\ &\leq d_4^{\frac{(1-\mu)k}{2}} \left( d_2^{\frac{1-\mu}{2}} \omega^{\frac{1-\mu}{2}}(\varphi) - \frac{1-\mu}{2} d_3 \sum_{j=1}^{k-1} d_4^{\frac{(\mu-1)j}{2}} (\theta_{m+j+1} - \theta_{m+j}) \right) \\ &\text{for } k > 0. \end{aligned}$$

Using these estimates and taking into account the constraint (13) imposed on the switching signal, it can be verified that, for any  $\Delta > 0$ , there exists  $T > 0$  such that if  $t_0 \geq 0$ ,  $\|\varphi\|_\tau < \Delta$ , then  $\omega(x_{\hat{t}}(\varphi, t_0)) \leq \tilde{\delta}_2$  for some  $\hat{t} \in [t_0, t_0 + T]$ . In addition, there exists  $\hat{k} \geq 1$  such that

$$d_2^{\frac{1-\mu}{2}} \tilde{\delta}_2^{\frac{1-\mu}{2}} - \frac{1-\mu}{2} d_3 \sum_{j=1}^{k-1} d_4^{\frac{(\mu-1)j}{2}} (\theta_{m+j+1} - \theta_{m+j}) \leq 0$$

for  $k \geq \hat{k}$  and all  $m \in \{0, 1, \dots\}$ . Therefore, if  $\omega(x_{\hat{t}}(\varphi, t_0)) < \tilde{\delta}_2$  for some  $\hat{t} \geq t_0$ , then  $\omega(x_t(\varphi, t_0)) \leq d_2 d_4^k \tilde{\delta}_2 / d_1$  for  $t \geq \hat{t}$ . This implies the delay-independent uniform ultimately boundedness of solutions of the system (9) with the following estimate of the ultimate boundedness domain:

$$\|x\|^2 + \|\dot{x}\|^2 \leq \frac{d_2}{d_1} d_4^k \tilde{\delta}_2,$$

which is necessary to prove.  $\square$

Note that the conditions (12) and (13) do not necessarily imply a dwell-time property (Liberzon, 2003) of  $\sigma(t)$ , but they can be used to determine how long the current subsystem may become active.

## 6. Examples

Consider a point of mass with a linear friction force proportional to velocity, a potential force shifting the point from the origin, a non-conservative rotating force, and a stabilizing potential delayed force, then the system (1) can be presented in the form

$$\begin{aligned} \ddot{x}(t) + b\dot{x}(t) - \|x(t)\|^2 x(t) + \rho \begin{pmatrix} x_1^3(t - \tau) \\ x_2^3(t - \tau) \\ x_3^3(t - \tau) \end{pmatrix} \\ + \|x(t)\|^{\lambda-1} \psi(t) \times x(t) = 0, \end{aligned} \quad (14)$$

where we assume that  $x(t), \dot{x}(t) \in \mathbb{R}^3$  (coordinates of the point and respective velocities),  $b$  and  $\rho$  are positive parameters,  $\psi : [0, +\infty) \rightarrow \mathbb{R}^3$  is a bounded continuous function of time,  $\lambda \geq 1$ ,  $\tau$  is a constant positive delay, and  $\times$  denotes the vector product. Hence,  $\Pi_1(x) = -\|x\|^4/4$ ,  $\Pi_2(x) = \rho(x_1^4 + x_2^4 + x_3^4)/4$  and assumptions 1 and 2 are satisfied for  $\rho > 3$ . For  $F(t, x) = \|x(t)\|^{\lambda-1} \psi(t) \times x(t)$  Assumption 3 is also verified with  $c = \sup_{t \geq 0} \|\psi(t)\|$ . Following the result of Theorem 1, we obtain that if  $\lambda > 2$ , then the equilibrium position (2) of (14) is asymptotically stable for any  $\tau > 0$ . Note that for  $\rho = 0$  or when there is no delayed part, the system (14) is unstable. For simulations we take:

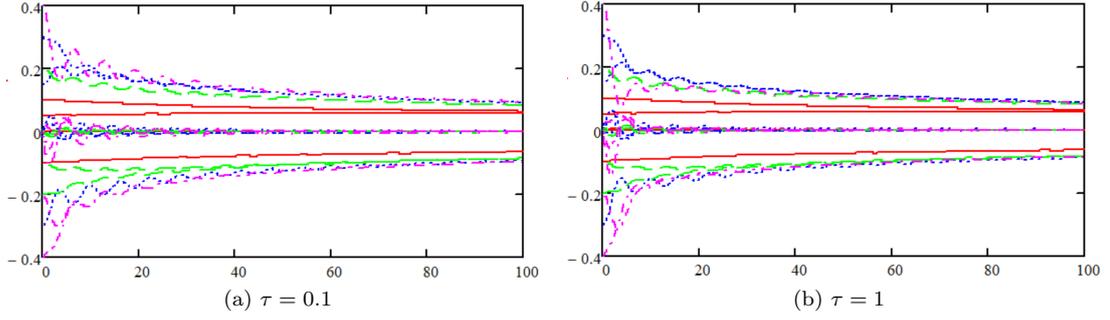
$$b = 2, \lambda = 3, \rho = 4, \psi(t) = [1 \ 0 \ -1]^\top \sin(t),$$

then the trajectories of  $x(t)$  and  $\dot{x}(t)$  for four different initial conditions are presented in Fig. 1 for two values of the delay ( $\tau = 0.1$  and  $\tau = 1$ ), demonstrating an asymptotic convergence to zero.

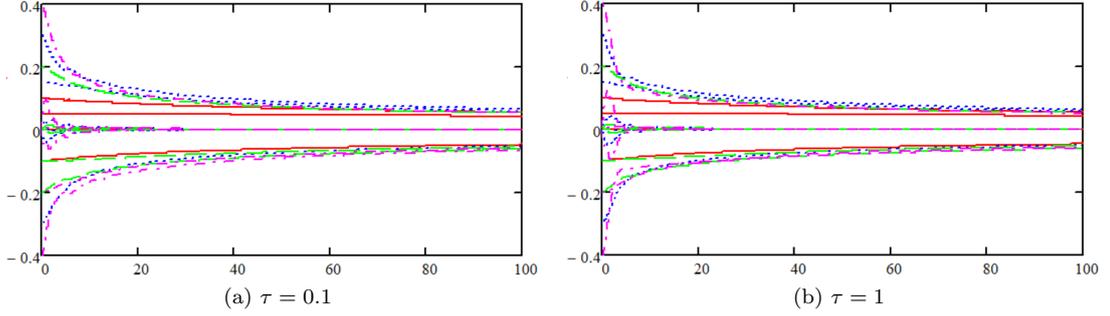
Next, consider the system (8) derived from (14):

$$\begin{aligned} \ddot{x}(t) + b\dot{x}(t) - \frac{\|x(t)\|^2 x(t)}{\sigma(t)} + \rho \begin{pmatrix} x_1^3(t - \tau) \\ x_2^3(t - \tau) \\ x_3^3(t - \tau) \end{pmatrix} \\ + \|x(t)\|^{\lambda_{\sigma(t)}-1} \psi_{\sigma(t)}(t) \times x(t) = 0, \end{aligned} \quad (15)$$

where  $\sigma : [0, +\infty) \rightarrow \{1, 2, 3, 4\}$  is an admissible switching signal, and the used notation is similar to (14) except that now  $\Pi_1^{(s)}(x) = -\frac{\|x\|^4}{4s}$  and  $F^{(s)}(t, x) = \|x(t)\|^{\lambda_s-1} \psi_s(t) \times$



**Figure 1.** The results of simulation for (14)



**Figure 2.** The results of simulation for (15)

$x(t)$  for  $\lambda_s > 2$  and a bounded continuous function  $\psi_s : [0, +\infty) \rightarrow \mathbb{R}^3$ ,  $s = 1, \dots, 4$ . It is straightforward to check that assumptions 5–7 are satisfied, then by Theorem 2 the equilibrium position (2) of (15) is asymptotically stable for any  $\tau > 0$ . For simulations we setup:

$$b = 2, \lambda_s = 2 + s, \rho = 4, \psi_s(t) = [-1 \ 1 \ 1]^\top \sin(st),$$

$$\sigma(t) = 1 + \text{ceil}(1 + \sin(t) + \cos(\pi t)),$$

then the trajectories of  $x(t)$  and  $\dot{x}(t)$  for four different initial conditions are presented in Fig. 2 for two values of the delay ( $\tau = 0.1$  and  $\tau = 1$ ).

The given results of numeric experiments confirm the theoretical findings of this work.

## 7. Conclusion

For a class of mechanical systems having dissipative, non-conservative and potential forces, the problems of delay-independent asymptotic stability of the origin and ultimate boundedness of the trajectories were investigated. It was assumed that the potential forces are formed by gradients of homogeneous functions, and delay is constant. The given stability conditions are direct for verification and do not require a sophisticated analysis, they are also weaker than existing in the literature (Zubov, 1962). To substantiate these results, a complete Lyapunov–Krasovskii functional was utilized. Further, these stability conditions were extended to a switched model of mechanical

system, and the cases of arbitrary and restricted commutation laws were explored by analyzing common and multiple Lyapunov–Krasovskii functionals, respectively. The efficiency of proposed stability conditions were demonstrated in numeric experiments.

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